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A PROOF OF THE HOWE DUALITY CONJECTURE

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to Professor Roger Howe
who started it all
on the occasion of his 70th birthday

1. Introduction

Let F be a nonarchimedean local field of characteristic not 2 and residue characteristic p. Let E be F itself or a quadratic field extension of F. For $\epsilon = \pm$, we consider a $-\epsilon$-Hermitian space W over E of dimension n and an ϵ-Hermitian space V of dimension m.

Let $G(W)$ and $H(V)$ denote the isometry group of W and V, respectively. Then the group $G(W) \times H(V)$ forms a dual reductive pair and possesses a Weil representation ω_ψ which depends on a nontrivial additive character ψ of F (and some other auxiliary data which we shall suppress for now). To be precise, when $E = F$ and one of the spaces, say V, is odd dimensional, one needs to consider the metaplectic double cover of $G(W)$; we shall simply denote this double cover by $G(W)$ as well. The various cases are tabulated in [GI, Section 3].

In the theory of local theta correspondence, one is interested in the decomposition of ω_ψ into irreducible representations of $G(W) \times H(V)$. More precisely, for any irreducible admissible representation π of $G(W)$, one may consider the maximal π-isotypic quotient of ω_ψ. This has the form $\pi \otimes \Theta_{W,V,\psi}(\pi)$ for some smooth representation $\Theta_{W,V,\psi}(\pi)$ of $H(V)$; we shall frequently suppress (W, V, ψ) from the notation if there is no cause for confusion. It was shown by Kudla [K1] that $\Theta(\pi)$ has finite length (possibly zero), so we may consider its maximal semisimple quotient $\theta(\pi)$. One has the following fundamental conjecture due to Howe [H1, H2]:

Howe Duality Conjecture for $G(W) \times H(V)$

(i) $\theta(\pi)$ is either 0 or irreducible.

(ii) If $\theta(\pi) = \theta(\pi') \neq 0$, then $\pi = \pi'$.

A concise reformulation is the following: for any irreducible π and π',

$$\text{(HD)} \qquad \dim \operatorname{Hom}_{H(V)}(\theta(\pi), \theta(\pi')) \leq \delta_{\pi,\pi'} := \begin{cases} 1, & \text{if } \pi \cong \pi'; \\ 0, & \text{if } \pi \ncong \pi'. \end{cases}$$

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We take note of the following theorem.

Theorem 1.1. *(i) If π is supercuspidal, then $\Theta(\pi)$ is either zero or irreducible (and thus is equal to $\theta(\pi)$). Moreover, for any irreducible supercuspidal π and π',*

$$\Theta(\pi) \cong \Theta(\pi') \neq 0 \Longrightarrow \pi \cong \pi'.$$

(ii) $\theta(\pi)$ is multiplicity-free.
(iii) If $p \neq 2$, the Howe duality conjecture holds.

The statement (i) is a classic theorem of Kudla [K1] (see also [MVW]), whereas (iii) is a well-known result of Waldspurger [W]. The statement (ii), on the other hand, is a result of Li-Sun-Tian [LST]. We note that the techniques for proving the three statements in the theorem are quite disjoint from each other. For example, the proof of (i) is based on arguments using the doubling seesaw and Jacquet modules of the Weil representation: these have become standard tools in the study of the local theta correspondence. The proof of (iii) is based on K-type analysis and uses various lattice models of the Weil representation. Finally, the proof of (ii) is based on an argument using the Gelfand-Kazhdan criterion for the (non)existence of equivariant distributions.

In this paper, we shall not assume any of the statements in Theorem 1.1. Indeed, the purpose of this paper is to give a simple proof of the Howe duality conjecture, following a strategy initiated by Howe in his Corvallis article [H1, Section 11] and using essentially the same tools in the proof of Theorem 1.1(i) [K1], as developed further in [MVW]. Thus, our main theorem is Theorem 1.2.

Theorem 1.2. *The Howe duality conjecture* (HD) *holds for the pair $G(W) \times H(V)$.*

Let us make a few remarks:

(1) From our previous brief sketch of the history of the Howe duality conjecture, the reader can discern two distinct lines of attack on the Howe duality conjecture, the genesis of which can both be found in [H1]. The first is a study of K-types, which was used to establish the conjecture in the archimedean cases and adapted to the p-adic case (with $p \neq 2$) by Howe and Waldspurger. The second is a doubling seesaw argument outlined in [H1, Section 11]. Quoting Howe [H1, Pg. 284]:

    From this doubling construction, we see that the space $(Y \otimes Y^\vee)_1$ defined earlier is described as a $2\tilde{G}'$-module by Theorem 9.2. By restriction we can investigate its structure as a $\tilde{G}' \times \tilde{G}'$-module. Doing so we find that the duality conjecture is certainly true if G or G' is compact, and in general is "almost" true. It remains in the general case to remove the "almost."

    It has taken almost 40 years to remove the "almost."

(2) The setup makes sense even when $E = F \times F$ is a split quadratic algebra, in which case the groups $G(W)$ and $H(V)$ are general linear groups. In that case, the Howe duality conjecture has been shown by Mínguez in [M]. As we shall see, the proof of Theorem 1.2 is essentially analogous to the one given by Mínguez.

(3) In an earlier paper [GT], we had extended Theorem 1.1(i) (with $\Theta(\pi)$ replaced by $\theta(\pi)$) from supercuspidal to tempered representations. Using this, we had shown the Howe duality conjecture for almost equal rank dual pairs. The argument in Section 4 of this paper (doubling seesaw) is the same as that in [GT, Section 2], but pushed to the limit beyond tempered representations.

On the other hand, the argument in Section 5 (Kudla's filtration) is entirely different from that in [GT, Section 3] and uses a key technique of Mínguez [M].

(4) We remark that in the papers [M1, M2, M3, M4], Muić has conducted detailed studies of the local theta correspondence for symplectic-orthogonal dual pairs. In [M1], for example, he explicitly determined the theta lift of discrete series representations π in terms of the Moeglin-Tadić classification and observed as a consequence the irreducibility of $\theta(\pi)$. The Moeglin-Tadić classification was conditional at that point, and we are not sure where it stands today. In [M3, M4], Muić proved various general properties of the theta lifting of tempered representations (such as the issue of whether $\Theta(\pi) = \theta(\pi)$) and obtained very explicit information about the theta lifting under the assumption of the Howe duality conjecture. The main tools he used are Jacquet modules analysis and Kudla's filtration. Since the Howe duality conjecture is a simple statement without reference to classification, it seems desirable to have a classification-free proof. Indeed, our result renders most results in [M3, M4] unconditional.

As is well known, there is another family of dual pairs associated to quaternionic Hermitian and skew-Hermitian spaces. (See [W] or [K2] for more details.) Our proof, unfortunately, does not apply to these quaternionic dual pairs, because we have made use of the MVW-involution $\pi \mapsto \pi^{\mathrm{MVW}}$ on the category of smooth representations of $G(W)$ and $H(V)$. For the same reason, the result of [LST] in Theorem 1.1(ii) is not known for these quaternionic dual pairs. Unlike the contragredient functor, which is contravariant in nature, the MVW-involution is covariant and has the property that $\pi^{\mathrm{MVW}} = \pi^\vee$ if π is irreducible. It was shown in [LnST] that such an involution does not exist for quaternionic unitary groups.

Nonetheless, even in the quaternionic case, our proof gives a partial result which is often sufficient for global applications. Namely, if π is an irreducible Hermitian representation (i.e., $\overline{\pi} = \pi^\vee$) and we let $\theta_{her}(\pi) \subset \theta(\pi)$ denote the submodule generated by irreducible Hermitian summands, then the results of Theorem 1.2 hold for Hermitian π's and with $\theta(\pi)$ replaced by $\theta_{her}(\pi)$. Namely we have Theorem 1.3.

Theorem 1.3. *Consider a quaternionic dual pair $G(W) \times H(V)$ and let π and π' be irreducible Hermitian representations of $G(W)$. Let $\theta_{her}(\pi) \subset \theta(\pi)$ be the submodule generated by irreducible Hermitian summands. Then we have*

$$\dim \mathrm{Hom}_{H(V)}(\theta_{her}(\pi), \theta_{her}(\pi')) \leq \delta_{\pi,\pi'}.$$

In particular, if π and π' are unitary, we have

$$\dim \mathrm{Hom}_{H(V)}(\theta_{unit}(\pi), \theta_{unit}(\pi')) \leq \delta_{\pi,\pi'},$$

where $\theta_{unit}(\pi) \subset \theta_{her}(\pi)$ consists of irreducible unitary summands of $\theta(\pi)$.

We give a proof of this theorem in the last section of the paper.

2. Basic notations and conventions

2.1. **Fields.** Throughout the paper, F denotes a nonarchimedean local field of characteristic different from 2 and residue characteristic p. Once and for all, we fix a nontrivial additive character ψ on F. Let E be F itself or a quadratic field extension of F. For $\epsilon = \pm 1$, we set

$$\epsilon_0 = \begin{cases} \epsilon \text{ if } E = F; \\ 0 \text{ if } E \neq F. \end{cases}$$

2.2. **Spaces.** Let

$$W = W_n = \text{a } -\epsilon\text{-Hermitian space over } E \text{ of dimension } n$$
$$V = V_m = \text{an } \epsilon\text{-Hermitian space over } E \text{ of dimension } m.$$

We also set

$$s_{m,n} = \frac{m - (n + \epsilon_0)}{2}.$$

2.3. **Groups.** We will consider the isometry groups associated to the pair (V, W) of $\pm\epsilon$-Hermitian spaces. More precisely, we set

$$G(W) = \begin{cases} \text{the metaplectic group } \mathrm{Mp}(W), \text{ if } W \text{ is symplectic and } \dim V \text{ is odd;} \\ \text{the isometry group of } W, \text{ otherwise.} \end{cases}$$

We define $H(V)$ similarly by switching the roles of W and V. Occasionally we write

$$G_n := G(W_n),$$
$$H_m := H(V_m).$$

For the general linear group, we shall write GL_n for the group $\mathrm{GL}_n(E)$. Also for a vector space X over E, we write $\det_{\mathrm{GL}(X)}$ or sometimes simply $\det_X$ for the determinant on $\mathrm{GL}(X)$.

2.4. **Representations.** For a p-adic group G, let $\mathrm{Rep}(G)$ denote the category of smooth representations of G and denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible smooth representations of G.

For a parabolic $P = MN$ of G, we have the normalized induction functor

$$\mathrm{Ind}_P^G : \mathrm{Rep}(M) \longrightarrow \mathrm{Rep}(G).$$

On the other hand, we have the normalized Jacquet functor

$$R_P : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(M).$$

If $\overline{P} = M\overline{N}$ denotes the opposite parabolic subgroup to P, we likewise have the functor $R_{\overline{P}}$. We shall frequently exploit the following two Frobenius' reciprocity formulas:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_M(R_P(\pi), \sigma) \quad \text{(standard Frobenius reciprocity)}$$

and

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \pi) \cong \mathrm{Hom}_M(\sigma, R_{\overline{P}}(\pi)) \quad \text{(Bernstein's Frobenius reciprocity).}$$

Moreover, Bernstein's Frobenius reciprocity is equivalent to the statement

$$R_P(\pi^\vee)^\vee \cong R_{\overline{P}}(\pi)$$

for any smooth representation π with contragredient $\pi^\vee$, where $\overline{P}$ denotes the opposite parabolic subgroup to P.

2.5. **Parabolic induction.** When G is a classical group, we shall use Tadić's notation for induced representations. Namely, for general linear groups, we set

$$\rho_1 \times \cdots \times \rho_a := \mathrm{Ind}_Q^{\mathrm{GL}_{n_1+\cdots+n_a}} \rho_1 \otimes \cdots \otimes \rho_a,$$

where Q is the standard parabolic subgroup with Levi subgroup $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_a}$. For a classical group such as $G(W)$, its parabolic subgroups are given as the stabilizers of flags of isotropic spaces. If X_t is a t-dimensional isotropic space of $W = W_n$ and we decompose

$$W = X_t \oplus W_{n-2t} \oplus X_t^*,$$

the corresponding maximal parabolic subgroup $Q(X_t) = L(X_t) \cdot U(X_t)$ has Levi factor $L(X_t) = \mathrm{GL}(X_t) \times G(W_{n-2t})$. If ρ is a representation of $\mathrm{GL}(X_t)$ and σ is a representation of $G(W_{n-2t})$, we write

$$\rho \rtimes \sigma = \mathrm{Ind}_{Q(X_t)}^{G(W)} \rho \otimes \sigma.$$

More generally, a standard parabolic subgroup Q of $G(W)$ has the Levi factor of the form $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_a} \times G(W_{n'})$, and we set

$$\rho_1 \times \cdots \times \rho_a \rtimes \sigma := \mathrm{Ind}_Q^{G(W)} \rho_1 \otimes \cdots \otimes \rho_a \otimes \sigma,$$

where ρ_i is a representation of GL_{n_i} and σ is a representation of $G(W_{n'})$. When $G(W) = \mathrm{Mp}(W)$ is a metaplectic group, we will follow the convention of [GS, Sections 2.2–2.5] for normalized parabolic induction.

We have the analogous convention for parabolic subgroups and induced representations of $H(V_m)$. For example, a maximal parabolic subgroup of $H(V_m)$ has the form $P(Y_t) = M(Y_t) \cdot N(Y_t)$ and is the stabilizer of a t-dimensional isotropic subspace Y_t of V_m.

To distinguish between representations of $G(W)$ and $H(V)$, we will normally use lowercase Greek letters such as π, σ, etc., to denote representations of $G(W)$, and uppercase Greek letters such as $\mathit{\Pi}, \mathit{\Sigma}$, etc., to denote representations of $H(V)$.

2.6. **Conjugation action.** Let X be a vector space over E of dimension n, and let c be the generator of $\mathrm{Gal}(E/F)$. For each representation ρ of $\mathrm{GL}(X)$, we define the c-conjugate ${}^c\rho$ of ρ as follows. First, fix a basis of X, which gives an isomorphism $\alpha : \mathrm{GL}(X) \cong \mathrm{GL}_n(E)$. The natural action of c on $\mathrm{GL}_n(E)$ induces an action on $\mathrm{GL}(X)$, which we write as $c \cdot g := \alpha^{-1} \circ c \circ \alpha(g)$ for $g \in \mathrm{GL}(X)$.

Then we define ${}^c\rho$ by

$${}^c\rho(g) := \rho(c \cdot g)$$

for $g \in \mathrm{GL}(X)$. Of course, a different choice of the basis gives a different α and thus a different automorphism $g \mapsto c \cdot g$ of $\mathrm{GL}(X)$. But these different automorphisms differ from each other by inner automorphisms of $\mathrm{GL}(X)$, and hence the isomorphism class of ${}^c\rho$ is independent of the choice of the basis. Of course, if $E = F$, then ${}^c\rho = \rho$.

2.7. **MVW.** In [MVW, p. 91], Moeglin, Vignéras and Waldspurger introduced a functor

$$\mathrm{MVW} : \mathrm{Rep}(G(W)) \longrightarrow \mathrm{Rep}(G(W)),$$

which is an involution and satisfies

$$\pi^{\mathrm{MVW}} = \pi^\vee \quad \text{if } \pi \text{ is irreducible.}$$

Unlike the contragredient functor, this MVW involution is covariant. It will be useful to observe that

$$(\rho \rtimes \sigma)^{\mathrm{MVW}} = {}^{c}\rho \rtimes \sigma^{\mathrm{MVW}}. \tag{2.1}$$

2.8. **Weil representations.** To consider the Weil representation of the pair $G(W) \times H(V)$, we need to specify extra data to give a splitting $G(W) \times H(V) \to \mathrm{Mp}(W \otimes V)$ of the dual pair. Such splittings were constructed and parametrized by Kudla [K2], and we shall use his convention here, as described in [GI, Sections 3.2 and 3.3]. In particular, a splitting is specified by fixing a pair of splitting characters $\boldsymbol{\chi} = (\chi_V, \chi_W)$, which are certain unitary characters of $E^\times$. Pulling back the Weil representation of $\mathrm{Mp}(W \otimes V)$ to $G(W) \times H(V)$ via the splitting, we obtain the associated Weil representation $\omega_{W,V,\boldsymbol{\chi},\psi}$ of $G(W) \times H(V)$. Note that the character χ_V satisfies

$$ {}^{c}\chi_V^{-1} = \chi_V \tag{2.2}$$

and likewise for χ_W. We shall frequently suppress $\boldsymbol{\chi}$ and ψ from the notation, and simply write $\omega_{W,V}$ for the Weil representation.

3. Some lemmas

In this section, we record a couple of standard lemmas which will be used in the proof of the main theorem. We also introduce the important notion of "occurring on the boundary."

3.1. **Kudla's filtration.** The first lemma describes the computation of the Jacquet modules of the Weil representation. This is a well-known result of Kudla [K1]; see also [MVW].

Lemma 3.1. *The Jacquet module $R_{Q(X_a)}(\omega_{W_n,V_m})$ has an equivariant filtration*

$$R_{Q(X_a)}(\omega_{W_n,V_m}) = R^0 \supset R^1 \supset \cdots \supset R^a \supset R^{a+1} = 0$$

whose successive quotients $J^k = R^k/R^{k+1}$ are described in [GI, Lemma C.2]. *More precisely,*

$$J^k = \mathrm{Ind}_{Q(X_{a-k},X_a)\times G(W_{n-2a})\times P(Y_k)}^{\mathrm{GL}(X_a)\times G(W_{n-2a})\times H(V_m)} \left(\chi_V|\det_{X_{a-k}}|^{\lambda_{a-k}} \otimes C_c^\infty(\mathrm{Isom}_{E,c}(X_k,Y_k)) \otimes \omega_{W_{n-2a},V_{m-2k}}\right),$$

where

- $\lambda_{a-k} = s_{m,n} + \frac{a-k}{2}$;
- $V_m = Y_k + V_{m-2k} + Y_k^*$ *with Y_k a k-dimensional isotropic space;*
- $X_a = X_{a-k} + X_k$ *and $Q(X_{a-k}, X_a)$ is the maximal parabolic subgroup of* $\mathrm{GL}(X_a)$ *stabilizing X_{a-k};*
- $\mathrm{Isom}_{E,c}(X_k, Y_k)$ *is the set of E-conjugate-linear isomorphisms from X_k to Y_k;*
- $\mathrm{GL}(X_k) \times \mathrm{GL}(Y_k)$ *acts on $C_c^\infty(\mathrm{Isom}_{E,c}(X_k,Y_k))$ as*

  $$((b,c)\cdot f)(g) = \chi_V(\det b)\chi_W(\det c)f(c^{-1}gb)$$

  for $(b,c) \in \mathrm{GL}(X_k) \times \mathrm{GL}(Y_k)$, $f \in C_c^\infty(\mathrm{Isom}_{E,c}(X_k,Y_k))$, and $g \in \mathrm{GL}_k$;
- $J^k = 0$ *for $k > \min\{a, q\}$, where q is the Witt index of V_m.*

In particular, the bottom piece of the filtration (if nonzero) is

$$J^a \cong \mathrm{Ind}_{\mathrm{GL}(X_a)\times G(W_{n-2a})\times P(Y_a)}^{\mathrm{GL}(X_a)\times G(W_{n-2a})\times H(V_m)} \left(C_c^\infty(\mathrm{Isom}_{E,c}(X_a,Y_a))\otimes\omega_{W_{n-2a},V_{m-2a}}\right).$$

3.2. **A degenerate principal series.** Another key ingredient used later is the decomposition of a certain degenerate principal series representation. Consider the "doubled" space $W+W^-$, where W^- is obtained from W by multiplying the form by -1. This doubled space contains the diagonally embedded

$$\Delta W \subset W + W^-$$

as a maximal isotropic subspace whose stabilizer $Q(\Delta W)$ is a maximal parabolic subgroup of $G(W+W^-)$ and has Levi factor $\mathrm{GL}(\Delta W)$. We may consider the degenerate principal series representation

$$I(s) = \mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)} \chi_V |\det|^s$$

of $G(W+W^-)$ induced from the character $\chi_V|\det|^s$ of $Q(\Delta W)$ (normalized induction).

The following lemma (see [KR]) describes the restriction of $I(s)$ to the subgroup $G(W)\times G(W)\subset G(W+W^-)$.

Lemma 3.2. *As a representation of $G(W)\times G(W^-)$, $I(s)$ possesses an equivariant filtration*

$$0\subset I_0(s)\subset I_1(s)\subset\cdots\subset I_{q_W}(s) = \mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V\cdot|\det|^s$$

whose successive quotients are given by

$$\begin{aligned} R_t(s) &= I_t(s)/I_{t-1}(s) \\ &= \mathrm{Ind}_{Q_t\times Q_t}^{G(W)\times G(W^-)}\Bigg(\left(\chi_V|\det_{X_t}|^{s+\frac{t}{2}}\boxtimes\chi_V|\det_{X_t}|^{s+\frac{t}{2}}\right) \\ &\qquad\qquad \otimes\left((\chi_V\circ\det_{W^-_{n-2t}})\otimes C_c^\infty(G(W_{n-2t}))\right)\Bigg). \end{aligned} \tag{3.1}$$

Here, the induction is normalized and

- q_W *is the Witt index of* W;
- Q_t *is the maximal parabolic subgroup of $G(W)$ stabilizing a t-dimensional isotropic subspace X_t of W, with Levi subgroup $\mathrm{GL}(X_t)\times G(W_{n-2t})$, where* $\dim W_{n-2t} = n-2t$;
- $G(W_{n-2t})\times G(W_{n-2t})$ *acts on $C_c^\infty(G(W_{n-2t}))$ by left-right translation.*

In particular,

$$R_0 = R_0(s) = (\chi_V\circ\det_{W^-})\otimes C_c^\infty(G(W))$$

is independent of s.

3.3. **Boundary.** Lemma 3.2 suggests a notion which plays an important role in this paper. For $\pi\in\mathrm{Irr}(G(W))$, we shall say that π occurs on the boundary of $I(s)$ if there exists $0<t\le q_W$ such that

$$\mathrm{Hom}_{G(W)}(R_t(s),\pi)\neq 0.$$

By Bernstein's Frobenius reciprocity, this is equivalent to

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(\chi_V|\det|^{s+\frac{t}{2}}, R_{\overline{Q}(X_t)}(\pi))\neq 0, \tag{3.2}$$

where $\overline{Q}(X_t) = L(X_t)\cdot\overline{U}(X_t)$ stands for the parabolic subgroup opposite to $Q(X_t)$. Dualizing and using the MVW involution, this is in turn equivalent to

$$\pi \hookrightarrow \chi_V|\det_{\mathrm{GL}(X_t)}|^{-s-\frac{t}{2}} \rtimes \sigma \tag{3.3}$$

for some irreducible representation σ of $G(W_{n-2t})$. This terminology is due to Kudla and Rallis, and the reader may consult [KR, Definition 1.3] for the explanation of the use of "boundary."

3.4. **Outline of proof.** With the basic notations introduced, we can now give a brief outline of the proof of Theorem 1.2. First of all, there is no loss of generality in assuming that

$$m \le n + \epsilon_0, \tag{3.4}$$

so that

$$s_{m,n} = \frac{m-(n+\epsilon_0)}{2} \le 0, \tag{3.5}$$

because otherwise one can switch the roles of $G(W)$ and $H(V)$. We shall assume this henceforth.

The proof proceeds by induction on $\dim W$, with the base case with $\dim W = 0$ being trivial. The inductive step is divided into two different parts.

The first part, which is given in Section 4, deals with the case when π does not occur on the boundary of $I(-s_{m,n})$, which covers "almost all" representations. The argument for this case has been sketched by Howe [H1, Section 11], using a seesaw dual pair and Lemma 3.2. For this part, the induction hypothesis is not used.

The second part of the proof, which is given in Section 5, deals with the case when π occurs on the boundary of $I(-s_{m,n})$. To use the induction hypothesis, we use Lemma 3.1 and a key idea of Mínguez [M] in his proof of the Howe duality conjecture for general linear groups.

In Section 6, we assemble the results of Sections 4 and 5 to complete the proof of Theorem 1.2.

4. A seesaw argument

In this section, we will give a proof of (HD) when at least one of π or $\pi' \in \mathrm{Irr}(G(W))$ does not occur on the boundary of $I(-s_{m,n})$ by assuming (3.4) or equivalently (3.5). As is mentioned at the end of the last section, though we prove (HD) by induction on $\dim W$, it turns out that for this case, one can prove (HD) without appealing to the induction hypothesis. Namely, we shall prove Theorem 4.1.

Theorem 4.1. *Assume that $m \le n+\epsilon_0$ and suppose that $\pi \in \mathrm{Irr}(G(W))$ does not occur on the boundary of $I(-s_{m,n})$. Then for any $\pi' \in \mathrm{Irr}(G(W))$,*

$$\dim \mathrm{Hom}_{H(V)}(\theta(\pi), \theta(\pi')) \le \delta_{\pi,\pi'}.$$

In particular, $\theta(\pi)$ is either zero or irreducible, and moreover for any irreducible π',

$$0 \ne \theta(\pi) \subset \theta(\pi') \Longrightarrow \pi \cong \pi'.$$

Proof. First, we consider the following seesaw diagram:

$$\begin{array}{ccc} G(W+W^-) & & H(V)\times H(V) \\ \Big| & \times & \Big| \\ G(W)\times G(W^-) & & H(V)^{\Delta}, \end{array}$$

where W^- denotes the space obtained from W by multiplying the form by -1, so that $G(W^-) = G(W)$. Given irreducible representations π and π' of $G(W)$, the seesaw identity [GI, Section 6.1] gives

$$\begin{aligned} &\mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,W+W^-}(\chi_W), \pi'\otimes\pi^\vee\chi_V) \\ &= \mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi')\otimes\Theta(\pi)^{MVW}, \mathbb{C}). \end{aligned} \tag{4.1}$$

Here $\Theta_{V,W+W^-}(\chi_W)$ denotes the big theta lift of the character χ_W of $H(V)$ to $G(W+W^-)$.

We analyze each side of the seesaw identity in turn. For the right-hand side (RHS), one has

$$\begin{aligned} \mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi')\otimes\Theta(\pi)^{\mathrm{MVW}}, \mathbb{C}) &\supseteq \mathrm{Hom}_{H(V)^\Delta}(\theta(\pi')\otimes\theta(\pi)^\vee, \mathbb{C}) \\ &= \mathrm{Hom}_{H(V)}(\theta(\pi'), \theta(\pi)). \end{aligned} \tag{4.2}$$

For the left-hand side (LHS) of the seesaw identity (4.1), we need to understand $\Theta_{V,W+W^-}(\chi_W)$. It is known that $\Theta_{V,W+W^-}(\chi_W)$ is irreducible (see [GI, Proposition 7.2]). Moreover, it was shown by Rallis that

$$\Theta_{V,W+W^-}(\chi_W) \hookrightarrow I(s_{m,n}) = \mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V|\det|^{s_{m,n}}. \tag{4.3}$$

Since $s_{m,n}\le 0$, there is a surjective map (see [GI, Proposition 8.2])

$$I(-s_{m,n}) \longrightarrow \Theta_{V,W+W^-}(\chi_W).$$

Hence the seesaw identity (4.1) gives

$$\mathrm{Hom}_{G(W)\times G(W)}(I(-s_{m,n}), \pi'\otimes\pi^\vee\chi_V) \supseteq \mathrm{Hom}_{H(V)}(\theta(\pi'), \theta(\pi)).$$

To prove the theorem, it suffices to prove that the LHS has dimension ≤ 1 with equality only if $\pi\cong\pi'$.

Applying Lemma 3.2, we claim that if π is not on the boundary of $I(-s_{m,n})$, the natural restriction map

$$\mathrm{Hom}_{G(W)\times G(W)}(I(-s_{m,n}), \pi'\otimes\pi^\vee\chi_V) \longrightarrow \mathrm{Hom}_{G(W)\times G(W)}(R_0, \pi'\otimes\pi^\vee\chi_V)$$

is injective. This will imply the theorem since the RHS has dimension ≤ 1, with equality if and only if $\pi = \pi'$.

To deduce the claim, it suffices to show that for each $0 < t \le q_W$,

$$\mathrm{Hom}_{G(W)\times G(W)}(R_t(-s_{m,n}), \pi'\otimes\pi^\vee\chi_V) = 0. \tag{4.4}$$

By Bernstein's Frobenius reciprocity, this Hom space is equal to

$$\begin{aligned} &\mathrm{Hom}_{L(X_t)\times L(X_t)}\Big(\big(\chi_V|\det|^{-s_{m,n}+\frac{t}{2}}\boxtimes\chi_V|\det|^{-s_{m,n}+\frac{t}{2}}\big) \\ &\qquad\otimes\Big((\chi_V\circ\det_{W^-_{n-2t}})\otimes C_c^\infty(G(W_{n-2t}))\Big),\ R_{\overline{Q}_t}(\pi')\otimes R_{\overline{Q}_t}(\pi^\vee\chi_V)\Big). \end{aligned}$$

Hence we deduce that Equation (4.4) holds if

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(\chi_V|\det|^{-s_{m,n}+\frac{t}{2}}, R_{\overline{Q}_t}(\pi^\vee\chi_V)) = 0.$$

By dualizing and Bernstein's Frobenius reciprocity, this is equivalent to

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(R_{Q_t}(\pi \cdot (\chi_V^{-1} \circ \det_W)), (\chi_V^{-1} \circ \det_{\mathrm{GL}(X_t)}) \cdot |\det|^{s_{m,n}-\frac{t}{2}}) = 0.$$

Now note that

$$\chi_V \circ \det_W|_{\mathrm{GL}(X_t)} = \chi_V^2 \circ \det_{\mathrm{GL}(X_t)}.$$

Hence the condition is equivalent to

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(R_{Q_t}(\pi), \chi_V|\det|^{s_{m,n}-\frac{t}{2}}) = 0.$$

Since this condition holds when π is not on the boundary of $I(-s_{m,n})$, Equation (4.4) is proved. This completes the proof of Theorem 4.1. $\square$

Finally, let us note that the argument also gives the following proposition, which we will use later.

Proposition 4.2. *Assume $m \leq n + \epsilon_0$. If $\pi \neq \pi' \in \mathrm{Irr}(G(W))$ are such that*

$$\mathrm{Hom}_{H(V)}(\theta_{W,V}(\pi), \theta_{W,V}(\pi')) \neq 0, \tag{4.5}$$

then there exists $t > 0$ such that

$$\pi \hookrightarrow \chi_V|\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \tau$$

and

$$\pi' \hookrightarrow \chi_V|\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \tau'$$

for some τ and $\tau' \in \mathrm{Irr}(G(W_{n-2t}))$.

Proof. If $\pi \neq \pi'$ but $\mathrm{Hom}_{H(V)}(\theta_{W,V}(\pi), \theta_{W,V}(\pi')) \neq 0$, then arguing as previously one must have $\mathrm{Hom}_{G(W)\times G(W)}(R_t(-s_{m,n}), \pi \otimes \pi'^{\vee}\chi_V) \neq 0$ for some $t > 0$, which gives the conclusion of the proposition. $\square$

Of course, the hypothesis of this proposition contradicts the Howe duality (HD), and hence is never satisfied. So what this proposition is saying is that if (HD) is to be violated as in (4.5), it must be violated by representations π and π' which occur on the boundary for "the same t."

5. Life on the boundary

After Theorem 4.1, we see that to prove Theorem 1.2 or equivalently (HD), it remains to consider the case when both π and π' occur on the boundary of $I(-s_{m,n})$, as in Proposition 4.2. In this section, we examine what life looks like on the boundary. We continue to assume that $m \leq n + \epsilon_0$, or equivalently that $s_{m,n} \leq 0$ (see (3.4) and (3.5)).

5.1. **An idea of Mínguez.** Since π occurs on the boundary of $I(-s_{m,n})$, we have

$$\pi \hookrightarrow \chi_V|\det_{\mathrm{GL}_{(X_t)}}|^{s_{m,n}-\frac{t}{2}} \rtimes \pi_0$$

for some $\pi_0 \in \mathrm{Irr}(G(W_{n-2t}))$ and some $t > 0$. By induction in stages, we have

$$\pi \hookrightarrow (\chi_V|-|^{s_{m,n}-t+\frac{1}{2}} \times \cdots \times \chi_V|-|^{s_{m,n}-\frac{1}{2}}) \rtimes \pi_0.$$

For simplicity, let us set

$$s_{m,n,t} = s_{m,n} - t + \frac{1}{2} < 0.$$

Now let us use a key idea of Mínguez [M]. Let $a > 0$ be maximal such that

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma,$$

where

$$1^{\times a} = \underbrace{1 \times \cdots \times 1}_{a \text{ times}} = \mathrm{Ind}_B^{\mathrm{GL}_a} 1 \otimes \cdots \otimes 1$$

and σ is an irreducible representation of $G(W_{n-2a})$. To simplify notation, let us set

$$\sigma_a := \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma. \tag{5.1}$$

Note that the representation $1^{\times a}$ is irreducible and generic by [Z, Theorem 9.7].

Observe that if $\pi \hookrightarrow \sigma_a$ with $a > 0$ maximal, then

$$\sigma \nsubseteq \chi_V |-|^{s_{m,n,t}} \rtimes \sigma'$$

for any σ'. In fact, the converse is also true, as we shall verify in Corollary 5.3(ii).

The chief reason for considering σ_a with $a > 0$ maximal is the following proposition.

Proposition 5.1. *Suppose $\sigma_a = \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$ is such that*

$$\sigma \nsubseteq \chi_V |-|^{s_{m,n,t}} \rtimes \sigma'$$

for any σ'. Then σ_a has a unique irreducible submodule.

The proof of Proposition 5.1 relies on the following key technical lemma, which we state in slightly greater generality as it may be useful for other purposes. The proof of the lemma is deferred to the Appendix.

Lemma 5.2. *Let ρ be a supercuspidal representation of $\mathrm{GL}_r(E)$ and consider the induced representation*

$$\sigma_{\rho,a} = \rho^{\times a} \rtimes \sigma$$

of $G(W_n)$ where σ is an irreducible representation of $G(W_{n-2ra})$. Assume that

(a) ${}^c\rho^\vee \neq \rho$;
(b) $\sigma \nsubseteq \rho \rtimes \sigma_0$ *for any* σ_0.

Then we have the following:

(i) One has a natural short exact sequence

$$0 \longrightarrow T \longrightarrow R_{\overline{Q}(X_{ra})}\sigma_{\rho,a} \longrightarrow ({}^c\rho^\vee)^{\times a} \otimes \sigma \longrightarrow 0,$$

and T does not contain any irreducible subquotient of the form $({}^c\rho^\vee)^{\times a} \rtimes \sigma'$ for any σ'. In particular, $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$ contains $({}^c\rho^\vee)^{\times a} \otimes \sigma$ with multiplicity one and does not contain any other subquotient of the form $({}^c\rho^\vee)^{\times a} \otimes \sigma'$. Likewise, $R_{Q(X_{ra})}\sigma_{\rho,a}$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity one and does not contain any other subquotient of the form $\rho^{\times a} \otimes \sigma'$.

(ii) The induced representation $\sigma_{\rho,a}$ has a unique irreducible submodule.

Proof of Proposition 5.1. We shall apply this lemma with

$$r = 1, \quad \rho = \chi_V |-|^{s_{m,n,t}}, \quad \text{and} \quad \sigma_a = \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma.$$

Here, condition (a) holds since $s_{m,n,t} < 0$, whereas condition (b) holds by the maximality of a. This proves Proposition 5.1. $\square$

We shall have another occasion to use Lemma 5.2 later on. We also note the following corollary.

Corollary 5.3. *Suppose that $\pi \hookrightarrow \sigma_a$ and $\sigma \nsubseteq \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'$ for any σ'.*

(i) If $\pi \hookrightarrow \delta_a := \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \delta$ for some δ, then $\delta \cong \sigma$.

(ii) Moreover, a is maximal with respect to the property that $\pi \hookrightarrow \delta_a$ for some irreducible δ.

Proof. By the exactness of the Jacquet functor, $R_{Q(X_a)}(\pi)$ is a submodule of $R_{Q(X_a)}(\sigma_a)$. By Lemma 5.2, it follows that $R_{Q(X_a)}(\pi)$ contains $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma$ with multiplicity one and does not contain any other subquotient of the form $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma'$ with $\sigma' \neq \sigma$. This key fact will imply both (i) and (ii).
(i) If $\pi \hookrightarrow \delta_a$, then $R_{Q(X_a)}(\pi)$ contains $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \delta$ as a quotient. By the key fact observed earlier, it follows that $\delta \cong \sigma$.
(ii) Suppose for the sake of contradiction that

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times (a+1)} \rtimes \sigma'$$

for some irreducible σ'. Then by induction in stages, one has

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \delta \quad \text{with } \delta = \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'.$$

By the Frobenius reciprocity, one has a nonzero equivariant map

$$R_{Q(X_a)}(\pi) \longrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \delta.$$

By the key fact observed earlier, the image of this nonzero map must be isomorphic to $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma$. Hence,

$$\sigma \hookrightarrow \delta = \chi_V|-|^{s_{m,n,t}} \rtimes \sigma',$$

which is a contradiction to the hypothesis of the corollary. □

5.2. **A key computation.** We are now ready to launch into a computation needed to complete the proof of Theorem 1.2 for the representations on the boundary. The following is the key proposition.

Proposition 5.4. *Assume $0 \neq \Pi \subset \theta(\pi)$ and Π is irreducible.*
(i) If

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

with a maximal (and for some σ), then

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma$$

for some Σ and where a is also maximal for Π.
(ii) Moreover, whenever Π is presented as a submodule as shown earlier, one has

$$0 \neq \mathrm{Hom}_{G_n \times H_m}(\omega_{W_n,V_m}, \pi \otimes \Pi) \hookrightarrow \mathrm{Hom}_{G_{n-2a} \times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma \otimes \Sigma),$$

so that $\Sigma \subseteq \theta_{W_{n-2a},V_{m-2a}}(\sigma)$.

Proof. (i) Since $0 \neq \Pi \subseteq \theta(\pi)$, we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{G_n \times H_m}(\omega_{W,V}, \pi \otimes \Pi) \\ &\hookrightarrow \mathrm{Hom}_{G_n \times H_m}(\omega_{W,V}, \sigma_a \otimes \Pi) \\ &= \mathrm{Hom}_{\mathrm{GL}(X_a) \times G_{n-2a} \times H_m}(R_{Q(X_a)}(\omega_{W,V}), \chi_V|-|^{s_{m,n,t}} 1^{\times a} \otimes \sigma \otimes \Pi), \end{aligned}$$

where we used the Frobenius reciprocity for the last step. Now the Jacquet module $R_{Q(X_a)}(\omega_{W,V})$ of the Weil representation is computed as in Lemma 3.1, which implies that there is a natural restriction map

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(R_{Q(X_a)}(\omega_{W,V}), \chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\otimes\sigma\otimes\varPi)\\ &\qquad\longrightarrow \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(J^a, \chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\otimes\sigma\otimes\varPi).\end{aligned}$$

We claim that this map is injective. To see this, it suffices to show that for all $0\le k<a$,

$$\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(J^k, \chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\otimes\sigma\otimes\varPi)=0.$$

By Lemma 3.1, this Hom space is equal to

$$\begin{aligned}&\mathrm{Hom}_{M(X_{a-k},X_a)\times G_{n-2a}\times H_m}\\ &\quad\Big(\mathrm{Ind}_{P(Y_k)}^{H(V_m)}\chi_V|\det_{X_{a-k}}|^{\lambda_{a-k}}\otimes C_c^\infty(\mathrm{Isom}_{E,c}(X_k,Y_k))\otimes\omega_{W_{n-2a},V_{m-2k}},\\ &\qquad\qquad\qquad R_{\overline{Q}(X_{a-k},X_a)}(\chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a})\otimes\sigma\otimes\varPi\Big),\end{aligned}$$

where $M(X_{a-k},X_a)$ is the Levi factor of the parabolic subgroup of $\mathrm{GL}(X_a)$ stabilizing X_{a-k}. Because $1^{\times a}$ is generic, the second representation in this Hom space has a nonzero Whittaker functional when viewed as a representation of $\mathrm{GL}(X_{a-k})$, and hence the first one must also have a nonzero Whittaker functional, which is possible only when $a-k=1$. Therefore, if this Hom space is nonzero, we must have $a-k=1$. But in that case, one has

$$\lambda_1 = s_{m,n}+\frac{1}{2} > s_{m,n}-t+\frac{1}{2} = s_{m,n,t},$$

so that the Hom space is zero even when $a-k=1$.

Therefore we have $J^a\neq 0$ and

$$\begin{aligned}0&\neq \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(J^a, \chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\otimes\sigma\otimes\varPi)\\ &=\mathrm{Hom}_{H_m}(\chi_W|-|^{-s_{m,n,t}}\cdot 1^{\times a}\rtimes\varTheta_{W_{n-2a},V_{m-2a}}(\sigma),\varPi).\end{aligned}$$

Dualizing and applying MVW along with (2.1) and (2.2), this shows that

$$\varPi\hookrightarrow\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times a}\rtimes(\varTheta_{W_{n-2a},V_{m-2a}}(\sigma)^\vee)^{\mathrm{MVW}},$$

and hence

$$\varPi\hookrightarrow\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times a}\rtimes\varSigma \tag{5.2}$$

for some irreducible representation $\varSigma$ of $H(V_{m-2a})$ which is a subquotient of the representation $(\varTheta_{W_{n-2a},V_{m-2a}}(\sigma)^\vee)^{MVW}$ and hence of $\varTheta_{W_{n-2a},V_{m-2a}}(\sigma)$.

To prove (i), it remains to show that in (5.2), the integer a is maximal for $\varPi$. Let $b\ge a$ be maximal such that

$$\varPi\hookrightarrow\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\rtimes\varSigma_0$$

for some irreducible representation $\varSigma_0$ of $H(V_{m-2b})$. Then we have

$$\begin{aligned}&0\neq\mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m},\pi\otimes\varPi)\\ &\quad\hookrightarrow\mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m},\pi\otimes(\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\rtimes\varSigma_0))\\ &\quad=\mathrm{Hom}_{G_n\times\mathrm{GL}(Y_b)\times H_{m-2b}}(R_{P(Y_b)}(\omega_{W_n,V_m}),\pi\otimes\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\otimes\varSigma_0).\end{aligned}\tag{5.3}$$

We can compute the Jacquet module $R_{P(Y_b)}(\omega_{W_n,V_m})$ by using Lemma 3.1 with the roles of $H(V_m)$ and $G(W_n)$ switched. But for this, it should be noted that the exponent λ_{b-k} (for $k<b$) in Lemma 3.1 satisfies

$$\lambda_{b-k} = -s_{m,n} + \frac{b-k}{2} > 0 > s_{m,n,t}. \tag{5.4}$$

Keeping this in mind, the last Hom space in (5.3) can be computed as

$$\begin{aligned}
&\mathrm{Hom}_{G_n\times \mathrm{GL}(Y_b)\times H_{m-2b}}(R_{P(Y_b)}(\omega_{W_n,V_m}), \pi\otimes\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\otimes \Sigma_0)\\
\hookrightarrow &\mathrm{Hom}_{G_n\times \mathrm{GL}(Y_b)\times H_{m-2b}}(J^b, \pi\otimes\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\otimes \Sigma_0)\\
= &\mathrm{Hom}_{\mathrm{GL}(X_b)\times G_{n-2b}\times H_{m-2b}}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\otimes \omega_{W_{n-2b},V_{m-2b}}, R_{\overline{Q}(X_b)}(\pi)\otimes \Sigma_0)\\
= &\mathrm{Hom}_{\mathrm{GL}(X_b)\times G_{n-2b}}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\otimes \Theta_{W_{n-2b},V_{m-2b}}(\Sigma_0), R_{\overline{Q}(X_b)}(\pi))\\
= &\mathrm{Hom}_{G_n}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\rtimes \Theta_{W_{n-2b},V_{m-2b}}(\Sigma_0), \pi),
\end{aligned}$$

where to obtain the second injection, we used the genericity of $1^{\times b}$ and (5.4) as before. Then again by dualizing and applying MVW, we have

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}}\cdot 1^{\times b}\rtimes \sigma_0$$

for some σ_0 which is a subquotient of $\Theta_{W_{n-2b},V_{m-2b}}(\Sigma_0)$. By the maximality of a, we conclude that $b\le a$ and hence $b=a$. This completes the proof of (i).
(ii) Suppose that Π is given as in (5.2) with a maximal and some Σ. Now that we know that $b=a$ in the proof of (i), we revisit the computations starting from (5.3),

$$\begin{aligned}
0 \neq &\mathrm{Hom}_{G_n\times H_m}\left(\omega_{W_n,V_m}, \pi\otimes\Pi\right)\\
&\hookrightarrow \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes \omega_{W_{n-2a},V_{m-2a}},\\
&\qquad\qquad R_{\overline{Q}(X_a)}(\pi)\otimes\Sigma)\\
&\hookrightarrow \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}\big(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes \omega_{W_{n-2a},V_{m-2a}},\\
&\qquad\qquad R_{\overline{Q}(X_a)}(\chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\rtimes\sigma)\otimes\Sigma\big).
\end{aligned}$$

To show the proposition, it suffices to show that the last Hom space embeds into

$$\mathrm{Hom}_{G_{n-2a}\times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma\otimes\Sigma).$$

To show this inclusion, we shall make use of Lemma 5.2. In Lemma 5.2(i), set $\sigma_{\rho,a}=\sigma_a$; namely, set $\rho=\chi_V|-|^{s_{m,n,t}}$. Tensoring the short exact sequence with Σ and then applying the functor

$$\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}(\chi_V\cdot|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\omega_{W_{n-2a},V_{m-2a}}, -),$$

one sees that the desired inclusion follows from the assertion:

$$\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}(\chi_V\cdot|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\omega_{W_{n-2a},V_{m-2a}}, T\otimes\Sigma)=0.$$

But this follows from Lemma 5.2(ii) which asserts that T does not contain any irreducible subquotient of the form

$$\chi_W|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\Sigma' \quad \text{for any } \Sigma'.$$

This completes the proof of (ii). $\square$

6. Proof of Theorem 1.2

We can now assemble the results of the last two sections and complete our proof of Theorem 1.2. As is mentioned in Section 3.4, we may assume that $m \le n+\epsilon_0$, or equivalently that $s_{m,n} \le 0$ (see (3.4) and (3.5)), because otherwise we may switch the roles of $G(W)$ and $H(V)$. Further, we shall argue by induction on $\dim W$. Thus, by induction hypothesis, we assume that Theorem 1.2 is known for dual pairs $G(W') \times H(V')$ with $\dim V' \le \dim W' + \epsilon_0 < n + \epsilon_0$.

6.1. **Irreducibility.** We first show the irreducibility of $\theta(\pi)$. If π does not occur on the boundary of $I(-s_{m,n})$, this follows from Theorem 4.1. Thus, we assume π occurs on the boundary, so that

$$\pi \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \sigma$$

for some $t > 0$ and some σ. Let us write

$$\pi \hookrightarrow \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

with a maximal. Let $\Pi \subseteq \theta(\pi)$ be an irreducible submodule. But by Proposition 5.4,

$$0 \ne \mathrm{Hom}_{G_n \times H_m}(\omega_{W_n,V_m}, \pi \otimes \Pi) \hookrightarrow \mathrm{Hom}_{G_{n-2a} \times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma \otimes \Sigma),$$

where $\Sigma \subset \theta_{W_{n-2a},V_{m-2a}}(\sigma)$ is an irreducible representation such that

$$\Pi \hookrightarrow \chi_W |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma,$$

where a is also maximal for Π. By the induction hypothesis, we deduce that $\Sigma = \theta_{W_{n-2a},V_{m-2a}}(\sigma)$, and hence

$$\Pi \hookrightarrow \chi_W |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \theta_{W_{n-2a},V_{m-2a}}(\sigma).$$

By Proposition 5.1, this induced representation has a unique submodule. This shows that $\theta(\pi)$ is an isotypic representation. Further, since

$$\dim \mathrm{Hom}_{G_{n-2a} \times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma \otimes \Sigma) = 1,$$

by the induction hypothesis, we conclude by Proposition 5.4(ii) that

$$\dim \mathrm{Hom}_{G_n \times H_m}(\omega_{W_n,V_m}, \pi \otimes \Pi) = 1.$$

This shows that Π occurs with multiplicity one in $\theta(\pi)$, so that $\theta(\pi)$ is irreducible.

6.2. **Disjointness.** It remains to prove that if $\theta(\pi) = \theta(\pi') = \Pi \ne 0$, then $\pi = \pi'$. By Proposition 4.2, this holds unless both π and π' occur on the boundary for the same t; namely, there exists $t > 0$ such that

$$\pi \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \tau$$

and

$$\pi' \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \tau'.$$

This means that we may write

$$\pi \hookrightarrow \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

and

$$\pi' \hookrightarrow \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a'} \rtimes \sigma'$$

with a and a' maximal (and for some σ and σ'). But then by Proposition 5.4(i) one must have $a = a'$, where a is maximal such that

$$\Pi \hookrightarrow \chi_W |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma \quad \text{for some } \Sigma.$$

Moreover, with Proposition 5.4(ii) and the induction hypothesis, we have

$$\theta_{W_{n-2a},V_{m-2a}}(\sigma) = \Sigma = \theta_{W_{n-2a},V_{m-2a}}(\sigma'),$$

so that $\sigma \cong \sigma'$. We then deduce by Proposition 5.1 that $\pi \cong \pi'$ is the unique irreducible submodule of $\sigma_a = \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$. This completes the proof of Theorem 1.2.

7. Quaternionic dual pairs

In this final section, we consider the case of the quaternionic dual pairs. As mentioned in the Introduction, due to the lack of the MVW involution, we are not able to prove the Howe duality conjecture in full generality. The best we can prove is Theorem 1.3, where we consider only Hermitian representations (i.e., those π such that $\overline{\pi} \cong \pi^\vee$, where $\overline{\pi}$ is the complex conjugate of π). The idea of the proof is essentially the same as for the nonquaternionic case. But in place of the MVW involution, we use the involution

$$\pi \mapsto \overline{\pi}.$$

In what follows, we will outline how to modify the proof.

7.1. **Setup.** Let us briefly recall the setup in the quaternionic case, with emphasis on the aspects which are different from before.

Let B be the unique quaternion division algebra over F. For $\epsilon = \pm$, let $W = W_n$ be a rank n B-module equipped with a $-\epsilon$-Hermitian form and V_m a rank m B-module equipped with an ϵ-Hermitian form. Then the product $G(W_n) \times H(V_m)$ of isometry groups is a dual pair, with a Weil representation $\omega_{W,V}$ associated to a pair of splitting characters (χ_V, χ_W). In this case, the characters χ_V and χ_W are simply (possibly trivial) quadratic characters determined by the discriminants of the corresponding spaces V and W.

For an isotropic subspace X_t of rank t over B, let $Q(X_t)$ be the stabilizer of X_t, which is a maximal parabolic subgroup of $G(W)$ with the Levi factor $L(X_t) \cong \mathrm{GL}(X_t) \times G(W_{n-2t})$, where $\mathrm{GL}(X_t) \cong \mathrm{GL}_t(B)$. We shall denote by $\det_{\mathrm{GL}(X_t)} : \mathrm{GL}(X_t) \longrightarrow F^\times$ the reduced norm map. Likewise, a maximal parabolic subgroup $P(Y_t)$ of $H(V_m)$ is the stabilizer of an isotropic subspace Y_t of V_m. As before, we have Tadić's notation for parabolic induction.

We set

$$s_{m,n} = m - n + \frac{\epsilon}{2}.$$

In the quaternionic case, we say that an irreducible representation π of $G(W_n)$ lies on the boundary of $I(-s_{m,n})$ if there exists $0 < t \le q_W$ such that

$$\pi \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-t} \rtimes \sigma$$

for some irreducible representation σ of $G(W_{n-2t})$. If $\overline{\pi} \cong \pi^\vee$, i.e., if π is Hermitian, then by dualizing and complex conjugating, we see that this is equivalent to

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(\chi_V |\det_{\mathrm{GL}(X_t)}|^{-s_{m,n}+t}, R_{\overline{Q}(X_t)}(\pi)) \neq 0.$$

7.2. **Nonboundary case.** We can now begin the proof of Theorem 1.3, starting with the case when the Hermitian representation π does not lie on the boundary. For the sake of proving Theorem 1.3, there is no loss of generality in assuming that $m < n - \frac{\epsilon}{2}$, so that $s_{m,n} < 0$.

Now one can verify that all the arguments in Section 4 continue to work for a Hermitian π, with the following modifications:

- The first place where the MVW involution is used in Section 4 is the seesaw identity (4.1). But one can see from the proof of the seesaw identity in [GI, Section 6.1] that one has
$$\mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,W+W^-}(\chi_W), \pi' \otimes \pi^\vee) = \mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi') \otimes \overline{\Theta(\pi)}, \mathbb{C}).$$
Then (4.2) can be written as
$$\begin{aligned}\mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi') \otimes \overline{\Theta(\pi)}, \mathbb{C}) &\supseteq \mathrm{Hom}_{H(V)^\Delta}(\theta_{her}(\pi') \otimes \theta_{her}(\pi)^\vee, \mathbb{C}) \\ &= \mathrm{Hom}_{H(V)}(\theta_{her}(\pi'), \theta_{her}(\pi)).\end{aligned}$$
- There is an evident analogue of Lemma 3.2 in the quaternionic case. The statement is as given in Lemma 3.2, except that the terms $|\det_{\mathrm{GL}(X_t)}|^{s+\frac{t}{2}}$ should be replaced by $|\det_{\mathrm{GL}(X_t)}|^{s+t}$.

With these provisions, the rest of the argument in Section 4 does not use the MVW involution, and hence applies to the quaternionic case without any modification. One also has the analogue of Proposition 4.2, with the exponent $s_{m,n} - \frac{t}{2}$ replaced by $s_{m,n} - t$.

7.3. **Boundary case.** Suppose that π is an irreducible Hermitian representation of $G(W)$ which lies on the boundary of $I(-s_{m,n})$. Then, for some $t > 0$, one has
$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$
with a maximal (and for some σ) and
$$s_{m,n,t} = s_{m,n} - 2t + 1 < 0.$$
Now one has an analogue of Lemma 5.2, based on the explicit Geometric Lemma in the quaternionic case (which is written down in the thesis of M. Hanzer [Han, Theorem 2.2.5]). Using this, one deduces the analogue of Proposition 5.1 by the same argument. However, it is essential to note the following lemma.

Lemma 7.1. *Suppose that*
$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$
with a maximal and for some σ. If π is Hermitian, so is σ.

Proof. To see this, starting from $\pi \hookrightarrow \rho^{\times a} \rtimes \sigma$ (with a maximal and ρ a real-valued one-dimensional character for which $\rho^\vee \neq \rho$), one deduces (by dualizing and complex-conjugating) that
$$(\overline{\rho}^\vee)^{\times a} \rtimes \overline{\sigma}^\vee \twoheadrightarrow \overline{\pi}^\vee \cong \pi.$$
Now note that for the case at hand, the supercuspidal representation ρ satisfies $\overline{\rho} = \rho$; indeed, ρ is a real-valued character for our application. Thus, Bernstein's Frobenius reciprocity implies that
$$(\rho^\vee)^{\times a} \otimes \overline{\sigma}^\vee \hookrightarrow R_{\overline{Q}(X_{ra})}(\pi).$$

However, the analogue of Lemma 5.2(i) says that the only irreducible subquotient of $R_{\overline{Q}(X_{ra})}(\pi)$ of the form $(\rho^\vee)^{\times a}\otimes\sigma_0$ is $(\rho^\vee)^{\times a}\otimes\sigma$. Hence, we see that $\overline{\sigma}^\vee \cong \sigma$. $\square$

Then one has the following analogue of Proposition 5.4.

Proposition 7.2. *Assume that π is an irreducible Hermitian representation of $G(W)$ and $0 \neq \Pi \subset \theta_{her}(\pi)$.*
(i) If

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

with a maximal (and for some σ, necessarily Hermitian by Lemma 7.1), then

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma$$

for some Σ (necessarily Hermitian by Lemma 7.1) and where a is also maximal for Π.
(ii) Moreover, whenever Π is presented as a submodule as shown previously, one has

$$0 \neq \mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m}, \pi\otimes\Pi) \hookrightarrow \mathrm{Hom}_{G_{n-2a}\times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma\otimes\Sigma),$$

so that $\Sigma \subseteq \theta_{W_{n-2a},V_{m-2a},her}(\sigma)$.

Proposition 7.2 is proved by the same argument as that for Proposition 5.4, using the analogue of Lemma 3.1 (see [MVW, Chapter 3, Section IV, Theorem 5, Page 70]). We only take note that in the statement of Lemma 3.1, the quantity λ_{a-k} should be equal to $s_{m,n}+a-k$ in the quaternionic case.

With the provisions given earlier, the rest of the proof goes through for the quaternionic case, which completes the proof of Theorem 1.3.

Appendix: Proof of Lemma 5.2

The goal of this appendix is to prove the technical Lemma 5.2. We restate the lemma here for the convenience of the reader.

Lemma 5.2. Let ρ be a supercuspidal representation of $\mathrm{GL}_r(E)$ and consider the induced representation

$$\sigma_{\rho,a} = \rho^{\times a} \rtimes \sigma$$

of $G(W_n)$ where σ is an irreducible representation of $G(W_{n-2ra})$. Assume that

(a) ${}^c\rho^\vee \neq \rho$;
(b) $\sigma \nsubseteq \rho \rtimes \sigma_0$ for any σ_0.

Then we have the following:

(i) One has a natural short exact sequence

$$0 \longrightarrow T \longrightarrow R_{\overline{Q}(X_{ra})}\sigma_{\rho,a} \longrightarrow ({}^c\rho^\vee)^{\times a}\otimes\sigma \longrightarrow 0,$$

and T does not contain any irreducible subquotient of the form $({}^c\rho^\vee)^{\times a} \rtimes \sigma'$ for any σ'. In particular, $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$ contains $({}^c\rho^\vee)^{\times a}\otimes\sigma$ with multiplicity one and does not contain any other subquotient of the form $({}^c\rho^\vee)^{\times a} \otimes \sigma'$. Likewise, $R_{Q(X_{ra})}\sigma_{\rho,a}$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity one and does not contain any other subquotient of the form $\rho^{\times a} \otimes \sigma'$.

(ii) The induced representation $\sigma_{\rho,a}$ has a unique irreducible submodule.

Proof. We shall use an explication of the Geometric Lemma of Bernstein-Zelevinsky due to Tadić [T, Lemmas 5.1 and 6.3]. (See [HM] for the metaplectic group.) Tadić's results imply that any irreducible subquotient $\delta \otimes \sigma'$ of $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$ is obtained in the following way.

For any partition $k_1+k_2+k_3 = ra$, write the semisimplification of the normalized Jacquet module of $\rho^{\times a}$ to the Levi subgroup $\mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \mathrm{GL}_{k_3}$ as a sum of $\delta_1 \otimes \delta_2 \otimes \delta_3$. Similarly, write the semisimplification of the normalized Jacquet module of σ to the Levi subgroup $\mathrm{GL}_{k_2} \times G(W_{n-2ra-2k_2})$ as a sum of $\delta_4 \otimes \sigma_5$. Then δ is a subquotient of $\delta_3 \times^c \delta_1^\vee \times^c \delta_4^\vee$, whereas σ' is a subquotient of $\delta_2 \otimes \sigma_5$.

For the case at hand, since ρ is supercuspidal, we can assume the partition of ra is of the form $rk_1 + rk_2 + rk_3 = ra$, and the (semisimplified) normalized Jacquet module of $\rho^{\times a}$ is the isotypic sum of $\rho^{\times k_1} \otimes \rho^{\times k_2} \otimes \rho^{\times k_3}$. Hence we see that for any irreducible subquotient $\delta \otimes \sigma'$ of $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$, δ is a subquotient of $\rho^{\times k_3} \times ({}^c\rho^{\times k_1})^\vee \times^c \delta_4^\vee$.

Now the irreducible subquotients of T correspond to those partitions with $k_2 > 0$ or $k_3 > 0$. (Note that the case $k_2 = k_3 = 0$ corresponds to the closed cell in $Q\backslash G/\overline{Q}$, which gives the third term in the short exact sequence.) The conditions (a) and (b) then imply that $\delta \neq ({}^c\rho^\vee)^{\times a}$. This proves the statements about T in (i). Now $Q(X_{ra})$ and $\overline{Q}(X_{ra})$ are conjugate in $G(W)$ by an element w which normalizes the Levi subgroup $L(X_{ra}) = \mathrm{GL}(X_{ra}) \times G(W_{n-2ra})$, acting as the identity on $G(W_{n-2ra})$ and via $g \mapsto {}^c({}^tg^{-1})$ on $\mathrm{GL}(X_{ra})$. Then one has

$$
{}^wR_{\overline{Q}(X_{ra})}(\sigma_{\rho,a}) = R_{Q(X_{ra})}(\sigma_{\rho,a}),
$$

where the LHS is the representation of the Levi $L(X_{ra})$ obtained by twisting $R_{\overline{Q}(X_{ra})}(\sigma_{\rho,a})$ by w. Hence, one deduces that $R_{Q(X_{ra})}(\sigma_{\rho,a})$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity one.

Finally, for (ii), let $\pi \subseteq \sigma_{\rho,a}$ be any irreducible submodule. Then the Frobenius reciprocity implies that the semisimplification of $R_{Q(X_{ra})}(\pi)$ contains $\rho^{\times a} \otimes \sigma$. Thus, if $\sigma_{\rho,a}$ contains more than one irreducible submodule, the exactness of the Jacquet functor implies that $R_{Q(X_{ra})}(\sigma_{\rho,a})$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity ≥ 2, which contradicts (i). □

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