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ON FEWNOMIALS, INTEGRAL POINTS, AND A TORIC VERSION OF BERTINI'S THEOREM

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1. Introduction

This paper is concerned with algebraic equations involving *fewnomials*, also sometimes called *sparse*, or *lacunary* polynomials. By this we mean that the number of *terms* is thought as being fixed, or bounded, whereas the degrees of these terms may vary, and similarly for the coefficients (though they are sometimes supposed to be fixed as well).

This context traces back to several different viewpoints and motivations. For instance, there are issues of reducibility (as in the well-known old theory of Capelli for binomials, and in more recent investigations for k-nomials, e.g. by Schinzel [18]). Sparse polynomials also occur when thinking of *complexity* in writing down an algebraic expression; see for instance Davenport's paper [8] (which also mentions issues related to the ones considered below). In turn, low complexity affects important geometrical or topological aspects (as in Khovanskiĭ's theory [13]).

One perspective and series of relevant questions appeared when Erdős and Rényi raised independently the following attractive conjecture: *Suppose that $g(x)$ is a (complex) polynomial such that $g(x)^2$ has at most l terms. Then the number of terms of $g(x)$ is bounded dependently only on l* [9]. It turned out that this problem was not innocuous as it might appear; indeed, for infinitely many l's the number of terms of $g(x)$ may be much larger than that of $g(x)^2$, in fact $> l^c$ for a $c > 1$, as was pointed out by Erdős himself [9, 18].

The conjecture was proved by Schinzel [17], actually for $g(x)^d$ for any given $d > 0$. Schinzel also extended the conjecture to compositions $p(g(x))$ for any given $p \in \mathbb{C}[x] \setminus \mathbb{C}$, which could not be dealt with by his methods. In turn, this was settled in [20].

1.1. **Main results.**

One of the main purposes of the present paper is to achieve a "final" result in the said direction, by treating general algebraic equations of the form $f(x, g(x)) = 0$, assuming that $f(x, y) \in \mathbb{C}[x, y]$ is a "fewnomial" in x and has an arbitrary degree in y; we then seek a bound for the number of terms of

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$g(x) \in \mathbb{C}[x]$. We shall indeed prove that such a bound exists and that it is actually uniform in the coefficients of f, recovering the above mentioned conclusions related to the Erdős-Rényi conjecture (in sharper form) as very special cases. For instance, we prove the following.

Theorem 1.1. *Let $f(x,y) \in \mathbb{C}[x,y]$ have l terms in x and be monic of degree $d > 0$ in y. If $g(x) \in \mathbb{C}[x]$ satisfies $f(x, g(x)) = 0$, then $g(x)$ has at most $B = B(d,l)$ terms.*

The Erdős-Rényi conjecture is re-obtained on taking $f(x,y) = y^2 - h(x)$ and also Schinzel's subsequent conjecture with $f(x,y) = p(y) - h(x)$ (moreover uniformly in the coefficients of p).

Results of this type are strongly related to other (apparently far) issues of arithmetic and geometric nature, as we now illustrate. First, we remark that a convenient point of view, adopted here, is to think of a (Laurent) fewnomial as *the restriction of a given regular function on a torus $\mathbb{G}_{\mathrm{m}}^l$ to a 1-parameter subgroup or coset.* Indeed, a regular function on $\mathbb{G}_{\mathrm{m}}^l$ is just a Laurent polynomial $f(t_1, \dots, t_l)$, whereas any connected 1-parameter subgroup (resp. coset) may be parametrized as $t_1 = x^{m_1}, \dots, t_l = x^{m_l}$ (resp. $t_1 = c_1 x^{m_1}, \dots, t_l = c_l x^{m_l}$) for integers $m_1, \dots, m_l$ (resp. and nonzero constants $c_1, \dots, c_l$). Hence, by substitution inside f, we obtain a Laurent polynomial in x whose number of terms is bounded independently of the subgroup or coset.[1]

In this view, the above theorem can be rephrased in the following equivalent form.

Theorem 1.2. *If $f \in \mathbb{C}[t_1, \dots, t_l, y] \setminus \mathbb{C}$ is monic in y and of degree at most d in each variable, if $n_1, \dots, n_l$ are natural numbers, and if $g(x) \in \mathbb{C}[x]$ satisfies*

$$f(x^{n_1}, \dots, x^{n_l}, g(x)) = 0, \tag{1.1}$$

then $g(x)$ has at most $B_1 = B_1(d,l)$ terms.

The numbers B, B_1 are actually effective, although we skip the details of such calculation. This leads, as we shall see, to a complete algorithmic description of all the possible solutions $g(x)$. Note that moreover the bound is independent of the coefficients of f, so that the conclusion remains valid if we use the substitution $t_i \mapsto \lambda_i x^{n_i}$ for some arbitrary numbers $\lambda_i \in \mathbb{C}$.

This viewpoint for instance suggests a generalization of the concept of fewnomial to the case of powers of abelian varieties.[2] A relevant result in this direction can be found in [14]. But, more important here, this is useful in the development of the proofs, and it also suggests a number of links with other topics. We will discuss in a moment those which appear to us more relevant.

We also point out the following *dichotomy between polynomials and rational functions*: we shall state a version of Theorem 1.2 for rational functions (see Theorem 2.2), where we drop the assumption that f is monic, and the conclusion will say that $g(x)$ can be written as the *ratio* of two polynomials with a bounded number of terms that are *not necessarily coprime*. An instance of this behavior

[1] Naturally, a similar interpretation holds for multivariate fewnomials; however, the issues may usually be reduced to the basic case of a single variable by substitution.

[2] None of the results of this paper are known in that case, and it seems of interest to ask whether an analogue of the Rényi-Erdős or Schinzel's conjecture is true in that context, already replacing $\mathbb{G}_{\mathrm{m}}$ with an elliptic curve.

is the cyclotomic polynomial $g(x) = 1 + \cdots + x^{n-1}$, which solves the equation $(x^n - 1) - g(x)(1 - x) = 0$ without being a fewnomial, but that in fact can be written as $g(x) = \frac{x^n-1}{x-1}$ (observe that the equation is indeed not monic). This phenomenon is intrinsic to the problem, and in fact many results here will be stated twice to account for both polynomials and rational functions.

The suitable statement for rational functions can be deduced straight away from Theorem 1.2; however, in this paper we shall actually proceed in the opposite direction, first proving a theorem for rational functions, and then recovering Theorem 1.2 (see the final argument in Section 8).

1.2. **Integral points on varieties over function fields.** Many attractive Diophantine problems concern the S-integers $\mathcal{O}_S$ and the S-units $\mathcal{O}_S^*$ in a number field K.[3] The latter may also be described as just the S-integral points for $\mathbb{G}_{\mathrm{m}}$. For instance, the *Mordell-Lang conjecture for tori* yields a description of the S-integral points on subvarieties W of $\mathbb{G}_{\mathrm{m}}^l$, that is the points on W with S-unit coordinates. Such a description follows from the S-unit theorem of Evertse, Schlickewei, and van der Poorten, while the general conjecture for tori has been a theorem of Laurent since the 1980s; see [2, Theorem 7.4.7].

Instead, much less is known for S-integral points on *finite covers* of $\mathbb{G}_{\mathrm{m}}^l$ (except for the case of curves). Take for instance the simple-looking equation $y^2 = 1 + x_1 + x_2$, to be solved with $x_1, x_2 \in \mathcal{O}_S^*$ and $y \in \mathcal{O}_S$. This represents a double cover of $\mathbb{G}_{\mathrm{m}}^2$, on which we seek the S-integral points. Alternatively, they may be described as the S-integral points for the affine variety obtained as the complement in $\mathbb{P}_2$ of two lines and a suitable conic (see [5]). Now, this is a divisor of degree 4 with normal crossings, so a celebrated conjecture of Vojta predicts that the solutions are not Zariski dense, but this has not yet been proved (see [2, Section 14.3]; this special case was proposed explicitly by Beukers in [1]).[4]

A related form of this problem has recently been proposed by Ghioca and Scanlon while studying the dynamical Mordell-Lang conjecture in positive characteristic. Specifically, for a given prime p, they ask about the integer solutions of $f(y) = c_1 p^{a_1} + \cdots + c_l p^{a_l}$, in the unknowns $y, a_1, \ldots, a_l$, where the polynomial f and the constants $c_1, \ldots, c_l$ are given. Since p^{a_i} are S-units, this is in turn a special case of seeking the integral points on the cover of $\mathbb{G}_{\mathrm{m}}^l$ given by $f(y) = x_1 + \cdots + x_l$.

The methods so far known do not suffice even to treat the former equation (see [7] for some special cases). Actually, the problem arises even in writing down what is expected to be the most general form of the solution. Note that any identity of the shape $f(g(x)) = c_1 x^{m_1} + \cdots + c_l x^{m_l}$, for a polynomial g, would produce solutions simply by setting $x = p^a$. Hence, it is a primary task to write down all such identities. Note also that such an identity (considered now over $\mathbb{C}$) represents an S-integral point on the said cover, but now relative to the function field $\mathbb{C}(x)$ and set $S = \{0, \infty\}$ [5]: in fact, the S-units of $\mathbb{C}(x)$ are precisely the monomials cx^m.

[3] We recall that $\mathcal{O}_S = \{x \in K : |x|_v \le 1\ \forall v \notin S\}$; for instance, for $K = \mathbb{Q}$, the S-units are those rationals with numerator and denominator made up only of primes in the finite set S.

[4] This is indeed a "borderline" case of Vojta's conjecture on integral points, one of the simplest but yet unsolved ones. See [5] for a proof in the function field context.

[5] Here, in accordance with quite a general principle, the integral points over a function field may be used to parametrize the integral points over a number field.

This example makes evident the connection of these topics on integral points with the topic of fewnomials (and with Erdős-Rényi and Schinzel's mentioned conjectures); indeed, in the case of the problem of Ghioca and Scanlon a complete description *in finite terms* of the relevant identities follows from Theorem 2 of [20].

The results of the present paper yield a corresponding description in a rather more general situation. Namely, in dealing with an arbitrary finite cover $\pi : W \to \mathbb{G}_{\mathrm{m}}^l$, they allow us to parametrize all the regular maps $\rho : \mathbb{G}_{\mathrm{m}} \to W$ (i.e., the S-integral points on W, with respect to the function field $\mathbb{C}(x)$ and set $S = \{0, \infty\}$).[6]

Theorem 1.3. *Let $\pi : W \to \mathbb{G}_{\mathrm{m}}^l$ be a finite map. Then there exists a finite set Ψ of regular maps $\psi : V \times \mathbb{G}_{\mathrm{m}}^s \to W$, with $s = s_\psi$ an integer and $V = V_\psi$ an affine algebraic variety, such that for every regular map $\rho : \mathbb{G}_{\mathrm{m}} \to W$ there exist a $\psi \in \Psi$, a point $\xi \in V_\psi(\mathbb{C})$, and a regular map $\gamma : \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}^s$ with $\rho = \psi_\xi \circ \gamma$.*

Here ψ_ξ denotes the restriction of ψ to $\{\xi\} \times \mathbb{G}_{\mathrm{m}}^s$. The special case $l = 2$ of this theorem appears as Theorem 5.1 in [4], in different phrasing and with a completely different (and somewhat involved) proof.

We therefore see that any "S-integral point" factors through a map $\psi_\xi : \mathbb{G}_{\mathrm{m}}^s \to W$ of *bounded degree*, in the sense that the inverse image of a hyperplane section of W has a bounded degree in $\mathbb{G}_{\mathrm{m}}^s \subset \mathbb{P}_s$. This can be expressed in terms of boundedness of the *heights* of the integral points. Such a conclusion, which is in a sense the best possible, proves Vojta's conjectures for W and the integral points in question.[7] As an application, we can prove the following corollary.

Corollary 1.4. *Suppose that the union of the images of all regular non-constant maps $\rho : \mathbb{G}_{\mathrm{m}} \to W$ is Zariski dense. Then the branch locus of π in $\mathbb{G}_{\mathrm{m}}^l$ is invariant by translation by an algebraic subgroup of positive dimension.*

This is a useful condition which fits within a classification of Kawamata (see, for instance, the remark after Theorem 2 of [6] or Section 5.5.5 of the recent book by Noguchi and Winkelmann [16]).

Moreover, this language also makes it more obvious how to prove the special case $l = 1$ (we thank one of the anonymous referees for pointing out this argument). Indeed, for $l = 1$ an integral point is a regular map $\rho : \mathbb{G}_{\mathrm{m}} \to W$ such that the composition $\pi \circ \rho : \mathbb{G}_{\mathrm{m}} \to W \to \mathbb{G}_{\mathrm{m}}$ is the map $x \mapsto \theta x^n$, that is an isogeny composed with a translation. Suppose that one such point exists. It follows easily that the normalization of W is in fact isomorphic to $\mathbb{G}_{\mathrm{m}}$ itself, and all the integral points can then be easily classified.

1.3. **A "Bertini Theorem" for covers of tori.** Consider again a (ramified) cover $\pi : W \to \mathbb{G}_{\mathrm{m}}^l$, by which we mean a dominant map of finite degree e from the irreducible algebraic variety $W/\mathbb{C}$. When $\mathbb{G}_{\mathrm{m}}^l$ is replaced by the affine space $\mathbb{A}^l$, a version of the Bertini irreducibility theorem asserts that for $l > 1$, if H is a "general" hyperplane in $\mathbb{A}^l$, the fiber $\pi^{-1}(H)$ is still irreducible. In the present context one may replace H by a general algebraic subgroup (or coset) of $\mathbb{G}_{\mathrm{m}}^l$ and ask about the same conclusion. Of course, a marked contrast with the Bertini case is

[6]The case of more general function fields or even more general sets S is not known to us and seems to present subtle difficulties; this happens already by taking $S = \{0, 1, \infty\}$. See [6] for some cases related to surfaces.

[7]See e.g. [2, Section 14] for a general formulation of Vojta's conjectures, especially over number fields. For brevity we omit here any further detail or example.

that the algebraic subgroups now form a discrete family, which prevents standard methods from working in this context. In [21, Theorem 3] a positive result was obtained, however, concerning irreducibility only above components of 1-parameter subgroups, and not above arbitrary cosets.

Now, the arguments and results of this paper (completely independent of [21]) directly lead to a toric analogue of Bertini's theorem without the said restriction.

Theorem 1.5. *Let W be a quasi-projective variety and $\pi : W \to \mathbb{G}_{\mathrm{m}}^l$ be a (complex) dominant rational map of finite degree e, and suppose that the pullback $[e]^*W$ is irreducible. Let $X \subseteq W$ be a proper algebraic subvariety such that $\pi_{|W\setminus X}$ is finite onto its image. Then there exists a finite union $\mathcal{E} = \mathcal{E}_{\pi,X}$ of proper algebraic subgroups of $\mathbb{G}_{\mathrm{m}}^l$ such that if H is a connected algebraic subgroup not contained in $\mathcal{E}$, then for all $\theta \in \mathbb{G}_{\mathrm{m}}^l$, $\pi^{-1}(\theta H) \setminus X$ is irreducible.*

Note that if π is already finite onto its image, then X may be omitted from the statement. However, as pointed out by an anonymous referee, to whom we are grateful for the correction, in the general case the subvariety X must be included in the statement; for instance, if $\pi : W \to \mathbb{G}_{\mathrm{m}}^2$ is the blowup of $\mathbb{G}_{\mathrm{m}}^2$ at a point, then the preimage of a coset passing through the point always contains the exceptional divisor. The hypothesis of irreducibility of the pullback is also a necessary condition.[8]

As for Theorem 1.2, the set $\mathcal{E}$ can be given a complete algorithmic description, which is rather uniform in the data. Consider the following particular case. Suppose that the variety W can be represented as the hypersurface $f(t_1, \ldots, t_l, y) = 0$, with π given by the projection on the first l coordinates. Assume moreover that f is a (Laurent) polynomial in the t_i's and monic in y. Under these assumptions, the map π is finite, and the conclusion of Theorem 1.2 gives the following strengthening.

Addendum to Theorem 1.5. *If W is the hypersurface defined by $f(t_1, \ldots, t_l, y) = 0$, where f is a Laurent polynomial in $t_1, \ldots, t_l$ and monic in y, and $\pi : W \to \mathbb{G}_{\mathrm{m}}^l$ is the projection onto the first l coordinates, then $X = \emptyset$ and the set $\mathcal{E}$ may be chosen dependently only on* $\deg(f)$.

As an application, we immediately obtain the following corollary, in which for a given integer $d > 1$ we let $K_d(x)$ denote the *Kronecker substitution* $K_d(x) = (x, x^d, \ldots, x^{d^{l-1}})$.

Corollary 1.6. *Let $f(t_1, \ldots, t_l, y)$ be a complex polynomial of degree $e > 0$ in y and such that $f(t_1^e, \ldots, t_l^e, y)$ is irreducible over $\mathbb{C}(t_1, \ldots, t_l)$. Then $f(K_d(x), y)$ is irreducible over $\mathbb{C}(x)$ for all integers d large enough in terms of* $\deg(f)$.

This had been obtained in [21] (with a completely different proof), however, without this uniformity, which was left as an open question.

1.4. **An application to composite rational functions.** One may propose an analogue for rational functions of the already mentioned conjectures of Erdős-Rényi and subsequent ones by Schinzel. Namely, let $f(x)$ be a given rational function and suppose that for a rational function $g(x)$, the composition $f(g(x))$ may be written as a ratio of two polynomials (not necessarily coprime) with at most l terms. *Is there*

[8]For instance, when π is an isogeny of $\mathbb{G}_{\mathrm{m}}^l$, the cover becomes reducible above every subgroup $\pi(H)$, for any torus H not containing the kernel K of π, since $\pi^{-1}(\pi(H)) = HK$.

a $B = B(f, l)$ such that $g(x)$ may be represented as the ratio of the polynomials with at most B terms? The present methods allow a positive solution of this problem as well, as follows.

Theorem 1.7. *If $f, g \in \mathbb{C}(x) \setminus \mathbb{C}$ are such that the composition $f(g(x))$ can be written as the ratio $P(x)/Q(x)$, where $P, Q \in \mathbb{C}[x]$ have altogether at most l terms, then there exist polynomials $p, q \in \mathbb{C}[x]$ with at most $B_2 = B_2(l)$ terms such that $g(x) = p(x)/q(x)$.*

Again, we stress that the pairs P, Q and p, q are not necessarily coprime. We also remark that we actually have full uniformity here in the rational function f, as the number B_2 only depends on l and not on $\deg(f)$ (this dependency can be removed thanks to a previous theorem proved by the first and last authors [12]; in the same paper, the function $f(x)$ is also described, on bounding its degree in terms of l only, unless $g(x)$ is of certain special shapes listed therein).

1.5. **Non-standard polynomials.** The notion of fewnomial and our main theorems can be translated naturally in the language of Robinson's non-standard analysis. We refer the reader to [10] for an introduction to the subject.

Here we just recall that in non-standard analysis one has a map * which sends the standard objects, such as $\mathbb{N}$ or $\mathbb{R}$, to their non-standard counterparts, in a way that preserves all first-order formulas. The easiest example of (non-trivial) map * is the one that sends any set S into the set of sequences with values in S (i.e., $S^{\mathbb{N}}$) modulo the equivalence relation defined by a fixed non-principal ultrafilter on $\mathbb{N}$ (i.e., $(a_n) \sim (b_n)$ if $\{n : a_n = b_n\}$ is in the ultrafilter). This introduces new, non-standard elements; for instance, the non-standard $^*\mathbb{N}$ contains an element ω, the equivalence class of the sequence $(n)_{n\in\mathbb{N}}$, which is different from any standard natural number.

Concerning our context, we note that the non-standard $^*(\mathbb{C}[x])$ contains "polynomials with infinitely many terms," such as

$$1 + x + x^2 + \cdots + x^{\omega-1} + x^{\omega}.$$

In fact, this is exactly the equivalence class of the sequence $(1 + x + \cdots + x^n)_{n\in\mathbb{N}}$.

We now define the *ring $\mathcal{F}$ of fewnomials* in $^*(\mathbb{C}[x])$ to be the subring of polynomials whose number of terms is actually finite,

$$\mathcal{F} := \{a_1 x^{n_1} + \cdots + a_l x^{n_l} \ : \ l \in \mathbb{N},\ a_i \in {}^*\mathbb{C},\ n_i \in {}^*\mathbb{N}\}.$$

In this language, statements about fewnomials become quite compact. As an instance of this phrasing, the Erdős-Rényi conjecture proved by Schinzel becomes the following: *if $g^2 \in \mathcal{F}$ for some $g \in {}^*(\mathbb{C}[x])$, then $g \in \mathcal{F}$.* Likewise, Theorem 1.1 translates to the following quite short statement.

Theorem *1.1. *The ring $\mathcal{F}$ is integrally closed in $^*(\mathbb{C}(x))$.*

This statement was proposed by Fornasiero before the results of this paper, together with its following immediate corollary (which is, in turn, a non-standard translation of Theorem 2.2, stated in the following section).

Corollary (Theorem *2.2). *The fraction field of $\mathcal{F}$ is relatively algebraically closed in $^*(\mathbb{C}(x))$.*

The latter conclusion is another example of the aforementioned behavior of rational functions, and indeed it corresponds to dropping the assumption that the polynomial is monic in Theorem 1.1.

It is rather easy to see that Theorems $^*1.1$ and 1.1 are indeed equivalent. For example, assume Theorem $^*1.1$ and suppose by contradiction that Theorem 1.1 is false. Then for some $d, l \in \mathbb{N}$ there should be a sequence $(g_n(x))$ of polynomials whose number of terms grows to infinity, while they also satisfy

$$f_n(x, g_n(x)) = 0,$$

where (f_n) is a sequence of polynomials with at most l terms, of degree at most d, and monic in the last variable.

But then the equivalence classes $^*g$ and $^*f$ of the above sequences satisfy

$$^*f(x, {}^*g(x)) = 0,$$

which means that $^*g(x)$ is integral over $\mathcal{F}$, while it lies in $^*(\mathbb{C}[x])$ and not in $\mathcal{F}$, a contradiction.

Although there are details to be worked out, we believe that also our proof of Theorem 1.1 can be translated rather naturally to a shorter argument in the nonstandard language; this is being investigated and may appear in a future paper. On the one hand, we would lose effectivity, but on the other, we may be able to avoid the use of the resolution of singularities and directly use the construction of the Puiseux series.

The main potential simplification comes from the fact that many notions, which in the proof depend on appropriately chosen parameters, become absolute. For example, the notion of being "small" with respect to a "large" number, which in our proof depends on a parameter ε to be chosen carefully, translates to being *infinitesimal* with respect to the second number.

1.6. **Fewnomials and unlikely intersections.** This instance does not directly use results of the present paper, but we still discuss it because it is far from being unrelated.

As already mentioned at the end of Section 1.1, several results here contain a dichotomy *lacunary polynomials* $\leftrightarrow$ *lacunary rational functions*, where by the latter terminology we mean rational functions which may be represented as ratios of fewnomials, possibly non-coprime, as in Theorem 1.7. Recall the standard example $(x^n-1)/(x-1)$, which shows that a *lacunary rational function* which is a polynomial is not necessarily a fewnomial. This gives rise to the following problem, also posed independently by Zieve.

Suppose that a rational function $r(x) \in \mathbb{C}(x)$ can be represented as the ratio $r(x) = g(x^{n_1}, \dots, x^{n_l})/h(x^{n_1}, \dots, x^{n_l})$, where the integers n_i vary, while g, h are *fixed* coprime polynomials in $\mathbb{C}[t_1, \dots, t_l]$. (In accordance with the viewpoint illustrated above, we are viewing $r(x)$ as the restriction of a fixed rational function g/h on $\mathbb{G}_{\mathrm{m}}^l$ to a one-dimensional algebraic subgroup which may vary.) One may ask the following question.

Question 1.8. *For which one-dimensional algebraic subgroups does $r(x)$ become a (Laurent)polynomial?*

For instance, the above example comes from $g = t_2 - 1$, $h = t_1 - 1$ on $\mathbb{G}_{\mathrm{m}}^2$; in this case it is easy to check that the only one-dimensional algebraic subgroups which make g/h a (Laurent) polynomial are given by $t_2 = t_1^n$ for integer n (as in the example).

Therefore, in particular, we have two coprime polynomials g, h such that they become non-coprime (or such that h becomes invertible) along the one-dimensional subtorus of $\mathbb{G}_{\mathrm{m}}^l$ parametrized by $t_i \mapsto x^{n_i}$. This kind of problem also appeared in a conjecture of Schinzel, which was later recognized as a special case of the more recent Zilber-Pink conjecture in the realm of the so-called *unlikely intersections*. See [22] for a discussion of this topic, especially Chapter 2. This conjecture of Schinzel was confirmed by Bombieri and the third author (see [18, Appendix]), and it was later refined with other methods, in collaboration also with Masser, in [3, Theorem 1.5], in a work proving the Zilber-Pink conjecture for intersections with one-dimensional subgroups.

These last results give an answer to the above question, showing that the relevant algebraic subgroups are contained in a *finite union* $\mathcal{E} = \mathcal{E}_{g,h}$ *of proper algebraic subgroups of* $\mathbb{G}_{\mathrm{m}}^l$. Given this, one may restrict to the subgroups in $\mathcal{E}$ and continue by induction to write down all the possibilities: it turns out that *the relevant one-dimensional algebraic subgroups are precisely those contained in a certain finite union* $\mathcal{E}'$ *of proper algebraic subgroups on which* g/h *becomes regular.*

It is to be remarked that the more general question in which *one-dimensional algebraic subgroups* are replaced by *one-dimensional algebraic cosets* does not admit a similar solution. This corresponds to the ratio $g(\theta_1 x^{n_1}, \ldots, \theta_l x^{n_l})/h(\theta_1 x^{n_1}, \ldots, \theta_l x^{n_l})$ being a polynomial, for integers n_i and nonzero constants θ_i. We do not know of any method able to deal with such a question in full generality.

Another connection to integral points was pointed out to us by one of the referees, and we are grateful for it. Starting from the rational function $r(t_1, \ldots, t_l) = g(t_1, \ldots, t_l)/h(t_1, \ldots, t_l)$, one can blow up the codimension-two subvariety of $\mathbb{G}_{\mathrm{m}}^l$ defined by the simultaneous vanishing of g and h, and remove the strict transform of the subvariety of $\mathbb{G}_{\mathrm{m}}^l$ defined by $h = 0$. Let W be the resulting variety. Then the one-dimensional (translates of) algebraic subgroups which are solutions to Question 1.8 correspond to regular maps $\mathbb{G}_{\mathrm{m}} \to W$. With this interpretation a solution for the problem of translates can possibly be given for $l = 2$ (the case of surfaces) under some normal-crossings conditions which will depend on h and are generically satisfied.

1.7. **Proof methods and quantitative issues.** The strategy of the proofs here follows only in part the pattern of [20]; this shall be outlined in more detail in Section 3 (before the formal arguments). The main technical issue is finding an appropriate way of expanding $g(x)$ as a kind of multivariate Puiseux series. This is done here by using first the theory of resolution of singularities to reduce to a rather regular case in which one can use multivariate analytic expansions. An earlier version of the proofs involved a different, more complicated construction of certain Puiseux-type expansions, but no use of resolution of singularities. The approach was dropped in favor of the present one for the sake of simplicity, but it may be of independent interest, and it can still be found in an earlier draft of this paper [11].

A by-product is a completely effective output of the proofs: *one can obtain effective estimates for the involved quantities, and effective parametrizations* (provided of course one deals with cases in which the fields and the equations which occur are finitely presented). However, we do not give here explicit bounds, which in any case would have the shape of highly iterated exponentials.[9]

2. Variations and reductions

2.1. **Variations of Theorem 1.2.** The following three statements are variations regarding irreducible factors and the dichotomy rational functions $\leftrightarrow$ polynomials mentioned in the Introduction.

The first one concerns factorizations.

Theorem 2.1. *If $f \in \mathbb{C}[t_1, \ldots, t_l, y]$ is monic in y and of degree at most d in each variable, if $n_1, \ldots, n_l$ are natural numbers, and if $g, h \in \mathbb{C}[x, y]$ are polynomials monic in y such that*

$$g(x, y)h(x, y) = f(x^{n_1}, \ldots, x^{n_l}, y), \tag{2.1}$$

then each coefficient of g (as a polynomial in y) has at most $B_3 = B_3(d, l)$ terms.

(By symmetry, a similar conclusion holds automatically for the coefficients of h.)

Note that we recover Theorem 1.2 on taking $(y - g(x))$ as the first factor. The converse deduction is also not difficult but shall be explained later. The other variations concern rational functions.

Warning. We stress again the point that whenever we write a rational function (even when it is a polynomial) as a quotient of two polynomials, we are not usually assuming that the numerator and denominator are coprime (recall the example $(x^n - 1)/(x - 1)$). This issue is related to not requiring that f is monic in y, as in the following statements.

Theorem 2.2. *If $f \in \mathbb{C}[t_1, \ldots, t_l, y] \setminus \mathbb{C}$ is a polynomial of degree at most d in each variable, if $n_1, \ldots, n_l$ are integers, and if $g(x) \in \mathbb{C}(x)$ is such that*

$$f(x^{n_1}, \ldots, x^{n_l}, g(x)) = 0, \tag{2.2}$$

then $g(x)$ is the ratio of two polynomials in $\mathbb{C}[x]$ with at most $B_4 = B_4(d, l)$ terms.

Theorem 2.3. *If $f \in \mathbb{C}[t_1, \ldots, t_l, y]$ is a polynomial of degree at most d in each variable, if $n_1, \ldots, n_l$ are integers, and if $g, h \in \mathbb{C}(x)[y]$ are such that*

$$g(x, y)h(x, y) = f(x^{n_1}, \ldots, x^{n_l}, y) \tag{2.3}$$

with g monic in y, then each coefficient of g (as a polynomial in y) is the ratio of two polynomials in $\mathbb{C}[x]$ with at most $B_5 = B_5(d, l)$ terms.

It is easy to see that Theorem 1.2 implies Theorem 2.2, but the converse deduction does not appear as straightforward (we stress yet again the point that the polynomial g in the conclusion of Theorem 2.2 is represented as a quotient of two polynomials which need not be coprime). In this paper, we actually prove Theorem 2.2 first, and then deduce Theorem 1.2 (and Theorem 2.1) via a general integrality argument.

[9] In the original cases of the Erdős-Rényi conjecture, doubly exponential bounds had been obtained by Schinzel [17], reduced later to a single exponential by Schinzel and the third author [19].

Remark 2.4. In all of the above statements, we may actually allow $n_1, \ldots, n_l$ to be negative and $g(x) \in \mathbb{C}[x, x^{-1}]$, with a similar conclusion.

We may also deduce that the fewnomials which arise can be parametrized with the same exponents. For instance, in Theorem 1.2, we can say that there are N and $G \in \mathbb{C}[t_1, \ldots, t_l]$, with N and $\deg(G)$ bounded in terms of d and l only, such that $g(x^N) = G(x^{n_1}, \ldots, x^{n_l})$.

For the sake of simplicity, we shall omit details about these further assertions.

2.2. **Reductions.** We are going to prove Theorem 2.2 first, and we can use some standard arguments to reduce the theorem to a simpler situation. In a moment, we shall reduce both Theorems 2.2 and 2.3 about rational functions to the case where f is monic in y, and $n_1, \ldots, n_l$ are non-negative. We obtain the following statements, in which the assumption is as in Theorem 1.2 (or Theorem 2.1), but the conclusion is as in Theorem 2.2 (resp. Theorem 2.3).

Proposition 2.5. *If $f \in \mathbb{C}[t_1, \ldots, t_l, y] \setminus \mathbb{C}$ is monic in y and of degree at most d in each variable, if $n_1, \ldots, n_l$ are natural numbers, and if $g(x) \in \mathbb{C}(x)$ is such that*

$$f(x^{n_1}, \ldots, x^{n_l}, g(x)) = 0, \tag{2.4}$$

then $g(x)$ is the ratio of two polynomials in $\mathbb{C}[x]$ with at most $B_6 = B_6(d, l)$ terms.

Note that since f is monic in y, it follows that $g(x)$ is actually a polynomial, but the conclusion only says that g is represented by a quotient of two polynomials which need not be coprime. Thus this proposition is a weak form of Theorem 1.2. It is similar for its corollary.

Proposition 2.6. *If $f \in \mathbb{C}[t_1, \ldots, t_l, y]$ is monic in y and of degree at most d in each variable, if $n_1, \ldots, n_l$ are natural numbers, and if $g, h \in \mathbb{C}(x)[y]$ are such that*

$$g(x, y)h(x, y) = f(x^{n_1}, \ldots, x^{n_l}, y) \tag{2.5}$$

with g monic in y, then each coefficient of g as a polynomial in y is the ratio of two polynomials in $\mathbb{C}[x]$ with at most $B_7 = B_7(d, l)$ terms.

Both are clearly special cases of the original Theorems 2.2 and 2.3. As we now show, it is not difficult to deduce the latter statements from them.

Note 2.7. It is important to note that *the following deductions are valid for each single l* (whereas the number d is changed in the course of the deductions). This is crucial in all our proofs that proceed by induction on l; namely, if we assume that one statement is true for a certain value of l and all possible d's, the other statements will follow as well for the same value of l and all possible d's.

Deduction of Theorem 2.2 *from Proposition* 2.5. Note first that Proposition 2.5 requires $n_1, \ldots, n_l$ to be natural numbers rather than integers. We may reduce to the case $n_i \geq 0$ by replacing, when necessary, t_i by t_i^{-1} and multiplying the resulting polynomial by t_i^d; after this transformation, the degree of f in each variable is still bounded by d. Therefore, we may assume that $n_i \geq 0$ for all i.

Write f as

$$f = \sum_{i=0}^{d} h_i(t_1, \ldots, t_l) y^i,$$

where the h_i's are polynomials of degree at most d in each variable. Let $e \leq d$ be the maximum integer such that $h_e(x^{n_1}, \ldots, x^{n_l})$ is not identically zero, and let $f_1 := \sum_{i=0}^{e} h_i(t_1, \ldots, t_l) y^i$.

We now consider the polynomial $f_2 := h_e^{e-1} f_1(t_1, \dots, t_l, y/h_e)$. Note that f_2 is monic in y, and it has degree at most $(e-1)d + d \leq d^2$ in each variable. Assuming Proposition 2.5, each rational root of $f_2(x^{n_1}, \dots, x^{n_l}, y)$ is the ratio of two polynomials with at most $B_6(d^2, l)$ terms. Multiplying each such root by h_e we obtain all the rational roots of $f(x^{n_1}, \dots, x^{n_l}, y)$, and therefore the rational solutions of (2.2). In particular, the solutions are ratios of polynomials with at most $B_4(d, l) := (d+1)^l B_6(d^2, l)$ terms, as desired. $\square$

Deduction of Theorem 2.3 *from Proposition* 2.6. We proceed as in the previous deduction to show that $B_5(d, l) := (d+1)^{dl} B_7(d^2, l)$ is a suitable value for B_5. $\square$

Moreover, as promised earlier, we can easily deduce Theorem 2.3 from Theorem 2.2. Thanks to the above reductions, it is sufficient to deduce Proposition 2.6 from Proposition 2.5.

Deduction of Proposition 2.6 *from Proposition* 2.5. Suppose that $p(x)$ is a coefficient of a monic irreducible factor of the polynomial,

$$\phi(x, y) := f(x^{n_1}, \dots, x^{n_l}, y).$$

Let us call $\alpha_1, \dots, \alpha_e$ the roots of this polynomial in an algebraic closure of $\mathbb{C}(x)$, with repetitions, where $e = \deg_y \phi = \deg_y f$. The polynomial $p(x)$ is, up to sign, an elementary symmetric polynomial in some of the roots. Let us denote the elementary symmetric polynomials as $\Sigma_j^k(z_1, \dots, z_k) := \sum_{1 \leq i_1 < \cdots < i_j \leq k} z_{i_1} \cdot \dots \cdot z_{i_j}$.

Up to reordering the roots, we may write

$$p(x) = \pm \Sigma_j^k(\alpha_1, \dots, \alpha_k)$$

for some $0 \leq j \leq k \leq e$. This implies that $p(x)$, up to sign, is a root of the monic polynomial

$$\psi_{jk}(x, y) := \prod_{1 \leq i_1 < \cdots < i_k \leq e} \left(y - \Sigma_j^k(\alpha_{i_1}, \dots, \alpha_{i_k}) \right).$$

But the coefficients of ψ_{jk} are now symmetric polynomials in the roots α_i, which implies that they are actually polynomials in the Σ_i^e's, i.e., the coefficients of ϕ. A rough estimate shows that the degree of each such polynomial in each Σ_i^e is bounded by $2^e \leq 2^d$.

This implies that we may find $f_{jk}(t_1, \dots, t_l, y) \in \mathbb{C}[t_1, \dots, t_l, y]$ monic in y and of degree at most $2^d d$ in each variable such that

$$f_{jk}(x^{n_1}, \dots, x^{n_l}, y) = \psi_{jk}(x, y).$$

Assuming Proposition 2.5, since $p(x)$ is a root of ψ_{jk}, it must be a ratio of two polynomials with at most $B_7(d, l) := B_6(2^d d, l)$ terms, as desired. $\square$

The exact same argument can also be used to show that Theorem 2.1 follows from Theorem 1.2.

Deduction of Theorem 2.1 *from Theorem* 1.2. We proceed as in the previous proof to show that $B_1(2^d d, l)$ is a suitable value for B_3. $\square$

2.3. **Further lemmas.** In the course of our proof, we will need on a few occasions to replace x with an auxiliary variable x_n such that $x_n^n = x$. In the next lemma, we show that these substitutions do not affect our statements, so they may be considered as immaterial.

Lemma 2.8. *Let $g(x)$ be a polynomial such that $g(x^n)$ can be written as the ratio of two polynomials with at most B terms. Then $g(x)$ is the ratio of two polynomials with at most B terms.*

Proof. Suppose that $g(x^n) = \frac{p(x)}{q(x)}$, where p and q are polynomials with at most B terms. Grouping the monomials whose degrees in x are in the same congruence class modulo n we may (uniquely) write

$$p(x) = p_0(x^n) + xp_1(x^n) + \cdots, \quad q(x) = q_0(x^n) + xq_1(x^n) + \cdots$$

with p_i, q_i polynomials with at most B terms as well.

But then, since $g(x^n)q(x) = p(x)$, we must have $g(x^n)q_i(x^n) = p_i(x^n)$ for all i, and in particular $g(x)q_i(x) = p_i(x)$. As at least one q_i is non-zero, we have found a representation of $g(x)$ as the ratio of two polynomials with at most B terms, as desired. □

Another easy reduction shows that if we find a $\mathbb{Z}$-linear relation with bounded coefficients between the exponents $n_1, \dots, n_l$, then we may actually remove one of the exponents. This is also crucial for our induction on l.

Lemma 2.9. *Suppose that we are under the hypothesis of Theorem 2.2, and that there are integers $h_1, \dots, h_l$, not all zero, and some $C > 0$ such that*

$$h_1 n_1 + \cdots + h_l n_l = 0, \ |h_i| \leq C.$$

Assume moreover that Theorem 2.2 has been proved for $(l-1)$ and any degree d. Then $g(x)$ is the ratio of two polynomials with at most $B_4(3dC, l-1)$ terms.

Proof. Without loss of generality, we may assume that $h_l > 0$. In this case, we take new variables $u_1, \dots, u_{l-1}$, we replace t_i in f with $u_i^{h_l}$ for $i = 1, \dots, l-1$ and t_l with $u_1^{-h_1} \cdots u_{l-1}^{-h_{l-1}}$, and we multiply the result by $(u_1 \cdots u_{l-1})^{dC}$. The resulting polynomial has degree at most $3dC$ in each variable, and it vanishes at $u_i = x^{n_i}$ and $y = g(x^{h_l})$.

Now, using the assumption about Theorem 2.2 and Lemma 2.8, $g(x)$ is the ratio of two polynomials with at most $B_5(3dC, l-1)$ terms. □

3. Introduction to the proof

In order to prove Theorem 2.2, we build up on the same technique of [20] but with the additional use of the theory of resolution of singularities to reduce to a sufficiently regular case. Indeed, the underlying expansions depend not quite on the variable x, but on the l variables $t_1, \dots, t_l$; it is well known that expansions of algebraic functions of several variables often depend on subtle geometric features.

For the sake of illustration, we explain the strategy of the proof in a simpler example where this combinatorial aspect is missing. We work by induction on l.

Say that, as in the original Erdős-Rényi conjecture (a special case of Theorem 1.2), we start with the polynomial

$$f(t_1,\dots,t_l,y)=y^2-c_0-c_1t_1-\cdots-c_lt_l.$$

For simplicity, we also assume that $c_0=1$.

If we want to prove that a rational root $g(x)$ of

$$f(x^{n_1},\dots,x^{n_l},y)=\phi(x,y)=y^2-1-c_1x^{n_1}-\cdots-c_lx^{n_l}$$

is the ratio of two polynomials with few terms, we may expand $g(x)$ with the binomial series; namely, letting $h(x):=c_1x^{n_1}+\cdots+c_lx^{n_l}$, we may easily obtain the multinomial expansion

$$g(x)=1+\frac{h(x)}{2}-\frac{h(x)^2}{8}+\cdots=\sum_{k_1=0}^{\infty}\cdots\sum_{k_l=0}^{\infty}c_{k_1,\dots,k_l}x^{k_1n_1+\cdots+k_ln_l}.$$

It is crucial that $k_1,\dots,k_l$ run through natural numbers. Assuming that $0<n_1\le n_2\le\cdots\le n_l$, if $n_1\ge\varepsilon n_l$ for some fixed $\varepsilon>0$, each exponent $k_1n_1+\cdots+k_ln_l$ is at least $(k_1+\cdots+k_l)\varepsilon n_l$. Since the degree of $g(x)$ must be $(n_l/2)$, we find that all terms must eventually cancel except possibly for those such that $(k_1+\cdots+k_l)\le 1/(2\varepsilon)$, leading to the bound $(2\varepsilon)^{-l+1}/l!$ for the number of terms.

This consideration always works for $l=1$ (with $\varepsilon=1$), and in particular we obtain the base case of our induction. However, in general we have no lower bound at all for n_1/n_l. To cope with this difficulty, the principle in [20] is that if some terms $n_1,\dots,n_p$ are very small compared to n_l, we can group together these small contributions as follows: we define

$$\delta(x)=1+c_1x^{n_1}+\cdots+c_px^{n_p},\ h_1(x)=c_{p+1}x^{n_{p+1}}+\cdots+c_lx^{n_l},$$

and we expand $g(x)$ as

$$g(x)=\sqrt{\delta(x)}\left(1+\frac{h_1(x)}{\delta(x)}\right)^{1/2}=\sqrt{\delta(x)}\left(1+\frac{h_1(x)}{2\delta(x)}-\frac{h_1(x)^2}{8\delta(x)^2}+\cdots\right). \tag{3.1}$$

As before, we can expand the powers of $h_1(x)$, which involve the large exponents only; however, the new coefficients will not be constants, as before, but actually functions in the hyperelliptic function field $\mathbb{C}(x,\delta(x)^{1/2})$. Despite this radically new feature, a theorem in Diophantine approximation over function fields (see Section 6) allows one to reduce to the inductive hypothesis at $p<l$, provided n_{p+1} is large enough, by which we mean that it is greater than εn_l for an absolute $\varepsilon>0$. Of course, for some $0\le p<l$ we must indeed have that n_p is small, whereas n_{p+1} is large, concluding the argument.

In the general case, we wish to apply the same approximation technique. However, a direct attempt at expanding $g(x)$ as a kind of multivariate series might fail; The main issue is that we may have monomials involving exponents that are combinations of $n_1,\dots,n_l$ with *negative* coefficients, in which case a combination of large exponents may become small, and it is not as easy any more to separate the big ones from the small ones. These obstacles appear when $g(0)$ is a non-simple root of $f(\mathbf{0}, y)$.[10]

We shall overcome these obstacles by applying a suitable monoidal transformation to our original equation. Although $g(0)$ might still be a non-simple root of $f(\mathbf{0}, y)$, the transformation will guarantee that $g(x)$ can still be expanded as in the original case. The choice of the monoidal transformation relies on the theory of resolution of singularities.[11]

4. Reduction to the regular case

In order to obtain our desired expansion of $g(x)$ as a "pseudo-analytic series," we prove that Proposition 2.5 can be further reduced to a special case in which the polynomial f is sufficiently regular.

Let $f \in \mathbb{C}[t_1, \ldots, t_l, y]$ be as in Proposition 2.5. Assume, as we may, that f is irreducible. Let $\mathbb{C}(t_1, \ldots, t_l, z)$ be the function field generated by the independent variables $t_1, \ldots, t_l$ and an algebraic function z such that $f(t_1, \ldots, t_l, z) = 0$.

Let W be a projective non-singular model of the function field $\mathbb{C}(t_1, \ldots, t_l, z)$. For each $i = 1, \ldots, l$, let $\mathcal{D}_i$ be the set of the irreducible components of the divisor of t_i, and let $\mathcal{D} := \bigcup_{i=1}^{l} \mathcal{D}_i$. By the known theory of resolution of singularities, after applying some blowups, we may further assume that *the divisors appearing in $\mathcal{D}$ are non-singular and have normal crossings.*

Let $\phi : \mathbb{P}_1 \to W$ be the unique non-constant map such that

- $t_i \circ \phi = x^{n_i}$ for all $i = 1, \ldots, l$;
- $z \circ \phi = g(x)$,

where x is the standard coordinate function $x : \mathbb{P}_1 \to \mathbb{P}_1$. Let $P := \phi(0)$.

Definition 4.1. Under the above notations, we say that a solution $n_1, \ldots, n_l, g(x)$ to (2.2), namely $f(x^{n_1}, \ldots, x^{n_l}, g(x)) = 0$, is *regular* if $t_1, \ldots, t_l$ are local parameters at P.

Note that one might reformulate the above notion in a more compact and geometric way by only referring to the map ϕ. However, we prefer to keep an explicit reference to the polynomial $g(x)$ and the numbers $n_1, \ldots, n_l$.

The purpose of this section is to show that it suffices to prove the conclusion of Proposition 2.5 in the special case in which the solution is regular.

In what follows, given a divisor D and a regular function u, we let $v_D(u)$ denote the order of u at D. Note that since f has degree at most d in each variable, we have $|v_D(t_i)| \leq d^l$ for all $D \in \mathcal{D}$.

Lemma 4.2. *Let $D_1, \ldots, D_m$ be the divisors in $\mathcal{D}$ on which P lies. Then either*

$$h_1 n_1 + \cdots + h_l n_l = 0$$

for some integers $h_1, \ldots, h_l \in \mathbb{Z}$ not all zero such that $|h_i| \leq (d^l l)^l$ or the matrix $(v_{D_i}(t_j))_{i,j}$ is invertible (and in particular $m = l$).

Proof. Since the divisors in $\mathcal{D}$ have normal crossings, we know at once that $m \leq l$. Assume that the matrix $(v_{D_i}(t_j))_{i,j}$ is not invertible; otherwise we are done. Then

[10] These issues are entirely avoided in the cases considered in [20], where multinomial expansions suffice.

[11] We recall that an earlier draft of this paper contained a different proof based on a careful construction of a Puiseux-type expansion rather than the resolution of singularities [11].

there are integers $h_1, \dots, h_l \in \mathbb{Z}$, not all zero, such that

$$h_1 v_{D_i}(t_1) + \cdots + h_l v_{D_i}(t_l) = 0$$

for all $i = 1, \dots, m$. Since $|v_{D_i}(t_j)| \leq d^l$ for all i, j, we may choose the integers h_i so that $|h_i| \leq (d^l l)^l$ by Siegel's lemma.

Let $u := t_1^{h_1} \cdot \dots \cdot t_l^{h_l}$. By the above observation, none of $D_1, \dots, D_m$ is a component of the divisor of u. On the other hand, the components of the divisor of u are in $\mathcal{D}$. It follows that no such component contains P, so u is regular at P and $u(P) \in \mathbb{C}^*$. Note moreover that $u \circ \phi = x^{h_1 n_1 + \cdots + h_l n_l}$. Therefore,

$$u(P) = x(P)^{h_1 n_1 + \cdots + h_l n_l} \in \mathbb{C}^*.$$

Since $x(P) = 0$, it immediately follows that $h_1 n_1 + \cdots + h_l n_l = 0$, reaching the desired conclusion. $\square$

Thanks to the above observation, in order to prove Proposition 2.5, it shall be sufficient to prove the following special version which has a few more hypotheses.

Proposition 4.3. *If $f \in \mathbb{C}[t_1, \dots, t_l, y] \setminus \mathbb{C}$ is monic in y, is irreducible, and is of degree at most d in each variable, and if $n_1, \dots, n_l \in \mathbb{N}^*$ and $g(x) \in \mathbb{C}(x)$ form a regular solution of*

$$f(x^{n_1}, \dots, x^{n_l}, g(x)) = 0,$$

then $g(x)$ is the ratio of two polynomials in $\mathbb{C}[x]$ with at most $B_8 = B_8(d, l)$ terms.

Note 4.4. As in Note 2.7, the following deduction is valid at every single l.

Deduction of Proposition 2.5 *from Proposition* 4.3. We work by induction on l. In particular, if $n_i = 0$ for some i, we may specialize the variable t_i to 1; if $l > 1$, we conclude by inductive hypothesis, while if $l = 1$, we simply note that we actually have $g(x) \in \mathbb{C}$. Therefore, we may assume that $n_i \neq 0$ for all $i = 1, \dots, l$. Moreover, we may replace f by an irreducible factor, and therefore assume directly that f is irreducible.

By Lemma 4.2, either P lies on l distinct divisors $D_1, \dots, D_l$ such that the matrix $(v_{D_i}(t_j))_{i,j}$ is invertible, or there is a relation $h_1 n_1 + \cdots + h_l n_l = 0$ with integers h_i not all zero and such that $|h_i| \leq (d^l l)^l$. In the latter case, we must have $l > 1$, and we may conclude by Lemma 2.9 and the inductive hypothesis. Therefore, we may assume to be in the former case.

Let $u_1, \dots, u_l$ be new independent variables, and set

$$f_1(u_1, \dots, u_l, y) := f\left(u_1^{v_{D_1}(t_1)} \cdot \dots \cdot u_l^{v_{D_1}(t_l)}, \dots, u_1^{v_{D_l}(t_1)} \cdot \dots \cdot u_l^{v_{D_l}(t_l)}, y\right).$$

Note that f_1 is a polynomial of degree at most $d^{l+1} l$ in each variable.

Let $(r_{i,j})_{i,j}$ be the inverse matrix of $(v_{D_i}(t_j))_{i,j}$ multiplied by its determinant Δ, so that its coefficients are all in $\mathbb{Z}$. Let $m_i := r_{1,i} n_1 + \cdots + r_{l,i} n_l$. By construction, we have

$$f_1(x^{m_1}, \dots, x^{m_l}, g(x^\Delta)) = f(x^{n_1 \Delta}, \dots, x^{n_l \Delta}, g(x^\Delta)) = 0.$$

In turn, we choose an irreducible factor f_2 of f_1 such that

$$f_2(x^{m_1}, \dots, x^{m_l}, g(x^\Delta)) = 0. \tag{4.1}$$

We claim that $m_1, \dots, m_l, g(x^\Delta)$ form a regular solution of this equation in the sense of Definition 4.1. Indeed, we may now assume that $u_1, \dots, u_l$ are algebraic functions in some algebraic closure of $\mathbb{C}(t_1, \ldots, t_l, z)$ such that

$$t_i = u_1^{v_{D_i}(t_1)} \cdot \cdots \cdot u_l^{v_{D_i}(t_l)}, \quad f_2(u_1, \ldots, u_l, z) = 0.$$

Let W' be a projective non-singular model of the function field $\mathbb{C}(u_1, \ldots, u_l, z)$, equipped with a surjective, finite map $\pi : W' \to W$. As at the beginning of the section, we may apply some blowups and assume that all the components of the divisors of the functions u_i are non-singular and have normal crossings.

Let $\phi' : \mathbb{P}_1 \to W'$ be the unique non-constant map such that $u_i \circ \phi' = x^{m_i}$ and $z \circ \phi' = g(x^\Delta)$. Let $P' := \phi'(0) \in W'$. Since by construction $\pi \circ \phi' = x^{n_i \Delta}$, it follows at once that $\pi(P') = P$. For $i = 1, \ldots, l$, let E_i be a component of $\pi^*(D_i)$ on which P' lies.

Finally, note that $u_i^\Delta = t_1^{r_{1,i}} \cdot \cdots \cdot t_l^{r_{l,i}}$. In particular, u_i^Δ can be factored as $u_i' \circ \pi$, where u_i' is a function on W. By construction, we have $v_{D_i}(u_j') = \Delta \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $w_1, \ldots, w_l$ be local parameters of $D_1, \ldots, D_l$, so that $u_i' \sim w_i^\Delta$ (in the sense of analytic equivalence). It follows at once that $u_i \sim w_i \circ \pi$; since the function field extension is generated by the functions u_i, each $w_i \circ \pi$ is also a local parameter of E_i, and in particular, u_i is a local parameter of E_i. Since the divisors E_i have normal crossings, this means that $u_1, \ldots, u_l$ are local parameters at P'.

In turn, $m_1, \ldots, m_l, g(x^\Delta)$ form a regular solution of (4.1), so $g(x^\Delta)$ can be written as the ratio of two polynomials with at most $B_8(d^{l+1}l, l)$ terms. By Lemma 2.8, $g(x)$ can also be written as the ratio of two polynomials with at most $B_8(d^{l+1}l, l)$ terms, concluding the argument. $\square$

5. From multivariate expansions to algebraic approximations

We now start our argument toward the proof of Proposition 4.3. From now up to Section 8, assume that we are working under the assumptions of Proposition 4.3, and in particular that $n_1, \ldots, n_l \in \mathbb{N}^*$ and $g(x) \in \mathbb{C}[x]$ form a regular solution of $f(x^{n_1}, \ldots, x^{n_l}, g(x)) = 0$, where $f \in \mathbb{C}[t_1, \ldots, t_l, y] \setminus \mathbb{C}$ is irreducible, is monic in y, and has degree at most d in each variable.

Under the notation of Section 4, regularity means that $t_1, \ldots, t_l$ are local parameters at $P = \phi(0)$, which means that there is an embedding of the regular functions at P into $\mathbb{C}[[t_1, \ldots, t_l]]$. Since moreover the function z is integral over $\mathbb{C}[t_1, \ldots, t_l]$, we obtain an embedding

$$\mathbb{C}[t_1, \ldots, t_l, z] \hookrightarrow \mathbb{C}[[t_1, \ldots, t_l]].$$

The fairly trivial, but crucial observation, is that for any $p = 0, \ldots, l-1$ we can also rewrite

$$\mathbb{C}[[t_1, \ldots, t_l]] \cong \mathbb{C}[[t_1, \ldots, t_p]][[t_{p+1}, \ldots, t_l]].$$

Therefore, fix one such $p = 0, \ldots, l-1$. We write $\mathbf{t}$ for the vector $(t_{p+1}, \ldots, t_l)$. If $\mathbf{k}$ is a vector of integers $\mathbf{k} = (k_{p+1}, \ldots, k_l)$, we write $\mathbf{t}^{\mathbf{k}} := t_{p+1}^{k_{p+1}} \cdot \cdots \cdot t_l^{k_l}$. With this notation, the above embedding yields a (unique) expansion

$$z = \sum_{\mathbf{k} \in \mathbb{N}^{l-p}} \alpha_{\mathbf{k}} \mathbf{t}^{\mathbf{k}},$$

where $\alpha_{\mathbf{k}} \in \mathbb{C}[[t_1, \ldots, t_p]]$. Recall that

$$f\left(t_1, \ldots, t_l, \sum_{\mathbf{k} \in \mathbb{N}^{l-p}} \alpha_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}\right) = 0. \tag{5.1}$$

We now specialize the above expansion along the curve $\phi(\mathbb{P}_1)$ and pull it back to $\mathbb{P}_1$. Recall that the ring of functions on $\mathbb{P}_1$ that are regular at the origin can be embedded (uniquely) into $\mathbb{C}[[x]]$. Under this embedding, the specialization and the pullback simply mean that we specialize at $t_i = x^{n_i}$ term by term.

We first specialize at $t_i = x^{n_i}$ for $i = 1, \ldots, p$. Since $n_i \neq 0$ for all i, each series $\alpha_{\mathbf{k}}$ converges at $t_i = x^{n_i}$ to a series $\tilde{\alpha}_{\mathbf{k}} \in \mathbb{C}[[x]]$, and we have

$$f\left(x^{n_1}, \ldots, x^{n_p}, t_{p+1}, \ldots, t_l, \sum_{\mathbf{k} \in \mathbb{N}^{l-p}} \tilde{\alpha}_{\mathbf{k}} \mathbf{t}^{\mathbf{k}}\right) = 0. \tag{5.2}$$

Likewise, we can further specialize at $t_i = x^{n_i}$ for $i = p+1, \ldots, l$. We then obtain the expansion

$$g(x) = \sum_{\mathbf{k} \in \mathbb{N}^{l-p}} \tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k} \cdot \mathbf{n}}, \tag{5.3}$$

where $\mathbf{n} = (n_{p+1}, \ldots, n_l)$ and $\mathbf{k} \cdot \mathbf{n}$ denotes the usual scalar product. Note that such expansion is convergent again since $n_i \neq 0$ for all i.

Equation (5.3) yields an expansion of $g(x)$ resembling an analytic expansion, but with coefficients that are themselves functions of x, providing the first ingredient toward the proof of Proposition 4.3. We now use (5.1) and (5.2) to deduce some bounds on the coefficients $\alpha_{\mathbf{k}}$ and $\tilde{\alpha}_{\mathbf{k}}$.

Proposition 5.1. *The coefficients $\alpha_{\mathbf{k}}$ generate a finite extension of degree at most d of $\mathbb{C}(t_1, \ldots, t_p)$. Similarly, the coefficients $\tilde{\alpha}_{\mathbf{k}}$ generate a finite extension of degree at most d of $\mathbb{C}(x)$.*

Proof. Suppose that the coefficients generate either an algebraic extension of degree greater than d or a non-algebraic extension. In both cases, we can apply Galois automorphisms over $\mathbb{C}(t_1, \ldots, t_p)$ to find at least $d+1$ distinct sequences of coefficients. In turn, such automorphisms extend naturally to $\mathbb{C}[[t_1, \ldots, t_l]]$ by leaving $t_{p+1}, \ldots, t_l$ fixed. Applying the automorphisms to (5.1), we find that the degree of f in the last variable should be at least $d+1$, a contradiction. The conclusion for the coefficients $\tilde{\alpha}_{\mathbf{k}}$ can be proved with a similar argument applied to (5.2) (just recall that f is monic in y, so it does not become trivial when specializing $t_i = x^{n_i}$). $\square$

For $L \in \mathbb{N}$, let F_L be the field generated by $\{\alpha_{\mathbf{k}} : |\mathbf{k}| \leq L\}$ over $\mathbb{C}(t_1, \ldots, t_p)$, where $|\mathbf{k}|$ is the 1-norm of $\mathbf{k} \in \mathbb{N}^{l-p}$, and $F_\infty := \bigcup_{L \in \mathbb{N}} F_L$. By Proposition 5.1, $[F_\infty : \mathbb{C}(x)] \leq d$. Similarly, let $\tilde{F}_L$ be the field generated by $\{\tilde{\alpha}_{\mathbf{k}} : |\mathbf{k}| \leq L\}$ over $\mathbb{C}(x)$, and $\tilde{F}_\infty := \bigcup_{L \in \mathbb{N}} F_L$. Again, $[\tilde{F}_\infty : \mathbb{C}(x)] \leq d$. Let h be the logarithmic height of the function field $\tilde{F}_\infty/\mathbb{C}$ normalized so that $h(x) = [\tilde{F}_\infty : \mathbb{C}(x)] \leq d$.

Lemma 5.2. *Let $\mathbf{k} \in \mathbb{N}$. Let $q_{\mathbf{k}} \in \mathbb{C}[t_1, \ldots, t_p, y]$ be an irreducible polynomial such that*

$$q_{\mathbf{k}}(t_1, \ldots, t_p, \alpha_{\mathbf{k}}) = 0.$$

If $\alpha_{\mathbf{k}} \neq 0$, then the degree of $q_{\mathbf{k}}$ in each variable is at most $C_1 \cdot |\mathbf{k}|$ for a suitable $C_1 = C_1(d, l)$.

Proof. This is a classical result, although usually only stated for the case $p = 1$, and either with $l - p = 1$ or with additional assumptions on the derivative in y of the polynomial f. For the sake of completeness, we sketch an argument that reduces the general case to $p = 1$, $l - p = 1$.

Suppose $p = 1$ and $l - p = 1$. In this case, the vector $\mathbf{k}$ is just a single natural number $k \in \mathbb{N}$ and the degree of q_k in t_1 coincides with the logarithmic height of α_k in the function field $F_\infty/\mathbb{C}$ (upon choosing an appropriate normalization). Then the height of α_k is at most $C \cdot k$ for a suitable $C = C(d)$; moreover, there is a finite set of places S, whose size can be bounded in terms of d only, such that all the coefficients α_k are S-integral. (See for instance [15, Lemma V.5], with the additional observation that the order of $\frac{\partial f}{\partial y}$ at $x = 0$ can also be bounded in terms of d only.)

The general case can be reduced to the above special case by specializing $t_i \mapsto \xi_i$ for $i = 2, \ldots, p$, and $t_j \mapsto \chi_j t_l$ for $j = p+1, \ldots, l$, where $\xi_i, \chi_j \in \mathbb{C}$ are algebraically independent over the field of definition of f. If $\beta_{\mathbf{k}}$ is the specialization of $\alpha_{\mathbf{k}}$, the coefficient of t_l^k in the specialized expansion is

$$\gamma_k = \sum_{|\mathbf{k}|=k} \beta_{\mathbf{k}} \boldsymbol{\chi}^{\mathbf{k}},$$

where $\boldsymbol{\chi} = (\chi_{p+1}, \ldots, \chi_l)$. By the previous argument, the height of γ_k is bounded by $C \cdot k$, and γ_k is S-integral for a suitable S of size bounded in terms of d only. Since S is contained among the poles of the functions $\beta_{\mathbf{k}}$, upon varying $\boldsymbol{\chi}$ one recovers that S does not depend on $\boldsymbol{\chi}$. Using sufficiently many independent values of $\boldsymbol{\chi}$, one can eliminate $\boldsymbol{\chi}$ and obtain that the height of each $\beta_{\mathbf{k}}$ is bounded by $C \cdot |S| \cdot |\mathbf{k}|$. It now suffices to note that

$$r_{\mathbf{k}}(t_1, y) := q_{\mathbf{k}}(t_1, \xi_2, \ldots, \xi_p, y)$$

satisfies $r_{\mathbf{k}}(t_1, \beta_{\mathbf{k}}) = 0$, and it is the product of at most $\deg_y(r_{\mathbf{k}}) \leq d$ conjugate irreducible factors (using Proposition 5.1), so $\deg_{t_1}(r_{\mathbf{k}}) \leq Cd \cdot |S| \cdot |\mathbf{k}|$. Since the degree of $q_{\mathbf{k}}$ in t_1 coincides with $\deg_{t_1}(r_{\mathbf{k}})$, the conclusion follows at once. □

Proposition 5.3. *For all $L \in \mathbb{N}$, there is a primitive element α of $F_L/\mathbb{C}(t_1, \ldots, t_p)$ such that, if $q \in \mathbb{C}[t_1, \ldots, t_p, y]$ is an irreducible polynomial such that*

$$q(t_1, \ldots, t_p, \alpha) = 0,$$

then the degree of q in each variable is at most $C_2 = C_2(d, l) \cdot L$. Moreover, we may assume that q is monic in y.

Proof. By a classical argument of Galois theory, for sufficiently generic coefficients $\lambda_{\mathbf{k}} \in \mathbb{C}$, the element

$$\alpha = \sum_{|\mathbf{k}| \leq L} \lambda_{\mathbf{k}} \alpha_{\mathbf{k}}$$

generates F_L over $\mathbb{C}(t_1, \ldots, t_p)$. The conclusion then follows by Lemma 5.2 and some elementary algebra. □

Proposition 5.4. *For all $\mathbf{k} \in \mathbb{N}^{l-p}$, if $\tilde{\alpha}_{\mathbf{k}} \neq 0$, then either $h(\tilde{\alpha}_{\mathbf{k}}) \leq C_1 d^2 (n_1 + \cdots + n_p) \cdot |\mathbf{k}|$ or there are integers $h_1, \ldots, h_p \in \mathbb{Z}$, not all zero, such that $|h_i| \leq 2C_1 \cdot |\mathbf{k}|$ and $h_1 n_1 + \cdots + h_p n_p = 0$.*

Proof. By Lemma 5.2, we have

$$q_{\mathbf{k}}(x^{n_1}, \ldots, x^{n_p}, \tilde{\alpha}_{\mathbf{k}}) = \tilde{q}_{\mathbf{k}}(x, \tilde{\alpha}_{\mathbf{k}}) = 0,$$

where $q_{\mathbf{k}}$ has degree at most $C_1 \cdot |\mathbf{k}|$ in each variable. Therefore, the specialized polynomial $\tilde{q}_{\mathbf{k}}$ has degree at most $C_1(n_1 + \cdots + n_p) \cdot |\mathbf{k}|$ in x. If $\tilde{q}_{\mathbf{k}} \neq 0$, then

$h(\tilde{\alpha}_{\mathbf{k}})$ is bounded by the height of the polynomial $q_{\mathbf{k}}(x^{n_1},\dots,x^{n_p},y)$, and the first conclusion follows by an easy estimate of such height.

Otherwise, if $\tilde{q}_{\mathbf{k}}$ vanishes, then (at least) two distinct terms of $q_{\mathbf{k}}$ become terms of the same degree in x when specialized. This immediately implies the second conclusion. ☐

Corollary 5.5. *For all $L \in \mathbb{N}$, either the genus of $\tilde{F}_L$ is bounded by $C_3(n_1+\cdots+n_p)\cdot L$ for some $C_3 = C_3(d,l)$ or there are integers $h_1,\dots,h_p \in \mathbb{Z}$, not all zero, such that $|h_i| \le 2C_1 \cdot |\mathbf{k}|$ and $h_1n_1+\cdots+h_pn_p = 0$.*

Proof. This follows at once from the estimate of Proposition 5.4 and the basic theory of function fields (for instance, by counting the ramification points). ☐

Remark 5.6. Recall that $\tilde{F}_L = \tilde{F}_\infty$ for all sufficiently large integers L. One can prove that this happens for all integers L larger than a number dependent on d only, showing that the above bound can be made independent of L (as for Lemma 5.2, this is usually proven only with additional assumptions on f). However, we will not need this additional uniformity.

6. Diophantine approximation

Now that we have found a suitable expansion of $g(x)$ as a convergent sum of algebraic functions, we proceed as in [20]. Recall the following lemma.

Lemma 6.1 ([20, Proposition 1])**.** *Let $E/\mathbb{C}$ be a function field in one variable, of genus $\mathfrak{g}$, let $\varphi_1,\dots,\varphi_n \in E$ be linearly independent over $\mathbb{C}$, and let $r \in \{0,1,\dots,n\}$. Let S be a finite set of places of E containing all the poles of $\varphi_1,\dots,\varphi_n$ and also all the zeros of $\varphi_1,\dots,\varphi_r$. Further, put $\sigma = \sum_{i=1}^n \varphi_i$. Then*

$$\sum_{v\in S}(v(\sigma) - \min_{i=1}^{n} v(\varphi_i)) \le \binom{n}{2}(\#S + 2\mathfrak{g} - 2) + \sum_{i=r+1}^{n} \deg(\varphi_i), \tag{6.1}$$

where $\deg(\varphi_i) = [E : \mathbb{C}(\varphi_i)]$.

A rather straightforward application of the above lemma to (5.3) yields the following (using the notations of Section 5).

Proposition 6.2. *Suppose that $0 < n_1 \le \cdots \le n_l$ and that $n_{p+1} \ge \varepsilon n_l$ for some given $\varepsilon > 0$. Then at least one of the following holds:*

1. *$g(x)$ is $\mathbb{C}$-linearly dependent on the set $\{\tilde{\alpha}_{\mathbf{k}} : |\mathbf{k}| \le \lceil \frac{2d^2+1}{\varepsilon} \rceil\}$,*
2. *$n_p \ge \varepsilon' n_l$ for some $\varepsilon' = \varepsilon'(d,l,\varepsilon)$,*
3. *there are $h_1,\dots,h_p \in \mathbb{Z}$, not all zero, such that $|h_i| \le 2C_1L$ and $h_1n_1 + \cdots + h_pn_p = 0$.*

Proof. We apply Lemma 6.1 with the following data.

Let $L := \lceil \frac{2d^2+1}{\varepsilon} \rceil$. Let $\varphi_1,\dots,\varphi_r$ be a $\mathbb{C}$-linear basis of the set $\{\tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}} : |\mathbf{k}| \le L\}$, and let $\varphi_{r+1} = \varphi_n = -g(x)$ (note that r is at most the number of vectors $\mathbf{k} \in \mathbb{N}^{l-p}$ such that $|\mathbf{k}| \le L$, which can be bounded in terms of l and L only). Let $E := \tilde{F}_L$. Let S be the set of zeros and poles of the functions $\tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}}$ and of the function x. Let v_0 be the place at $x = 0$ induced by the embedding of E into $\mathbb{C}[[x]]$.

Thanks to the observations of Section 5, either we get conclusion (3) or we have the following bounds:

- the zeros and poles of each $\tilde{\alpha}_{\mathbf{k}}$ are at most $2C_1 d^2 p n_p L$ by Proposition 5.4;
- the zeros and poles of $x^{\mathbf{k}\cdot\mathbf{n}}$, which include the poles of $g(x)$, are at most $2[F_L : \mathbb{C}(x)] \leq 2d$ by Proposition 5.1;
- the genus of F_L is at most $C_3 p n_p L$ by Corollary 5.5;
- $v_0(\sigma) > L n_{p+1} \geq L\varepsilon n_l$ by the expansion (5.3).

Finally, note that for all $v \in S$, $v(\sigma) - \min_{i=1}^n v(\varphi_i)$ is non-negative, and that by simple degree considerations, $0 \leq v_0(g(x)), \deg_x(g(x)) \leq dn_l$. In particular, $\min_{i=1}^n(v_0(\varphi_i)) \leq dn_l$.

Applying Lemma 6.1 and using these bounds, we either reach conclusion (1), or we obtain the following inequality:

$$L\varepsilon n_l - dn_l \leq \binom{n}{2}(2C_1 d^2 p n_p L + 2d + 2C_3 p n_p L) + d^2 n_l,$$

which in turn yields

$$n_l \leq n_l \cdot \left(L\varepsilon - 2d^2\right) \leq \binom{n}{2}(2C_1 d^2 pL + 2d + 2C_3 pL) \cdot n_p,$$

proving conclusion (2). □

7. The case of linear dependence

Note that outcome (1) of Proposition 6.2 is that $g(x)$ is $\mathbb{C}$-linearly dependent on $\{\tilde{\alpha}_{\mathbf{k}} : |\mathbf{k}| \leq L\}$ for a certain $L \in \mathbb{N}$. In this section, we study what happens when this is the case.

Let $L \in \mathbb{N}$. Fix α to be a primitive element of $F_L/\mathbb{C}(t_1, \ldots, t_p)$ given by Proposition 5.3, with the corresponding irreducible polynomial $q \in \mathbb{C}[t_1, \ldots, t_p, y]$. Let also $e = [F_L : \mathbb{C}(t_1, \ldots, t_p)]$. Then for all $|\mathbf{k}| \leq L$ we can write

$$\alpha_{\mathbf{k}} = \sum_{i=0}^{e-1} q_{i,\mathbf{k}} \alpha^i,$$

where $q_{i,\mathbf{k}} \in \mathbb{C}(t_1, \ldots, t_p)$.

Lemma 7.1. *The degree of $q_{i,\mathbf{k}}$ in each variable is at most C_4 for some $C_4 = C_4(d, l, L)$.*

Proof. Let $\sigma : F_L \to \overline{\mathbb{C}(t_1, \ldots, t_p)}$ be an embedding of F_L into an algebraic closure of $\mathbb{C}(t_1, \ldots, t_p)$. Then

$$\sigma(\alpha_{\mathbf{k}}) = \sum_{i=0}^{e-1} q_{i,\mathbf{k}} \sigma(\alpha)^i.$$

Since the matrix $(\sigma(\alpha)^i)_{i,\sigma}$ is invertible, the desired bound follows from the bound of Proposition 5.3 and some elementary algebra. □

Proposition 7.2. *Suppose that $g(x)$ is $\mathbb{C}$-linearly dependent on $\{\tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}} : |\mathbf{k}| \leq L\}$. Assume that Proposition 4.3 is true for $l = p$. Then either $h_1 n_1 + \cdots + h_p n_p = 0$ for some integers $h_1, \ldots, h_p$, not all zero, such that $|h_i| \leq 2C_4$, or $g(x)$ can be written as the ratio of two polynomials with at most $C_5 = C_5(d, l, L)$ terms.*

Proof. Assume first that for some $\mathbf{k}$ with $|\mathbf{k}| \le L$, $q_{i,\mathbf{k}}(x^{n_1}, \dots, x^{n_p})$ is not well defined, by which we mean that some denominator of $q_{i,\mathbf{k}}$ vanishes on $(x^{n_1}, \dots, x^{n_p})$. In turn, two distinct terms of such a denominator must have the same degree when specialized, which means that

$$h_1 n_1 + \cdots + h_p n_p = h'_1 n_1 + \cdots + h'_p n_p$$

for some integers h_i, h'_i such that $|h_i|, |h'_i| \le C_4$ for all i, and $h_i \ne h'_i$ for at least one i. We thus reach the former conclusion.

Otherwise, we specialize α at $t_i = x^{n_i}$ for $i = 1, \dots, p$, which is possible since α is by construction integral over $\mathbb{C}[t_1, \dots, t_p]$. This yields a $\tilde{\alpha} \in \tilde{F}_L$. By the above argument, we can also specialize each $q_{i,\mathbf{k}}$, yielding the following:

$$\tilde{\alpha}_{\mathbf{k}} = \sum_{i=0}^{e-1} q_{i,\mathbf{k}}(x^{n_1}, \dots, x^{n_p}) \tilde{\alpha}^i.$$

In particular, $\tilde{\alpha}$ is a primitive element of $\tilde{F}_L$ over $\mathbb{C}(x)$.

By assumption of linear dependence, we have

$$g(x) = \sum_{|\mathbf{k}| \le L} \lambda_{\mathbf{k}} \tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}} = \sum_{|\mathbf{k}| \le L} \lambda_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}} \sum_{i=0}^{e-1} q_{i,\mathbf{k}}(x^{n_1}, \dots, x^{n_p}) \tilde{\alpha}^i$$

for some numbers $\lambda_{\mathbf{k}} \in \mathbb{C}$.

Let $[\mathbb{C}(x, \tilde{\alpha}) : \mathbb{C}(x)] = e' \le e$. If $e' = e$, then $\tilde{\alpha}^0, \dots, \tilde{\alpha}^{e-1}$ are $\mathbb{C}(x)$-linearly independent, so we must have

$$g(x) = \sum_{|\mathbf{k}| \le L} \lambda_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}} q_{0,\mathbf{k}}(x^{n_1}, \dots, x^{n_p}),$$

and the latter conclusion follows.

Otherwise, if $\tilde{\alpha}^0, \dots, \tilde{\alpha}^{e-1}$ are not $\mathbb{C}(x)$-linearly independent, it means that $[\mathbb{C}(x, \tilde{\alpha}) : \mathbb{C}(x)] = e' < e$. Note that $\tilde{\alpha}$ is a root of

$$q(x^{n_1}, \dots, x^{n_p}, \tilde{\alpha}) = 0.$$

By hypothesis, we may assume that Proposition 4.3 holds for $l = p$, and in particular we may apply Theorem 2.3. It follows that the irreducible factor of $q(x^{n_1}, \dots, x^{n_p}, y)$ of which $\tilde{\alpha}$ is a root has coefficients that can be written as ratios of polynomials with at most $B_5(d, p)$ terms. In turn, we may rewrite each power $\tilde{\alpha}^j$, for $j \ge e'$, as

$$\tilde{\alpha}^j = \sum_{i=0}^{e'-1} r_{i,j} \tilde{\alpha}^i,$$

where each $r_{i,j} \in \mathbb{C}(x)$ can be written as the ratio of two polynomials whose number of terms is bounded in terms of d and p only.

Therefore, we have

$$g(x) = \sum_{|\mathbf{k}| \le L} \lambda_{\mathbf{k}} x^{\mathbf{k}\cdot\mathbf{n}} \sum_{i=0}^{e'-1} \left(q_{i,\mathbf{k}}(x^{n_1}, \dots, x^{n_p}) + \sum_{j=e'}^{e} q_{j,\mathbf{k}}(x^{n_1}, \dots, x^{n_p}) r_{i,j} \right) \tilde{\alpha}^i.$$

Since $\tilde{\alpha}^0, \dots, \tilde{\alpha}^{e'-1}$ are $\mathbb{C}(x)$-linearly independent, we may now conclude as in the case $e' = e$. $\square$

8. Proof of the main theorem

Finally, we can prove Proposition 4.3. We shall then prove that it implies Theorem 1.2.

Proof of Proposition 4.3. First of all, we may directly assume that $0 < n_1 \leq \cdots \leq n_l$: indeed, we may simply rearrange $t_1, \ldots, t_l$ as required.

We then work by primary induction on $l \in \mathbb{N}^*$ and secondary reverse induction on $p = l-1, \ldots, 0$. Our inductive hypothesis at stage (l, p) reads as follows:

(1) either $g(x)$ is the ratio of two polynomials with at most $B_9(d, l, p)$ terms
(2) or $n_p \geq \varepsilon_p n_l$ for some $\varepsilon_p = \varepsilon_p(d, l)$.

Note that in the case $(l, p) = (1, 0)$, the conclusion is trivial: the expansion (5.3) is just

$$g(x) = \sum_{k \in \mathbb{N}} \tilde{\alpha}_k x^{k n_l},$$

where $\tilde{\alpha}_k \in \mathbb{C}$. Since the degree of g in x is at most dn_l, by simple degree considerations, $g(x)$ is a polynomial with at most $d+1$ terms, reaching conclusion (1).

Similarly, in the case $(l, 0)$, the expansion is of the type

$$g(x) = \sum_{\mathbf{k} \in \mathbb{N}^l} \tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k} \cdot \mathbf{n}},$$

where $\tilde{\alpha}_{\mathbf{k}} \in \mathbb{C}$. Since again the degree of g in x is at most dn_l, the only terms appearing on the right hand side satisfy $\mathbf{k} \cdot \mathbf{n} \leq dn_l$. On the other hand, $\mathbf{k} \cdot \mathbf{n} \geq |\mathbf{k}| n_1 \geq |\mathbf{k}| \varepsilon_0 n_l$. Therefore,

$$|\mathbf{k}| \leq \frac{d}{\varepsilon_0}.$$

In turn, this implies that the number of terms of $g(x)$ can be bounded in terms of l and $\varepsilon_0 = \varepsilon_0(d, l)$, so in terms of d and l only, reaching again conclusion (1).

We now wish to prove the general case for arbitrary (l, p). Assume either that $p = l-1$ or that we have already proven stage $(l, p+1)$. Proposition 6.2 yields three possible conclusions: in the first case, we obtain that $g(x)$ is $\mathbb{C}$-linearly dependent on $\{\tilde{\alpha}_{\mathbf{k}} x^{\mathbf{k} \cdot \mathbf{n}} : |\mathbf{k}| \leq L\}$ for a suitably chosen L; in the second case, we reach conclusion (2) straightaway; in the third case, we obtain that $h_1 n_1 + \cdots + h_p n_p = 0$ for some integers h_i of bounded size, in which case we reach conclusion (1) by Lemma 2.9 and the inductive hypothesis.

For the first case, we then apply Proposition 7.2, and either we reach conclusion (1) immediately or we find again that $h_1 n_1 + \cdots + h_p n_p = 0$ for integers h_i of bounded size, so we reach conclusion (1) again by Lemma 2.9 and the inductive hypothesis. This concludes our induction. □

Chasing back the series of deductions, this finally proves Theorem 2.2. The proof of Theorem 1.2 now follows the same argument found in [20].

Proof of Theorem 1.2. By Theorem 2.2, a rational function $g(x) \in \mathbb{C}(x)$ such that

$$f(x^{n_1}, \ldots, x^{n_l}, g(x)) = 0$$

can always be written as the ratio of two polynomials, say $g_1(x)$ and $g_2(x)$, with at most B_4 terms.

As in [20], we may exploit this information to show that we may explicitly parametrize all such polynomials g_1, g_2. Indeed, for $\mathbf{k} \in \mathbb{N}^l$, let $|\mathbf{k}|_\infty$ be the maximum absolute value of its entries; for $r = 1, 2$, if

$$g_r(x) = \sum_{k=1}^{B_4} b_{rk} x^{n_{rk}}$$

and, writing $\mathbf{t}^\mathbf{k}$ for $t_1^{k_1} \cdot \ldots \cdot t_l^{k_l}$,

$$f(t_1, \ldots, t_l, y) = \sum_{i=0}^{d} \sum_{|\mathbf{k}|_\infty \leq d} a_{i\mathbf{k}} y^i \mathbf{t}^\mathbf{k},$$

we have that

$$\sum_{i=0}^{d} \sum_{|\mathbf{k}|_\infty \leq d} a_{i\mathbf{k}} \left(\sum_{k=1}^{B_4} b_{1k} x^{n_{1k}} \right)^i \left(\sum_{k=1}^{B_4} b_{2k} x^{n_{2k}} \right)^{d-i} x^{\mathbf{n} \cdot \mathbf{k}} = 0. \tag{8.1}$$

We now expand all the involved products to get monomials of the shape γx^μ, where γ is a monomial in the coefficients $a_{i\mathbf{k}}$ and b_{rk}, and μ is a positive $\mathbb{Z}$-linear combination of the exponents n_{rk} and n_i. In order to satisfy (8.1), we can recognize two types of conditions.

(I) The first type concerns the exponents μ of x. We can partition the monomials γx^μ by grouping the ones with the same μ. For each set of the partition, the corresponding expressions of μ must have the same value, producing several vanishing homogeneous linear forms with integer coefficients in the n_i, n_{rk}. Note that the coefficients of such linear forms are bounded in terms of d only. Moreover, since the number of possible partitions is bounded in terms of d and l, there is a bound on the number of resulting linear equations.

(II) For a fixed partition of the monomials γx^μ with the same μ as in (I), the sum of their coefficients must be zero. This yields an affine algebraic variety whose coordinates correspond to the coefficients b_{rk}.

Each solution $g_1(x), g_2(x)$ of (8.1) yields a solution to a linear equation as in (I) and a point on the corresponding algebraic variety given in (II). Vice versa, each solution to a linear equation as in (I) and a point on the corresponding algebraic variety in (II) yield two polynomials $g_1(x), g_2(x)$ satisfying (8.1).

Suppose now that we fix a set of linear equations as in (I), given by a partition of the exponents, and a point in the algebraic variety found in (II), but we let the exponents n_{rk} vary among all the possible solutions. Since the (vector) solutions of such a system of linear equations span a subgroup of $\mathbb{Z}^{2B_4}$, we may in fact find a $\mathbb{Z}$-basis, say with $s \leq 2B_4$ elements, whose entries are bounded only in terms of d and l; we may then write each solution as linear combinations of these basis vectors, with integer coefficients $u_1, \ldots, u_s$. After this substitution, we may rewrite the resulting polynomials g_1 and g_2 as

$$g_r(x) = \tilde{g}_r(x^{u_1}, \ldots, x^{u_s}), \ r = 1, 2$$

and f as

$$f(x^{n_1}, \ldots, x^{n_l}, y) = \tilde{f}(x^{u_1}, \ldots, x^{u_s}, y),$$

where $\tilde{f}$, $\tilde{g}_1$, and $\tilde{g}_2$ are certain Laurent polynomials in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_s^{\pm 1}, y]$. Note that moreover their degrees are bounded in terms of the basis vectors and hence may be bounded in terms of d and l only.

Now, the equality

$$\tilde{f}\left(x^{u_1},\dots,x^{u_s},\frac{\tilde{g}_1(x^{u_1},\dots,x^{u_s})}{\tilde{g}_2(x^{u_1},\dots,x^{u_s})}\right)=0$$

is satisfied for *all* $u_1,\dots,u_s$ in $\mathbb{Z}$, and therefore we actually have that

$$\tilde{f}\left(z_1,\dots,z_s,\frac{\tilde{g}_1(z_1,\dots,z_s)}{\tilde{g}_2(z_1,\dots,z_s)}\right)=0.$$

Since $\tilde{f}$ is monic in y, this implies that $\frac{\tilde{g}_1}{\tilde{g}_2}$ is integral over $\mathbb{C}[z_1^{\pm 1},\dots,z_s^{\pm 1}]$, and therefore it is a Laurent polynomial in $\mathbb{C}[z_1^{\pm 1},\dots,z_s^{\pm 1}]$; moreover, the number of terms as a Laurent polynomial is bounded dependently on d and l because the degree of $\tilde{f}$ is likewise bounded.

Therefore, since any $g(x)$ satisfying (1.1) can be obtained using the above procedure, we have that $g(x)$ must be a Laurent polynomial in $\mathbb{C}[x^{\pm 1}]$ with a number of terms bounded dependently on d and l. Now, since $g(x)$ is integral over $\mathbb{C}[x]$, then all of its monomials have non-negative degree, and therefore it is a polynomial with a bounded number of terms, as desired. $\square$

9. Proofs of the remaining assertions

From Theorem 2.2 we can now deduce the various statements given in Section 1 with a relatively small effort.

We first prove Theorem 1.3 and its Corollary 1.4 on integral points, i.e., regarding the regular maps $\rho:\mathbb{G}_{\mathrm{m}}\to W$ for a given finite cover $W\to\mathbb{G}_{\mathrm{m}}^l$.

Proof of Theorem 1.3. We first note that it suffices to prove the conclusion for a finite set of regular functions y on W. Therefore, we may assume that W may be represented as the hypersurface $f(t_1,\dots,t_l,y)=0$, where f is monic in y, it is Laurent in the t_i's, and π is the projection onto the first l coordinates.

With this proviso, we go to the proof. A regular map $\rho:\mathbb{G}_{\mathrm{m}}\to W$ may be represented in the form $x\mapsto(\theta_1x^{m_1},\dots,\theta_rx^{m_l},g(x))$, where $\theta_i\in\mathbb{C}^*$, $m_i\in\mathbb{Z}$, and $g\in\mathbb{C}[x,x^{-1}]$. Thus $f(\theta_1x^{m_1},\dots,\theta_rx^{m_l},g(x))=0$.

By Theorem 2.2, and using the same argument of the proof of Theorem 1.2, we see that each choice of the coefficients θ_i and of the polynomial $g(x)$ corresponds to an integer solution of a system of linear equations (I) and to a point on an algebraic variety (II).

Now, for each system (I), let s be the rank of its solution space, and let V be the corresponding algebraic variety (II). By construction, we obtain a map $\psi: V\times\mathbb{G}_{\mathrm{m}}^s\to W$. The above comment on θ_i and $g(x)$ implies that there is map $\gamma:\mathbb{G}_{\mathrm{m}}\to\mathbb{G}_{\mathrm{m}}^s$, given by the solution of the system (I) corresponding to $g(x)$, and a point $\xi\in V$ corresponding to the coefficients of $g(x)$ and the θ_i's, such that in fact $\rho=\psi_\xi\circ\gamma$. Since the number of possible systems, and therefore of maps ψ, is bounded in terms of d and l, this yields the desired conclusion. $\square$

Proof of Corollary 1.4. We first remark a few things about the conclusion of Theorem 1.3. First, we observe that since a regular map from $\mathbb{G}_{\mathrm{m}}^s$ to $\mathbb{G}_{\mathrm{m}}$ is a monomial, each $\pi\circ\psi: V\times\mathbb{G}_{\mathrm{m}}^s\to\mathbb{G}_{\mathrm{m}}^l$ is of the shape $\{\xi\}\times(z_1,\dots,z_s)\mapsto(c_1(\xi)\mu_1,\dots,c_l(\xi)\mu_l)$, for non-vanishing functions c_i on V and pure monomials μ_i in the z_j. Also, since the map $(z_1,\dots,z_s)\mapsto(\mu_1,\dots,\mu_l)$ is a homomorphism, after an automorphism of $\mathbb{G}_{\mathrm{m}}^s$ it factors as a projection $\mathbb{G}_{\mathrm{m}}^s \to \mathbb{G}_{\mathrm{m}}^t$ times a homomorphism with a finite kernel; hence, $t \leq l$, and we may in fact take $s = t \leq l$. (Indeed, the map $\psi : V \times \mathbb{G}_{\mathrm{m}}^t \times \mathbb{G}_{\mathrm{m}}^{s-t} \to W$ sends $\{\xi\} \times \{\eta\} \times \mathbb{G}_{\mathrm{m}}^{s-t}$ to a fiber of π, which is finite; hence this image is constant, and we may remove $\mathbb{G}_{\mathrm{m}}^{s-t}$ from the picture.)

Then, after pullback of π by an isogeny, we may assume that $\mathbb{G}_{\mathrm{m}}^t$ embeds in $\mathbb{G}_{\mathrm{m}}^l$ on the first t coordinates. Therefore, we can see that the map ψ yields a family of translates of $\mathbb{G}_{\mathrm{m}}^t$ parametrized by V, and corresponding regular sections of π over each of them.

Turning back to the proof, we note that the hypothesis combined with Theorem 1.3 imply immediately that one of the maps $\psi \in \Psi$ is dominant. Therefore, the composition $\pi \circ \psi : V \times \mathbb{G}_{\mathrm{m}}^s \to \mathbb{G}_{\mathrm{m}}^l$ is regular and dominant, and (by the previous remarks) we may even suppose that it is expressed in the shape $\pi \circ \psi(\{\xi\} \times (z_1, \ldots, z_s)) = (c_1(\xi)z_1, \ldots, c_s(\xi)z_s, c_{s+1}(\xi)\mu_{s+1}, \ldots, c_l(\xi)\mu_l)$ where μ_i are monomials in $z_1, \ldots, z_s$, $c_1, \ldots, c_l$ are non-vanishing regular functions on V, and $s \geq 1$.

If we fix a point $\xi \in V$, the restriction of $\pi \circ \psi$ to $\{\xi\} \times \mathbb{G}_{\mathrm{m}}^s$ is an isogeny, and therefore unramified. We define $R \subset W$ as the ramification divisor of π, and $S = \pi(R) \subset \mathbb{G}_{\mathrm{m}}^l$ as the branch locus. Let, for $z \in \mathbb{G}_{\mathrm{m}}^{l-s}$, $K_z := \pi^{-1}(\mathbb{G}_{\mathrm{m}}^s \times \{z\})$. Note that K_z may be reducible, even for all z. However, the image of $\pi \circ \psi$ restricted to $\{\xi\} \times \mathbb{G}_{\mathrm{m}}^s$ is of the shape $\mathbb{G}_{\mathrm{m}}^s \times \{\phi(\xi)\}$ (where ϕ is a certain regular map $\phi : V \to \mathbb{G}_{\mathrm{m}}^{l-s}$), and the map is essentially an isogeny and is finite. Then we have that $\psi(V \times \mathbb{G}_{\mathrm{m}}^s) \cap K_z$ consists of a finite union of components C of K_z such that $\pi(C) = \mathbb{G}_{\mathrm{m}}^s \times \{z\}$.

Since ψ is dominant, it follows easily (by counting dimensions) that $\psi(V \times \mathbb{G}_{\mathrm{m}}^s)$ can miss a whole component of K_z only for z in a proper closed subset E of $\mathbb{G}_{\mathrm{m}}^{l-s}$. On the other hand, since the said map is essentially an isogeny, R cannot meet its image, so $R \cap K_z$ is contained in the components missed by $\psi(V \times \mathbb{G}_{\mathrm{m}}^s) \cap K_z$.

Therefore $R \cap K_z$ can be nonempty only for $z \in E$, and then the projection of S to $\mathbb{G}_{\mathrm{m}}^{l-s}$ is contained in E. Since S has pure codimension 1 in $\mathbb{G}_{\mathrm{m}}^l$, it follows that S is a union of cosets of $\mathbb{G}_{\mathrm{m}}^s$ and is therefore invariant by multiplication by $\mathbb{G}_{\mathrm{m}}^s$. $\square$

The proof of the toric version of Bertini's Theorem 1.5 follows a similar pattern.

Proof of Theorem 1.5. Let us assume first that W is representable as an open dense subset of the hypersurface $f(t_1, \ldots, t_l, y) = 0$, where f is an irreducible complex polynomial, and π is the projection onto the first l coordinates.

Let us analyze a factorization $f(\theta_1 x^{n_1}, \ldots, \theta_l x^{n_l}, y) = g(x, y)h(x, y)$ with integers n_i and polynomials (Laurent in x) g, h, monic in y. By Theorem 2.2, and proceeding as in the proof of Theorem 1.2, we can see that the pairs g, h correspond to solutions of suitable linear systems (I) and to points on the corresponding affine algebraic varieties (II).

Now, fix a system (I) and a point on the algebraic variety (II). As before, if s is the rank of the solution space, we can easily obtain the following factorization:

$$f(\theta_1\mu_1, \ldots, \theta_l\mu_l, y) = \tilde{g}(z_1, \ldots, z_s, y)\tilde{h}(z_1, \ldots, z_s, y), \tag{9.1}$$

where $\mu_1, \ldots, \mu_l$ are (Laurent) monomials in $z_1, \ldots, z_s$ and $\tilde{g}$, $\tilde{h}$ are polynomials (Laurent in the z_i's) and monic in y.

Now, suppose the monomials $\mu_1, \dots, \mu_l$ are multiplicatively independent. This means that the homomorphism $\phi : \mathbb{G}_{\mathrm{m}}^s \to \mathbb{G}_{\mathrm{m}}^l$ given by $\phi(z_1, \dots, z_s) = (\mu_1, \dots, \mu_l)$ is surjective. By simple general theory, it must factor as a composition of a projection $\mathbb{G}_{\mathrm{m}}^l \times \mathbb{G}_{\mathrm{m}}^{s-l} \to \mathbb{G}_{\mathrm{m}}^l$ and an isogeny ψ of $\mathbb{G}_{\mathrm{m}}^l$. But then the identity (9.1) shows that the pullback $\psi^* W$ is reducible; now, it is known and not too difficult to prove that this implies that $[e]^* W$ is already reducible (see [21], Proposition 2.1), against the assumptions.

Therefore, we may assume that in all cases the sets of monomials μ_i so obtained are multiplicatively dependent; hence they satisfy an identical relation $\mu_1^{e_1} \cdots \mu_l^{e_l} = 1$ for integer exponents e_i, not all zero and depending only on the linear form chosen in (I). In particular, the vector $(e_1, \dots, e_l)$ takes altogether only finitely many values.

Since the μ_i's are pure monomials in the z_h, we may assume that the e_i's are coprime. The multiplicative relation defines a certain proper connected algebraic subgroup E of $\mathbb{G}_{\mathrm{m}}^l$, while the corresponding factorization implies that $\pi^{-1}(\theta E)$ is reducible for $\theta = (\theta_1, \dots, \theta_l)$. Therefore, the original one-dimensional torus parametrized by $(x^{n_1}, \dots, x^{n_l})$ is contained in E. We now let $\mathcal{E}$ be the union of all finitely many sub-tori E which arise in this way. Note that $\mathcal{E}$ can be chosen dependently only on $\deg(f)$.

Now, assume that $\pi^{-1}(\theta H)$ is reducible, for a certain $\theta \in \mathbb{G}_{\mathrm{m}}^l$ and a certain torus H of dimension $t \geq 1$. If $(u_1, \dots, u_t) \mapsto (\nu_1, \dots, \nu_l)$ is a parametrization of H by monomials ν_i in the u_h, then the polynomial $f(\theta_1 \nu_1, \dots, \theta_l \nu_l, y) = 0$ is reducible (over $\mathbb{C}(u_1, \dots, u_t)$). Hence, simply by specialization, the polynomial $f(\theta_1 x^{n_1}, \dots, \theta_l x^{n_l}, y)$ must be reducible for all integer vectors $(n_1, \dots, n_l)$ such that the torus $(x^{n_1}, \dots, x^{n_l})$ is contained in H. But then any such torus must be contained in some E as above; it is now easy to see that H itself must be contained in $\mathcal{E}$, proving the desired conclusion.

To complete the proof, consider a general quasi-projective variety W. After replacing W with $W \setminus X$ for a suitable proper subvariety X, we may assume that $\pi : W \to \mathbb{G}_{\mathrm{m}}^l$ is finite onto its image. We note that we may cover W with finitely many (open dense) affine charts, such that for any two points of W there is a chart containing both of them; since π is finite over its image, we may further assume that each chart can be represented as an open dense subset of the hypersurface $f(t_1, \dots, t_l, y) = 0$ for some f. We then observe that if $\pi^{-1}(\theta H)$ has at least two irreducible components, for some subgroup $H < \mathbb{G}_{\mathrm{m}}^l$ and some $\theta \in \mathbb{G}_{\mathrm{m}}^l$, then there is at least one affine chart intersecting both components, and the conclusion follows by the previous case. □

Finally, the only remaining statement is the analogue for composite rational functions of Schinzel's conjecture, namely Theorem 1.7.

Proof of Theorem 1.7. Let l be given and let $f(x) = g(h(x))$ be as in the statement. We write $f(x) = P(x)/Q(x)$ with $P(x) = p_1 x^{n_1} + \cdots + p_l x^{n_l}, Q(x) = q_1 x^{n_1} + \cdots + q_l x^{n_l}, P(x), Q(x) \in \mathbb{C}[x]$. If we put $d = 2016 \cdot 5^l$, we know by the main theorem of [12] that $\deg g \leq d$ unless we are in the exceptional situation of that theorem, where our statement is trivially true. Therefore we may write $g(x) = A(x)/B(x)$ where $A(x) = a_0 + a_1 x + \cdots + a_d x^d, B(x) = b_0 + b_1 x + \cdots + b_d x^d$ are two (coprime) polynomials in $\mathbb{C}[x]$. From $f(x) = g(h(x))$ we therefore get

$$A(h(x))Q(x) - B(h(x))P(x) = 0.$$

We then define

$$f(t_1,\ldots,t_l,y)=A(y)(q_1t_1+\cdots+q_lt_l)-B(y)(p_1t_1+\cdots+p_lt_l)\in\mathbb{C}[t_1,\ldots,t_l,y].$$

This is a polynomial of degree at most d in each variable. An application of Theorem 2.2 shows at once that there exists a number $B_2=B_2(l)=B_4(d,l)$ such that $h(x)\in\mathbb{C}(x)$, which satisfies $f(x^{n_1},\ldots,x^{n_l},h(x))=0$, is the ratio of two polynomials in $\mathbb{C}[x]$ with most B_2 terms, as desired. □

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