THE VLASOV-POISSON-LANDAU SYSTEM IN A PERIODIC BOX

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1. INTRODUCTION

In the absence of magnetic effects, the dynamics of charged dilute particles (e.g., electrons and ions) is described by the Vlasov–Poisson–Landau system:

$$\begin{aligned}
 (1) \quad & \partial_t F_+ + v \cdot \nabla_x F_+ + \frac{e_+}{m_+} E \cdot \nabla_v F_+ = Q(F_+, F_+) + Q(F_-, F_+), \\
 & \partial_t F_- + v \cdot \nabla_x F_- - \frac{e_-}{m_-} E \cdot \nabla_v F_- = Q(F_+, F_-) + Q(F_-, F_-), \\
 & F_{\pm}(0, x, v) = F_{0,\pm}(x, v).
 \end{aligned}$$

Here $F_{\pm}(t, x, v) \geq 0$ are the spatially periodic number density functions for the ions (+) and electrons (−) respectively, at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in [-\pi, \pi]^3 = \mathbf{T}^3$, velocity $v = (v_1, v_2, v_3) \in \mathbf{R}^3$, and $e_{\pm}$, $m_{\pm}$ the magnitude of their charges and masses, c the speed of light. The collision between charged particles is given by

$$(2) \quad Q(G_1, G_2)(v) = \frac{c_{12}}{m_1} \nabla_v \cdot \int_{\mathbf{R}^3} \Phi(v - v') \left\{ \frac{G_1(v') \nabla_v G_2(v)}{m_1} - \frac{G_2(v) \nabla_{v'} G_1(v')}{m_2} \right\} dv',$$

where Φ is the famous Landau (Fokker–Planck) kernel [G1]:

$$(3) \quad \Phi(u) = \frac{1}{|u|} \left(I - \frac{u \otimes u}{|u|^2} \right).$$

The constant $c_{12} = 2\pi e_1^2 e_2^2 \ln \Lambda$, $\ln \Lambda = \ln(\frac{\lambda_D}{b_0})$, where $\lambda_D = (\frac{T}{4\pi n_e e^2})^{1/2}$ is the Debye shielding distance and $b_0 = \frac{e^2}{3T}$ a typical ‘distance of closest approach’ for a thermal particle [H]. The self-consistent electrostatic field $E(t, x) = -\nabla \phi$, and the electric potential ϕ satisfies the Poisson equation:

$$(4) \quad -\Delta \phi = 4\pi \rho = 4\pi \int_{\mathbf{R}^3} \{e_+ F_+ - e_- F_-\} dv, \quad \int_{\mathbf{T}^3} \phi(t, x) dx = 0.$$

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It is well known that for classical solutions to the Vlasov-Poisson-Landau system, the following conservation laws of mass, total momentum, and total energy hold:

$$\begin{aligned}\frac{d}{dt} \int_{\mathbf{T}^3 \times \mathbf{R}^3} m_{\pm} F_{\pm}(t) &= 0, \\ \frac{d}{dt} \left\{ \int_{\mathbf{T}^3 \times \mathbf{R}^3} v(m_+ F_+(t) + m_- F_-(t)) \right\} &= 0, \\ \frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbf{T}^3 \times \mathbf{R}^3} |v|^2 (m_+ F_+(t) + m_- F_-(t)) + \frac{1}{8\pi} \int_{\mathbf{T}^3} |E(t)|^2 \right\} &= 0.\end{aligned}$$

Moreover, we also have the following celebrated H-Theorem of Boltzmann:

$$(5) \quad \frac{d}{dt} \left\{ \int_{\mathbf{T}^3 \times \mathbf{R}^3} (F_+(t) \ln F_+(t) + F_-(t) \ln F_-(t)) \right\} \leq 0.$$

It is our purpose in this article to construct unique global solutions for the Vlasov-Poisson-Landau system (1) and (4) near global Maxwellians:

$$\mu_+(v) = \frac{n_0}{e_+} \left(\frac{m_+}{2\pi\kappa T_0} \right)^{3/2} e^{-m_+|v|^2/(2\kappa T_0)}, \quad \mu_-(v) = \frac{n_0}{e_-} \left(\frac{m_-}{2\pi\kappa T_0} \right)^{3/2} e^{-m_-|v|^2/(2\kappa T_0)}.$$

For notational simplicity, we normalize all constants in the Vlasov-(Poisson)-Landau system to be one. Accordingly, we normalized the Maxwellian as

$$(6) \quad \mu(v) \equiv \mu_+(v) = \mu_-(v) = e^{-|v|^2}.$$

We define the standard perturbation $f_{\pm}(t, x, v)$ to μ as

$$(7) \quad F_{\pm} = \mu + \sqrt{\mu} f_{\pm}.$$

Letting $f(t, x, v) = \begin{pmatrix} f_+ \\ f_- \end{pmatrix}(t, x, v)$, the Vlasov-Poisson-Landau system for the perturbation now takes the form

$$(8) \quad \{\partial_t + v \cdot \nabla_x \pm E \cdot \nabla_v\} f_{\pm} \mp 2\{E \cdot v\} \sqrt{\mu} + L_{\pm} f = \pm \{E \cdot v\} f_{\pm} + \Gamma_{\pm}(f, f),$$

$$(9) \quad -\Delta \phi = \int \sqrt{\mu} [f_+ - f_-] dv$$

with $\int_{\mathbf{T}^3} \phi dx = 0$. For any $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, the linearized collision operator Lg in (8) is given by the vector

$$(10) \quad Lg \equiv \begin{pmatrix} L_+ g \\ L_- g \end{pmatrix} \equiv -\frac{1}{\sqrt{\mu}} \begin{pmatrix} 2Q(\mu, \sqrt{\mu} g_1) + Q(\sqrt{\mu} \{g_1 + g_2\}, \mu) \\ 2Q(\mu, \sqrt{\mu} g_2) + Q(\sqrt{\mu} \{g_1 + g_2\}, \mu) \end{pmatrix}.$$

For $g = [g_1, g_2]$ and $h = [h_1, h_2]$, the nonlinear collision operator $\Gamma(g, h)$ is given by the vector

$$(11) \quad \Gamma(g, h) \equiv \begin{pmatrix} \Gamma_+(g, h) \\ \Gamma_-(g, h) \end{pmatrix} \equiv \frac{1}{\sqrt{\mu}} \begin{pmatrix} Q(\sqrt{\mu} g_1, \sqrt{\mu} h_1) + Q(\sqrt{\mu} g_2, \sqrt{\mu} h_1) \\ Q(\sqrt{\mu} g_1, \sqrt{\mu} h_2) + Q(\sqrt{\mu} g_2, \sqrt{\mu} h_2) \end{pmatrix}.$$

By assuming that initially $F_{0\pm}$ has the same mass, total momentum and total energy as the steady state μ , we can then rewrite the conservation laws in terms of

the perturbation f as

$$(12) \quad \int_{\mathbf{T}^3 \times \mathbf{R}^3} f_+(t) \sqrt{\mu} \equiv \int_{\mathbf{T}^3 \times \mathbf{R}^3} f_-(t) \sqrt{\mu} \equiv 0,$$

$$(13) \quad \int_{\mathbf{T}^3 \times \mathbf{R}^3} v \{f_+(t) + f_-(t)\} \sqrt{\mu} \equiv 0,$$

$$(14) \quad \int_{\mathbf{T}^3 \times \mathbf{R}^3} |v|^2 \{f_+(t) + f_-(t)\} \sqrt{\mu} \equiv - \int_{\mathbf{T}^3} |E(t)|^2.$$

In an attempt [G1]–[G2] to construct a global smooth solution near Maxwellian for the Vlasov-Poisson-Landau system, the author developed a nonlinear energy method for a general dissipative problem:

$$(15) \quad \partial_t g + \mathcal{L}g = \mathcal{N}(g).$$

We denote $\|\cdot\|_2$ to be the L^2 norm. Upon taking the L^2 inner product with g , we obtain

$$(16) \quad \frac{1}{2} \frac{d}{dt} \|g\|_2^2 + (\mathcal{L}g, g) = (\mathcal{N}(g), g).$$

Global solutions with small L^2 norm can be constructed if one can identify a dissipation rate $|||\cdot|||$ such that the following estimates can be established:

$$(17) \quad (\mathcal{L}g, g) \gtrsim |||g|||^2,$$

$$(18) \quad (\mathcal{N}(g), g) \lesssim \|g\|_2 \cdot |||g|||^2.$$

For $\|g\|_2 \ll 1$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|g\|_2^2 + \delta |||g|||^2 \leq 0,$$

with some $\delta > 0$. This implies a global uniform bound for $\|g\|_2$ and hence stability. Several remarks are needed: (1) Usually higher-order Sobolev norms are needed to close the argument. (2) Such a dissipation rate $|||\cdot|||$ satisfying both (17) and (18), if it exists, usually is unique up to a constant. The proof of (17) and (18) can be very challenging [G1], [GrS1]–[GrS2]. (3) Such an approach is designed to treat the case when $|||\cdot|||$ is weaker, or it cannot be compared with $\|\cdot\|_2$ so that a standard spectra analysis and semigroup approach is difficult to apply. (4) In many interesting applications, such an $(\mathcal{L}g, g)$ cannot control the full $|||g|||^2$, and one has to control the missing part via further study of the nonlinear equation (15). In particular, in the context of Landau or Boltzmann equations, the linear collision operator L as in (10) can be shown to be positive definite ([G1], [SG1]–[SG3]) along the nonlinear dynamics even if it has a kernel.

Such a flexible approach turns out to be robust, leading to constructions of global solutions to several different applied PDEs [G1]–[G5], [GS], [GrS1]–[GrS2], [SG1]–[SG3], [GT] and [Ha]. Unfortunately, despite these advances and many attempts, the original motivation in this program, the stability of the Maxwellian for the Vlasov-Poisson-Landau system has remained out of reach, as pointed out in [SG3]. For other work related to the Landau equation from different approaches, see [AB], [AV], [CDH], [HY], [L], [V], and [Z1]–[Z2] among others. Let $w(v) \geq 1$ be a weight function and $\|\cdot\|_{2,w}$ denote the weighted $L^2(\mathbf{T}^3 \times \mathbf{R}^3)$ norm. We define

$$(19) \quad \|f\|_{\sigma,w} \equiv \|f\|_{H_w^\sigma} \equiv \|\langle v \rangle^{-\frac{1}{2}} f\|_{2,w} + \|\langle v \rangle^{-\frac{3}{2}} \nabla_v f \cdot \frac{v}{|v|}\|_{2,w} + \|\langle v \rangle^{-\frac{1}{2}} \nabla_v f \times \frac{v}{|v|}\|_{2,w}$$

with $\langle v \rangle = \sqrt{1 + |v|^2}$. It is well known that $\|f\|_\sigma \equiv \|f\|_{\sigma,1}$ captures the dissipation for the Landau kernel [DL], [G1]. There are two intrinsic difficulties [G1] associated with the Landau kernel: the vanishing factors $\langle v \rangle^{-\frac{1}{2}}$ and $\langle v \rangle^{-\frac{3}{2}}$ make $\|f\|_\sigma$ ‘soft’, and their different vanishing rates along different directions make $\|f\|_\sigma$ nonisotropic for $\nabla_v f$.

There are two major mathematical difficulties in the study of stability in the Vlasov-Poisson-Landau system. The first difficulty is created by the (innocent looking!) nonlinear term $E \cdot v f_\pm$ in (8). Such a term comes from the factor $\sqrt{\mu}$ in our linearization $F_\pm = \mu + \sqrt{\mu} f_\pm$, which is the only known choice so far capturing the linear dissipation rate of L , a linear analog to the entropy production in the fundamental Boltzmann’s H-theorem. Hence, the presence of the term $E \cdot v f_\pm$ is a basic feature of interaction between the electric field and the particles in the near Maxwellian context. Upon multiplying $f_\pm$ one hopes to bound

$$(20) \quad \int |v E f^2|$$

in terms of $\|f\|_2 \cdot \|f\|_\sigma^2$ (19). The electric field $E = E_f$ behaves nicely, but the extra velocity factor v makes the control by $\|f\|_\sigma^2$ (which only controls $\|\langle v \rangle^{-1/2} f\|_2^2$ in (19)) impossible. In a simpler model problem [G2], the Landau collision is replaced by the hard-sphere interaction so that $\int |v| f^2$ can be exactly bounded by the dissipation rate of the hard-sphere kernel. In a relativistic Landau collision kernel [SG3], [DL], the relativistic counterpart is $\int |E \cdot \hat{p} f^2|$, where the velocity $\hat{p}$ is bounded, and $\|f\|_2^2$ is controlled by the relativistic Landau dissipation. These two facts led to the resolution for the relativistic Landau problem [SG3].

The second main difficulty stems from controlling the velocity v derivative of f . Taking the v derivative of (8), we estimate $\|\nabla_v f\|_2^2$ via the standard energy method. In this process, the free streaming term $v \cdot \nabla_x f$ produces a quadratic term as

$$(21) \quad \int \nabla_x f \cdot \nabla_v f.$$

It is well known that $\nabla_v f$ can produce growth in time in the kinetic theory. To estimate (21) by the norm (19) is tricky, again due to its negative weight. In the absence of the electric field E [G1], the author designed a weighted norm to overcome this difficulty with more v derivatives associated with more negative velocity weights. The first step was to take pure x derivatives (with no weight and no electric field E) and to bound

$$\int_0^t \|\nabla_x f\|_\sigma^2.$$

In the next step, $\|\langle v \rangle^{-1} \nabla_v f\|_2^2$ (instead of $\|\nabla_v f\|_2^2$) was estimated through the v derivative of (8), which contains a dissipation rate of $\|\nabla_v f\|_{\sigma, \langle v \rangle^{-1}}^2$. The weighted mixed terms (21) could be bounded by (see (19)):

$$\int \langle v \rangle^{-2} |\nabla_x f \cdot \nabla_v f| \leq C_\varepsilon \int_0^t \|\nabla_x f\|_\sigma^2 + \varepsilon \int_0^t \|\nabla_v f\|_{\sigma, \langle v \rangle^{-1}}^2,$$

which could be closed for ε small. Such a weight disparity between x and v derivatives of f has played an important role in treating soft potentials [G3], [GrS1]–[GrS2]. Unfortunately, in the presence of E , this strategy fails completely because $\int_0^t \|\nabla_x f\|_\sigma^2$ cannot be estimated independently of $\nabla_v f$ at the first step. In fact,

taking x derivatives of (8) produces a new contribution

$$(22) \quad \int E \nabla_x f \cdot \nabla_v f.$$

Now this cannot be controlled with a norm for $\nabla_v f$ with a negative weight $\langle v \rangle^{-1}$.

The key to overcoming the first difficulty (20) is to realize that, instead of treating $E \cdot v f_{\pm}$ as a second-order perturbation in (8), we need to combine or cancel it with the linear streaming term $v \cdot \nabla_x f$ in (8), which also contains an extra v factor! From $E = -\nabla_x \phi$, upon multiplying by $e^{\pm \phi}$, we can rewrite

$$(23) \quad e^{\pm \phi} [v \cdot \nabla_x f_{\pm} \pm \nabla_x \phi \cdot v f_{\pm}] = v \cdot \nabla_x \{e^{\pm \phi} f_{\pm}\}.$$

Indeed, such a perfect derivative leads to no contribution in the integration (see (78)). Even though such a spatial weight destroys the exact energy structure from the Poisson equation, fortunately, new error contributions are of the type $\int \nabla_x \phi \cdot v \sqrt{\mu} f_{\pm} (e^{\pm 2\phi} - 1)$, which can be controlled if ϕ is small. Our observation works for all forces given by a potential.

To overcome the second main difficulty and to control the v derivative of f , we need to design a new weight function in v . In light of (22), we need to assign the same weight functions for both $\nabla_x f$ and $\nabla_v f$, which seems to contradict the weight disparity for controlling (21). We observe crucially that $\|f\|_{\sigma}$ contains a (weak) v derivative of f . Hence $\nabla_v f$ can also be viewed, not as a part of $\|\nabla_v f\|_{\sigma}$, but as a part of $\|f\|_{\sigma, \langle v \rangle^2}$ with no v derivative, with an extra stronger weight $\langle v \rangle^2$. In fact, thanks to (19), we can estimate (21) by

$$\int \nabla_x f \cdot \nabla_v f \leq \|\nabla_x f\|_{\sigma} \|f\|_{\sigma, \langle v \rangle^2}$$

provided $\|f\|_{\sigma, \langle v \rangle^2}$ is controlled at an earlier step. The weight disparity is not between x and v derivatives, but between $\|f\|_{\sigma, \langle v \rangle^2}$ and $\|\nabla_x f\|_{\sigma}$. That is, fewer derivatives of f should require stronger weights. Since higher spatial derivatives are associated with weaker velocity weights in our norm (19), more careful analysis is needed for spatial Sobolev imbedding to close the energy estimate, especially when we take one derivative (Step 2 in the proof of Proposition 7). Such a cascade of weights takes advantage of the crucial feature of the Landau operator: a weak gain of ∇_v . It is interesting to use such a weight for the full inverse power law without angular cutoff [GrS1]–[GrS2].

For notational simplicity, we use $\|\cdot\|_p$ to denote L^p norms with weight $w(v)$ in $\mathbf{T}^3 \times \mathbf{R}^3$ or $\mathbf{T}^3$, and $\|\cdot\|_{p,w}$ for L^p norms with weight $w(v)$. Letting the multi-indices α and β be $\alpha = [\alpha_1, \alpha_2, \alpha_3]$, $\beta = [\beta_1, \beta_2, \beta_3]$, we define $\partial_{\beta}^{\alpha} \equiv \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}$. If each component of θ is not greater than that of the $\bar{\theta}$'s, we denote by $\theta \leq \bar{\theta}$; $\theta < \bar{\theta}$ means $\theta \leq \bar{\theta}$, and $|\theta| < |\bar{\theta}|$. We define the velocity weight

$$(24) \quad w(\alpha, \beta)(v) \equiv e^{\frac{q|v|^2}{2}} \langle v \rangle^{2(l-|\alpha|-|\beta|)}, \quad l \geq |\alpha| + |\beta|, \quad 0 \leq q < 1.$$

The presence of $e^{\frac{q|v|^2}{2}}$ would lead to stretched exponential decay. Recall (19). Let the instant energy and dissipation rate be:

$$(25) \quad \mathcal{E}_{m;l,q}(f)(t) \equiv \sum_{|\alpha|+|\beta| \leq m} \sum_{\pm} \|\partial_{\beta}^{\alpha} f_{\pm}(t)\|_{2, w(\alpha, \beta)}^2,$$

$$(26) \quad \mathcal{D}_{m;l,q}(f)(t) \equiv \sum_{|\alpha|+|\beta| \leq m} \sum_{\pm} \|\partial_{\beta}^{\alpha} f_{\pm}(t)\|_{\sigma, w(\alpha, \beta)}^2.$$

Remark 1. We remark that from the definition, the exponential factor $e^{\frac{q|v|^2}{2}}$ does not change with the number of derivatives. However, if $\mathcal{E}_{m;l,q}(f) < +\infty$ (or $\mathcal{D}_{m;l,q}(f) < +\infty$), $l \geq m$, the polynomial weight $\langle v \rangle^{2(l-|\alpha|-|\beta|)}$ does depend on the number of derivatives $|\alpha| + |\beta|$. In fact, if $\alpha = \beta = 0$ with no derivatives, then weight $\langle v \rangle^{2l}$ is the strongest. On the other hand, if $|\alpha| + |\beta| = m$, the weight $\langle v \rangle^{2(l-m)}$ is the weakest. We therefore have a cascade of velocity weights in (25) and (26) so that fewer derivatives of f demand stronger velocity weights.

It is important to note that due to the presence of (20), all our estimates can only be obtained with a (nonlinear) exponential weight of $e^{\pm(1+q)\phi_f}$ (see (80) to (90) and Lemma 15). But if the electric potential $\|\phi_f\|_\infty$ is bounded (as we shall prove), such weighted norms are equivalent to $\mathcal{E}_{m;l,q}(f)$ and $\mathcal{D}_{m;l,q}(f)$, and we need to use $\mathcal{E}_{m;l,q}(f)$ and $\mathcal{D}_{m;l,q}(f)$ without the nonlinear weight to close our continuity argument. Our main result is as follows.

Theorem 2. *Assume that f_0 satisfies the conservation laws (12), (13), (14) with $F_{0,\pm}(x, v) = \mu + \sqrt{\mu}f_{0,\pm}(x, v) \geq 0$. There exists a sufficiently small $M > 0$ such that if*

$$\mathcal{E}_{2;2,0}(f_0) \leq M,$$

then there exists a unique global solution $f(t, x, v)$ to the Vlasov-Poisson-Landau system (8) and (9) with $F_\pm(t, x, v) = \mu + \sqrt{\mu}f_\pm(t, x, v) \geq 0$.

(1) If $\mathcal{E}_{2;l,q}(f_0) < +\infty$ for $l \geq 2$ and $q \geq 0$, then there exists $C_l > 0$ such that

$$(27) \quad \sup_{0 \leq s \leq \infty} \mathcal{E}_{2;l,q}(f(s)) + \int_0^\infty \mathcal{D}_{2;l,q}(f(s))ds \leq C_l \mathcal{E}_{2;l,q}(f_0).$$

Furthermore,

$$(28) \quad \|\partial_t \phi(t)\|_\infty + \|\nabla_x \phi(t)\|_\infty + \|f(t)\|_2 \leq C_l (1+t)^{-2l+2} \mathcal{E}_{2;l,0}(f_0),$$

$$(29) \quad \|\partial_t \phi(t)\|_\infty + \|\nabla_x \phi(t)\|_\infty + \|f(t)\|_2 \leq C_l e^{-C_l t^{2/3}} \mathcal{E}_{2;l,q}(f_0) \quad \text{for } q > 0.$$

(2) In addition, if $\mathcal{E}_{m;l,q}(f_0) < \infty$ for any $l > 2$, $l \geq m \geq 2$, $q \geq 0$, there exists an increasing continuous function $P_{m,l}(\cdot)$ with $P_{m,l}(0) = 0$ such that the unique solution satisfies

$$(30) \quad \sup_{0 \leq t \leq \infty} \mathcal{E}_{m;l,q}(f(t)) + \int_0^\infty \mathcal{D}_{m;l,q}(f(s))ds \leq P_{m,l}(\mathcal{E}_{m;l,q}(f_0)).$$

The continuous function $P_{m,l}(\cdot)$ can be determined inductively on m . We remark that we only require $\mathcal{E}_{2;2,0}(f_0)$ to be small, but for $l > 2$ and $m \geq 2$, $\mathcal{E}_{m;l,q}(f_0)$ can be arbitrarily large. An H^2 type of construction was first carried out in [GrS1]–[GrS2] for the Boltzmann equation with a noncutoff inverse power collision kernel. Note from the Sobolev imbedding, L^∞ is not necessarily bounded by $\mathcal{E}_{2;2,0}(f_0)$. We also note that the estimate (30) is uniform in time, so that we verify the bounds in [DV] and more decay can be obtained (see also [SG1]–[SG2]). Such an estimate (30) also gives a natural approximation mechanism to establish the gain of smoothness for $t > 0$ [CDH].

Throughout the paper, we introduce the notation $A \lesssim B$ ($A \gtrsim B$ and $A \sim B$) if A is bounded by B up to a universal constant C which *does not* depend on either l or m .

The introduction of the spatial weight function $e^{\pm(1+q)\phi}$ creates a new analytical difficulty: we need to control (see (122) and (123))

$$(31) \quad \int_0^\infty [||\partial_t \phi(s)||_\infty + ||\nabla_x \phi(s)||_\infty] ds$$

to close the global energy estimate for $\mathcal{E}_{2;2,0}(f)$. Note that (31) is different from the dissipation estimate $\int_0^\infty \mathcal{D}_{2;2,0}(f(s)) ds < \infty$ since

$$||\partial_t \phi(s)||_\infty + ||\nabla_x \phi(s)||_\infty \lesssim \sqrt{\mathcal{D}_{2;2,0}(f(s))},$$

and $\mathcal{D}_{2;2,0}(f(s))$ is expected to be small most of the time. It is a typical difficulty that $\int_0^\infty \sqrt{\mathcal{D}_{2;2,0}(f(s))} ds < \infty$ cannot be derived directly from the energy-dissipation estimate (27). We need to make an interplay (see (151)) between the decay estimate of (28) (with $l = 2, q = 0$) and the energy estimate (27). This is different from the program in [SG1]–[SG2], in which the energy estimate can be closed alone first and the decay is obtained after. In fact, the proof of (28) is intertwined with (27) with $l = 2$, and we are able to close a differential inequality (150) with up to only one spatial derivative. This leads to a decay rate of

$$||\partial_t \phi(s)||_\infty + ||\nabla_x \phi(s)|| \sim \frac{1}{s^2}$$

in terms of (27) with $l = 2$, which is sufficient to close the estimate. The strong decay rate of $\frac{1}{s^2}$ is a consequence of the periodic box $\mathbf{T}^3$, and it remains an open question if a sufficient decay rate can be obtained for the Vlasov-Poisson-Landau system in the whole space $\mathbf{R}^3$.

The last novelty of the paper is the proof of high moments and high regularity (30) with only small $\mathcal{E}_{2;2,0}(f_0)$ initially. It is important to note that we require $l \geq m$, the total number of derivatives in $w(\alpha, \beta)$ (24). This is because the starting point of our method is to control pure x derivatives of f without any weight, which demands that $l \geq m$ at the highest level of derivatives. It is also important for our analysis to require $w(\alpha, \beta) \geq 1$. This implies that $||f||_{2,w} \leq \mathcal{E}_{m;l,q}(f_0)$, which cannot be small for all l ! Therefore it is natural to assume $\mathcal{E}_{m;l,q}(f_0)$ is only finite but not small for $l > 2$. The hope is to obtain, at the highest-order derivatives,

$$(32) \quad \int |w^2(\alpha, \beta) \Gamma(\partial_\beta^\alpha f, f) \partial_\beta^\alpha f| + \int |w^2(\alpha, \beta) \Gamma(f, \partial_\beta^\alpha f) \partial_\beta^\alpha f| \\ \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)} ||\partial_\beta^\alpha f(t)||_{\sigma, w(\alpha, \beta)}^2,$$

so that the smallness of $\mathcal{E}_{2;2,0}(f_0)$ would be sufficient. Unfortunately, due to the combination of the nonlocal feature of the Landau operator as well as the velocity weight $w(\alpha, \beta)$, as in Lemma 10 [SG2], a term such as

$$\sqrt{\mathcal{E}_{2;l,q}(f)} ||\partial_\beta^\alpha f(t)||_\sigma ||\partial_\beta^\alpha f(t)||_{\sigma, w(\alpha, \beta)}$$

will occur in the upper bound for (32). Since $\sqrt{\mathcal{E}_{2;l,q}(f)}$ is large for $l > 2$, we cannot absorb the whole product by the dissipation rate of $||\partial_\beta^\alpha f(t)||_{\sigma, w(\alpha, \beta)}^2$. In Proposition 7, we are able to move the weight function from $\sqrt{\mathcal{E}_{2;l,q}(f)}$ to $||\partial_\beta^\alpha f(t)||_\sigma$, and the price to pay is an additional contribution of

$$C_l \mathcal{D}_{2;l,q}(f) ||\partial_\beta^\alpha f(t)||_{2, w(\alpha, \beta)}^2.$$

Even though $C_l \mathcal{D}_{2;l,q}(f)$ can be very large, but $\|\partial_\beta^\alpha f(t)\|_{2,w(\alpha,\beta)}^2$ belongs to the instant energy not the dissipation rate, therefore we can still control this via the Gronwall lemma and the fact that $\int_0^t \mathcal{D}_{2;l,q}(f)(s)ds < \infty$. A new splitting of the domain needs to be designed to achieve this goal. We believe that our method will lead to new estimates for solutions recently constructed in [GrS1]–[GrS2] for the inverse power law. Finally, the presence of E also makes the construction of the local solutions more delicate, and we need to employ fractional Sobolev spaces to gain compactness of the approximate solutions. The estimate for high derivatives and high moments is also needed to control the term vEf even for the justification of (23).

Our paper is organized as follows. In section 2, new refined estimates are developed to cope with nonlinear terms in (8). In section 3, local-in-time solutions are constructed via estimates of the instant energy (25) and dissipation rate (26). In section 4, decay estimates (28) and (29) are obtained to bootstrap into global-in-time solutions.

2. BASIC ESTIMATES

We use $\langle \cdot, \cdot \rangle$ to denote the standard L^2 inner product in $\mathbf{R}_v^3$ for a pair of functions $g = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$ and $h = \begin{pmatrix} h_+ \\ h_- \end{pmatrix}$:

$$(33) \quad \langle f, h \rangle = \langle f_+, h_+ \rangle + \langle f_-, h_- \rangle.$$

We also denote $|\cdot|_{2,w}$ and $|\cdot|_{\sigma,w}$ to be the corresponding w -weighted L^2 and H^σ norms (19) in $\mathbf{R}_v^3$.

Recall the linear Landau operator L in (10). We first recall a basic property of L (see [H], [G1], [SG3] for a proof).

Lemma 3. *We have $\langle Lg, h \rangle = \langle Lh, g \rangle$, $\langle Lg, g \rangle \geq 0$, and $Lg = 0$ if and only if $g = Pg$, with P being the $L_v^2(\mathbf{R}^3)$ projection with respect to the vector L^2 inner product (33) onto the null space of L :*

$$(34) \quad \text{span} \left\{ \sqrt{\mu} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sqrt{\mu} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_i \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, |v|^2 \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\},$$

with $1 \leq i \leq 3$. Moreover

$$(35) \quad \langle Lg, g \rangle \gtrsim |\{I - P\}g|_\sigma^2 \equiv \sum_{\pm} |\{I - P\}g|_\pm|_\sigma^2,$$

where $|\{I - P\}g|_\pm$ are two components of $\{I - P\}g$.

Lemma 4. *Let $w(v) \geq 0$. Then for any $\eta > 0$,*

$$\begin{aligned} \sup_x |g(x, \cdot)|_w &\lesssim \|g|_{2,w}\|_{H^{\frac{7}{4}}} \lesssim \eta \sum_{|\gamma|=2} \|\partial^\gamma g\|_{2,w} + C_\eta \|g\|_{2,w}, \\ \sup_x |g(x, \cdot)|_{\sigma,w} &\lesssim \|g|_{\sigma,w}\|_{H^{\frac{7}{4}}} \lesssim \eta \sum_{|\gamma|=2} \|\partial^\gamma g\|_{\sigma,w} + C_\eta \|g\|_{\sigma,w}, \\ \|g\|_{L_x^4(L_w^2)} &\lesssim \|g|_{2,w}\|_{H^{\frac{3}{4}}} \lesssim \eta \sum_{|\gamma|=1} \|\partial^\gamma g\|_{2,w} + C_\eta \|g\|_{2,w}, \\ \|g\|_{L^4(H_w^\sigma)} &\lesssim \|g|_{\sigma,w}\|_{H^{\frac{3}{4}}} \lesssim \eta \sum_{|\gamma|=1} \|\partial^\gamma g\|_{\sigma,w} + C_\eta \|g\|_{\sigma,w}. \end{aligned}$$

We note that $L^\infty(\mathbf{T}^3) \subset H^{\frac{7}{4}}(\mathbf{T}^3)$, $L^4(\mathbf{T}^3) \subset H^{\frac{3}{4}}(\mathbf{T}^3)$. Moreover, $H^2(\mathbf{T}^3) \subset H^{\frac{7}{4}}(\mathbf{T}^3)$, and $H^1(\mathbf{T}^3) \subset H^{\frac{3}{4}}(\mathbf{T}^3)$ compactly, which gives rise to the arbitrarily small constant η . The lemma then follows from the general functional Sobolev inequality of H^k ; see [GrS1]–[GrS2].

We also need the following version of the Gronwall Lemma.

Lemma 5. *Let $A(t), B(t), y(t) \geq 0$ satisfy $y(t) \leq \int_0^t A(s)y(s)ds + B(t)$. Then*

$$y(t) \leq e^{\int_0^t A(s)ds} \int_0^t A(s)B(s)ds + B(t).$$

The following lemma is the key to treating the streaming term $\nabla_v f \cdot \nabla_x f$ via our dissipation rate (26). Denote

$$(36) \quad \delta_\beta^{\mathbf{e}_i} = 1 \quad \text{if} \quad \mathbf{e}_i \leq \beta; \quad \text{or} \quad \delta_\beta^{\mathbf{e}_i} = 0, \quad \text{otherwise.}$$

Lemma 6. *We have*

$$\int w^2(\alpha, \beta) \delta_\beta^{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f \partial_\beta^\alpha f \lesssim \|\delta_\beta^{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^\alpha f\|_{\sigma, w(\alpha, \beta-\mathbf{e}_i)} \|\partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f_\pm\|_{\sigma, w(\alpha+\mathbf{e}_i, \beta-\mathbf{e}_i)}.$$

Proof. Recalling (24), we note that $w(\alpha + \mathbf{e}_i, \beta - \mathbf{e}_i)(v) = w(\alpha, \beta)(v)$ and $w(\alpha, \beta - \mathbf{e}_i)(v) = \langle v \rangle^2 w(\alpha, \beta)(v)$. We rewrite from (36) $\partial_\beta^\alpha = \partial_{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^\alpha \delta_\beta^{\mathbf{e}_i}$ and we use (19) for $\partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f_\pm$ to get

$$\begin{aligned} & \int w^2(\alpha, \beta) \delta_\beta^{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f \partial_\beta^\alpha f \\ & \leq \int |w(\alpha, \beta) \langle v \rangle^{1/2} \delta_\beta^{\mathbf{e}_i} \partial_{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^\alpha f_\pm| |w(\alpha, \beta) \langle v \rangle^{-1/2} \partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f_\pm| \\ & \leq \|w(\alpha, \beta - \mathbf{e}_i) \langle v \rangle^{-3/2} \delta_\beta^{\mathbf{e}_i} \partial_{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^\alpha f_\pm\|_2 \|\partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i} f_\pm\|_{\sigma, w(\alpha, \beta)}, \end{aligned}$$

where $\langle v \rangle^{1/2} w(\alpha, \beta) = \langle v \rangle^{-3/2} w(\alpha, \beta - \mathbf{e}_i)(v)$. We now use (19) again for $\partial_{\beta-\mathbf{e}_i}^\alpha f_\pm$ to conclude the proof:

$$\|w(\alpha, \beta - \mathbf{e}_i) \langle v \rangle^{-3/2} \delta_\beta^{\mathbf{e}_i} \partial_{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^\alpha f_\pm\|_2 \lesssim \|\delta_\beta^{\mathbf{e}_i} \partial_{\beta-\mathbf{e}_i}^\alpha f_\pm\|_{\sigma, w(\alpha, \beta-\mathbf{e}_i)}. \quad \square$$

The following proposition is a refined estimate for the nonlinear collision term $\Gamma(g, g)$ in [G1], [SG1]–[SG2]. The key improvement is that the factors in front of the highest-order dissipation rate are bounded by only $\sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)}$.

Proposition 7. (1) *For $|\alpha| + |\beta| = m \leq 2$, recall $w = w(\alpha, \beta)$ in (24). We have*

$$\begin{aligned} \langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2 \rangle & \lesssim C_l \sum_{\bar{\beta} \leq \beta_1 \leq \beta} |\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma, w} |\partial_\beta^\alpha g_2|_{2, w} \\ (37) \quad & + \sum_{\substack{\alpha_1 \leq \alpha, \\ \bar{\beta} \leq \beta_1 \leq \beta}} [|\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma, w} + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_\sigma |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2, w}] |\partial_\beta^\alpha g_2|_{\sigma, w}. \end{aligned}$$

Moreover, for any $\eta > 0$, there exists $C_{l,\eta} > 0$ such that

$$\begin{aligned}
 (38) \quad & \int_{\mathbf{T}^3} |\langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2 \rangle| dx \\
 & \lesssim (\mathcal{E}_{2;2,0}(g_1) + \eta) \mathcal{D}_{2;l,q}(g_2) + C_{l,\eta} \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_1|_\sigma \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{2;l,q}(g_2) \\
 & \quad + C_l \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(g_1)} \sqrt{\mathcal{E}_{2;l,q}(g_2)} \\
 & \quad + C_l \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_2|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(g_1)} \sqrt{\mathcal{D}_{2;l,q}(g_2)}.
 \end{aligned}$$

(2) For $|\alpha| + |\beta| = m \geq 3$, we have

$$\begin{aligned}
 (39) \quad & \langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2 \rangle \\
 & \lesssim \sum_{\alpha_1 \leq \alpha, \beta_1 \leq \beta} C_\alpha^{\alpha_1} C_\beta^{\beta_1} C_l \sum_{\tilde{\beta} \leq \beta_1} |\partial_{\tilde{\beta}}^{\alpha_1} g_1|_\sigma |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} |\partial_{\tilde{\beta}}^{\alpha_1} g_2|_{2,w} \\
 & \quad + \sum_{|\alpha_1|+|\tilde{\beta}| \leq \frac{|\alpha|+|\beta|}{2}} C_\alpha^{\alpha_1} C_\beta^{\beta_1} [|\partial_{\tilde{\beta}}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} + |\partial_{\tilde{\beta}}^{\alpha_1} g_1|_\sigma |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2,w}] |\partial_{\tilde{\beta}}^{\alpha_1} g_2|_{\sigma,w} \\
 & \quad + \sum_{|\alpha_1|+|\tilde{\beta}| \geq \frac{|\alpha|+|\beta|}{2}} C_\alpha^{\alpha_1} C_\beta^{\beta_1} [|\partial_{\tilde{\beta}}^{\alpha_1} g_1|_{2,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_\sigma + |\partial_{\tilde{\beta}}^{\alpha_1} g_1|_{\sigma,w} |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2] |\partial_{\tilde{\beta}}^{\alpha_1} g_2|_{\sigma,w} \\
 & \quad + \sum_{|\alpha_1|+|\tilde{\beta}| \geq \frac{|\alpha|+|\beta|}{2}} C_\alpha^{\alpha_1} C_\beta^{\beta_1} C_{l,\alpha,\beta} |\partial_{\tilde{\beta}}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} |\partial_{\tilde{\beta}}^{\alpha_1} g_2|_{\sigma,w}.
 \end{aligned}$$

Furthermore, for any $\eta > 0$, there exists $C_{l,m,\eta} > 0$ such that

$$\begin{aligned}
 (40) \quad & \int_{\mathbf{T}^3} |\langle w^2(\alpha, \beta) \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2 \rangle| dx \\
 & \lesssim \{ \sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)} \} \{ \|\partial_\beta^\alpha g_1\|_{\sigma,w}^2 + \|\partial_\beta^\alpha g_2\|_{\sigma,w}^2 \} \\
 & \quad + \eta \{ \sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)} + 1 \} \sum_{\substack{|\alpha'|+|\beta'|=m \\ \beta' \leq \beta}} \{ \|\partial_{\beta'}^{\alpha'} g_1\|_{\sigma,w}^2 + \|\partial_{\beta'}^{\alpha'} g_2\|_{\sigma,w}^2 \} \\
 & \quad + C_{l,m,\eta} \sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \right\} \\
 & \quad \times \{ \mathcal{E}_{m;l,q}(g_1) + \mathcal{E}_{m;l,q}(g_2) \} \\
 & \quad + C_{l,m,\eta} \{ \mathcal{E}_{m-1;l,q}(g_1) + \mathcal{E}_{m-1;l,q}(g_2) + 1 \} \{ \mathcal{D}_{m-1;l,q}(g_1) + \mathcal{D}_{m-1;l,q}(g_2) \}.
 \end{aligned}$$

We remark that from (24), (26) and Lemma 4,

$$\begin{aligned}
 (41) \quad & \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} g|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \lesssim \sum_{|\alpha'|+|\beta'|\leq 1, |\gamma|\leq 1} \|\partial^{\gamma} \partial_{\beta'}^{\alpha'} g\|_{\sigma}^2 \lesssim \mathcal{D}_{2;2,0}(g), \\
 & \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} g|_2 \right\|_{H^{\frac{3}{4}}}^2 \lesssim \sum_{|\alpha'|+|\beta'|\leq 1, |\gamma|\leq 1} \|\partial^{\gamma} \partial_{\beta'}^{\alpha'} g\|_2^2 \lesssim \mathcal{E}_{2;2,0}(g), \\
 & \sum_{|\alpha'|+|\beta'|\leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} g|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \right\} \lesssim \sum_{\substack{|\alpha'|+|\beta'|\leq [\frac{m}{2}] \\ |\gamma|\leq 1}} \|\partial^{\gamma} \partial_{\beta'}^{\alpha'} g\|_{\sigma, w(\alpha'+\gamma, \beta')}^2 \\
 & \lesssim \mathcal{D}_{[\frac{m}{2}]+1; l, q}(g)
 \end{aligned}$$

(replacing $H^{\frac{3}{4}}$ by H^1), which are sufficient for our estimates for global solutions. However, for the construction of the local solutions, such sharper estimates are important for compactness of the approximate solutions.

Proof. Recall Lemma 10 in [SG2] and Theorem 3 in [G1]. We first separate:

$$(42) \quad \partial_i[w^2] = \{2qv_i w^2\} + \left\{ \frac{4(l-|\alpha|-|\beta|)}{1+|v|^2} v_i w^2 \right\}.$$

We can express $\langle w^2 \partial_{\beta}^{\alpha} \Gamma[g_1, g_2], \partial_{\beta}^{\alpha} g_2 \rangle = \sum C_{\alpha}^{\alpha_1} C_{\beta}^{\beta_1} \times G_{\alpha_1, \beta_1}$ with G_{α_1, β_1} as

$$(43) \quad -\langle w^2 \{\Phi^{ij} * \partial_{\beta_1} [\mu^{1/2} \partial^{\alpha_1} g_1]\} \partial_j \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2, \partial_i \partial_{\beta}^{\alpha} g_2 \rangle$$

$$(44) \quad -(1+2q) \langle w^2 \{\Phi^{ij} * \partial_{\beta_1} [v_i \mu^{1/2} \partial^{\alpha_1} g_1]\} \partial_j \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2, \partial_{\beta}^{\alpha} g_2 \rangle$$

$$(45) \quad + \langle w^2 \{\Phi^{ij} * \partial_{\beta_1} [\mu^{1/2} \partial_j \partial^{\alpha_1} g_1]\} \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2, \partial_i \partial_{\beta}^{\alpha} g_2 \rangle$$

$$(46) \quad + (1+2q) \langle w^2 \{\Phi^{ij} * \partial_{\beta_1} [v_i \mu^{1/2} \partial_j \partial^{\alpha_1} g_1]\} \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2, \partial_{\beta}^{\alpha} g_2 \rangle$$

$$(47) \quad - \left\langle \frac{4(l-|\alpha|-|\beta|)}{1+|v|^2} w^2 \{\Phi^{ij} * \partial_{\beta_1} [v_i \mu^{1/2} \partial^{\alpha_1} g_1]\} \partial_j \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2, \partial_{\beta}^{\alpha} g_2 \right\rangle$$

$$(48) \quad + \left\langle \frac{2(l-|\alpha|-|\beta|)}{1+|v|^2} w^2 \{\Phi^{ij} * \partial_{\beta_1} [\mu^{1/2} v_i \partial_j \partial^{\alpha_1} g_1]\} \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2, \partial_{\beta}^{\alpha} g_2 \right\rangle$$

with double summations over $1 \leq i, j \leq 3$ and $\partial_{\mathbf{e}_i} = \partial_{v_i} = \partial_i$.

Step 1. Estimate of (47) and (48).

We note that there is a (large) factor $4(l-|\alpha|-|\beta|)$ in both (47) and (48). We follow exactly as in the proof of Lemma 10 in [SG2]. From $\Phi^{ij} * \mu^{1/8} \leq \frac{1}{1+|v|}$ and the Cauchy-Schwarz inequality,

$$|\Phi^{ij} * \partial_{\beta_1} [v_i \mu^{1/2} \partial_j \partial^{\alpha_1} g_1]| + |\Phi^{ij} * \partial_{\beta_1} [v_i \mu^{1/2} \partial^{\alpha_1} g_1]| \lesssim \langle v \rangle^{-1} \sum_{\bar{\beta} \leq \beta_1} |\partial_{\bar{\beta}}^{\alpha} g_1|_{\sigma},$$

where we have used the exponential weight $v_i \mu^{1/2}$ to bound by $|\partial_{\beta}^{\alpha} g_1|_{\sigma}$ in (19). Therefore

$$\begin{aligned} & 4(l - |\alpha| - |\beta|) \left| \left\langle \frac{w^2}{\langle v \rangle^2} \Phi^{ij} * \{ \partial_{\beta_1} [v_i \mu^{1/2} \partial^{\alpha_1} g_1] \partial_j \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2, \partial_{\beta}^{\alpha} g_2 \} \right\rangle \right| \\ & \quad + 4(l - |\alpha| - |\beta|) \left| \left\langle \frac{w^2}{\langle v \rangle^2} \Phi^{ij} * \{ \partial_{\beta_1} [v_i \mu^{1/2} \partial_j \partial^{\alpha_1} g_1] \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2, \partial_{\beta}^{\alpha} g_2 \} \right\rangle \right| \\ & \leq C_{l,m} |\partial_{\beta}^{\alpha_1} g_1|_{\sigma} \int \frac{w^2}{\langle v \rangle^3} (|\partial_j \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2| + |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|) |\partial_{\beta}^{\alpha} g_2| dv \\ & \leq C_{l,m} \sum_{\beta} |\partial_{\beta}^{\alpha_1} g_1|_{\sigma} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_{\sigma, w} |\partial_{\beta}^{\alpha} g_2|_{2, w}, \end{aligned}$$

where we have used $|w \langle v \rangle^{-3} \partial_j \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_2 \lesssim |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_{\sigma, w}$ by (19). This concludes the control of (47) and (48) via the first terms on the right-hand sides of both (37) and (39).

Step 2. Proof of (37) and (38) for $|\alpha| + |\beta| = m \leq 2$.

In light of step 1, we only need to bound (43) to (46), which are precisely bounded as in Lemma 10 of [SG2] leading to (37). Note that one can eliminate the weight function in the g_1 factor due to the exponential decay factor $\mu^{1/2}$ in the Landau integral.

To conclude (38), we need to take the x integration of (37) and we need to separate three cases.

If $|\alpha| + |\beta| = 0$, $w = w(0, 0)$, we have from (37),

$$\begin{aligned} & \int_{\mathbf{T}^3} \langle w^2 \Gamma[g_1, g_2], g_2 \rangle \\ & \lesssim C_l \sup_x |g_1|_{\sigma} \int_{\mathbf{T}^3} |g_2|_{\sigma, w} |g_2|_{2, w} + \sup_x |g_1|_2 \int_{\mathbf{T}^3} |g_2|_{\sigma, w}^2 + \sup_x |g_1|_{\sigma} \int_{\mathbf{T}^3} |g_2|_{2, w} |g_2|_{\sigma, w} \\ & \lesssim C_l \| |g_1|_{\sigma} \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2; l, q}(g_2)} \sqrt{\mathcal{E}_{2; l, q}(g_2)} + \sqrt{\mathcal{E}_{2; 2, 0}(g_1)} \mathcal{D}_{2; l, q}(g_2) \\ & \lesssim (\sqrt{\mathcal{E}_{2; 2, 0}(g_1)} + \eta) \mathcal{D}_{2; l, q}(g_2) + C_{l, \eta} \| |g_1|_{\sigma} \|_{H^{\frac{7}{4}}}^2 \mathcal{E}_{2; l, q}(g_2). \end{aligned}$$

We have used (25), (26) and Lemma 4:

$$(49) \quad \sup_x |g_1|_{\sigma} \lesssim \| |g_1|_{\sigma} \|_{H^{\frac{7}{4}}} \quad \text{and} \quad \sup_x |g_1|_2 \lesssim \sqrt{\mathcal{E}_{2; 2, 0}(g_1)}.$$

If $|\alpha| + |\beta| = 1$, either $(\alpha_1, \beta_1) = 0$ or $(\alpha - \alpha_1, \beta - \beta_1) = 0$. In the case $(\alpha_1, \beta_1) = 0$, we use (49) to bound (37) similarly

$$\begin{aligned} \int_{\mathbf{T}^3} \langle w^2 \Gamma[g_1, g_2], g_2 \rangle & \lesssim C_l \sup_x |g_1|_{\sigma} \int_{\mathbf{T}^3} |\partial_{\beta}^{\alpha} g_2|_{\sigma, w} |\partial_{\beta}^{\alpha} g_2|_{2, w} \\ & \quad + \sup_x |g_1|_2 \int_{\mathbf{T}^3} |\partial_{\beta}^{\alpha} g_2|_{\sigma, w}^2 + \sup_x |g_1|_{\sigma} \int_{\mathbf{T}^3} |\partial_{\beta}^{\alpha} g_2|_{2, w} |\partial_{\beta}^{\alpha} g_2|_{\sigma, w} \\ & \lesssim (\sqrt{\mathcal{E}_{2; 2, 0}(g_1)} + \eta) \mathcal{D}_{2; l, q}(g_2) + C_{l, \eta} \| |g_1|_{\sigma} \|_{H^{\frac{7}{4}}}^2 \mathcal{E}_{2; l, q}(g_2). \end{aligned}$$

On the other hand, if $(\alpha - \alpha_1, \beta - \beta_1) = 0$, we take $L^4 - L^4 - L^2$ and use Lemma 4 to bound (37) by

$$(50) \quad \begin{aligned} C_l \int_{\mathbf{T}^3} |\partial_{\bar{\beta}}^{\alpha} g_1|_{\sigma} |g_2|_{\sigma, w} |\partial_{\bar{\beta}}^{\alpha} g_2|_{2, w} + \int_{\mathbf{T}^3} [|\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 |g_2|_{\sigma, w} + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} |g_2|_{2, w}] |\partial_{\bar{\beta}}^{\alpha} g_2|_{\sigma, w} \\ \lesssim C_l \left\| |\partial_{\bar{\beta}}^{\alpha} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}} \|\partial^{\gamma} g_2\|_{\sigma, w} \|\partial_{\bar{\beta}}^{\alpha} g_2\|_{2, w} + \|\partial^{\gamma} \partial_{\bar{\beta}}^{\alpha_1} g_1\|_2 \|\partial^{\gamma} g_2\|_{\sigma, w} \|\partial_{\bar{\beta}}^{\alpha} g_2\|_{\sigma, w}, \end{aligned}$$

where $|\gamma| \leq 1$. Note that $\|\partial^{\gamma} \partial_{\bar{\beta}}^{\alpha_1} g_1\|_2 \lesssim \sqrt{\mathcal{E}_{2;2,0}(g_1)}$. For $w = w(\alpha, \beta)$,

$$\|\partial^{\gamma} g_2\|_{\sigma, w} \lesssim \sqrt{\mathcal{D}_{1,l,q}(g_2)} \lesssim \sqrt{\mathcal{D}_{2;l,q}(g_2)}$$

from (25) and (26). Therefore, (50) is further bounded by

$$\begin{aligned} C_l \left\| |\partial_{\bar{\beta}}^{\alpha} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \sqrt{\mathcal{E}_{2;l,q}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_1)} \mathcal{D}_{2;l,q}(g_2) \\ \lesssim (\sqrt{\mathcal{E}_{2;2,0}(g_1)} + \eta) \mathcal{D}_{2;l,q}(g_2) + C_{l,\eta} \left\| |\partial_{\bar{\beta}}^{\alpha} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{2;l,q}(g_2). \end{aligned}$$

We thus complete the case $|\alpha| + |\beta| = 1$.

We remark that we cannot take $\sup_x |g_2|_{\sigma, w(\alpha, \beta)}$ and $\sup_x |g_2|_{2, w(\alpha, \beta)}$ in this case, because $w(\alpha, \beta)$ now is associated with $|\alpha| + |\beta| = 1$, which is stronger than the w allowed for second-order derivatives from (24): for $|\gamma| \geq 2$,

$$w(\gamma, 0) < w(\alpha, \beta).$$

As a consequence,

$$\sum_{|\gamma| \leq 2} \|\partial^{\gamma} g_2\|_{\sigma, w(\alpha, \beta)} \quad \text{and} \quad \sum_{|\gamma| \leq 2} \|\partial^{\gamma} g_2\|_{2, w(\alpha, \beta)}$$

cannot be bounded by $\sqrt{\mathcal{D}_{2;l,q}(g_2)}$ or $\sqrt{\mathcal{E}_{2;l,q}(g_2)}$.

We now consider the third case of $|\alpha| + |\beta| = 2$. We first consider

$$(51) \quad \text{either } \alpha_1 = \bar{\beta} = 0 \quad \text{or } \alpha - \alpha_1 = \beta - \beta_1 = 0.$$

We take L_x^{∞} of the term without derivatives and L_x^2 of the other two terms. From (24),

$$(52) \quad w(\alpha, \beta) \leq w(\gamma + \alpha_1, \bar{\beta}) \quad \text{or} \quad w(\alpha, \beta) \leq w(\gamma + \alpha - \alpha_1, \beta - \beta_1)$$

for $|\gamma| \leq 2$, respectively. Therefore for any g ,

$$\begin{aligned} \sup_x |g|_{\sigma} &\lesssim \|g\|_{H^{\frac{7}{4}}}, \quad \text{and} \quad \sup_x |g|_2 \lesssim \sqrt{\mathcal{E}_{2;2,0}(g)}, \\ \sup_x |g|_{\sigma, w} &\lesssim \sum_{|\gamma| \leq 2} \|\partial^{\gamma} g\|_{\sigma, w(\alpha, \beta)} \lesssim \sum_{|\gamma| \leq 2} \|\partial^{\gamma} g\|_{\sigma, w(\gamma, 0)} \lesssim \sqrt{\mathcal{D}_{2;l,q}(g)}, \\ (53) \quad \sup_x |g|_{2, w} &\lesssim \sum_{|\gamma| \leq 2} \|\partial^{\gamma} g\|_{2, w(\alpha, \beta)} \lesssim \sum_{|\gamma| \leq 2} \|\partial^{\gamma} g\|_{2, w(\gamma, 0)} \lesssim \sqrt{\mathcal{E}_{2;l,q}(g)}. \end{aligned}$$

If $\alpha_1 = \bar{\beta} = 0$, by Lemma 4 and (53), we bound (37) by

$$\begin{aligned} & \int_{\mathbf{T}^3} |\langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2 \rangle| \\ & \lesssim C_l \| |g_1|_\sigma \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \sqrt{\mathcal{E}_{2;l,q}(g_2)} \\ & \quad + \sqrt{\mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{2;l,q}(g_2)} + \| |g_1|_\sigma \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{E}_{2;2,0}(g_2)} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \\ & \lesssim C_l \| |g_1|_\sigma \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \sqrt{\mathcal{E}_{2;l,q}(g_2)} + \sqrt{\mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{2;l,q}(g_2)}. \end{aligned}$$

We note that

$$(54) \quad \| |g_1|_\sigma \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \sqrt{\mathcal{E}_{2;l,q}(g_2)} \leq \eta \mathcal{D}_{2;l,q}(g_2) + C_\eta \| |g_1|_\sigma \|_{H^{\frac{7}{4}}}^2 \mathcal{E}_{2;l,q}(g_2).$$

This completes the case of $\alpha = \bar{\beta} = 0$. On the other hand, if $\alpha - \alpha_1 = \beta - \beta_1 = 0$, by Lemma 4 and (53), we bound (37) by

$$\begin{aligned} & C_l \sqrt{\mathcal{D}_{2,0,0}(g_1)} \| |g_2|_{\sigma,w} \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{E}_{2;l,q}(g_2)} \\ & + \sqrt{\mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{2;l,q}(g_2)} + \sqrt{\mathcal{D}_{2,0,0}(g_1)} \| |g_2|_{2,w} \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \\ & \lesssim C_l \sqrt{\mathcal{D}_{2;2,0}(g_1)} \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(g_2)} \\ & \quad + \sqrt{\mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{2;l,q}(g_2)} \\ & \quad + \sqrt{\mathcal{D}_{2;2,0}(g_1)} \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_2|_{2, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \end{aligned}$$

since $\frac{w(\alpha',\beta')}{\langle v \rangle^2} = w(\alpha, \beta)$ for $|\alpha'| + |\beta'| = 1$ and $|\alpha| + |\beta| = 2$. We conclude (38) in this case.

If $(\alpha_1, \bar{\beta}) \neq 0$ and $(\alpha - \alpha_1, \beta - \beta_1) \neq 0$, now $|\alpha_1| + |\bar{\beta}| = 1$ and $|\alpha - \alpha_1| + |\beta - \beta_1| = 1$ since $|\alpha| + |\beta| = 2$. We take $L^4 - L^4 - L^2$ for (37) to get:

$$\begin{aligned} & \langle w^2 \partial_\beta^\alpha \Gamma[g_1, g_2], \partial_\beta^\alpha g_2 \rangle \\ & \lesssim C_l \sum_{|\gamma|\leq 1} \left\| |\partial_\beta^{\alpha_1} g_1|_\sigma \right\|_{H^{\frac{3}{4}}} \|\partial^\gamma \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2\|_{\sigma,w} \|\partial_\beta^\alpha g_2\|_{2,w} \\ & \quad + \sum_{|\gamma|\leq 1} [\|\partial^\gamma \partial_\beta^{\alpha_1} g_1\|_2 \|\partial^\gamma \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2\|_{\sigma,w} \\ & \quad + \left\| |\partial_\beta^{\alpha_1} g_1|_\sigma \right\|_{H^{\frac{3}{4}}} \|\partial^\gamma \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2\|_{2,w}] \|\partial_\beta^\alpha g_2\|_{\sigma,w} \\ & \lesssim C_l \left\| |\partial_\beta^{\alpha_1} g_1|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;l,q}(g_2)} \sqrt{\mathcal{E}_{2;l,q}(g_2)} + \sqrt{\mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{2;l,q}(g_2)}. \end{aligned}$$

This completes the case of $|\alpha| + |\beta| = 2$ and thus concludes step 2 by (54).

Step 3. Proof of (39).

In the case of $|\alpha_1| + |\bar{\beta}| \leq \frac{|\alpha|+|\beta|}{2}$, the estimate again follows exactly as in Lemma 10 of [SG2] for (43) to (46).

The case of $|\alpha_1| + |\bar{\beta}| \geq \frac{|\alpha|+|\beta|}{2}$ is most delicate as we need to avoid a contribution of $|\partial_\beta^\alpha g_1|_\sigma |g_2|_{2,w} |\partial_\beta^\alpha g_2|_{\sigma,w}$ in Lemma 10 of [SG2]. Our goal is to ‘move’ the weight w out of $|g_2|_{2,w}$ to $|\partial_\beta^\alpha g_1|_\sigma$. To accomplish this, we follow exactly the proof of Lemma

10 of [SG2] but with a different splitting of the phase space v, v' , depending on l and m :

$$(55) \quad \{w \leq 2\}, \quad \left\{ \left| \frac{|v'|}{|v|} - 1 \right| \leq \varepsilon_{l,m}, \quad w \geq 2 \right\} \quad \text{and} \quad \left\{ \left| \frac{|v'|}{|v|} - 1 \right| \geq \varepsilon_{l,m}, \quad w \geq 2 \right\}$$

with $\varepsilon_{l,m} < 1$ satisfying ($l \geq |\alpha| + |\beta|$ from (24)):

$$(56) \quad \frac{q\varepsilon_{l,m}}{(1 - \varepsilon_{l,m})^2} < \frac{1}{4} \quad \text{and} \quad (1 - \varepsilon_{l,m})^{-2(l - |\alpha| - |\beta|)} \leq 2.$$

To represent a smooth version of the splitting (55), we choose a smooth cutoff function $\chi(s) = 1$ for $s \leq 1$ and $\chi(s) \equiv 0$ for $s \geq 2$. We then decompose the unity

$$(57) \quad \begin{aligned} 1 &= \chi_1 + \chi_2 + \chi_3 \\ &\equiv \chi\left(\frac{w}{2}\right) + (1 - \chi\left(\frac{w}{2}\right))\chi\left(\frac{2\left|\frac{|v'|}{|v|} - 1\right|}{\varepsilon_{l,m}}\right) + (1 - \chi\left(\frac{w}{2}\right))\left\{1 - \chi\left(\frac{2\left|\frac{|v'|}{|v|} - 1\right|}{\varepsilon_{l,m}}\right)\right\} \end{aligned}$$

in the integrand from (43) to (46) and estimate each of these three partitions.

For the first part $\chi(\frac{w}{2})$, we have $\chi(\frac{w}{2}) \leq \mathbf{1}_{|w| \leq 4}$. We follow the proof of case 1 in Lemma 10 in [SG2] with an upper bound of terms (43) to (46) with $|\alpha_1| + |\bar{\beta}| \geq \frac{|\alpha| + |\beta|}{2}$ as

$$(58) \quad \begin{aligned} &[|\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_{w \leq 4}|_{\sigma, w} + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_{w \leq 4}|_{2, w}] |\partial_{\bar{\beta}}^{\alpha} g_2|_{\sigma, w} \\ &\lesssim |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{2, w} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_{\sigma} |\partial_{\bar{\beta}}^{\alpha} g_2|_{\sigma, w} + |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma, w} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_2 |\partial_{\bar{\beta}}^{\alpha} g_2|_{\sigma, w}, \end{aligned}$$

where we have used the important fact $w \leq 4$ and the property

$$w(\alpha, \beta) \geq 1.$$

We remark that even if $q > 0$ and $l = 0$ in (24):

$$w(\alpha, \beta) = e^{\frac{q}{2}|v|^2} \langle v \rangle^{-2|\alpha| - 2|\beta|}, \quad w \geq c_{\alpha, \beta} \rightarrow 0$$

as $|\alpha| + |\beta| \rightarrow \infty$ so that a large constant of $\frac{1}{c_{\alpha, \beta}}$ appears in front of

$$|\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma, w} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_2 |\partial_{\bar{\beta}}^{\alpha} g_2|_{\sigma, w}$$

in (58), when we bound $|\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma}$ by $|\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma, w}$. In particular

$$\frac{1}{c_{\alpha, \beta}} |\partial_{\bar{\beta}}^{\alpha} g_1|_{\sigma, w} |g_2|_2 |\partial_{\bar{\beta}}^{\alpha} g_2|_{\sigma, w}$$

cannot be absorbed by the dissipation rate (26). This illustrates the importance of the choice of $l \geq |\alpha| + |\beta|$ to guarantee $w \geq 1$.

For the second part in (57), we have

$$(1 - \chi\left(\frac{w}{2}\right))\chi\left(\frac{2\left|\frac{|v'|}{|v|} - 1\right|}{\varepsilon_{l,m}}\right) \leq \mathbf{1}_{\left\{\left|\frac{|v'|}{|v|} - 1\right| \leq \varepsilon_{l,m}, \quad |w| \geq 2\right\}}.$$

We shall swap the weight $w(v)$ by $w(v')$ in this case. Since $(1 - \varepsilon_{l,m})|v| \leq |v'| \leq (1 + \varepsilon_{l,m})|v|$, we deduce from (56),

$$\begin{aligned}
 \frac{w(v)}{w(v')} &= e^{\frac{q}{2}|v|^2 - \frac{q}{2}|v'|^2} \left(\frac{1 + |v|^2}{1 + |v'|^2} \right)^{l - |\alpha| - |\beta|} \\
 &\leq e^{\frac{q}{2}|v'|^2 [\frac{1}{(1 - \varepsilon_{l,m})^2} - 1]} \left(\frac{1 + \frac{|v'|^2}{(1 - \varepsilon_{l,m})^2}}{1 + |v'|^2} \right)^{l - |\alpha| - |\beta|} \\
 &\leq e^{\frac{q\varepsilon_{l,m}}{(1 - \varepsilon_{l,m})^2}|v'|^2} (1 - \varepsilon_{l,m})^{-2(l - |\alpha| - |\beta|)} \\
 (59) \quad &\leq 2e^{\frac{1}{4}|v'|^2},
 \end{aligned}$$

where $1 + \frac{|v'|^2}{(1 - \varepsilon_{l,m})^2} \leq \frac{1 + |v'|^2}{(1 - \varepsilon_{l,m})^2}$. In the Case 2 of the proof of Lemma 10 of [SG2], due to exponential decay of $v'e^{\frac{1}{2}|v'|^2}$ in the Landau kernel which dominates $2e^{\frac{1}{4}|v'|^2}$, by (59) we can move the weight function $w(v) = w(v') \times \frac{w(v)}{w(v')}$ to g_1 to obtain bounds ($w \geq 2$) for (43) to (46) as

$$[|\partial_{\beta}^{\alpha_1} g_1|_{2,w} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_{\sigma} + |\partial_{\beta}^{\alpha_1} g_1|_{\sigma,w} |\partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2|_2] |\partial_{\beta}^{\alpha} g_2|_{\sigma,w}.$$

For the third part in (57), we have $\left\{ \left| \frac{|v'|}{|v|} - 1 \right| \geq \frac{\varepsilon_{l,m}}{2}, \quad |w| \geq 2 \right\}$ so that

$$(60) \quad |v' - v| \geq ||v'| - |v|| \geq \frac{\varepsilon_{l,m}}{2}|v| \quad \text{and} \quad |v| \geq \theta_{l,m} > 0$$

from (24). As in step 3 in the proof in Lemma 10 in [SG2], we make integration by parts of $\partial_j = \partial_{v'_j}$ in (45) and (46). Since

$$\begin{aligned}
 \mu^{1/2} \partial_j \partial^{\alpha_1} g_1 &= -\partial_j \mu^{1/2} \partial^{\alpha_1} g_1 + \partial_j \{ \mu^{1/2} \partial^{\alpha_1} g_1 \}, \\
 \mu^{1/2} v'_i \partial_j \partial^{\alpha_1} g_1 &= -\partial_j \{ v'_i \mu^{1/2} \} \partial^{\alpha_1} g_1 + \partial_j \{ v'_i \mu^{1/2} \partial^{\alpha_1} g_1 \},
 \end{aligned}$$

we rewrite (43) to (46) as

$$\begin{aligned}
 (61) \quad & - \int \left\{ \int \Phi^{ij}(v - v') \chi_3 \partial_{\beta_1} [\mu^{1/2} \partial^{\alpha_1} g_1](v') dv' \right\} w^2 \partial_j \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_i \partial_{\beta}^{\alpha} g_2(v) \\
 & - (1 + 2q) \int \left\{ \int \Phi^{ij}(v - v') \chi_3 \partial_{\beta_1} [v'_i \mu^{1/2} \partial^{\alpha_1} g_1](v') dv' \right\} w^2 \partial_j \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_{\beta}^{\alpha} g_2(v) \\
 & - \int \left\{ \int \Phi^{ij}(v - v') \chi_3 \partial_{\beta_1} [\partial_j \{ \mu^{1/2} \} \partial^{\alpha_1} g_1] dv' \right\} w^2 \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_i \partial_{\beta}^{\alpha} g_2(v) \\
 & - \int \left\{ \int \Phi^{ij}(v - v') \partial_j \chi_3 \partial_{\beta_1} [\mu^{1/2} \partial^{\alpha_1} g_1] dv' \right\} w^2 \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_i \partial_{\beta}^{\alpha} g_2(v) \\
 & - (1 + 2q) \int \left\{ \int \Phi^{ij}(v - v') \chi_3 \partial_{\beta_1} [\partial_j \{ v'_i \mu^{1/2} \} \partial^{\alpha_1} g_1] dv' \right\} \\
 & - (1 + 2q) \int \left\{ \int \Phi^{ij}(v - v') \partial_j \chi_3 \partial_{\beta_1} [v'_i \mu^{1/2} \partial^{\alpha_1} g_1] dv' \right\} w^2 \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_{\beta}^{\alpha} g_2(v) \\
 & - \int \left\{ \int \partial_j \Phi^{ij}(v - v') \chi_3 \partial_{\beta_1} [\mu^{1/2} \partial^{\alpha_1} g_1] dv' \right\} w^2 \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_i \partial_{\beta}^{\alpha} g_2(v) \\
 & - (1 + 2q) \int \left\{ \int \partial_j \Phi^{ij}(v - v') \chi_3 \partial_{\beta_1} [v'_i \mu^{1/2} \partial^{\alpha_1} g_1] dv' \right\} w^2 \partial_{\beta - \beta_1}^{\alpha - \alpha_1} g_2(v) \partial_{\beta}^{\alpha} g_2(v).
 \end{aligned}$$

By (57) and the fact that $\chi(\frac{w}{2})$ does not depend on v'_j ,

$$\begin{aligned} |\partial_{v'_j} \chi_3| &= \frac{1}{\varepsilon_{l,m}} \left| \chi' \left(\frac{2 \left| \frac{|v'|}{|v|} - 1 \right|}{\varepsilon_{l,m}} \right) \partial_{v'_j} \left\{ 2 \left| \frac{|v'|}{|v|} - 1 \right| \right\} (1 - \chi(\frac{w}{2})) \right| \\ &\leq \frac{1}{\varepsilon_{l,m}} \left| \chi' \left(\frac{2 \left| \frac{|v'|}{|v|} - 1 \right|}{\varepsilon_{l,m}} \right) \frac{2 \frac{1}{|v|}}{\left| \frac{|v'|}{|v|} - 1 \right|} (1 - \chi(\frac{w}{2})) \right| \\ &\lesssim C_{l,m} \end{aligned}$$

since $|v|$ and $\left| \frac{|v'|}{|v|} - 1 \right|$ are both bounded from below.

In (60), the case of $|v'| - |v| \geq \frac{\varepsilon_{l,m}}{2}|v|$ implies that $|v'| \geq (1 + \frac{\varepsilon_{l,m}}{2})|v|$. Together with the lower bound of $|v|$ and a strong exponential decay from $\mu^{1/2}(v')$, (61) is clearly bounded by $C_{l,m} |\partial_{\beta}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} |\partial_{\beta}^{\alpha} g_2|_{\sigma,w}$. On the other hand, for the remaining case of $|v'| - |v| \leq -\frac{\varepsilon_{l,m}}{2}|v|$, we have $|v'| \leq (1 - \frac{\varepsilon_{l,m}}{2})|v|$, and we can repeat the same argument as in Case 3 of the proof for Lemma 10 [SG2] to bound the first six terms in (61) by

$$C_{l,m} |\partial_{\beta}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} |\partial_{\beta}^{\alpha} g_2|_{\sigma,w}.$$

Since $|\partial_j \Phi^{ij}(v - v')| \lesssim \frac{1}{|v-v'|^2} \lesssim \frac{1}{\langle v \rangle^2}$ in this region, from (19), we can bound the last two terms in (61) again by

$$\begin{aligned} C_{l,m} |\partial_{\beta}^{\alpha_1} g_1|_2 \int \frac{1}{\langle v \rangle^2} |w \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2| \{ |w \partial_i \partial_{\beta}^{\alpha} g_2| + |w \partial_{\beta}^{\alpha} g_2| \} \\ \lesssim C_{l,m} |\partial_{\beta}^{\alpha_1} g_1|_2 \langle v \rangle^{-1/2} |w \partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2 \{ \langle v \rangle^{-3/2} |w \partial_i \partial_{\beta}^{\alpha} g_2|_2 + \langle v \rangle^{-3/2} |w \partial_{\beta}^{\alpha} g_2|_2 \} \\ \lesssim C_{l,m} |\partial_{\beta}^{\alpha_1} g_1|_2 |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} |\partial_{\beta}^{\alpha} g_2|_{\sigma,w}. \end{aligned}$$

We therefore complete Step 3 and conclude the proof of (39).

Step 4. Proof of (40).

We shall separate three cases in (39). The first case is either $\alpha_1 = \alpha, \beta = \bar{\beta}$ or $\alpha - \alpha_1 = \alpha$, and $\beta - \beta_1 = \beta$. Indeed we now have $C_{\alpha}^{\alpha_1} C_{\beta}^{\beta_1} = 1$. Recall (51) and (52) in this case. We note that $m \geq 3$ so that we can apply the L_x^{∞} estimate to terms without derivatives, and then take corresponding $L^{\infty} - L^2 - L^2$ for the x integration. By (53), we obtain an upper bound of (39):

$$\begin{aligned} C_l [\| |g_1|_{\sigma} \|_{H^{\frac{7}{4}}} \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} + \| \partial_{\beta}^{\alpha} g_1 \|_{\sigma} \| |g_2|_{\sigma,w} \|_{H^{\frac{7}{4}}}] \sqrt{\mathcal{E}_{m;l,q}(g_2)} \\ + [\sqrt{\mathcal{E}_{2;2,0}(g_1)} \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} + \| |g_1|_{\sigma} \|_{H^{\frac{7}{4}}} \sqrt{\mathcal{E}_{m;l,q}(g_2)}] \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} \\ + [\sqrt{\mathcal{E}_{m;l,q}(g_1)} \| |g_2|_{\sigma} \|_{H^{\frac{7}{4}}} + \| \partial_{\beta}^{\alpha} g_1 \|_{\sigma,w} \sqrt{\mathcal{E}_{2;2,0}(g_2)}] \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} \\ + C_{l,m} \sqrt{\mathcal{E}_{m;l,0}(g_1)} \| |g_2|_{\sigma,w} \|_{H^{\frac{7}{4}}} \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} \\ \lesssim \{ \sqrt{\mathcal{E}_{2;2,0}(g_1)} + \sqrt{\mathcal{E}_{2;2,0}(g_2)} + \eta \} \{ \| \partial_{\beta}^{\alpha} g_1 \|_{\sigma,w}^2 + \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w}^2 \} \\ + C_{l,m,\eta} \{ \| |g_1|_{\sigma,w} \|_{H^{\frac{7}{4}}}^2 + \| |g_2|_{\sigma,w} \|_{H^{\frac{7}{4}}}^2 \} \{ \mathcal{E}_{m;l,q}(g_1) + \mathcal{E}_{m;l,q}(g_2) \}. \end{aligned}$$

We have used the fact that $w \geq 1$ and $l \geq 2$, $q \geq 0$. We note that

$$\begin{aligned}
 & \| |g_1|_{\sigma,w} \|_{H^{\frac{7}{4}}}^2 + \| |g_2|_{\sigma,w} \|_{H^{\frac{7}{4}}}^2 \\
 & \lesssim \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma,w(\alpha,\beta)} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma,w(\alpha,\beta)} \right\|_{H^{\frac{3}{4}}}^2 \\
 (62) \quad & \lesssim \sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2,
 \end{aligned}$$

as $[\frac{m}{2}] \geq 1$ and $w(\alpha, \beta) \leq \frac{w(\alpha', \beta')}{\langle v \rangle^2}$ for $|\alpha| + |\beta| \geq |\alpha'| + |\beta'| + 1$ from (24). We thus conclude the first case.

The second case is when $2 \leq |\alpha_1| + |\bar{\beta}| \leq m-2$ and $2 \leq |\alpha - \alpha_1| + |\beta - \beta_1| \leq m-2$. In this case, we shall take $L^4 - L^4 - L^2$ and by Lemma 4 to find an upper bound of (39):

$$\begin{aligned}
 & C_{l,m} \left\| |\partial_{\beta}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \| \partial_{\beta}^{\alpha} g_2 \|_{2,w} \\
 & + C_{l,m} \left[\left\| |\partial_{\beta}^{\alpha_1} g_1|_2 \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \right. \\
 & \quad \left. + \left\| |\partial_{\beta}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2,w} \right\|_{H^{\frac{3}{4}}} \right] \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} \\
 (63) \quad & + C_{l,m} \left[\left\| |\partial_{\beta}^{\alpha_1} g_1|_{2,w} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma} \right\|_{H^{\frac{3}{4}}} \right. \\
 & \quad \left. + \left\| |\partial_{\beta}^{\alpha_1} g_1|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2 \right\|_{H^{\frac{3}{4}}} \right] \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w} \\
 & + C_{l,m} \left\| |\partial_{\beta}^{\alpha_1} g_1|_2 \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w}.
 \end{aligned}$$

For the last three terms, we use $H^1 \subset H^{\frac{3}{4}}$ and the facts

$$\begin{aligned}
 & \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g|_{2,w} \right\|_{H^1} + \left\| |\partial_{\beta}^{\alpha_1} g|_{2,w} \right\|_{H^1} \lesssim \mathcal{E}_{m-1;l,q}(g), \\
 & \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g|_{\sigma,w} \right\|_{H^1} + \left\| |\partial_{\beta}^{\alpha_1} g|_{\sigma,w} \right\|_{H^1} \lesssim \mathcal{D}_{m-1;l,q}(g)
 \end{aligned}$$

to obtain a desired upper bound of

$$\eta \| \partial_{\beta}^{\alpha} g_2 \|_{\sigma,w}^2 + C_{l,m,\eta} \{ \mathcal{E}_{m-1;l,q}(g_1) + \mathcal{E}_{m-1;l,q}(g_2) \} \{ \mathcal{D}_{m-1;l,q}(g_1) + \mathcal{D}_{m-1;l,q}(g_2) \}.$$

For the first term in (63), we combine the factor of smaller total derivatives (less than $[\frac{m}{2}]$) with $\| \partial_{\beta}^{\alpha} g_2 \|_{2,w}$ to obtain an upper bound:

$$\begin{aligned}
 (64) \quad & \eta \sum_{[\frac{m}{2}] \leq |\alpha'|+|\beta'| \leq m-1} \left\{ \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma,w} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}}^2 \right\} \\
 & + C_{l,m,\eta} \sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma,w} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}}^2 \right\} \| \partial_{\beta}^{\alpha} g_2 \|_{2,w}^2 \\
 & \leq \eta \sum_{|\alpha'|+|\beta'|=m} \{ \| \partial_{\beta'}^{\alpha'} g_1 \|_{\sigma,w}^2 + \| \partial_{\beta'}^{\alpha'} g_2 \|_{\sigma,w}^2 \} + \eta \{ \mathcal{D}_{m-1;l,q}(g_1) + \mathcal{D}_{m-1;l,q}(g_2) \} \\
 & + C_{l,m,\eta} \sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma, \frac{w'}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, \frac{w'}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \right\} \| \partial_{\beta}^{\alpha} g_2 \|_{2,w}^2,
 \end{aligned}$$

where $w' = w(\alpha', \beta')$ and $w(\alpha, \beta) \leq \frac{w'(\alpha', \beta')}{\langle v \rangle^2}$ for $|\alpha| + |\beta| \geq |\alpha'| + |\beta'| + 1$ by (24). We therefore conclude the second case.

We now consider the last case of the following possibilities: $1 = |\alpha_1| + |\bar{\beta}|$, $|\alpha_1| + |\bar{\beta}| = m - 1$, or $1 = |\alpha - \alpha_1| + |\beta - \beta_1|$, $|\alpha - \alpha_1| + |\beta - \beta_1| = m - 1$. Note that in this case the weight function satisfies:

$$w(\alpha, \beta) \leq w(\alpha_1 + \gamma, \bar{\beta}), \quad w(\alpha, \beta) \leq w(\alpha - \alpha_1 + \gamma, \beta_1),$$

for $|\gamma| \leq 1$. We can bound (39) by taking $L^4 - L^4 - L^2$ as

$$\begin{aligned}
 & C_{l,m} \sum \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{2,w} \\
 & + C_{l,m} \sum_{|\alpha|+|\bar{\beta}|\leq 1} \left[\left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \right. \\
 & + \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2,w} \right\|_{H^{\frac{3}{4}}} \left. \right] \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{\sigma,w} \\
 & + C_{l,m} \sum_{|\alpha-\alpha_1|+|\beta-\beta_1|\leq 1} \left[\left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{2,w} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma} \right\|_{H^{\frac{3}{4}}} \right. \\
 & + \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_2 \right\|_{H^{\frac{3}{4}}} \left. \right] \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{\sigma,w} \\
 (65) \quad & + C_{l,m} \sum_{|\alpha-\alpha_1|+|\beta-\beta_1|\leq 1} \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_2 \right\|_{H^{\frac{3}{4}}} \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}} \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{\sigma,w}.
 \end{aligned}$$

The first term above can be estimated exactly as in (64). Note that from (26),

$$(66) \quad \sum_{|\alpha|+|\bar{\beta}|\leq 1} \left\| |\partial_{\bar{\beta}}^{\alpha_1} g|_2 \right\|_{H^{\frac{3}{4}}} \lesssim \mathcal{E}_{2;2,0}(g).$$

By Lemma 4, the second term in (65) is bounded by a desired upper bound:

$$\begin{aligned}
 & \eta \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{\sigma,w}^2 + C_{l,m,\eta} \mathcal{E}_{2;2,0}(g_1) \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{\sigma,w} \right\|_{H^{\frac{3}{4}}}^2 \\
 & + C_{l,m,\eta} \sum_{|\alpha|+|\bar{\beta}|\leq 1} \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \left\| |\partial_{\beta-\beta_1}^{\alpha-\alpha_1} g_2|_{2,w} \right\|_{H^{\frac{3}{4}}}^2 \\
 & \lesssim \eta \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{\sigma,w}^2 + C_{l,m,\eta} \eta_1 \mathcal{E}_{2;2,0}(g_1) \sum_{|\alpha'|+|\beta'|=m} \left\| \partial_{\beta'}^{\alpha'} g_2 \right\|_{\sigma,w(\alpha',\beta')}^2 \\
 & + C_{l,m,\eta,\eta_1} \mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{m-1;l,q}(g_2) + C_{l,m,\eta} \sum_{|\alpha|+|\bar{\beta}|\leq 1} \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{m;l,q}(g_2) \\
 & \lesssim \eta \left\| \partial_{\bar{\beta}}^{\alpha} g_2 \right\|_{\sigma,w}^2 + \eta \mathcal{E}_{2;2,0}(g_1) \sum_{|\alpha'|+|\beta'|=m} \left\| \partial_{\beta'}^{\alpha'} g_2 \right\|_{\sigma,w(\alpha',\beta')}^2 \\
 & + C_{l,m,\eta} \sum_{|\alpha|+|\bar{\beta}|\leq 1} \left\| |\partial_{\bar{\beta}}^{\alpha_1} g_1|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{m;l,q}(g_2) + C_{l,m,\eta} \mathcal{E}_{2;2,0}(g_1) \mathcal{D}_{m-1;l,q}(g_2),
 \end{aligned}$$

with η_1 still smaller ($w(\alpha, \beta) \leq w(\alpha', \beta')$).

Similarly, the second and the third terms in (65) are bounded by a desired upper bound of $(w = w(\alpha, \beta))$:

$$\begin{aligned} & \eta \|\partial_\beta^\alpha g_2\|_{\sigma, w}^2 + C_{l, m, \eta} \left\{ \sum_{|\alpha'| + |\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, w} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{m; l, q}(g_1) \right. \\ & \quad \left. + \sum_{|\alpha - \alpha_1| + |\beta - \beta_1| \leq 1} \left\| |\partial_\beta^{\alpha_1} g_1|_{\sigma, w} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{2; 2, 0}(g_2) \right\} \\ & \lesssim \eta \|\partial_\beta^\alpha g_2\|_{\sigma, w}^2 + C_{l, m, \eta} \sum_{|\alpha'| + |\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} g_2|_{\sigma, w} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{m; l, q}(g_1) \\ & \quad + \eta \mathcal{E}_{2; 2, 0}(g_2) \sum_{|\alpha'| + |\beta'| = m} \left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma, w(\alpha', \beta')} \right\|_{H^{\frac{3}{4}}}^2 + C_{l, m, \eta} \mathcal{E}_{2; 2, 0}(g_2) \mathcal{D}_{m-1; l, q}(g_1), \end{aligned}$$

where we have applied compact imbedding again to $\left\| |\partial_{\beta'}^{\alpha'} g_1|_{\sigma, w} \right\|_{H^{\frac{3}{4}}}^2$. By (62), we complete the proof of the proposition. $\square$

In order to obtain decay for the electric field, we now treat purely spatial derivatives with more details about $\int \langle \partial^\alpha \Gamma_\pm(f, f), \partial^\alpha f_\pm \rangle$.

Lemma 8. (1) If $|\alpha| = 0, 1, 2$, then

$$\int \langle \partial^\alpha \Gamma_\pm(f, f), \partial^\alpha f_\pm \rangle dx \lesssim \sqrt{\mathcal{E}_{2; 2, 0}(f)} \sum_{|\alpha'| \leq |\alpha|} \|\partial^{\alpha'} f\|_\sigma^2.$$

(2) If $|\alpha| = m \geq 3$, then for any $\eta > 0$ there exists $C_{m, \eta} > 0$ such that

$$\begin{aligned} \int |\langle \partial^\alpha \Gamma_\pm(f, f), \partial^\alpha f_\pm \rangle| dx & \lesssim [\sqrt{\mathcal{E}_{2; 2, 0}(f)} + \eta] \sum_{|\alpha'| = m} \|\partial^{\alpha'} f\|_\sigma^2 \\ & \quad + C_{m, \eta} [\mathcal{D}_{2; 2, 0}(f) \mathcal{E}_{m; l, q}(f) + \mathcal{E}_{m-1; l, q}(f) \mathcal{D}_{m-1; l, q}(f)]. \end{aligned}$$

Proof. Since there is no weight w , we apply Theorem 3 in [G1] with $w = 1$:

$$(67) \quad \int \langle \partial^\alpha \Gamma_\pm(f, f), \partial^\alpha f_\pm \rangle dx \lesssim \sum_{\alpha_1 \leq \alpha} C_{\alpha_1}^\alpha \int |\partial^{\alpha - \alpha_1} f|_2 |\partial^{\alpha_1} f|_\sigma |\partial^\alpha f|_\sigma dx.$$

We now estimate case by case according to $|\alpha|$.

Assume $|\alpha| = 0$. We can take $\sup_x |f|_2 \lesssim \sqrt{\mathcal{E}_{2; 2, 0}(f)}$ by (25) and $L^\infty - L^2 - L^2$ in (67) to obtain an upper bound of

$$\int |f|_2 |f|_\sigma |f|_\sigma dx \lesssim \sqrt{\mathcal{E}_{2; 2, 0}(f)} \|f\|_\sigma^2.$$

Assume $|\alpha| = 1$. We take $L^4 - L^4 - L^2$ in (67) if $\alpha_1 = 0$, and $L^\infty - L^2 - L^2$ in (67) if $|\alpha_1| = 1$. By Lemma 4 with $\eta = 1$ and $w = 1$, we obtain an upper bound of

$$\sum_{\alpha_1 \leq \alpha} \int |\partial^{\alpha - \alpha_1} f|_2 |\partial^{\alpha_1} f|_\sigma |\partial^\alpha f|_\sigma dx \lesssim \sqrt{\mathcal{E}_{2; 2, 0}(f)} \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_\sigma^2.$$

Assume $|\alpha| = 2$. If $\alpha_1 = 0$, we take $L^2 - L^\infty - L^2$; if $|\alpha_1| = 1$, we take $L^4 - L^4 - L^2$; if $|\alpha_1| = 2$, we take $L^\infty - L^2 - L^2$ in (67) respectively. By Lemma 4 with $w = 1$

and $\eta = 1$, we obtain

$$\sum_{\alpha_1 \leq \alpha} \int |\partial^{\alpha-\alpha_1} f|_2 |\partial^{\alpha_1} f|_\sigma |\partial^\alpha f|_\sigma dx \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)} \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_\sigma^2.$$

Combining with the cases $|\alpha| = 0, 1, 2$, we conclude part (1) of the lemma.

Assume $|\alpha| = m \geq 3$. We need to be careful with the (large) constant $C_\alpha^{\alpha_1}$. We single out the terms with highest-order derivatives with either $\alpha_1 = 0$ or $\alpha_1 = \alpha$. By taking L^∞ on the term without any derivatives with $\sup_x |f|_\sigma \lesssim \sqrt{\mathcal{D}_{2;2,0}(f)}$, $\sup_x |f|_\sigma \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)}$, we apply Lemma 4 with $w = 1$ to find an upper bound of (67):

$$\begin{aligned} & \int [|f|_2 |\partial^\alpha f|_\sigma^2 + |\partial^\alpha f|_2 |f|_\sigma |\partial^\alpha f|_\sigma] dx \\ & \leq \sqrt{\mathcal{E}_{2;2,0}(f)} \|\partial^\alpha f\|_\sigma^2 + \sqrt{\mathcal{D}_{2;2,0}(f)} \|\partial^\alpha f\|_2 \|\partial^\alpha f\|_\sigma \\ & \leq [\sqrt{\mathcal{E}_{2;2,0}(f)} + \eta] \|\partial^\alpha f\|_\sigma^2 + C_\eta \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f). \end{aligned}$$

For the remaining cases of $1 \leq |\alpha_1| \leq |\alpha| - 1$, we always take $L^4 - L^4 - L^2$ and use Lemma 4 with a small constant η (depending on α). By singling out the cases of $|\alpha_1| = m - 1$ and $|\alpha_1| = 1$, and combining the rest of the lower-order terms together, we bound (67) by

$$\begin{aligned} & \eta C_m \sum_{|\gamma|=1} \|\partial^{\alpha-\alpha_1} \partial^\gamma f\|_2 \|\partial^{\alpha_1} \partial^\gamma f\|_\sigma \|\partial^\alpha f\|_\sigma + C_{m,\eta} \|\partial^{\alpha-\alpha_1} f\|_2 \|\partial^{\alpha_1} f\|_\sigma \|\partial^\alpha f\|_\sigma \\ & \leq \eta C_m [\sqrt{\mathcal{E}_{2;2,0}(f)} \sum_{|\gamma|=1, |\alpha_1|=m-1} \|\partial^\gamma \partial^{\alpha_1} f\|_\sigma^2 \\ & \quad + \sqrt{\mathcal{D}_{2;2,0}(f)} \sum_{|\gamma|=1, |\alpha_1|=1} \|\partial^\gamma \partial^{\alpha-\alpha_1} f\|_2 \|\partial^\alpha f\|_\sigma] \\ & \quad + C_{m,\eta} \sqrt{\mathcal{E}_{m-1;l,q}(f)} \sqrt{\mathcal{D}_{m-1;l,q}(f)} \|\partial^\alpha f\|_\sigma \\ & \leq \eta [C_m \sqrt{\mathcal{E}_{2;2,0}(f)} + 1] \sum_{|\alpha|=m} \|\partial^\alpha f\|_\sigma^2 \\ & \quad + C_{m,\eta} [\mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) + \mathcal{E}_{m-1;l,q}(f) \mathcal{D}_{m-1;l,q}(f)]. \end{aligned}$$

We thus conclude the lemma by further adjusting η (depending on m). $\square$

Next we estimate the other nonlinear terms with the electric field.

Lemma 9. Let $|\alpha| + |\beta| = m \geq 1$. For $\alpha_1 < \alpha$ with $w = w(\alpha, \beta)$ defined in (24),

$$\begin{aligned} & \int |w^2 \partial_\beta^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \nabla_v \partial_\beta^{\alpha_1} f_\pm| + \int |w^2 \partial_\beta^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \partial_\beta [v \partial^{\alpha_1} f_\pm]| \\ & \lesssim \eta \|\partial_\beta^\alpha f_\pm\|_{\sigma,w}^2 + C_\eta \left\| |f_\pm|_{\sigma, \frac{w(0,0)}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \|\partial^\alpha \nabla_x^2 \phi\|_2^2 \\ & \quad + C_\eta [\|\nabla_x^2 \phi\|_{H^{m-1}}^2 + \|\nabla_x^4 \phi\|_2^2] \mathcal{D}_{m-1;l,q}(f). \end{aligned}$$

Proof. We first treat the first term in the lemma. Noting that $\alpha_1 < \alpha$,

$$(68) \quad w(\alpha, \beta) = w(\alpha_1, \beta) \langle v \rangle^{-2|\alpha|+2|\alpha_1|} = w(\alpha_1, \beta) \langle v \rangle^{-2[|\alpha|-|\alpha_1|-1]}.$$

From (19), we obtain

$$\begin{aligned}
 & - \int w^2 \partial_\beta^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \nabla_v \partial_\beta^{\alpha_1} f_\pm \\
 & \lesssim \int |w \langle v \rangle^{-1/2} \partial_\beta^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \langle v \rangle^{-2[|\alpha|-|\alpha_1|-1]} w(\alpha_1, \beta) \langle v \rangle^{-3/2} \nabla_v \partial_\beta^{\alpha_1} f_\pm| \\
 (69) \quad & \lesssim \int |\partial_\beta^\alpha f_\pm|_{\sigma, w} |\partial^{\alpha-\alpha_1} \nabla_x \phi| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^{2[|\alpha|-|\alpha_1|-1]}}} dx.
 \end{aligned}$$

We now separate three cases.

Case 1. $|\alpha_1| + |\beta| = m - 1$ or $m = 1$. If $m = 1$, $|\alpha| - |\alpha_1| = 1$. Otherwise, since $|\alpha_1| + |\beta| + |\alpha - \alpha_1| = m$ and $\alpha_1 < \alpha$, we also have $|\alpha| - |\alpha_1| = 1$ if $|\alpha_1| + |\beta| = m - 1$. Taking $L^2 - L^\infty - L^2$ in (69) yields a desired upper bound:

$$||\partial_\beta^\alpha f_\pm||_{\sigma, w} ||\nabla_x^4 \phi||_2 \left\| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, w(\alpha_1, \beta)} \right\|_2 \lesssim \eta ||\partial^\alpha f_\pm||_{\sigma, w}^2 + C_\eta ||\nabla_x^4 \phi||_2^2 \mathcal{D}_{m-1; l, q}(f).$$

For the rest of the cases, we take $L^2 - L^4 - L^4$ of (69) to get an upper bound:

$$(70) \quad ||\partial_\beta^\alpha f_\pm||_{\sigma, w} ||\partial^{\alpha-\alpha_1} \nabla_x \phi||_{H^{\frac{3}{4}}} \left\| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^{2[|\alpha|-|\alpha_1|-1]}}} \right\|_{H^{\frac{3}{4}}}.$$

Case 2. $|\alpha - \alpha_1| = m \geq 2$. We have $\alpha_1 = \beta = 0$ now. Since $|\alpha| - |\alpha_1| - 1 \geq 1$ now, we further estimate (70) by

$$\begin{aligned}
 & ||\partial^\alpha f_\pm||_{\sigma, w} ||\partial^\alpha \nabla_x \phi||_{H^{\frac{3}{4}}} \left\| |f_\pm|_{\sigma, \frac{w(0,0)}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \\
 & \lesssim \eta ||\partial^\alpha f_\pm||_{\sigma, w}^2 + C_\eta \left\| |f_\pm|_{\sigma, \frac{w(0,0)}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 ||\partial^\alpha \nabla_x^2 \phi||_2^2.
 \end{aligned}$$

Case 3. $|\alpha_1| + |\beta| \leq m - 2$ and $|\alpha - \alpha_1| \leq m - 1$. By $|\alpha_1| + |\alpha - \alpha_1| + |\beta| = m$, we deduce $|\alpha - \alpha_1| \geq 2$. Hence $\langle v \rangle^{-2[|\alpha|-|\alpha_1|-1]} \leq \langle v \rangle^{-2}$. We estimate (70) as

$$\begin{aligned}
 & ||\partial_\beta^\alpha f_\pm||_{\sigma, w} ||\partial^{\alpha-\alpha_1} \nabla_x \phi||_{H^{\frac{3}{4}}} \left\| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \\
 & \lesssim \eta ||\partial_\beta^\alpha f_\pm||_{\sigma, w}^2 + C_\eta ||\partial^{\alpha-\alpha_1} \nabla_x \phi||_{H^1}^2 \left\| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^2}} \right\|_{H^1} \\
 & \lesssim \eta ||\partial_\beta^\alpha f_\pm||_{\sigma, w}^2 + C_\eta ||\nabla^2 \phi||_{H^{m-1}}^2 \mathcal{D}_{m-1; l, q}(f),
 \end{aligned}$$

since $1 \leq |\alpha - \alpha_1| \leq m - 1$ and $||\partial^{\alpha-\alpha_1} \nabla_x \phi||_{H^1}^2 \lesssim ||\nabla^2 \phi||_{H^{m-1}}^2$. Here we have used the fact that $\frac{w(\alpha_1, \beta)}{\langle v \rangle^2} \leq w(\alpha_1 + \gamma, \beta)$ for $|\gamma| \leq 1$ from (24) so that

$$\begin{aligned}
 \left\| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^2}} \right\|_{H^1} &= \sum_{|\gamma| \leq 1} \left\| |\partial^\gamma \partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^2}} \right\|_2 \\
 &\lesssim \sum_{|\gamma| \leq 1} \left\| |\partial^\gamma \partial_\beta^{\alpha_1} f_\pm|_{\sigma, w(\alpha_1 + \gamma, \beta)} \right\|_2 \\
 &\lesssim \mathcal{D}_{m-1; l, q}(f).
 \end{aligned}$$

We now turn to the second term in the lemma. Similarly, by (68), $w(\alpha, \beta) = w(\alpha_1, \beta - \mathbf{e}_i) \langle v \rangle^{-4} \langle v \rangle^{-2[|\alpha| - |\alpha_1| - 1]}$, and we have from (19):

$$\begin{aligned}
& \int w^2 \partial_\beta^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \partial_\beta [v \partial^{\alpha_1} f_\pm] \\
& \leq \left| \int w^2 \partial_\beta^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot v \partial_\beta^{\alpha_1} f_\pm \right| + C_\beta \left| \int w^2 \partial_\beta^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \delta_\beta^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha_1} f_\pm \right| \\
& \leq \int |w \langle v \rangle^{-\frac{1}{2}} \partial_\beta^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \frac{w(\alpha_1, \beta)}{\langle v \rangle^{2[|\alpha| - |\alpha_1| - 1]}} \langle v \rangle^{-\frac{3}{2}} \partial_\beta^{\alpha_1} f_\pm| \\
& \quad + C_\beta \int |w \langle v \rangle^{-\frac{1}{2}} \partial_\beta^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \frac{w(\alpha_1, \beta - \mathbf{e}_i)}{\langle v \rangle^{2[|\alpha| - |\alpha_1|]}} \langle v \rangle^{-\frac{3}{2}} \delta_\beta^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha_1} f_\pm| \\
& \leq C_m \int |\partial_\beta^\alpha f_\pm|_{\sigma, w(\alpha, \beta)} |\partial^{\alpha - \alpha_1} \nabla_x \phi| |\partial_\beta^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta)}{\langle v \rangle^{2[|\alpha| - |\alpha_1| - 1]}}} \\
& \quad + C_m \int |\partial_\beta^\alpha f_\pm|_{\sigma, w(\alpha, \beta)} |\partial^{\alpha - \alpha_1} \nabla_x \phi| |\delta_\beta^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta - \mathbf{e}_i)}{\langle v \rangle^{2[|\alpha| - |\alpha_1|]}}} .
\end{aligned}$$

The first term is estimated exactly as in (70). For the second term, we note that $|\beta| \geq 1$ so that $|\alpha - \alpha_1| \leq m - 1$ and $|\alpha_1| + |\beta| - 1 \leq m - 2$. Since $\alpha_1 < \alpha$,

$$|\delta_\beta^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta - \mathbf{e}_i)}{\langle v \rangle^{2[|\alpha| - |\alpha_1|]}}} \leq |\delta_\beta^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha_1} f_\pm|_{\sigma, \frac{w(\alpha_1, \beta - \mathbf{e}_i)}{\langle v \rangle^2}} .$$

We thus can apply case 3 above (to α and $\beta - \mathbf{e}_i$) to complete the proof. $\square$

We also need more precise estimates for purely spatial derivatives of $E \cdot \nabla_v f_\pm$.

Lemma 10. *Let $-\Delta \phi = \int \sqrt{\mu} [f_+ - f_-] dv$ with $\int \phi = 0$.*

(1) *For $|\alpha| = 1, 2$, then for $\alpha_1 < \alpha$,*

$$\begin{aligned}
& \int \left| \int \partial^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \nabla_v \partial^{\alpha_1} f_\pm dv \right| dx + \int \left| \int \partial^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_\pm dv \right| dx \\
& \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)} \sum_{|\alpha'| \leq |\alpha|} \|\partial^{\alpha'} f\|_\sigma^2 .
\end{aligned}$$

(2) *For $|\alpha| = m \geq 3$, then for $\alpha_1 < \alpha$, any $\eta > 0$,*

$$\begin{aligned}
& \int \left| \int \partial^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot \nabla_v \partial^{\alpha_1} f_\pm dv \right| dx + \int \left| \int \partial^\alpha f_\pm \partial^{\alpha - \alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_\pm dv \right| dx \\
& \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)} \|\partial^\alpha f\|_\sigma^2 + \eta \sum_{|\alpha'| = |\alpha|} \|\partial^{\alpha'} f\|_\sigma^2 + C_{m,\eta} \mathcal{D}_{m-1;l,0}(f) \mathcal{E}_{m-1;l,0}(f) .
\end{aligned}$$

Proof. We first perform integration by parts in v to get

$$\begin{aligned}
 & \int \left| \int \partial^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \nabla_v \partial^{\alpha_1} f_\pm dv \right| + \int \left| \int \partial^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_\pm dv \right| \\
 &= \int \left| \int \nabla_v \partial^\alpha f_\pm \cdot \partial^{\alpha-\alpha_1} \nabla_x \phi \partial^{\alpha_1} f_\pm dv \right| dx \\
 (71) \quad &+ \int \left| \int \partial^\alpha f_\pm \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_\pm dv \right| dx \\
 &\leq \int [|\langle v \rangle^{-\frac{3}{2}} \partial^\alpha \nabla_v f_\pm| + |\langle v \rangle^{-\frac{1}{2}} \partial^\alpha f_\pm|] |\partial^{\alpha-\alpha_1} \nabla_x \phi| |\langle v \rangle^{\frac{3}{2}} \partial^{\alpha_1} f_\pm| \\
 &\lesssim \int |\partial^\alpha f_\pm|_\sigma |\partial^{\alpha-\alpha_1} \nabla_x \phi| |\langle v \rangle^{3/2} \partial^{\alpha_1} f_\pm|_2 dx.
 \end{aligned}$$

Here we have used the norm (19) for $\partial^\alpha f_\pm$.

If $|\alpha| = 1$, then since $\alpha_1 < \alpha$, so $\alpha_1 = 0$. From the elliptic estimate:

$$||\partial^\alpha \nabla_x \phi||_4 \lesssim ||\partial^\alpha \nabla_x^2 \phi||_2 = ||\int \sqrt{\mu} \partial^\alpha f_\pm||_2 \lesssim ||\partial^\alpha f||_\sigma$$

and $\sum_{|\gamma| \leq 1} ||\langle v \rangle^{3/2} \partial^\gamma f_\pm||_2 \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)}$. We take $L^2 - L^4 - L^4$ to get

$$\begin{aligned}
 (72) \quad (71) &\lesssim ||\partial^\alpha f||_\sigma ||\partial^\alpha \nabla_x \phi||_4 \sum_{|\gamma| \leq 1} ||\langle v \rangle^{3/2} \partial^\gamma f_\pm||_2 \\
 &\lesssim ||\partial^\alpha f||_\sigma^2 \times \sum_{|\gamma| \leq 1} ||\langle v \rangle^{3/2} \partial^\gamma f_\pm||_2 \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)} ||\partial^\alpha f||_\sigma^2.
 \end{aligned}$$

If $|\alpha| = 2$, the case $\alpha_1 = 0$ is treated as in (72) and we only need to treat the case of $|\alpha_1| = 1$. Note that $||\langle v \rangle^{3/2} \partial^{\alpha_1} f_\pm||_2 \lesssim \sqrt{\mathcal{E}_{2;2,0}(f)}$. We take $L^2 - L^\infty - L^2$ in (71) to obtain

$$\begin{aligned}
 (71) &\lesssim \int |\partial^\alpha f_\pm|_\sigma |\partial^{\alpha-\alpha_1} \nabla_x \phi| |\langle v \rangle^{3/2} \partial^{\alpha_1} f_\pm|_2 dx \\
 &\lesssim ||\partial^\alpha f_\pm||_\sigma ||\nabla_x \partial^{\alpha-\alpha_1} \phi||_\infty ||\langle v \rangle^{3/2} \partial^{\alpha_1} f_\pm||_2 \\
 (73) \quad &\lesssim \sum_{|\alpha'| \leq |\alpha|} ||\partial^{\alpha'} f_\pm||_\sigma^2 \sqrt{\mathcal{E}_{2;2,0}(f)}.
 \end{aligned}$$

Here we have used the elliptic estimate ($|\alpha_1| = 1$):

$$||\nabla_x \partial^{\alpha-\alpha_1} \phi||_\infty \lesssim ||\nabla_x^2 \phi||_\infty \lesssim \sum_{|\alpha'| \leq |\alpha|} ||\partial^{\alpha'} f||_\sigma$$

with $|\alpha| = 2$. This completes the proof of the first part of the lemma.

If $|\alpha| = m \geq 3$, the case $|\alpha_1| = 0$ is again treated in (72). When $|\alpha_1| = m-1$, as in (73), by (25), $||\langle v \rangle^{3/2} \partial^{\alpha_1} f_\pm||_2 \lesssim \mathcal{E}_{m-1;l,q}(f)$, and $||\nabla_x^2 \phi||_\infty \lesssim \sqrt{\mathcal{D}_{2;2,0}(f)}$. We take $L^2 - L^\infty - L^2$ in (71) to get

$$\begin{aligned}
 (71) &\lesssim ||\partial^\alpha f_\pm||_\sigma ||\nabla_x^2 \phi||_\infty ||\langle v \rangle^{3/2} \partial^{\alpha_1} f_\pm||_2 \\
 &\lesssim \eta ||\partial^\alpha f_\pm||_\sigma^2 + C_\eta \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m-1;l,q}(f).
 \end{aligned}$$

When $1 \leq |\alpha_1| \leq m-2$, by (25) and (26),

$$\begin{aligned} \|\partial^{\alpha-\alpha_1} \nabla_x \phi\|_4 &\lesssim \|\partial^{\alpha-\alpha_1} \nabla_x^2 \phi\|_2 \lesssim \|\partial^{\alpha-\alpha_1} f\|_2 \lesssim \sqrt{\mathcal{D}_{m-1;l,0}(f)}, \\ \sum_{|\gamma| \leq 1} \|\langle v \rangle^{3/2} \partial^{\alpha_1+\gamma} f_{\pm}\|_2 &\lesssim \sqrt{\mathcal{E}_{m-1;l,0}(f)}. \end{aligned}$$

We obtain by taking $L^2 - L^4 - L^4$ in (71):

$$\begin{aligned} \|\partial^{\alpha} f_{\pm}\|_{\sigma} \|\partial^{\alpha-\alpha_1} \nabla_x \phi\|_4 \sum_{|\gamma| \leq 1} \|\langle v \rangle^{3/2} \partial^{\alpha_1+\gamma} f_{\pm}\|_2 \\ \lesssim \eta \|\partial^{\alpha} f_{\pm}\|_{\sigma}^2 + C_{m,\eta} \mathcal{D}_{m-1;l,0}(f) \mathcal{E}_{m-1;l,0}(f). \end{aligned}$$

This completes the proof of the lemma. $\square$

3. LOCAL SOLUTIONS

Our goal is to construct a unique local-in-time solution to the Vlasov-Poisson-Landau system (8) and (9) if $\mathcal{E}_{2;2,0}(f_0)$ is sufficiently small. The construction is based on a uniform energy estimate for a sequence of iterating approximate solutions. We first note, by direct computations, $\sum_{ij} \partial_{ij} \Phi^{ij} = 8\pi\delta$ and the Landau collision operator has the following nondivergent form of [G1]:

$$Q(G_1, G_2) = \{\Phi^{ij} * G_1\} \partial_{ij} G_2 + 8\pi G_1 G_2.$$

We start with

$$(74) \quad F^0(t, x, v) = \mu \quad \text{or} \quad f^0 \equiv 0.$$

To preserve the positivity for F^{n+1} , we design the following iterating sequence of $F_{\pm}^{n+1}$ as [SG3]:

$$\begin{aligned} (75) \quad [\partial_t + v \cdot \nabla_x \mp \nabla_x \phi^n \cdot \nabla_v] F_{\pm}^{n+1} &= Q(F_{\pm}^n, F_{\pm}^{n+1}) - 8\pi F_{\pm}^n (F_{\pm}^{n+1} - F_{\pm}^n) \\ &\quad + Q(F_{\mp}^n, F_{\pm}^{n+1}) - 8\pi F_{\mp}^n (F_{\pm}^{n+1} - F_{\pm}^n) \\ &= \Phi^{ij} * [F_{\pm}^n + F_{\mp}^n] \partial_{ij} F_{\pm}^{n+1} + 8\pi \{F_{\pm}^n\}^2 + 8\pi F_{\mp}^n F_{\pm}^n, \\ \Delta \phi^{n+1} &= - \int (F_{+}^{n+1} - F_{-}^{n+1}) dv \end{aligned}$$

with $\int_{\mathbf{T}^3} \phi^{n+1} = 0$. We note that $F_{\pm}^n \geq 0$ implies that $F_{\pm}^{n+1} \geq 0$ from (75) by the Maximum Principle. We now rewrite the above iteration in the perturbation form of $F^{n+1} = \mu + \sqrt{\mu} f^{n+1}$:

$$\begin{aligned} (76) \quad [\partial_t + v \cdot \nabla_x \mp \nabla_x \phi^n \cdot \nabla_v] f_{\pm}^{n+1} &- A f_{\pm}^{n+1} \pm \nabla_x \phi^n \cdot v f_{\pm}^{n+1} \\ &= \mp 2 \nabla_x \phi^n \cdot v \sqrt{\mu} + K_{\pm} f^n + \Gamma_{\pm}(f^n, f^{n+1}) \\ &\quad - 8\pi (f_{\pm}^n + f_{\mp}^n) \sqrt{\mu} (f_{\pm}^{n+1} - f_{\pm}^n) - 16\pi \mu (f_{\pm}^{n+1} - f_{\pm}^n) \\ &- \Delta \phi^{n+1} = \int (f_{+}^{n+1} - f_{-}^{n+1}) \sqrt{\mu} dv. \end{aligned}$$

with $f^{n+1}|_{t=0} = f_0$. Here for $g = \begin{pmatrix} g_+ \\ g_- \end{pmatrix}$, we denote as in [G1], [SG1]–[SG2]:

$$\begin{aligned} Ag_{\pm} &= \frac{2}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} g_{\pm}), \\ K_{\pm} g &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} [g_{\pm} + g_{\mp}], \mu). \end{aligned}$$

To solve such an f^{n+1} , we can add an artificial dissipation

$$\varepsilon \{A^1 f_{\pm}^{n+1} + \Delta_x (1 + |v|^2) f^{n+1}\}$$

with

$$A^1 g_{\pm} = \frac{2}{\sqrt{\mu}} Q(\mu, \sqrt{\mu} g_{\pm}) \quad \text{with } \Phi^1(u) = |u| \left(I - \frac{u \otimes u}{|u|^2} \right) \text{ in (3).}$$

This choice makes the problem strongly parabolic in both x and v with a strong bound in v which justifies the moments estimates [SG2]. We shall construct f^{n+1} as $\varepsilon \rightarrow 0$ with uniform bound in ε . The procedure is standard and for notational brevity, we ignore such a regularization. We take $\partial_{\beta}^{\alpha}$ of the Landau-Poisson system (setting $\varepsilon = 0$):

$$\begin{aligned} & [\partial_t + v \cdot \nabla_x \mp \nabla_x \phi^n \cdot \nabla_v] \partial_{\beta}^{\alpha} f_{\pm}^{n+1} \pm [\nabla_x \phi^n \cdot v \partial_{\beta}^{\alpha} f_{\pm}^{n+1}] - \partial_{\beta}^{\alpha} A f_{\pm}^{n+1} \\ &= -\delta_{\beta}^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha + \mathbf{e}_i} f_{\pm}^{n+1} \\ &\quad \pm \sum_{\alpha_1 < \alpha} C_{\alpha}^{\alpha_1} \{ \partial^{\alpha - \alpha_1} \nabla_x \phi^n \cdot \nabla_v \partial_{\beta}^{\alpha_1} f_{\pm}^{n+1} - \partial^{\alpha - \alpha_1} \nabla_x \phi^n \cdot \partial_{\beta} [v \partial^{\alpha_1} f_{\pm}^{n+1}] \} \\ &\quad \mp 2 \partial_{\beta}^{\alpha} [\nabla_x \phi^n \cdot v \sqrt{\mu}] + \partial_{\beta}^{\alpha} K_{\pm} f^n + \partial_{\beta}^{\alpha} \Gamma_{\pm}(f^n, f^{n+1}) \\ &\quad - 8\pi \partial_{\beta}^{\alpha} [\sqrt{\mu} (f_{\pm}^n + f_{\mp}^n)(f_{\pm}^{n+1} - f_{\pm}^n)] - 16\pi \partial_{\beta}^{\alpha} [\mu (f_{\pm}^{n+1} - f_{\pm}^n)]. \end{aligned}$$

We multiply by $e^{(2q \pm 2)\phi_n} w^2 \partial_{\beta}^{\alpha} f_{\pm}^{n+1}$ with w in (24) to get:

$$\begin{aligned} (77) \quad & [e^{\pm 2(q+1)\phi_n} w^2 \partial_{\beta}^{\alpha} f_{\pm}^{n+1}] \times \{ [\partial_t + v \cdot \nabla_x \mp \nabla_x \phi^n \cdot \nabla_v] \partial_{\beta}^{\alpha} f_{\pm}^{n+1} \pm [\nabla_x \phi^n \cdot v \partial_{\beta}^{\alpha} f_{\pm}^{n+1}] \} \\ &= \frac{d}{dt} \left\{ \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2}{2} \right\} \mp (q+1) \phi_t^n e^{\pm 2(q+1)\phi_n} w^2 (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2 \\ &\quad + v \cdot \nabla_x \left\{ \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2}{2} \right\} \mp (q+1) v \cdot \nabla_x \phi^n e^{\pm 2(q+1)\phi_n} (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2 w^2 \\ &\quad \mp \nabla_x \phi^n \cdot \nabla_v \left\{ \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2}{2} \right\} \\ &\quad \pm q \nabla_x \phi^n \cdot v (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2 e^{\pm 2(q+1)\phi_n} w^2 \\ &\quad \pm \left\{ \frac{2(l - |\alpha| - |\beta|)}{1 + |v|^2} \right\} \nabla_x \phi^n \cdot v e^{\pm 2(q+1)\phi_n} w^2 (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2 \\ &\quad \pm \nabla_x \phi^n \cdot v (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2 w^2 e^{\pm 2(q+1)\phi_n}, \end{aligned}$$

where we have used (42). Our weight function is so designed such that there is an exact cancellation for the high momentum contributions:

$$\begin{aligned}
 & \mp (q+1)v \cdot \nabla_x \phi^n e^{\pm 2(q+1)\phi_n} (\partial_\beta^\alpha f_\pm^{n+1})^2 w^2 \\
 & \pm q \nabla_x \phi^n \cdot v e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2 \\
 & \pm \nabla_x \phi^n \cdot v (\partial_\beta^\alpha f_\pm^{n+1})^2 w^2 e^{\pm 2(q+1)\phi_n} \\
 & = 0.
 \end{aligned}
 \tag{78}$$

Therefore, we can rewrite (77) as:

$$\begin{aligned}
 & \frac{d}{dt} \left\{ \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2}{2} \right\} \mp (q+1) \phi_t^n e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2 \\
 & + v \cdot \nabla_x \left\{ \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2}{2} \right\} \\
 & \mp \nabla_x \phi^n \cdot \nabla_v \left\{ \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2}{2} \right\} \\
 & \pm \left\{ \frac{2(l - |\alpha| - |\beta|)}{1 + |v|^2} \right\} \nabla_x \phi^n \cdot v e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2.
 \end{aligned}
 \tag{79}$$

Upon integrating over $\mathbf{T}^3 \times \mathbf{R}^3$, and combining terms we obtain:

$$\frac{d}{dt} \left\{ \int \frac{e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2}{2} \right\} - \int \langle w^2 \partial_\beta^\alpha A f^{n+1}, \partial_\beta^\alpha f^{n+1} \rangle
 \tag{80}$$

$$= - \int e^{\pm 2(q+1)\phi_n} w^2 \delta_{\beta - \mathbf{e}_i}^\alpha \partial_{\beta - \mathbf{e}_i}^{\alpha + \mathbf{e}_i} f_\pm^{n+1} \partial_\beta^\alpha f_\pm^{n+1}
 \tag{81}$$

$$\pm \sum_{\alpha_1 < \alpha} C_{\alpha}^{\alpha_1} \int e^{\pm 2(q+1)\phi_n} w^2 \partial_\beta^\alpha f_\pm^{n+1} \partial^{\alpha - \alpha_1} \nabla_x \phi^n \cdot \nabla_v \partial_{\beta}^{\alpha_1} f_\pm^{n+1}
 \tag{82}$$

$$\mp \sum_{\alpha_1 < \alpha} C_{\alpha}^{\alpha_1} \int e^{\pm 2(q+1)\phi_n} w^2 \partial_\beta^\alpha f_\pm^{n+1} \partial^{\alpha - \alpha_1} \nabla_x \phi^n \cdot \partial_\beta [v \partial_{\beta}^{\alpha_1} f_\pm^{n+1}]
 \tag{83}$$

$$\mp \int \left[\frac{2(l - |\alpha| - |\beta|)}{1 + |v|^2} \nabla_x \phi^n \cdot v - (q+1) \phi_t^n \right] e^{\pm 2(q+1)\phi_n} w^2 (\partial_\beta^\alpha f_\pm^{n+1})^2
 \tag{84}$$

$$+ \int w^2 (e^{\pm 2(q+1)\phi_n} - 1) \partial_\beta^\alpha f_\pm^{n+1} \partial_\beta^\alpha A f_\pm^{n+1}
 \tag{85}$$

$$\mp 2 \int e^{\pm 2(q+1)\phi_n} w^2 \nabla_x \partial^\alpha \phi^n \cdot \partial_\beta [v \sqrt{\mu}] \partial_\beta^\alpha f_\pm^{n+1}
 \tag{86}$$

$$+ \int w^2 e^{(2q \pm 2)\phi_n} \partial_\beta^\alpha K_\pm f^n \partial_\beta^\alpha f_\pm^{n+1}
 \tag{87}$$

$$+ \int w^2 e^{\pm 2(q+1)\phi_n} \partial_\beta^\alpha \Gamma_\pm(f^n, f^{n+1}) \partial_\beta^\alpha f_\pm^{n+1}
 \tag{88}$$

$$- 8\pi \int w^2 e^{\pm 2(q+1)\phi_n} \partial_\beta^\alpha [\sqrt{\mu} (f_\pm^n + f_\mp^n) (f_\pm^{n+1} - f_\mp^{n+1})] \partial_\beta^\alpha f_\pm^{n+1}
 \tag{89}$$

$$- 16\pi \int w^2 e^{\pm 2(q+1)\phi_n} \partial_\beta^\alpha [\mu (f_\pm^{n+1} - f_\pm^n)] \partial_\beta^\alpha f_\pm^{n+1}.
 \tag{90}$$

We now derive estimates for (80) to (90).

Lemma 11. Assume for M sufficiently small that

$$\mathcal{E}_{2;2,0}(f^n) \leq M.$$

(1) We have

(91)

$$\begin{aligned} & \mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) ds \\ & \leq \frac{1}{8} \int_0^t \mathcal{D}_{2;l,q}(f^n) ds + C_l [\mathcal{E}_{2;l,q}(f_0) + \int_0^t \sum_{|\alpha| \leq 2} \{ \|\mu^{\frac{q-1}{8}} \partial^\alpha f^n\|_2^2 + \|\mu^{\frac{q-1}{8}} \partial^\alpha f^{n+1}\|_2^2 \}] \\ & \quad + C_l \int_0^t \left[\sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}}^2 + \|\nabla_x \phi^n\|_\infty + \|\partial_t \phi^n\|_\infty \right] \mathcal{E}_{2;l,q}(f^{n+1}) \\ & \quad + C_l \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;l,q}(f^n)} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\ & \quad + C_l \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;l,q}(f^n)} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} \\ & \quad + C_l \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \sqrt{\mathcal{D}_{2;l,q}(f^n)} \left\| |\partial_{\beta'}^{\alpha'} (f_\pm^{n+1} - f_\pm^n)|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\ & \quad + C_l \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1} - f^n)} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\ & \quad + C_l \int_0^t \sqrt{\mathcal{E}_{2;l,q}(\mu\{f^{n+1} - f^n\})} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}. \end{aligned}$$

(2) For $m \geq 3$, we have

(92)

$$\begin{aligned} & \mathcal{E}_{m;l,q}(f^{n+1}, \phi^n) + \int_0^t \mathcal{D}_{m;l,q}(f^{n+1}) ds \\ & \leq \frac{1}{8} \int_0^t \mathcal{D}_{m;l,q}(f^n) ds + C_l \mathcal{E}_{m;l,q}(f_0) \\ & \quad + C_{l,m} \int_0^t \sum_{|\alpha|=m} \{ \|\mu^{\frac{q-1}{8}} \partial^\alpha f^n\|_2^2 + \|\mu^{\frac{q-1}{8}} \partial^\alpha f^{n+1}\|_2^2 \} \\ & \quad + C_{l,m} \int_0^t \left[\sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \right\} \right. \\ & \quad \quad \left. + \|\nabla \phi^n\|_\infty + \|\partial_t \phi^n\|_\infty \right] \\ & \quad \times [\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})] \\ & \quad + C_{l,m} \int_0^t \sum_{|\alpha'|+|\beta'| \leq [\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{m;l,q}(f^{n+1} - f^n)} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \end{aligned}$$

$$\begin{aligned}
& + C_{l,m} \int_0^t \sum_{|\alpha'|+|\beta'|\leq[\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'}(f^{n+1} - f^n)|_{\sigma} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{m;l,q}(f^n)} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \\
& + C_{l,m} \int_0^t \sqrt{\mathcal{E}_{m;l,q}(\mu\{f^{n+1} - f^n\})} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \\
& + C_{l,m} \int_0^t [\mathcal{D}_{m-1;l,q}(f^n) + \mathcal{D}_{m-1;l,q}(f^{n+1})][\mathcal{E}_{m-1;l,q}(f^n) + \mathcal{E}_{m-1;l,q}(f^{n+1}) + 1].
\end{aligned}$$

We remark that the exponential factor $\mu^{\frac{q-1}{8}}$ above is chosen for convenience.

Proof. First note that $\Delta\phi^n = \int \sqrt{\mu}(f_+^n - f_-^n)dv$ and $\int \phi^n = 0$ so that $\|\phi^n\|_{\infty} \lesssim M$ by the elliptic estimate and

$$e^{\pm 2(q+1)\phi^n} \lesssim 1.$$

We shall estimate term by term from (80) to (90). In the second term of (80), we apply both Lemmas 8 and 9 in [SG2]. For $\beta = 0$,

$$(93) \quad - \int \langle w^2(\alpha, 0) \partial^{\alpha} A f_{\pm}^{n+1}, \partial^{\alpha} f_{\pm}^{n+1} \rangle dx \gtrsim \|\partial^{\alpha} f^{n+1}\|_{\sigma, w(\alpha, 0)}^2 - C_m \|\chi \partial^{\alpha} f^{n+1}\|_2^2,$$

$\chi(v)$ being a general cutoff function in v . For $\beta \neq 0$, for any $\eta > 0$, we have

$$\begin{aligned}
& - \int \langle w^2(\alpha, \beta) \partial_{\beta}^{\alpha} A f_{\pm}^{n+1}, \partial_{\beta}^{\alpha} f_{\pm}^{n+1} \rangle dx \\
(94) \quad & \gtrsim \|\partial_{\beta}^{\alpha} f^{n+1}\|_{\sigma, w(\alpha, \beta)}^2 - \eta \mathcal{D}_{m;l,q}(f^{n+1}) - C_{l,\eta} \sum_{\beta' < \beta} \|\partial_{\beta'}^{\alpha} f^{n+1}\|_{\sigma, w(\alpha, \beta')}^2.
\end{aligned}$$

From Lemma 6, we have for any $\eta > 0$ and $\beta \geq \mathbf{e}_i$,

$$\begin{aligned}
(81) \quad & \lesssim \|\delta_{\beta}^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha} f^{n+1}\|_{\sigma, w(\alpha, \beta - \mathbf{e}_i)} \|\partial_{\beta - \mathbf{e}_i}^{\alpha + \mathbf{e}_i} f_{\pm}^{n+1}\|_{\sigma, w(\alpha + \mathbf{e}_i, \beta - \mathbf{e}_i)} \\
(95) \quad & \leq \eta \mathcal{D}_{m;l,q}(f^{n+1}) + C_{\eta} \|\delta_{\beta}^{\mathbf{e}_i} \partial_{\beta - \mathbf{e}_i}^{\alpha} f_{\pm}^{n+1}\|_{\sigma, w(\alpha, \beta - \mathbf{e}_i)}.
\end{aligned}$$

To estimate (82) and (83), from the elliptic estimate and our assumption,

$$\begin{aligned}
\|\nabla^4 \phi^n\|_2 & \lesssim \|f^n\|_{H^2} \lesssim \sqrt{M}, \\
\|\nabla^2 \phi^n\|_{H^{m-1}} & \lesssim \left\| \int \sqrt{\mu} f^n \right\|_{H^{m-1}} \lesssim \sqrt{\mathcal{E}_{m-1;m-1,0}(f^n)}.
\end{aligned}$$

We deduce from Lemma 9 that

$$\begin{aligned}
(96) \quad (82) + (83) & \lesssim \eta \|\partial_{\beta}^{\alpha} f^{n+1}\|_{\sigma, w}^2 + C_{\eta} \left\| |f_{\pm}^{n+1}|_{\sigma, \frac{w(0,0)}{(v)^2}} \right\|_{H^{\frac{3}{4}}}^2 \mathcal{E}_{m;m,0}(f^n) \\
& + C_{\eta} [\mathcal{E}_{m-1;m-1,0}(f^n) + M] \mathcal{D}_{m-1;l,q}(f^{n+1}).
\end{aligned}$$

In particular, if $m = 2$, we have $\mathcal{E}_{m;m,0}(f^n) \leq M$ and by (24),

$$\left\| |f_{\pm}^{n+1}|_{\sigma, \frac{w(0,0)}{(v)^2}} \right\|_{H^1}^2 \lesssim \sum_{|\gamma| \leq 1} \|\partial^{\gamma} f_{\pm}^{n+1}\|_{\sigma, w(\gamma, 0)}^2 \lesssim \sum_{|\alpha|+|\beta| \leq 2} \|\partial_{\beta}^{\alpha} f_{\pm}^{n+1}\|_{\sigma, w(\alpha, \beta)}^2,$$

so that

$$(97) \quad (82) + (83) \lesssim (\eta + C_{\eta} M) \sum_{|\alpha|+|\beta| \leq 2} \|\partial_{\beta}^{\alpha} f_{\pm}^{n+1}\|_{\sigma, w}^2.$$

Next, we easily control

$$(98) \quad (84) \leq C_{l,m} \int_0^t \{ \|\phi_t^n\|_{\infty} + \|\nabla_x \phi^n\|_{\infty} \} w^2 (\partial_{\beta}^{\alpha} f_{\pm}^{n+1})^2.$$

To estimate (85), since $(e^{\pm 2(q+1)\phi^n} - 1) \lesssim \sqrt{M}$, we control from Lemmas 8-9 of [SG2]: for any $\eta > 0$,

$$(99) \quad (85) \lesssim \sqrt{M} \|\partial_\beta^\alpha f^{n+1}\|_{\sigma, w(\alpha, \beta)}^2 + C_{l, m} \sqrt{M} \sum_{|\alpha| \leq m} \|\partial^\alpha f^{n+1} \chi\|_2^2 \\ + \eta D_{m; l, q}(f^{n+1}) + C_{l, m, \eta} \sum_{\beta' < \beta} \|\partial_{\beta'}^\alpha f^{n+1}\|_{\sigma, w(\alpha, \beta')}^2.$$

In (86), since $0 \leq q < 1$ in (24), via repeated integration by parts in v , we can move all the v derivatives ∂_β out of $\partial_\beta^\alpha f^{n+1}$ to the factor $\partial_\beta[v\sqrt{\mu}w^2]$, so that

$$(100) \quad (86) \lesssim C_{l, m} \int \mu^{\frac{q-1}{8}} [|\partial^\alpha f^{n+1}|^2 + |\partial^\alpha f^n|^2] dv dx.$$

By Lemma 8 in [SG2], we have

$$(101) \quad (87) \lesssim \{\eta \sqrt{\mathcal{D}_{m; l, q}(f^n)} + C_{l, m, \eta} \sum_{|\alpha| \leq m} \|\chi \partial^\alpha f^n\|_2\} \sqrt{\mathcal{D}_{m; l, q}(f^{n+1})} \\ \lesssim \eta [\mathcal{D}_{m; l, q}(f^n) + \mathcal{D}_{m; l, q}(f^{n+1})] + C_{l, m, \eta} \sum_{|\alpha| \leq m} \|\chi \partial^\alpha f^n\|_2^2.$$

For (89), we first take a v integration to get

$$|w \partial_\beta^\alpha \{\mu^{\frac{1}{4}}(f_\pm^n + f_\mp^n)\} \{\mu^{\frac{1}{4}}(f_\pm^{n+1} - f_\mp^{n+1})\}|_2 \\ \lesssim \sum_{|\alpha'| + |\beta'| \leq m} |w \partial_{\beta - \beta'}^{\alpha - \alpha'} \{\mu^{\frac{1}{4}}(f_\pm^n + f_\mp^n)\} \partial_{\beta'}^{\alpha'} \{\mu^{\frac{1}{4}}(f_\pm^{n+1} - f_\mp^{n+1})\}|_2 \\ \lesssim \sum_{|\alpha'| + |\beta'| \leq m} |w \partial_{\beta - \beta'}^{\alpha - \alpha'} \{\mu^{\frac{1}{4}}(f_\pm^n + f_\mp^n)\}|_{L_v^4} |\partial_{\beta'}^{\alpha'} \{\mu^{\frac{1}{4}}(f_\pm^{n+1} - f_\mp^{n+1})\}|_{L_v^4}.$$

Here we have assumed $|\alpha - \alpha'| + |\beta - \beta'| \geq |\alpha'| + |\beta'|$ and combined the weight w with $\partial_{\beta - \beta'}^{\alpha - \alpha'}$. Due to the exponential decaying factor $\mu^{\frac{1}{4}}$, by (19) and the Sobolev imbedding of $\dot{H}^1(\mathbf{R}_v^3) \subset L^6(\mathbf{R}_v^3)$, we can further bound above by

$$|w \partial_{\beta - \beta'}^{\alpha - \alpha'} f^n|_\sigma |\partial_{\beta'}^{\alpha'} (f_\pm^{n+1} - f_\mp^n)|_\sigma.$$

By further taking $L^4 - L^4$ or $L^2 - L^\infty$ in the x integration and by Sobolev imbedding, we can bound above by

$$\sum_{|\alpha - \alpha'| + |\beta - \beta'| \geq |\alpha'| + |\beta'|} \left\| |w \partial_{\beta - \beta'}^{\alpha - \alpha'} f^n|_\sigma |\partial_{\beta'}^{\alpha'} (f_\pm^{n+1} - f_\mp^n)|_\sigma \right\|_2 \\ \lesssim C_l \sum_{|\alpha'| + |\beta'| \leq [\frac{m}{2}]} \sqrt{\mathcal{D}_{m; l, q}(f^n)} \left\| |\partial_{\beta'}^{\alpha'} (f_\pm^{n+1} - f_\mp^n)|_\sigma \right\|_{H^{\frac{3}{4}}}.$$

The case of $|\alpha - \alpha'| + |\beta - \beta'| \leq |\alpha'| + |\beta'|$ is similarly bounded by

$$C_l \sum_{|\alpha'| + |\beta'| \leq [\frac{m}{2}]} \sqrt{\mathcal{D}_{m; l, q}(f^{n+1} - f^n)} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}}.$$

Finally, (90) is bounded by $\sqrt{\mathcal{E}_{2; l, q}(\mu\{f^{n+1} - f^n\})} \sqrt{\mathcal{E}_{2; l, q}(f^{n+1})}$.

We now prove the first part (1) of the lemma. Applying Proposition 7 to (88) with $g_1 = f^n$, $g_2 = f^{n+1}$, we choose η small and M (independent of l !) even

smaller such that $C_\eta M \ll 1$ in (97). By (24) and (26), $|\partial_{\beta'}^{\alpha'} f|_\sigma \lesssim |\partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}}$ for $|\alpha'| + |\beta'| \leq 1$. We collect terms to get

$$\begin{aligned}
& \|\partial_\beta^\alpha f^{n+1}\|_{2,w}^2 + \int_0^t \|\partial_\beta^\alpha f^{n+1}\|_{\sigma,w}^2 \\
& \lesssim C_l \|\partial_\beta^\alpha f^{n+1}(0)\|_w^2 + \eta \int_0^t \{\mathcal{D}_{2;l,q}(f^n) + \mathcal{D}_{2;l,q}(f^{n+1})\} \\
& + C_{l,\eta} \int_0^t \sum_{|\alpha| \leq 2} [\|\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}\|_2^2 + \|\mu^{\frac{1-q}{8}} \partial^\alpha f^n\|_2^2] \\
& + C_l \int_0^t \left\{ \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}}^2 + \|\nabla_x \phi^n\|_\infty + \|\partial_t \phi^n\|_\infty \right\} \mathcal{E}_{2;l,q}(f^{n+1}) \\
& + C_l \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
& + C_l \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} \\
& + C_{l,\eta} \int_0^t \sum_{|\alpha'| \leq 2, |\beta'| < |\beta|} \|\partial_{\beta'}^\alpha f^{n+1}\|_{\sigma, w(\alpha', \beta')}^2 \\
& + C_m \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1} - f^n)} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
& + C_m \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \sqrt{\mathcal{D}_{2;l,q}(f^n)} \left\| |\partial_{\beta'}^{\alpha'} (f^{n+1} - f^n)|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
& + C_m \int_0^t \sqrt{\mathcal{E}_{2;l,q}(\mu\{f^{n+1} - f^n\})} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}.
\end{aligned}$$

Here we have bounded the cutoff function χ by $\mu^{\frac{1-q}{4}}$. As in [G1], we can get rid of the contribution of $C_{l,\eta} \int_0^t \sum_{|\alpha'| \leq 2, |\beta'| < |\beta|} \|\partial_{\beta'}^\alpha f^{n+1}\|_{\sigma, w(\alpha', \beta')}^2$ which has fewer v derivatives. Using an induction starting from $|\beta| = 0, 1, 2$ with different small values of η each time, we conclude that for any $\eta > 0$ (different value of C_l) that

$$\begin{aligned}
& \sum_{|\alpha|+|\beta| \leq 2} \|\partial_\beta^\alpha f^{n+1}\|_{2,w}^2 + \int_0^t \sum_{|\alpha|+|\beta| \leq m} \|\partial_\beta^\alpha f^{n+1}\|_{\sigma,w}^2 \\
& \lesssim \eta \int_0^t \sum_{|\alpha|+|\beta| \leq 2} \|\partial_\beta^\alpha f^{n+1}\|_{\sigma,w}^2 + C_{l,\eta} \sum_{|\alpha|+|\beta| \leq m} \|\partial_\beta^\alpha f^{n+1}(0)\|_{2,w}^2 \\
& + C_l \int_0^t \left\{ \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}}^2 + \|\nabla_x \phi^n\|_\infty + \|\partial_t \phi^n\|_\infty \right\} \mathcal{E}_{2;l,q}(f^{n+1}) \\
& + C_{l,\eta} \int_0^t \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}
\end{aligned}$$

$$\begin{aligned}
& + C_{l,\eta} \int_0^t \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} \\
& + C_{l,\eta} \int_0^t \sum_{|\alpha|\leq 2} [\|\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}\|_2^2 + \|\mu^{\frac{1-q}{8}} \partial^\alpha f^n\|_2^2] \\
& + C_m \int_0^t \sum_{|\alpha'|+|\beta'|\leq 1} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1} - f^n)} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
& + C_m \int_0^t \sum_{|\alpha'|+|\beta'|\leq 1} \sqrt{\mathcal{D}_{2;l,q}(f^n)} \left\| |\partial_{\beta'}^{\alpha'} (f^{n+1} - f^n)|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
& + C_m \int_0^t \sqrt{\mathcal{E}_{2;l,q}(\mu\{f^{n+1} - f^n\})} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})}.
\end{aligned}$$

We sum over $|\alpha| + |\beta| \leq 2$, then choose η sufficiently small (but fixed) to conclude the first part of our lemma. We remark that M is also small but fixed.

For $|\alpha| + |\beta| = m \geq 3$, we simply put terms such as $\sum_{\beta' < \beta} \|\partial_{\beta'}^\alpha f^{n+1}\|_{\sigma, w(\alpha, \beta')}^2$ into $\mathcal{D}_{m-1;l,q}(f^{n+1})$. We apply Proposition 7 to (88) and collect terms to get, for M sufficiently small (independent of m and l)

$$\begin{aligned}
& \|\partial_\beta^\alpha f^{n+1}\|_{2,w}^2 + \int_0^t \|\partial_\beta^\alpha f^{n+1}\|_{\sigma,w}^2 \\
& \lesssim \eta \int_0^t [\mathcal{D}_{m;l,q}(f^n) + \mathcal{D}_{m;l,q}(f^{n+1})] + C_{l,m,\eta} \|\partial_\beta^\alpha f^{n+1}(0)\|_{2,w}^2 \\
& + C_{l,m,\eta} \int_0^t \sum_{|\alpha|=m} [\|\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}\|_2^2 + \|\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}\|_2^2] \\
& + C_{l,m} \int_0^t [\sum_{|\alpha'|+|\beta'|\leq [\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \right\} \\
& \quad + \|\nabla \phi^n\|_\infty + \|\partial_t \phi^n\|_\infty] \\
& \quad \times \{ \mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1}) \} \\
& + C_{l,m,\eta} \int_0^t \sum_{|\alpha'|+|\beta'|\leq [\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'} f^n|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{m;l,q}(f^{n+1} - f^n)} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \\
& + C_{l,m,\eta} \int_0^t \sum_{|\alpha'|+|\beta'|\leq [\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'} (f^{n+1} - f^n)|_\sigma \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{m;l,q}(f^n)} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \\
& + C_{l,m,\eta} \int_0^t \sqrt{\mathcal{E}_{m;l,q}(\mu\{f^{n+1} - f^n\})} \sqrt{\mathcal{E}_{m;l,q}(f^{n+1})} \\
& + C_{l,m} \int_0^t \{ \mathcal{E}_{m-1;l,q}(f^n) + \mathcal{E}_{m-1;l,q}(f^{n+1}) + 1 \} \{ \mathcal{D}_{m-1;l,q}(f^n) + \mathcal{D}_{m-1;l,q}(f^{n+1}) \}.
\end{aligned}$$

We therefore conclude the lemma by setting η sufficiently small. $\square$

We are now ready to construct local-in-time solutions by showing uniform bounds for f^n which require the use of fractional Sobolev norms.

Lemma 12. Assume $f_0 \in C_c^\infty$ such that $F_0 = \mu + \sqrt{\mu} f_0 > 0$ with (74).

(1) *There exist small constants $0 < T \leq 1$ and $M > 0$, such that if $\mathcal{E}_{2;2,0}(f_0, \phi_0)$ is sufficiently small, then*

$$(102) \quad \mathcal{E}_{2;2,0}(f^{n+1}) + \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})(s) ds \leq M.$$

(2) *$\{f^n\}$ is Cauchy in $L^\infty([0, T], L_{x,v}^2)$.*

(3) *Assume (102). There exists $C_l > 0$ such that for $0 \leq t \leq T$:*

$$(103) \quad \mathcal{E}_{2;l,q}(f^{n+1})(t) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1})(s) ds \leq C_l \mathcal{E}_{2;l,q}(f_0).$$

(4) *Assume (102). For $m \geq 3$, there exists an increasing continuous function $P_{m,l}$ with $P_{m,l}(0) = 0$ such that for $0 \leq t \leq T$:*

$$(104) \quad \mathcal{E}_{m;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{m;l,q}(f^{n+1}) ds \leq P_{m,l}(\mathcal{E}_{m;l,q}(f_0)).$$

(5) *$\{F^n \geq 0\}$.*

Proof. To prove part (1) of the lemma, we apply (91) with an induction over n . Assume (102) is valid for $k = 0, 1, 2, \dots, n$. Recall that $\rho^n = \int \sqrt{\mu} [f_+^n - f_-^n] dv$, $j^n = \int v \sqrt{\mu} [f_+^n - f_-^n] dv$. We now note that from the continuity equation of

$$\rho_t^n + \nabla_x \cdot j^n = 0,$$

we have

$$(105) \quad -\Delta \nabla_x \phi^n = \nabla_x \rho^n, \quad \Delta \partial_t \phi^n = \nabla_x j^n,$$

and

$$(106) \quad \|\partial_t \phi^n\|_\infty + \|\nabla_x \phi^n\|_\infty \lesssim |\partial_t \int [f_+^n - f_-^n] \sqrt{\mu} dv|_2 + |\partial_x \int [f_+^n - f_-^n] \sqrt{\mu} dv|_2.$$

We therefore deduce that for $t \leq T \leq 1$,

$$\int_0^T \{\|\phi_t^n\|_\infty + \|\nabla_x \phi^n\|_\infty\} ds \lesssim \sqrt{\mathcal{E}_{2;2,0}(f^n)} \leq \sqrt{M}.$$

It follows that

$$\int_0^t \sum \{ \|\mu^{\frac{1-q}{8}} \partial^\alpha f^{n+1}\|_2^2 + \|\mu^{\frac{1-q}{8}} \partial^\alpha f^n\|_2^2 \} \lesssim \int_0^t [\mathcal{E}_{2;l,q}(f^n) + \mathcal{E}_{2;l,q}(f^{n+1})].$$

Moreover,

$$(107) \quad \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \lesssim \sqrt{\mathcal{D}_{2;2,0}(f^{n+1})},$$

$$\left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \lesssim \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})}.$$

We now take $l = 2$, $q = 0$ in w from Lemma 11. By (41) and Cauchy-Schwarz' inequality, we further collect and arrange terms in (91):

$$\begin{aligned} & \mathcal{E}_{2;2,0}(f^{n+1}) + \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})ds \\ & \leq \frac{1}{8} \int_0^t \mathcal{D}_{2;2,0}(f^n)ds + \frac{1}{16} \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})ds + C\mathcal{E}_{2;2,0}(f_0) \\ & \quad + C \int_0^t [\mathcal{D}_{2;2,0}(f^n) + \mathcal{E}_{2;2,0}(f^n) + 1]\mathcal{E}_{2;2,0}(f^{n+1}) + C \int_0^t \mathcal{E}_{2;2,0}(f^n). \end{aligned}$$

By using the induction hypothesis for f^n , for M and T both small, it is standard to conclude that

$$\mathcal{E}_{2;2,0}(f^{n+1}) + \int_0^t \mathcal{D}_{2;2,0}(f^{n+1})ds \leq \frac{4M}{5} + C\mathcal{E}_{2;2,0}(f_0) < M.$$

We thus deduce part (1) of the lemma for $\mathcal{E}_{2;2,0}(f_0)$ sufficiently small.

We now prove part (2). It is standard to prove that $\{f^n\}$ is Cauchy in L^2 in light of the strong bound obtained in part (1). We take the difference in (76) to obtain

$$\begin{aligned} & [\partial_t + v \cdot \nabla_x \mp \nabla_x \phi^n \cdot \nabla_v][f_\pm^{n+1} - f_\pm^n] - A_\pm[f_\pm^{n+1} - f_\pm^n] \\ & \pm \nabla_x \phi^n \cdot v[f_\pm^{n+1} - f_\pm^n] \mp [\nabla_x \phi^n - \nabla_x \phi^{n-1}]\nabla_v f_\pm^n \pm [\nabla_x \phi^n - \nabla_x \phi^{n-1}] \cdot v f_\pm^n \\ & = \mp 2[\nabla_x \phi^n - \nabla_x \phi^{n-1}] \cdot v \sqrt{\mu} + K_\pm[f^n - f^{n-1}] \\ & \quad + \Gamma_\pm(f^{n+1} - f^n, f^n) + \Gamma_\pm(f^n, f^n - f^{n-1}) \\ & \quad - 8\pi\mu[f_\pm^n(f_\pm^{n+1} - f_\pm^n) - f_\pm^{n-1}(f_\pm^n - f_\pm^{n-1})] \\ & \quad - 16\pi\sqrt{\mu}([f_\pm^{n+1} - f_\pm^n] - [f_\pm^n - f_\pm^{n-1}]), \\ & \Delta[\phi^{n+1} - \phi^n] = - \int ([f_+^{n+1} - f_+^n] - [f_-^{n+1} - f_-^n])dv, \end{aligned}$$

We take the inner product with $e^{\pm 2\phi^n}(f^{n+1} - f^n)$ (f^{n+1} and f^n have the same initial value). Noticing

$$\begin{aligned} & \int |\mu[f_\pm^n(f_\pm^{n+1} - f_\pm^n) - f_\pm^{n-1}(f_\pm^n - f_\pm^{n-1})]||f^{n+1} - f^n| \\ & \lesssim \int_{\mathbf{T}^3} |f^n|_4 |f^{n+1} - f^n|_\sigma^2 + \int_{\mathbf{T}^3} |f^{n-1}|_2 |f^n - f^{n-1}|_\sigma |f^{n+1} - f^n|_\sigma \\ & \lesssim \sqrt{\mathcal{E}_{2;2,0}(f^n)} \|f^{n+1} - f^n\|_\sigma^2 + \sqrt{\mathcal{E}_{2;2,0}(f^{n-1})} \|f^n - f^{n-1}\|_\sigma \|f^{n+1} - f^n\|_\sigma, \end{aligned}$$

we apply Lemmas 10, 8 with zero derivatives above. By taking $\sqrt{\mathcal{E}_{2;2,0}(f^n)}$ sufficiently small, and $e^{\pm 2\phi^n} \sim 1$, we can conclude that

$$\begin{aligned} & \|f^{n+1} - f^n\|_2^2(t) + \int_0^t \|f^{n+1} - f^n\|_\sigma^2 \\ & \leq \frac{1}{8} \int_0^t \|f^n - f^{n-1}\|_\sigma^2(s) + C \int_0^t \{\|f^{n+1} - f^n\|_2^2 + \|f^n - f^{n-1}\|_2^2\}. \end{aligned}$$

By taking T sufficiently small, we obtain

$$\begin{aligned} & \max \left\{ \sup_{0 \leq t \leq T} \|f^{n+1} - f^n\|_2^2(t), \int_0^T \|f^{n+1} - f^n\|_\sigma^2 \right\} \\ & \leq \frac{1}{4} \left\{ \sup_{0 \leq t \leq T} \|f^{n+1} - f^n\|_2^2(t) + \int_0^T \|f^n - f^{n-1}\|_\sigma^2 \right\}. \end{aligned}$$

By an iteration of n , $\sup_{0 \leq t \leq T} \|f^{n+1} - f^n\|_2^2(t) + \int_0^T \|f^{n+1} - f^n\|_\sigma^2 \leq \frac{1}{2^n}$, and $\{f^n\}$ is a Cauchy sequence in $L^\infty(L_{x,v}^2)$. We denote f to be the limit of f^n . We remark, however, unlike [G1], that it is impossible to establish that f^n is Cauchy with respect to $\sqrt{\mathcal{E}_{2,2,0}}$, due to the presence of the term $[\nabla_x \phi^n - \nabla_x \phi^{n-1}] \nabla_v f_\pm^n$.

We now turn to part (3) of the lemma for which we have to make use of the sharper estimates of $H^{\frac{3}{4}}$ norms. We first observe in (91), by (19), $\left\| |\mu^{\frac{1}{4}} \partial_{\beta'}^{\alpha'} g|_{H^1} \right\|_{H^{\frac{3}{4}}} \lesssim \left\| |\partial_{\beta'}^{\alpha'} g|_\sigma \right\|_{H^{\frac{3}{4}}}$. Moreover, for any $\varepsilon > 0$, we claim that

$$\begin{aligned} (108) \quad & \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \\ & \leq \frac{\varepsilon}{2} \sum_{\substack{|\alpha'|+|\beta'| \leq 1 \\ |\gamma| \leq 1}} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^1} + C_\varepsilon \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{L^2} \\ & \leq \varepsilon \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} + C_\varepsilon \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})}. \end{aligned}$$

In fact, $\frac{w(\alpha', \beta')}{\langle v \rangle^2} = w(\alpha' + \gamma, \beta)$ in the first term, and we have decomposed

$$\frac{w(\alpha', \beta')}{\langle v \rangle^2} = \mathbf{1}_{\{|v| \geq \frac{1}{\eta}\}} + \mathbf{1}_{\{|v| \leq \frac{1}{\eta}\}}$$

in the second term to further estimate $(\frac{w(\alpha', \beta')}{\langle v \rangle^2} \mathbf{1}_{\{|v| \leq \frac{1}{\eta}\}} \leq C_\eta)$:

$$\begin{aligned} & C_\varepsilon \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{L^2} \\ & \leq C_\varepsilon \sum_{|\alpha'|+|\beta'| \leq 1} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2} \mathbf{1}_{\{|v| \geq \frac{1}{\eta}\}}} \right\|_{L^2} + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2} \mathbf{1}_{\{|v| \leq \frac{1}{\eta}\}}} \right\|_{L^2} \right\} \\ & \leq C_\varepsilon \eta^2 \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} + C_{\varepsilon, \eta} \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})} \\ & \leq \frac{\varepsilon}{2} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} + C_\varepsilon \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})} \end{aligned}$$

by choosing $\eta = \eta(\varepsilon)$ even smaller. Similarly,

$$(109) \quad \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \leq \varepsilon \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} + C_\varepsilon \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})}.$$

Therefore, in (91), by choosing ε sufficiently small,

$$\begin{aligned}
 C_l & \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
 & + C_l \sum_{|\alpha'|+|\beta'|\leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{2, \frac{w(\alpha', \beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} \\
 & \leq 2\varepsilon C_l \sqrt{\mathcal{D}_{2;2,0}(f^n)} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
 & \quad + C_{\varepsilon,l} \sqrt{\mathcal{D}_{2;2,0}(f^n)} \{ \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})} \sqrt{\mathcal{E}_{2;l,q}(f^{n+1})} \\
 & \quad + \sqrt{\mathcal{E}_{2;2,0}(f^{n+1})} \sqrt{\mathcal{D}_{2;l,q}(f^{n+1})} \} \\
 & \leq \frac{1}{8} \mathcal{D}_{2;l,q}(f^{n+1}) + \varepsilon C_l \mathcal{D}_{2;2,0}(f^n) \mathcal{E}_{2;l,q}(f^{n+1}) + C_{\varepsilon,l} [\mathcal{D}_{2;2,0}(f^n) + 1] \mathcal{E}_{2;2,0}(f^{n+1}).
 \end{aligned}$$

Here C_l can be chosen as a generic constant independent of ε . Since from (102),

$$\begin{aligned}
 C_{\varepsilon,l} \int_0^T [\mathcal{D}_{2;2,0}(f^n) + 1] \mathcal{E}_{2;2,0}(f^{n+1}) \\
 \leq C_{\varepsilon,l} \sup_{0 \leq t \leq T} \mathcal{E}_{2;2,0}(f^{n+1}) \int_0^T [\mathcal{D}_{2;2,0}(f^n) + 1] \leq C_{\varepsilon,l} \mathcal{E}_{2;2,0}(f^0),
 \end{aligned}$$

collecting terms in (91), we can rewrite (91) as

$$\begin{aligned}
 (110) \quad & \mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) ds \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{2;l,q}(f^n) ds + C_{\varepsilon,l} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^t A_n(s) [\mathcal{E}_{2;l,q}(f^{n+1}) + \mathcal{E}_{2;l,q}(f^n)],
 \end{aligned}$$

where

$$\begin{aligned}
 (111) \quad A_n(s) \equiv & \sum_{|\alpha'|+|\beta'|\leq 1} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \right\} \\
 & + \|\nabla_x \phi^n\|_{\infty} + \|\partial_t \phi^n\|_{\infty} + 1 + \varepsilon \mathcal{D}_{2;2,0}(f^n).
 \end{aligned}$$

The key step to prove (103) is to replace $A_n(s)$ by a fixed, integrable function. First of all, in light of parts (1) and (2), we shall prove as $n \rightarrow \infty$,

$$\begin{aligned}
 (112) \quad & \int_0^T \sum_{|\alpha'|+|\beta'|\leq 1} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 \right\} + \|\nabla_x \phi^n\|_{\infty} + \|\partial_t \phi^n\|_{\infty} \\
 & \rightarrow \int_0^T \left\{ \sum_{|\alpha'|+|\beta'|\leq 1} 2 \left\| |\partial_{\beta'}^{\alpha'} f|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 + \|\nabla_x \phi\|_{\infty} + \|\partial_t \phi\|_{\infty} \right\}.
 \end{aligned}$$

In fact, since $f^n \rightarrow f$ in L^2 from part (1), we deduce from (102):

$$\mathcal{E}_{2;2,0}(f) + \int_0^T \mathcal{D}_{2;2,0}(f) \leq M.$$

Moreover, by (105) and (106),

$$\begin{aligned}
 & \max_{0 \leq t \leq} \|\partial_t \phi^n - \partial_t \phi\|_\infty + \|\nabla_x \phi^n - \nabla_x \phi\|_\infty \\
 & \lesssim [\|\nabla_x j^n - \nabla_x j\|_2 + \|\nabla_x \rho^n - \nabla_x \rho\|_2] \\
 (113) \quad & \lesssim \|f^n - f\|_{H^1} \rightarrow 0,
 \end{aligned}$$

from (102) and $f^n \rightarrow f$ in L^2 .

We separate $|v| \geq R$ and $|v| \leq R$ to get, for any $\eta > 0$, by (24) and (26) in $\mathcal{D}_{2;l,q}(\cdot)$,

$$\begin{aligned}
 (114) \quad & \int_0^T \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_\sigma \right\|_{H^{\frac{3}{4}}}^2 \\
 & \leq \eta \int_0^T \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_\sigma \right\|_{H^1}^2 + C_\eta \int_0^T \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_\sigma \right\|_2^2 \\
 & \leq \eta \int_0^T \sum_{\substack{|\alpha'|+|\beta'| \leq 1 \\ |\gamma| \leq 1}} \|\partial^\gamma \partial_{\beta'}^{\alpha'} f^n - \partial^\gamma \partial_{\beta'}^{\alpha'} f\|_\sigma^2 + C_\eta \int_0^T \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_\sigma \right\|_2^2 \\
 & \lesssim \eta \left[\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f) \right] \\
 & \quad + \frac{C_\eta}{R^4} \int_0^T \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, w(\alpha', \beta')} \right\|_{2, w(\alpha', \beta')}^2 \\
 & \quad + C_\eta \int_0^T \|[\nabla_v f^n - \nabla_v f] \chi_{(|v| \leq 2R)}\|_{H^1}^2 \\
 & \lesssim \eta \left[\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f) \right] + \frac{C_\eta}{R^4} \left[\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f) \right] \\
 & \quad + C_{\eta, R} \int_0^T \|[\nabla_v f^n - \nabla_v f] \chi_{(|v| \leq 2R)}\|_{H^1}^2,
 \end{aligned}$$

where χ is a cutoff function, and $w(\alpha', \beta') \geq \langle v \rangle^2$. In light of

$$\int_0^T \mathcal{D}_{2;2,0}(f^n) + \int_0^T \mathcal{D}_{2;2,0}(f) < \infty,$$

and by (26), $\int_0^T \|[\nabla_v f^n - \nabla_v f] \chi_{(|v| \leq 2R)}\|_{H^2}^2$ is uniformly bounded. Hence

$$\lim_{n \rightarrow \infty} \int_0^T \|[\nabla_v f^n - \nabla_v f] \chi_{(|v| \leq 2R)}\|_{H^1}^2 \rightarrow 0$$

from $f^n \rightarrow f$ in L^2 . We thus deduce that

$$\lim_{n \rightarrow \infty} \int_0^T \sum_{|\alpha'|+|\beta'| \leq 1} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 = 0$$

by first choosing η sufficiently small, then R sufficiently large and letting $n \rightarrow \infty$. We thus complete the proof of (112).

Now from (112), for any ε small, $\int_0^T \mathcal{D}_{2;2,0}(f) \lesssim M$ in (111), we can find $N(\varepsilon)$ such that for $n \geq N$,

$$(115) \quad \int_0^T [A_n(s) - A(s)] ds < C_l \varepsilon$$

with

$$A(s) \equiv \sum_{|\alpha'|+|\beta'| \leq 1} 2 \left\| |\partial_{\beta'}^{\alpha'} f|_{\sigma} \right\|_{H^{\frac{3}{4}}}^2 + \|\nabla_x \phi\|_{\infty} + \|\partial_t \phi\|_{\infty} + 1.$$

Our strategy of establishing (103) is to separate two cases of $n \geq N$ and $n \leq N$. For $n \leq N$, by (110),

$$\begin{aligned} & \mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) ds \\ & \leq \frac{1}{4} \int_0^t \mathcal{D}_{2;l,q}(f^n) ds + C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^t A_n(s) [\mathcal{E}_{2;l,q}(f^{n+1}) + \mathcal{E}_{2;l,q}(f^n)] \end{aligned}$$

for $1 \leq n \leq N-1$. We apply the Gronwall Lemma 5 to $\mathcal{E}_{2;l,q}(f^{n+1})(t)$ with $A = A_n(s)$ and

$$B = - \int_0^s \mathcal{D}_{2;l,q}(f^{n+1}) d\tau + \frac{1}{4} \int_0^s \mathcal{D}_{2;l,q}(f^n) d\tau + C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^t A_n(s) \mathcal{E}_{2;l,q}(f^n)$$

to obtain

$$\begin{aligned} \mathcal{E}_{2;l,q}(f^{n+1})(t) & \leq e^{\int_0^t A_n(s) ds} \int_0^t A_n(s) \\ & \quad \times \left\{ - \int_0^s \mathcal{D}_{2;l,q}(f^{n+1}) d\tau + \frac{1}{4} \int_0^s \mathcal{D}_{2;l,q}(f^n) d\tau \right. \\ & \quad \left. + [C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^s A_n(s) \mathcal{E}_{2;l,q}(f^n)] \right\} \\ & \quad - \int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) d\tau + \frac{1}{4} \int_0^t \mathcal{D}_{2;l,q}(f^n) d\tau \\ & \quad + C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^t A_n(s) \mathcal{E}_{2;l,q}(f^n). \end{aligned}$$

By the boundedness of $\int_0^T A_n(s)$, $\int_0^T A_n(s) \mathcal{E}_{2;l,q}(f^n) \leq C \sup_{0 \leq t \leq T} \mathcal{E}_{2;l,q}(f^n)$. Collecting terms, dropping $-\int_0^s \mathcal{D}_{2;l,q}(f^{n+1}) d\tau$ above, we use induction over n to get

$$\begin{aligned}
 & \sup_{0 \leq t \leq T} \mathcal{E}_{2;l,q}(f^{n+1})(t) + \int_0^T \mathcal{D}_{2;l,q}(f^{n+1}) d\tau \\
 & \leq C_l \left\{ \sup_{0 \leq t \leq T} \mathcal{E}_{2;l,q}(f^n) + \int_0^T \mathcal{D}_{2;l,q}(f^n) d\tau \right\} + C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) \\
 & \leq C_l^2 \left\{ \sup_{0 \leq t \leq T} \mathcal{E}_{2;l,q}(f^{n-1}) + \int_0^T \mathcal{D}_{2;l,q}(f^{n-1}) d\tau \right\} + [1 + C_l] C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) \\
 & \dots \\
 & \leq C_l^{n+1} \left\{ \sup_{0 \leq t \leq T} \mathcal{E}_{2;l,q}(f^0) + \int_0^T \mathcal{D}_{2;l,q}(f^0) d\tau \right\} + [1 + \dots + C_l^n] C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) \\
 (116) \quad & \leq C_l^{N+1} C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0),
 \end{aligned}$$

as $f^0 \equiv 0$ from (74). This concludes (103) of the case for $n \leq N$.

For $n \geq N$, we define for $k \geq N$,

$$\mathcal{X}_k(t) = \max_{N \leq n \leq k} \mathcal{E}_{2;l,q}(f^n),$$

and our goal is to show a uniform bound for $\mathcal{X}_k(t)$. By (110) and (115), for $n+1 \leq k$, we replace A_n by A to get

$$\begin{aligned}
 & \mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) ds \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{2;l,q}(f^n) ds + C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^t A_n(s) \mathcal{X}_k(t) \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{2;l,q}(f^n) ds + C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \int_0^t A(s) \mathcal{X}_k(t) + C_l \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t).
 \end{aligned}$$

We iterate such an inequality $n+1$ back to n until reaching $n = N$ to get

$$\begin{aligned}
 & \mathcal{E}_{2;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) ds \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{2;l,q}(f^n) ds + C_l \{ C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + \int_0^t A(s) \mathcal{X}_k(s) + \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t) \} \\
 & \leq \frac{1}{4^2} \int_0^t \mathcal{D}_{2;l,q}(f^{n-1}) ds \\
 & \quad + [1 + \frac{1}{4}] C_l \left\{ C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + \int_0^t A(s) \mathcal{X}_k(s) + \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t) \right\} \\
 & \dots
 \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4^{n-N}} \int_0^t \mathcal{D}_{2;l,q}(f^N) ds \\
&\quad + \sum_0^{n+1} \frac{1}{4^j} C_l \left\{ C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + \int_0^t A(s) \mathcal{X}_k(s) + \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t) \right\} \\
&\leq C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + 2C_l \left\{ \int_0^t A(s) \mathcal{X}_k(t) + \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t) \right\},
\end{aligned}$$

from the estimate for $\int_0^t \mathcal{D}_{2;l,q}(f^N) ds$. We now take $\max_{N \leq n \leq k}$ to get

$$\mathcal{X}_k(t) \leq C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \left\{ \int_0^t A(s) \mathcal{X}_k(s) + \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t) \right\}.$$

From the Gronwall Lemma 5, we obtain from $\int_0^T A(s) ds < +\infty$:

$$\mathcal{X}_k(t) \leq C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0) + C_l \varepsilon \sup_{0 \leq t \leq T} \mathcal{X}_k(t).$$

With ε sufficiently small we obtain

$$\sup_{0 \leq t \leq T} \mathcal{X}_k(t) \leq C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0).$$

Plugging this into (117) yields

$$\int_0^t \mathcal{D}_{2;l,q}(f^{n+1}) ds \leq C_{l,\varepsilon} \mathcal{E}_{2;l,q}(f_0),$$

and we complete part (3) and the proof of (103).

We now turn to part (4) of the lemma and (104). We shall use an induction over m . We assume that (104) is valid for $m-1$ and all l, q :

$$\begin{aligned}
(118) \quad &\sup_{0 \leq t \leq T} \mathcal{E}_{m-1;l,q}(f^{n+1}(t)) + \int_0^T \mathcal{D}_{m-1;l,q}(f^{n+1}) ds \\
&\leq P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0))
\end{aligned}$$

for some increasing function $P_{m-1,l}(0) = 0$. Clearly, this is valid for $m-1 = 2$ in light of (103).

We follow the same argument as in the proof of (103). Collecting terms and using the induction hypothesis (118), we summarize (92) as

$$\begin{aligned}
 & \mathcal{E}_{m;l,q}(f^{n+1}) + \int_0^t \mathcal{D}_{m;l,q}(f^{n+1}) ds \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{m;l,q}(f^n) ds + C_{l,m} \mathcal{E}_{m;l,q}(f_0) \\
 & \quad + C_{l,m} \int_0^t A_{n,m}(s) [\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})] \\
 & \quad + C_{l,m} \int_0^t [\mathcal{D}_{m-1;l,q}(f^n) + \mathcal{D}_{m-1;l,q}(f^{n+1})] [\mathcal{E}_{m-1;l,q}(f^n) + \mathcal{E}_{m-1;l,q}(f^{n+1}) + 1] \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{m;l,q}(f^n) ds + C_{l,m} \mathcal{E}_{m;l,q}(f_0) \\
 & \quad + C_{l,m} \int_0^t A_{n,m}(s) [\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})] \\
 & \quad + C_{l,m} [P_{m-1,l}^2(\mathcal{E}_{m-1;l,q}(f_0)) + P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0))] \\
 & \leq \frac{1}{4} \int_0^t \mathcal{D}_{m;l,q}(f^n) ds + P_{m,l}(\mathcal{E}_{m;l,q}(f_0)) \\
 & \quad + C_{l,m} \int_0^t A_{n,m}(s) [\mathcal{E}_{m;l,q}(f^n) + \mathcal{E}_{m;l,q}(f^{n+1})],
 \end{aligned}$$

where $P_{m,l}(z) \equiv C_{l,m} [P_{m-1,l}^2(z) + P_{m-1,l}(z)] + z$ and

$$\begin{aligned}
 A_{n,m} \equiv & \sum_{|\alpha'|+|\beta'|\leq[\frac{m}{2}]} \left\{ \left\| |\partial_{\beta'}^{\alpha'} f^n|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \left\| |\partial_{\beta'}^{\alpha'} f^{n+1}|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \right\} \\
 & + \|\nabla \phi^n\|_{\infty} + \|\partial_t \phi^n\|_{\infty} + 1.
 \end{aligned}$$

As in the proof of (103), the major step is to show that

$$(119) \quad \lim_{n \rightarrow \infty} \int_0^T A_{n,m}(s) ds \rightarrow \int_0^T A_m(s) ds$$

with

$$A_m \equiv 2 \sum_{|\alpha'|+|\beta'|\leq[\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 + \|\nabla \phi\|_{\infty} + \|\partial_t \phi\|_{\infty} + 1.$$

Note that $\int_0^T A_m < \infty$ by (118). Similarly, as in (114),

$$\begin{aligned}
 & \int_0^T \sum_{|\alpha'|+|\beta'|\leq[\frac{m}{2}]} \left\| |\partial_{\beta'}^{\alpha'} f^n - \partial_{\beta'}^{\alpha'} f|_{\sigma, \frac{w(\alpha',\beta')}{\langle v \rangle^2}} \right\|_{H^{\frac{3}{4}}}^2 \\
 (120) \quad & \leq \eta \left[\int_0^T \mathcal{D}_{m-1;l,q}(f^n) + \int_0^T \mathcal{D}_{m-1;l,q}(f) \right] \\
 & \quad + \frac{C_{\eta}}{R^4} \left[\int_0^T \mathcal{D}_{m-1;l,q}(f^n) + \int_0^T \mathcal{D}_{m-1;l,q}(f) \right] \\
 & \quad + C_{\eta,R} \left[\int_0^T \|\nabla_v f^n - \nabla_v f\| \chi_{(|v|\leq 2R)} \right]_{H^{\frac{m}{2}}}^2,
 \end{aligned}$$

where χ is a smooth cutoff function. We note that $\lceil \frac{m}{2} \rceil \leq m - 2$ for $m \geq 3$. From (118), $\int_0^T \|[\nabla_v f^n - \nabla_v f] \chi_{(|v| \leq 2R)}\|_{H^{m-1}}^2$ is bounded for fixed R and

$$\lim_{n \rightarrow \infty} \int_0^T \|[\nabla_v f^n - \nabla_v f] \chi_{(|v| \leq 2R)}\|_{H^{\lceil \frac{m}{2} \rceil}}^2 \rightarrow 0.$$

We therefore conclude that (119) is valid and (104) is proven exactly as in part (3).

To prove part (5) of the lemma, we use an induction on n . Assume that $F^n \geq 0$ for $n \leq k$ ($F^0 = \mu$ is chosen to be positive). Note that from (76) we constructed f^{k+1} , which induces F^{k+1} to satisfy the equivalent (75), our special design of the approximation scheme. Note that in (75), $F_{\pm}^k \geq 0$ so that $\Phi^{ij} * [F_{\pm}^k + F_{\mp}^k]$ is a nonnegative matrix and $\{F_{\pm}^k\}^2 \geq 0$, $F_{\mp}^k F_{\pm}^k \geq 0$ in

$$\Phi^{ij} * [F_{\pm}^k + F_{\mp}^k] \partial_{ij} F_{\pm}^{k+1} + 8\pi \{F_{\pm}^k\}^2 + 8\pi F_{\mp}^k F_{\pm}^k.$$

Therefore, we can use the Maximum Principle for F^{k+1} to conclude that $F^{k+1} \geq 0$ and the induction is complete. $\square$

We summarize the local well-posedness as $n \rightarrow \infty$.

Theorem 13. *Assume that $\mathcal{E}_{2,2,0}(f_0)$ is sufficiently small. Then there exist $0 < T \leq 1$ and $M > 0$ small such that there is a unique solution $F = \mu + \sqrt{\mu}f > 0$ with*

$$\mathcal{E}_{2,2,0}(f)(t) + \int_0^t \mathcal{D}_{2,2,0}(f)(s) ds \lesssim \mathcal{E}_{2,2,0}(f_0) \leq M.$$

In general, if $0 \leq t \leq T$, there exists an increasing continuous function $P_{m,l}(\cdot)$ with $P_{m,l}(0) = 0$ such that

$$\mathcal{E}_{m;l,q}(f)(t) + \int_0^t \mathcal{D}_{m;l,q}(f)(s) ds \leq P_{m,l}(\mathcal{E}_{m;l,q}(f_0)).$$

The uniqueness is standard with the strong bound $\mathcal{E}_{2,2,0}(f)(t) < \infty$. To establish Theorem 13 for $\mathcal{E}_{m;l,q}(f_0) < \infty$ and $\mu + \sqrt{\mu}f_0 \geq 0$, we first choose a velocity cutoff function χ such that $(1 - \frac{1}{R})f_0\chi(\frac{|v|}{R})$ has compact support in v and $\mathcal{E}_{m;l,q}(f_0\chi(\frac{|v|}{R})) < \infty$. We then choose a smooth approximation of $(1 - \frac{1}{R})f_0\chi(\frac{|v|}{R})$ as f_0^k . For fixed R , we can choose f_0^k such that

$$(121) \quad \mathcal{E}_{m;l,q} \left(\left(1 - \frac{1}{R}\right) f_0 \chi\left(\frac{|v|}{R}\right) - f_0^k \right) \rightarrow 0.$$

We therefore can construct a solution for the system with initial condition f_0^k thanks to Lemma 12. We finally take limits as $k \rightarrow \infty$ and $R \rightarrow \infty$ to construct a solution for the desired initial datum f_0 .

4. TIME DECAY AND GLOBAL SOLUTION

In this section, we establish our main theorem. We first summarize the mixed x and v derivative estimates by applying Lemma 11 with $f^n = f^{n+1} = f$ and $\phi^n = \phi$:

Lemma 14. *Let $f_0 \in C_c^\infty$ and assume f is the solution constructed in Theorem 13. Assume for M sufficiently small that*

$$\mathcal{E}_{2,2,0}(f) \leq M.$$

(1) We have

(122)

$$\begin{aligned} & \mathcal{E}_{2;l,q}(f) + \int_0^t \mathcal{D}_{2;l,q}(f) ds \\ & \leq C_l \{ \mathcal{E}_{2;l,q}(f_0) + \int_0^t \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_\sigma^2 + \int_0^t [\mathcal{D}_{2;2,0}(f) + \|\nabla \phi\|_\infty + \|\partial_t \phi\|_\infty] \mathcal{E}_{2;l,q}(f) ds \}. \end{aligned}$$

(2) If $m \geq 3$, we have

$$\begin{aligned} & \mathcal{E}_{m;l,q}(f) + \int_0^t \mathcal{D}_{m;l,q}(f) ds \\ (123) \quad & \leq C_{l,m} \{ \mathcal{E}_{2;l,q}(f_0) + \int_0^t [\mathcal{D}_{m-1;l,q}(f) + \|\nabla_x \phi\|_\infty + \|\partial_t \phi\|_\infty] \mathcal{E}_{m;l,q}(f) \\ & \quad + \int_0^t \sum_{|\alpha|=m} \|\partial^\alpha f\|_\sigma^2 + \int_0^t [\mathcal{E}_{m-1;l,q}(f) + 1] \mathcal{D}_{m-1;l,q}(f) \}. \end{aligned}$$

We note that since $f_0 \in C_c^\infty$, the sequence f_n in Lemma 12 satisfies

$$\sup_n \mathcal{E}_{m;l,q}(f^n) < +\infty$$

for any l and any m (the bound may depend on l, m). This implies that f^n is compact in all $\mathcal{E}_{m;l,q}$ norms so that we can take $n \rightarrow \infty$ in Lemma 11 so that terms with $f^{n+1} - f^n$ drop. We also have used (41), (107), (120) with f , and the fact that $\sum_{|\alpha| \leq 2} \|\mu^{\frac{\alpha-1}{8}} \partial^\alpha f\|^2 \lesssim \|\partial^\alpha f\|_\sigma^2$.

We next investigate the pure x derivatives to control $\int_0^t \|\partial^\alpha f\|_\sigma^2$ above. We take ∂^α of the Vlasov-Poisson-Landau system (8):

$$\begin{aligned} & [\partial_t + v \cdot \nabla_x \mp \nabla_x \phi \cdot \nabla_v] \partial^\alpha f_\pm \pm 2 \nabla_x \partial^\alpha \phi \cdot v \sqrt{\mu} + L_\pm \partial^\alpha f \\ & = \mp \nabla_x \phi \cdot v \partial^\alpha f_\pm \mp \sum_{\alpha_1 < \alpha} C_{\alpha_1}^{\alpha_1} \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_\pm \\ & \quad \pm \sum_{\alpha_1 < \alpha} C_{\alpha_1}^{\alpha_1} \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \nabla_v \partial^{\alpha_1} f_\pm + \partial^\alpha \Gamma_\pm(f, f). \end{aligned}$$

Lemma 15. Let $f_0 \in C_c^\infty$ and assume f is the solution constructed in Theorem 13 with $\mathcal{E}_{2;2,0}(f) \leq M$. Then for $|\alpha| = 0, 1, 2$:

$$\begin{aligned} & \frac{d}{dt} \left[\int \sum_{\pm} \frac{e^{\pm 2\phi} (\partial^\alpha f_\pm)^2}{2} + \int |\nabla \partial^\alpha \phi|^2 \right] + \int \langle L \partial^\alpha f, \partial^\alpha f \rangle \\ (124) \quad & \lesssim \sum_{\pm} \int |\phi_t| (\partial^\alpha f_\pm)^2 + \sqrt{M} \sum_{|\alpha'| \leq |\alpha|} \|\partial^{\alpha'} f\|_\sigma^2. \end{aligned}$$

For $|\alpha| = m \geq 3$, we have for any $\eta > 0$,

$$\begin{aligned} & \frac{d}{dt} \left[\int \sum_{\pm} \frac{e^{\pm 2\phi} (\partial^\alpha f_\pm)^2}{2} + \int |\nabla \partial^\alpha \phi|^2 \right] + \int \langle L \partial^\alpha f, \partial^\alpha f \rangle \\ & \lesssim \sum_{\pm} \int |\phi_t| |\partial^\alpha f_\pm|^2 + \sqrt{M} \int |\partial^\alpha f_\pm|_\sigma^2 + \eta \sum_{|\alpha|=m} \|\partial^{\alpha'} f\|_\sigma^2 \\ (125) \quad & + C_{m,\eta} [\mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) + \{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f)]. \end{aligned}$$

Proof. We sum over $\pm$ from (80) to (90) with $f^n = f^{n+1} = f$ and with $w = 1$ ($l = |\alpha| + |\beta|$). Note that we can combine $-A - K = L$. From the continuity equation $\partial^\alpha \rho_t + \nabla_x \cdot \partial^\alpha j = 0$ (with $\rho = \int \sqrt{\mu}[f_+ - f_-]dv$ and $j = \int v\sqrt{\mu}[f_+ - f_-]dv$)

$$-2 \int \nabla_x \partial^\alpha \phi \cdot v \sqrt{\mu} [\partial^\alpha f_+ - \partial^\alpha f_-] = \frac{d}{dt} \int |\nabla \partial^\alpha \phi|^2,$$

and we deduce:

$$(126) \quad \frac{d}{dt} \left[\int \sum_{\pm} \frac{e^{\pm 2\phi} (\partial^\alpha f_{\pm})^2}{2} + \int |\nabla \partial^\alpha \phi|^2 \right] + \int \langle L \partial^\alpha f, \partial^\alpha f \rangle$$

$$(127) \quad = \sum_{\pm} \int e^{\pm 2\phi} \phi_t (\partial^\alpha f_{\pm})^2$$

$$(128) \quad + 2 \sum_{\pm} \int e^{\pm 2\phi} \nabla_x \partial^\alpha \phi \cdot v \sqrt{\mu} \partial^\alpha f_{\pm} (e^{\pm 2\phi} - 1)$$

$$(129) \quad + \sum_{\pm} \int (1 - e^{\pm 2\phi}) \partial^\alpha f_{\pm} L_{\pm} \partial^\alpha f$$

$$(130) \quad + \sum_{\pm} \int e^{\pm 2\phi} \partial^\alpha f_{\pm} \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot \nabla_v \partial^{\alpha_1} f_{\pm}$$

$$(131) \quad - \sum_{\pm, \alpha_1 < \alpha} C_{\alpha}^{\alpha_1} \int e^{\pm 2\phi} \partial^\alpha f_{\pm} \partial^{\alpha-\alpha_1} \nabla_x \phi \cdot v \partial^{\alpha_1} f_{\pm}.$$

We now estimate each term from (127) to (131). From $|1 - e^{\pm \phi}| \lesssim \|\phi\|_{\infty} \lesssim \sqrt{\mathcal{E}_{2,2,0}(f)} \lesssim \sqrt{M}$, $\|\nabla_x \partial^\alpha \phi\|_2 \lesssim \|\partial^\alpha f\|_{\sigma}$ and $\|v \mu^{1/4} \partial^\alpha f\|_2 \lesssim \|\partial^\alpha f\|_{\sigma}$, then clearly

$$(127) \lesssim \sqrt{M} \|\partial^\alpha f\|_{\sigma}^2.$$

By Lemma 5 of [G1], we have

$$(128) \lesssim \sqrt{M} \|\partial^\alpha f\|_{\sigma}^2.$$

Applying Lemma 8 to (129) and Lemma 10 to (130) and (131), we conclude the proof. $\square$

We now establish a positivity of L in a ‘differential form’ [G3]. Recalling (8), we rewrite

$$(132) \quad \begin{aligned} \{\partial_t + v \cdot \nabla_x\} f_{\pm} \mp 2\{E \cdot v\} \sqrt{\mu} + L_{\pm} f &= N_{\pm}(f) \\ &\equiv \mp E \cdot \nabla_v f_{\pm} \pm \{E \cdot v\} f_{\pm} + \Gamma_{\pm}(f, f). \end{aligned}$$

Proposition 16. Assume that for $0 \leq t \leq T$, f is the solution to the Vlasov-Poisson-Landau system (8) and (9) with

$$\mathcal{E}_{2,2,0}(f(t)) \leq M,$$

sufficiently small. Then for $m \geq 0$, there exists a function $G(t)$ with

$$(133) \quad G(t) \lesssim \sqrt{\sum_{|\alpha|=m} \|\partial^\alpha f\|^2} \sqrt{\sum_{|\alpha|=m} \|\partial^\alpha \nabla_x P f\|^2}$$

such that

$$(134) \quad \sum_{|\alpha|=m} [||\nabla_x \partial^\alpha P f||_2^2 + ||\partial^\alpha \nabla_x E||_2^2] \\ \lesssim \frac{d}{dt} G(t) + \sum_{|\alpha|=m} [||\nabla_x \partial^\alpha (I - P)f||_\sigma^2 + ||\partial^\alpha (I - P)f||_\sigma^2 + ||\partial^\alpha N_{||}||^2].$$

Here $\partial^\alpha N_{||}$ denotes the L_v^2 projection of $\partial^\alpha N_\pm(f)$ with respect to the subspace generated by $[\sqrt{\mu}, v_i \sqrt{\mu}, v_i v_j \sqrt{\mu}, v_i |v|^2 \sqrt{\mu}]$. Furthermore, for ε small,

$$(135) \quad \sum_{|\alpha|=m} \int \langle L \nabla_x \partial^\alpha f, \nabla_x \partial^\alpha f \rangle dx \\ \gtrsim \varepsilon \sum_{|\alpha|=m} \{ ||\nabla_x \partial^\alpha f||_\sigma^2 - ||(I - P) \partial^\alpha f||_\sigma^2 - ||\partial^\alpha N_{||}||_2^2 \} - \varepsilon \frac{dG}{dt}.$$

Proof. The proof of this lemma is now standard in light of methods developed in [G2], [G3] (with $\varepsilon = 1$). We denote the kernel of L as

$$Pf = \begin{pmatrix} a_+(t, x) \sqrt{\mu} \\ a_-(t, x) \sqrt{\mu} \end{pmatrix} + \begin{pmatrix} b(t, x) \cdot v \sqrt{\mu} \\ b(t, x) \cdot v \sqrt{\mu} \end{pmatrix} + \begin{pmatrix} c(t, x) |v|^2 \sqrt{\mu} \\ c(t, x) |v|^2 \sqrt{\mu} \end{pmatrix},$$

so that $||\nabla_x \partial^\alpha b||_2^2 + ||\nabla_x \partial^\alpha c||_2^2 + ||\nabla_x \partial^\alpha a_\pm||_2^2 \sim ||P \nabla_x \partial^\alpha f||_2^2$.

The first step is to use local conservation laws to estimate the temporal derivatives of $P \nabla_x \partial^\alpha f$ in terms of spatial derivatives. Recalling (8) and (132) we denote

$$\{\partial_t + v \cdot \nabla_x\} f_\pm + L_\pm f = \pm 2 \{E \cdot v\} \sqrt{\mu} + N_\pm(f).$$

Upon taking the vector inner product with $\sqrt{\mu} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \sqrt{\mu} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $|v|^2 \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (the null space of L ; see (34)), we obtain local conservations of masses, total momentum and total energy as in (6.5) in [G3]:

$$(136) \quad (\rho_0 \partial_t a_\pm + \rho_2 \partial_t c) + \frac{\rho_2 \nabla_x \cdot b}{3} = \langle N_\pm, \sqrt{\mu} \rangle, \\ \frac{2\rho_2 \partial_t b}{3} + \frac{\rho_2 \nabla_x [a_+ + a_-]}{3} + \frac{2\rho_4 \nabla_x c}{3} \\ = \langle -v \cdot \nabla_x (I - P)f, v \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle + \langle N, v \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle,$$

$$(137) \quad \partial_t (\rho_2 [a_+ + a_-] + 2\rho_4 c) + \frac{2\rho_4 \nabla_x \cdot b}{3} = \langle N - v \cdot \nabla_x (I - P)f, \sqrt{\mu} \begin{pmatrix} |v|^2 \\ |v|^2 \end{pmatrix} \rangle.$$

Here $\rho_i = \int |v|^i \mu dv$, $N = \begin{pmatrix} N_+ \\ N_- \end{pmatrix}$, and $\pm E \cdot v \sqrt{\mu}$ makes no contribution in the process. Subtract the $+$ from $-$ parts in (136), then take

$$\rho_4 \times [(136)_+ + (136)_-] - \rho_2 \times (137)$$

to get:

$$\begin{aligned} \rho_0 \partial_t [a_+ - a_-] &= \langle N_+ - N_-, \sqrt{\mu} \rangle, \\ (\rho_0 \rho_4 - \rho_2^2) \partial_t [a_+ + a_-] &= \langle N_+ + N_-, [\rho_4 - \rho_2 |v|^2] \sqrt{\mu} \rangle \\ &\quad + \rho_2 \langle v \cdot \nabla_x (I - P)f, |v|^2 \sqrt{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle. \end{aligned}$$

Since $\rho_0\rho_4 - \rho_2^2 > 0$, $\rho_0 > 0$, we can solve for $\partial_t a_\pm$ and $\partial_t \partial^\alpha a_\pm$. Taking ∂^α of (136) to (137), by the definition of $N_{||}$, we get:

$$(138) \quad \|\partial_t \partial^\alpha a\|_2 \lesssim \|(I - P)\nabla_x \partial^\alpha f\|_2 + \|\partial^\alpha N_{||}\|_2,$$

$$(139) \quad \|\partial_t \partial^\alpha c\|_2 \lesssim \|\nabla_x \partial^\alpha b\|_2 + \|(I - P)\nabla_x \partial^\alpha f\|_\sigma + \|\partial^\alpha N_{||}\|_2,$$

$$(140) \quad \|\partial_t \partial^\alpha b\|_2 \lesssim \|\nabla_x \partial^\alpha a\|_2 + \|\nabla_x \partial^\alpha c\|_2 + \|(I - P)\nabla_x \partial^\alpha f\|_\sigma + \|\partial^\alpha N_{||}\|_2.$$

The next step is to use so-called macroscopic equations to estimates $a_\pm, b$ and c . In fact, following the same procedures as in Lemma 6.1 [G3] (with $\varepsilon = 1$), for b and c in (139) and (140), we obtain for any $\eta > 0$:

$$(141) \quad \begin{aligned} \|\nabla_x \partial^\alpha b\|_2^2 + \|\nabla_x \partial^\alpha c\|_2^2 &\leq \frac{dG_{bc}}{dt} + \eta \|P\nabla_x \partial^\alpha f\|_2^2 \\ &\quad + C_\eta [\|\nabla \partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha N_{||}\|_2], \end{aligned}$$

with some $G_{bc} \lesssim \|\partial^\alpha f\| \cdot \|\partial^\alpha \nabla_x P f\|_2$. In the macroscopic equation (6.10) of [G3], $\nabla_x a$ should be replaced by $\nabla_x a_\pm \mp E$. Taking $\nabla_x \cdot$ and the inner product with $a_\pm$ with revised (6.10) in [G3], and using the fact that $\partial^\alpha E = -\nabla \partial^\alpha \phi$ [G4],

$$-\Delta \partial^\alpha \phi = \int [\partial^\alpha f_+ - \partial^\alpha f_-] \sqrt{\mu} dv = \langle \partial^\alpha f, \sqrt{\mu} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = [\partial^\alpha a_+ - \partial^\alpha a_-] \rho_0,$$

we deduce that

$$(142) \quad \begin{aligned} \|\nabla_x \partial^\alpha a_\pm\|_2^2 + \|\partial^\alpha E\|_2^2 \\ \leq \frac{dG_a}{dt} + C [\|\nabla_x \partial^\alpha b\|_2^2 + \|\nabla \partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha N_{||}\|_2], \end{aligned}$$

for some $G_a \lesssim \|\partial^\alpha f\| \cdot \|\partial^\alpha \nabla_x P f\|_2$. Assume $C \geq 1$. We take $2C \times (141) + (142)$ to absorb $C\|\nabla_x \partial^\alpha b\|_2^2$ from the right-hand side:

$$\begin{aligned} C\|\nabla_x \partial^\alpha b\|_2^2 + 2C\|\nabla_x \partial^\alpha c\|_2^2 + \|\nabla_x \partial^\alpha a_\pm\|_2^2 + \|\partial^\alpha E\|_2^2 \\ \leq \frac{d[2CG_{bc} + G_a]}{dt} + 2\eta C \|P\nabla_x \partial^\alpha f\|_2^2 \\ + (2CC_\eta + C) [\|\nabla \partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha N_{||}\|_2]. \end{aligned}$$

By choosing η sufficiently small, we obtain

$$\begin{aligned} \|\nabla_x \partial^\alpha b\|_2^2 + \|\nabla_x \partial^\alpha c\|_2^2 + \|\nabla_x \partial^\alpha a_\pm\|_2^2 + \|\partial^\alpha E\|_2^2 \\ \lesssim \frac{d[2CG_{bc} + G_a]}{dt} + \|\nabla \partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha (I - P)f\|_\sigma^2 + \|\partial^\alpha N_{||}\|_2^2. \end{aligned}$$

We therefore deduce the lemma by summing over $|\alpha| = m$.

The proof of (135) follows from (35) and (134):

$$\begin{aligned}
& \sum_{|\alpha|=m} \int \langle L \nabla_x \partial^\alpha f, \nabla_x \partial^\alpha f \rangle dx \\
& \gtrsim \sum_{|\alpha|=m} \|(I - P) \nabla_x \partial^\alpha f\|_\sigma^2 \\
& = (1 - \varepsilon) \sum_{|\alpha|=m} \|(I - P) \nabla_x \partial^\alpha f\|_\sigma^2 + \varepsilon \sum_{|\alpha|=m} \|(I - P) \nabla_x \partial^\alpha f\|_\sigma^2 \\
& \gtrsim (1 - \varepsilon) \sum_{|\alpha|=m} \|(I - P) \nabla_x \partial^\alpha f\|_\sigma^2 + \varepsilon \sum_{|\alpha|=m} \|P \nabla_x \partial^\alpha f\|_\sigma^2 - \varepsilon \frac{dG}{dt} \\
& \quad - \varepsilon \sum_{|\alpha|=m} \|(I - P) \partial^\alpha f\|_\sigma^2 - \varepsilon \sum_{|\alpha|=m} \|\partial^\alpha N_{||}(f)\|_2^2 \\
& \gtrsim \varepsilon \sum_{|\alpha|=m} \|\nabla_x \partial^\alpha f\|_\sigma^2 - \varepsilon \frac{dG}{dt} - \varepsilon \sum_{|\alpha|=m} \|(I - P) \partial^\alpha f\|_\sigma^2 - \varepsilon \sum_{|\alpha|=m} \|\partial^\alpha N_{||}(f)\|_2^2
\end{aligned}$$

for $1 - \varepsilon \geq \varepsilon$. $\square$

The following proposition establishes the crucial decay to obtain global solutions. It is important to only use up to first-order derivatives of f to extract strong decay with an $\mathcal{E}_{2;2,0}(t)$ bound.

Proposition 17. *Assume that for $0 \leq t \leq T$, $\sup_{0 \leq t \leq T} \mathcal{E}_{2;2,0}(t) \leq M$ is sufficiently small, and*

$$(143) \quad \int_0^T \|\phi_t(s)\|_\infty ds \leq 1.$$

Assume that the conservation laws (12), (13) and (14) are valid. Then there exists $C_l > 0$ such that

$$\begin{aligned}
& \|\nabla_{t,x} \phi(t)\|_\infty + \sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|_2 \lesssim C_l \left\{ 1 + \frac{t}{4l-4} \right\}^{-2l+2} \sup_{0 \leq s \leq T} \sqrt{\mathcal{E}_{2;l,0}(f(s))}, \\
(144) \quad & \|\nabla_{t,x} \phi(t)\|_\infty + \sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|_2 \lesssim e^{-C_l t^{\frac{2}{3}}} \sup_{0 \leq s \leq T} \sqrt{\mathcal{E}_{2;l,q}(f(s))}.
\end{aligned}$$

Proof. Summing over $|\alpha| \leq 1$ in (124), by Gronwall's inequality, we have

$$\begin{aligned}
& \frac{d}{dt} \left[e^{-C \int_0^t \|\phi_t\|_\infty(s) ds} \left\{ \sum_{|\alpha| \leq 1} \frac{e^{\pm 2\phi} (\partial^\alpha f_\pm)^2}{2} + \sum_{|\alpha| \leq 1} \int |\nabla \partial^\alpha \phi|^2 \right\} \right] \\
& + e^{-C \int_0^t \|\phi_t\|_\infty(s) ds} \sum_{|\alpha| \leq 1} \int \langle L \partial^\alpha f, \partial^\alpha f \rangle dx \\
& \lesssim \sqrt{M} e^{-C \int_0^t \|\phi_t\|_\infty(s) ds} \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_\sigma^2
\end{aligned}$$

for some constant $C > 0$. Note that $e^{-\int_0^t \|\phi_t\|_\infty(s) ds} \sim 1$. We treat $\int \langle L \partial^\alpha f, \partial^\alpha f \rangle$ and the summation of $\alpha = 0$ and $|\alpha| = 1$ separately. For $\alpha = 0$, we applying the

standard (35) of L to get

$$(145) \quad \int \langle Lf, f \rangle dx \gtrsim \|(I - P)f\|_\sigma^2.$$

On the other hand, for $|\alpha| = 1$, we bound from below $\int \langle L\partial^\alpha f, \partial^\alpha f \rangle$ by using (135) without derivatives (i.e. $|\alpha| = 0$ in (135)) to get

$$(146) \quad \sum_{|\alpha|=1} \int \langle L\partial^\alpha f, \partial^\alpha f \rangle \gtrsim \varepsilon \sum_{|\alpha|=0} \{ \|\nabla_x f\|_\sigma^2 - \|(I - P)f\|_\sigma^2 - \|N_{||}(f)\|_2^2 \} - \varepsilon \frac{dG}{dt}.$$

For ε sufficiently small, combining (145) and (146), we obtain

$$(147) \quad \begin{aligned} \sqrt{M} \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_\sigma^2 &\gtrsim \frac{d}{dt} \left[e^{-\int_0^t \|\phi_t\|_\infty(s) ds} \left\{ \int \sum_{|\alpha| \leq 1} \frac{e^{\pm 2\phi} (\partial^\alpha f_\pm)^2}{2} + \sum_{|\alpha| \leq 1} \int |\nabla \partial^\alpha \phi|^2 \right\} \right] \\ &\quad + \|(I - P)f\|_\sigma^2 + \varepsilon \sum_{|\alpha|=1} \|\partial^\alpha f\|_\sigma^2 - \varepsilon \|(I - P)f\|_\sigma^2 - \varepsilon \frac{dG}{dt} - \varepsilon \|N_{||}\|_2^2 \\ &\gtrsim \frac{d}{dt} \left[e^{-\int_0^t \|\phi_t\|_\infty(s) ds} \left\{ \int \sum_{|\alpha| \leq 1} \frac{e^{\pm 2\phi} (\partial^\alpha f_\pm)^2}{2} + \sum_{|\alpha| \leq 1} \int |\nabla \partial^\alpha \phi|^2 \right\} - \varepsilon G(t) \right] \\ &\quad + \varepsilon \{ \|(I - P)f\|_\sigma^2 + \sum_{|\alpha|=1} \|\partial^\alpha f\|_\sigma^2 \} - \varepsilon \|N_{||}\|_2^2. \end{aligned}$$

By (132), it is clear from Lemma 7 of [G1] that $\|N_{||}\|_2^2 \lesssim M \|f\|_\sigma^2$. From arguments used in [G2] and the fact that $\|Pf\|_\sigma \sim \|Pf\|_2$, we obtain

$$\|Pf\|_\sigma \lesssim \|\nabla_x Pf\|_\sigma,$$

thanks to the conservation laws (12), (13), (14) with $|\int Pf| \lesssim \sqrt{M} \|Pf\|_2$ and the Poincaré inequality. Since $e^{\pm 2\phi} \lesssim 1$ and $\|\nabla \partial^\alpha \phi\|_2 \lesssim \|\partial^\alpha f\|_2$, by (133) with $m = 0$, we can choose ε small but fixed such that

$$(148) \quad \begin{aligned} Y(t) &\equiv e^{-\int_0^t \|\phi_t\|_\infty ds} \sum_{|\alpha| \leq 1} \left\{ \int \frac{e^{\pm 2\phi} (\partial^\alpha f_\pm)^2}{2} + \int |\nabla \partial^\alpha \phi|^2 \right\} - \varepsilon G(t) \\ &\sim \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_2^2. \end{aligned}$$

By (105) and (106) with $\phi^n = \phi$ and $f^n = f$, we have

$$(149) \quad \|\partial_t \phi(t)\|_\infty + \|\nabla_x \phi(t)\|_\infty + \sum_{|\alpha| \leq 1} \|\partial^\alpha f(t)\|_2 \lesssim \sqrt{Y(t)}.$$

With such an ε , we therefore conclude from (147) that

$$\dot{Y} + 2\delta \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_\sigma^2 \lesssim \sqrt{M} \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_\sigma^2,$$

where $\delta = \delta(\varepsilon)$. For M sufficiently small, we finally have

$$(150) \quad \dot{Y} + \delta \sum_{|\alpha| \leq 1} \|\partial^\alpha f\|_\sigma^2 \leq 0.$$

We now establish polynomial decay by applying the interpolation method developed in [SG1]. We observe from Hölder's inequality with $p = \frac{4l-3}{4l-4}$ and $q = 4l-3$:

$$\begin{aligned} \left\{ \int |\partial^\alpha f|^2 \right\} &= \left\{ \int \frac{1}{\langle v \rangle^{\frac{4l-4}{4l-3}}} \langle v \rangle^{\frac{4l-4}{4l-3}} |\partial^\alpha f|^2 \right\} \\ &\leq \left\{ \int \frac{1}{\langle v \rangle} |\partial^\alpha f|^2 \right\}^{\frac{4l-4}{4l-3}} \left\{ \int \langle v \rangle^{4l-4} |\partial^\alpha f|^2 \right\}^{\frac{1}{4l-3}}. \end{aligned}$$

From (19) and an interpolation, we obtain:

$$\begin{aligned} \|\partial^\alpha f\|_\sigma^2 &\gtrsim \int \frac{1}{\langle v \rangle} |\partial^\alpha f|^2 dv \\ &\gtrsim \left\{ \int |\partial^\alpha f|^2 \right\}^{\frac{4l-3}{4l-4}} \left\{ \int \langle v \rangle^{4l-4} |\partial^\alpha f|^2 \right\}^{-\frac{1}{4l-4}} \\ &\gtrsim \left\{ \int |\partial^\alpha f|^2 \right\}^{\frac{4l-3}{4l-4}} \left\{ \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(f(s)) \right\}^{-\frac{1}{4l-4}} \end{aligned}$$

as $4l-4 > 0$. We therefore have for some other $\delta > 0$,

$$\frac{d}{dt} Y + \delta Y^{1+\frac{1}{4l-4}} \left\{ \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(f(s)) \right\}^{-\frac{1}{4l-4}} \leq 0.$$

It thus follows that

$$Y^{-1-\frac{1}{4l-4}} \frac{d}{dt} Y + \delta \left\{ \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(s) \right\}^{-\frac{1}{4l-4}} \leq 0$$

and by integrating over time, we obtain:

$$(4l-4) \{ Y^{-\frac{1}{4l-4}}(0) - Y^{-\frac{1}{4l-4}}(t) \} \leq -\delta t \left\{ \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(f(s)) \right\}^{-\frac{1}{4l-4}}.$$

But from (148), $Y(0) \lesssim \mathcal{E}_{2;l,0}(0) \leq \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(f(s))$, we have

$$\begin{aligned} (4l-4) Y^{-\frac{1}{4l-4}}(t) &\gtrsim \delta t \left\{ \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(f(s)) \right\}^{-\frac{1}{4l-4}} + (4l-4) \{ Y^{-\frac{1}{4l-4}}(0) \} \\ &\gtrsim \{ \delta t + (4l-4) \} \left\{ \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,0}(f(s)) \right\}^{-\frac{1}{4l-4}}. \end{aligned}$$

By (149), we conclude the polynomial decay by taking the $(4l-4)$ -th power (positive) of both sides.

We now prove the stretched exponential decay (144) by the splitting method in [SG2]. For any $\theta > 0$,

$$\begin{aligned}
 \|\partial^\alpha f\|_\sigma^2 &\gtrsim \int \frac{1}{\langle v \rangle} |\partial^\alpha f|^2 dv = \int_{|v| \leq \theta t^{1/3}} + \int_{|v| \geq \theta t^{1/3}} \\
 &\geq \frac{t^{-1/3}}{\theta} \int_{|v| \leq \theta t^{1/3}} |\partial^\alpha f|^2 dv \\
 &= \frac{t^{-1/3}}{\theta} \int |\partial^\alpha f|^2 dv - \frac{t^{-1/3}}{\theta} \int_{|v| \geq \theta t^{1/3}} |\partial^\alpha f|^2 dv \\
 &\gtrsim \frac{t^{-1/3}}{\theta} Y(t) - \frac{t^{-1/3}}{\theta} \int_{|v| \geq \theta t^{1/3}} |\partial^\alpha f|^2 dv.
 \end{aligned}$$

By (150), for some other $\delta > 0$, $\frac{t^{-1/3}}{\theta} \leq |v|$ and $e^{q|v|^2} e^{-q\theta^2 t^{2/3}} \geq 1$, so that

$$\begin{aligned}
 \frac{d}{dt} Y + \frac{\delta t^{-1/3}}{\theta} Y(t) &\leq \frac{t^{-1/3}}{\theta} \int_{|v| \geq \theta t^{1/3}} |\partial^\alpha f|^2 dv \\
 &\leq \frac{t^{-1/3}}{\theta} \int e^{q|v|^2} e^{-q\theta^2 t^{2/3}} |\partial^\alpha f|^2 dv \\
 &\leq \frac{t^{-1/3}}{\theta} e^{-q\theta^2 t^{2/3}} \int e^{q|v|^2} |\partial^\alpha f|^2 dv \\
 &\leq \frac{t^{-1/3}}{\theta} e^{-q\theta^2 t^{2/3}} \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,q}(f(s)).
 \end{aligned}$$

We therefore have

$$\begin{aligned}
 \{e^{\frac{3\delta t^{2/3}}{2\theta}} Y(t)\}' &\leq e^{\frac{3\delta t^{2/3}}{2\theta}} \frac{t^{-1/3}}{\theta} e^{-q\theta^2 t^{2/3}} \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,q}(f(s)) \\
 &\leq e^{\frac{3\delta t^{2/3}}{2\theta} - q\theta^2 t^{2/3}} \frac{t^{-1/3}}{\theta} \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,q}(f(s)).
 \end{aligned}$$

By (148), $Y(0) \lesssim \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,q}(f(s))$, we obtain by integrating from 0 to t :

$$\begin{aligned}
 Y(t) &\leq \left\{ 1 + e^{-\frac{3\delta t^{2/3}}{2\theta}} \int_0^t e^{\frac{3\delta s^{2/3}}{2\theta} - q\theta^2 s^{2/3}} \frac{s^{-1/3}}{\theta} ds \right\} \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,q}(f(s)) \\
 &\leq C_l e^{-\frac{3\delta t^{2/3}}{2\theta}} \sup_{0 \leq s \leq T} \mathcal{E}_{2;l,q}(f(s)),
 \end{aligned}$$

if θ is so large that $\int_0^\infty e^{[\frac{3\delta}{2\theta} - q\theta^2]s^{2/3}} \frac{s^{-1/3}}{\theta} ds < \infty$. □

We are now ready to prove the main Theorem 2.

Proof. We first choose smooth initial data $f_0 \in C_c^\infty$ and $F_0 = \mu + \sqrt{\mu} f_0 > 0$.

Step 1. Global Small $\mathcal{E}_{2;2,0}$ Solutions.

We denote

(151)

$$T_* = \sup_{t \geq 0} \left\{ \mathcal{E}_{2;2,0}(f)(t) + \int_0^t \mathcal{D}_{2;2,0}(f)(s) ds \leq M \text{ and } \int_0^t \|\nabla_{t,x} \phi(s)\|_\infty ds \leq \sqrt{M} \right\}.$$

Clearly $T_* > 0$ if $\mathcal{E}_{2;2,0}(f_0)$ is sufficiently small from Theorem 13. Our goal is to show that $T_* = \infty$ if we further choose $\mathcal{E}_{2;2,0}(f_0)$ small.

In (124), by $\int_0^{T_*} \|\partial_t \phi\|_\infty(s) ds \leq 1$, we use the standard Gronwall lemma to get from (35):

$$\begin{aligned} & \sum_{|\alpha| \leq 2} \left\{ |\partial^\alpha f|^2 + \int |\nabla \partial^\alpha \phi|^2 \right\} + \sum_{|\alpha| \leq 2} \int_0^t \|(I-P)\partial^\alpha f\|_\sigma^2 \\ & \lesssim \mathcal{E}_{2;2,0}(f_0) + \sqrt{M} \int_0^t \mathcal{D}_{2;0,0}(f) ds. \end{aligned}$$

As in (147), we note that $\|\partial^\alpha N_\pm\| \lesssim \sqrt{M} \mathcal{D}_{2;2,0}(f)$ for $|\alpha| \leq 2$ from Lemma 7 in [G1] and $-\Delta \phi = \int [f_+ - f_-] \sqrt{\mu} dv$. We apply Proposition 16 for $1 \leq |\alpha| \leq 2$ with a fixed and small ε in (135) so that

$$\sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_2^2 + \sum_{1 \leq |\alpha| \leq 2} \int_0^t \|\partial^\alpha f\|_\sigma^2 + \int_0^t \|(I-P)f\|_\sigma^2 \lesssim \mathcal{E}_{2;2,0}(f_0) + \sqrt{M} \int_0^t \mathcal{D}_{2;2,0}(f) ds.$$

Thanks to the conservation laws (12), (13) and (14), $\|Pf\|_\sigma \lesssim \|\nabla_x Pf\|_\sigma$ as before, and we deduce

$$\sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_2^2 + \sum_{|\alpha| \leq 2} \int_0^t \|\partial^\alpha f\|_\sigma^2 \lesssim \mathcal{E}_{2;2,0}(f_0) + \sqrt{M} \int_0^t \mathcal{D}_{2;2,0}(f) ds.$$

For $\sqrt{M}$ sufficiently small,

$$(152) \quad \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_2^2 + \sum_{|\alpha| \leq 2} \int_0^t \|\partial^\alpha f\|_\sigma^2 \lesssim \mathcal{E}_{2;2,0}(f_0).$$

We take a large constant $C \times (152) + (122)$ to absorb $\int_0^t \sum_{|\alpha| \leq 2} \|\partial^\alpha f\|_\sigma^2 ds$:

$$\begin{aligned} & \mathcal{E}_{2;l,q}(f) + \int_0^t \mathcal{D}_{2;l,q}(f) ds \\ & \lesssim C_l [\mathcal{E}_{2;l,q}(f_0) + \int_0^t [\|\partial_t \phi(s)\|_\infty + \|\nabla_x \phi(s)\|_\infty + \mathcal{D}_{2;2,0}(f)] \mathcal{E}_{2;l,q}(f) ds]. \end{aligned}$$

Since $\int_0^{T_*} [\|\partial_t \phi(s)\|_\infty + \|\nabla_x \phi(s)\|_\infty + \mathcal{D}_{2;2,0}(f)] ds \leq 1$, Lemma 5 implies that

$$(153) \quad \mathcal{E}_{2;l,q}(f) + \int_0^t \mathcal{D}_{2;l,q}(f) ds \lesssim C_l \mathcal{E}_{2;l,q}(f_0).$$

In particular, we choose $l = 2$ and $q = 0$ to conclude that

$$\mathcal{E}_{2;2,0}(f)(t) + \int_0^t \mathcal{D}_{2;2,0}(f) ds \lesssim \mathcal{E}_{2;2,0}(f_0).$$

Combining this bound with Proposition 17, we obtain

$$\int_0^t \{ \|\nabla_x \phi(s)\|_\infty + \|\partial_t \phi(s)\|_\infty \} ds \lesssim \sqrt{\sup_{0 \leq s \leq T_*} \mathcal{E}_{2;2,0}(f(s))} \int_0^t \frac{ds}{1+s^2} \lesssim \sqrt{\mathcal{E}_{2;2,0}(f_0)}.$$

Upon choosing the initial condition $\mathcal{E}_{2;2,0}(f_0)$ even smaller, we deduce that for $0 \leq t \leq T_*$,

$$\mathcal{E}_{2;2,0}(f(t)) + \int_0^t \mathcal{D}_{2;2,0}(f) ds \leq \frac{M}{2} < M$$

and

$$\int_0^{T_*} \{ \|\nabla_x \phi(s)\|_\infty + \|\partial_t \phi(s)\|_\infty \} ds \leq \frac{M}{2} < M.$$

This implies that $T_* = \infty$ and the solution is global.

Step 2. Higher Moments and Higher Regularity.

We shall prove this via an induction of the total derivatives $|\alpha| + |\beta| = m$. By (153), clearly the theorem is valid for $m = 2$.

Assume $|\alpha| = m - 1$ is valid for (30). Summing over $|\alpha| = m$ in (125), by (35), we deduce that

$$\begin{aligned} & \sum_{|\alpha|=m} \left\{ \|\partial^\alpha f\|_2^2 + \int \|\nabla \partial^\alpha \phi\|^2 \right\} + \sum_{|\alpha|=m} \int_0^t \|(I - P)\partial^\alpha f\|_\sigma^2 \\ & \lesssim \mathcal{E}_{m;l,q}(f_0) + (\sqrt{M} + \eta) \int_0^t \sum_{|\alpha|=m} \|\partial^\alpha f\|_\sigma^2 \\ & \quad + \int_0^t \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) + C_{m,\eta} \int_0^t \{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f). \end{aligned}$$

We now integrate (135) with $m - 1$ from 0 to t . We note that from Lemma 7 of [G1] and (132),

$$\begin{aligned} \sum_{|\alpha| \leq m-1} \|\partial^\alpha N\|_2^2 & \lesssim C_{m,\eta} \{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f), \\ \sum_{|\alpha| \leq m-1} \|\partial^\alpha (I - P)f\|_\sigma^2 & \lesssim \mathcal{D}_{m-1;l,q}(f). \end{aligned}$$

From (133) with $m - 1$, we have

$$\begin{aligned} G(t) & \lesssim \sum_{|\alpha|=m} \|\partial^\alpha f(t)\|^2 + \sum_{|\alpha|=m-1} \|\partial^\alpha f(t)\|^2 \\ & \lesssim \sum_{|\alpha|=m} \|\partial^\alpha f(t)\|^2 + \mathcal{E}_{m-1;l,q}(f(t)) \\ & \lesssim \sum_{|\alpha|=m} \|\partial^\alpha f(t)\|^2 + P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0)) \end{aligned}$$

by the induction hypothesis. Choosing ε small in (135), we obtain:

$$\begin{aligned} & \sum_{|\alpha|=m} \|\partial^\alpha f\|_2^2 + \sum_{|\alpha|=m} \int_0^t \|\partial^\alpha f\|_\sigma^2 \\ & \lesssim \mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0)) \\ & \quad + (\sqrt{M} + \eta) \int_0^t \sum_{|\alpha|=m} \|\partial^{\alpha'} f\|_\sigma^2 + \int_0^t \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) \\ & \quad + C_{l,m,\eta} \int_0^t \{1 + \mathcal{E}_{m-1;l,q}(f)\} \mathcal{D}_{m-1;l,q}(f) \\ & \lesssim \mathcal{E}_{m;l,q}(f_0) + (\sqrt{M} + \eta) \sum_{|\alpha|=m} \|\partial^{\alpha'} f\|_\sigma^2 + \int_0^t \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f) \\ & \quad + C_{l,m,\eta} [1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0)). \end{aligned}$$

Here $P_{m-1,l}$ is a continuous, increasing function from the induction hypothesis. For M, η sufficiently small,

$$\begin{aligned}
 & \sum_{|\alpha|=m} \left\{ |\partial^\alpha f|^2 + \int |\nabla \partial^\alpha \phi|^2 \right\} + \sum_{|\alpha|=m} \int_0^t \|\partial^\alpha f\|_\sigma^2 \\
 & \lesssim C_{l,m} [1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] [\mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
 (154) \quad & + \int_0^t \mathcal{D}_{2;2,0}(f) \mathcal{E}_{m;l,q}(f),
 \end{aligned}$$

where we have used

$$\mathcal{E}_{m-1;l,q}(f_0) \leq \mathcal{E}_{m;l,q}(f_0), \quad P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0)) \leq P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0)).$$

Multiplying a large constant $C \times (154) + (123)$ to absorb $\int_0^t \sum_{|\alpha|=m} \|\partial^\alpha f\|_\sigma^2$ in (123), we obtain:

$$\begin{aligned}
 & \mathcal{E}_{m;l,q}(f) + \int_0^t \mathcal{D}_{m;l,q}(f) \\
 & \lesssim C_{l,m} [1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] [\mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
 & + C_{l,m} \int_0^t \mathcal{D}_{m-1;l,q}(f) \mathcal{E}_{m;l,q}(f).
 \end{aligned}$$

We use Gronwall Lemma 5 with $\int_0^t \mathcal{D}_{m-1;l,q}(f) ds \lesssim P_{m-1,l}(\mathcal{E}_{m-1;l,q}(f_0))$ to get

$$\begin{aligned}
 & \mathcal{E}_{m;l,q}(f) + \int_0^t \mathcal{D}_{m;l,q}(f) \\
 & \lesssim C_{l,m} e^{C_{l,m} P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))} [1 + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
 & \quad \times [\mathcal{E}_{m;l,q}(f_0) + P_{m-1,l}(\mathcal{E}_{m;l,q}(f_0))] \\
 & \equiv P_{m,l}(\mathcal{E}_{m;l,q}(f_0)).
 \end{aligned}$$

This concludes the theorem for $f_0 \in C_c^\infty$. For a general datum $f_0 \in \mathcal{E}_{m;l,q}$ we can use a sequence of smooth approximation f_0^k as in (121) and take a limit. $\square$

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