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ON THE VANISHING RATE OF SMOOTH CR FUNCTIONS

GIUSEPPE DELLA SALA AND BERNHARD LAMEL

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Abstract. Let M be a lineally convex hypersurface of $\mathbb{C}^n$ of finite type, $0 \in M$. Then there exist non-trivial smooth CR functions on M that are *flat* at 0, i.e. whose Taylor expansion about 0 vanishes identically. Our aim is to characterize the rate at which flat CR functions can decrease without vanishing identically. As it turns out, non-trivial CR functions cannot decay arbitrarily fast, and a possible way of expressing the critical rate is by comparison with a suitable exponential of the modulus of a local peak function.

1. Introduction and statement of results

Let $M \subset \mathbb{C}^N$ be a smooth hypersurface containing 0. We recall that the space of germs of CR functions at 0, which we denote by $C^\infty_{CR}(M,0)$, is the space of germs at 0 of smooth funtions on M which are annihilated by the CR vector fields. In a recent paper [1] (for the general case of integrable structures see [4]) we showed that if a peak function at 0 exists, then the "Borel map"

$$T_0 \colon C^\infty_{CR}(M,0) \to \mathbb{C}[[Z_1,\dots,Z_N]]$$

is onto (and possesses a continuous inverse). It is a natural question to determine the kernel of T_0, i.e. describe (germs of) flat CR functions. In this paper, we shall find a critical rate of decay for such flat functions for the case of a lineally convex hypersurface.

In order to introduce our main result, we first discuss a particular example. Let M denote the Lewy hypersurface, given as

$$\operatorname{Im} w = \|z\|^2.$$

It is well known that every CR function α defined near 0 on M extends holomorphically to a one-sided neighbourhood of $0 \in M$; i.e. $\alpha(z,w)$ is a holomorphic function for $\operatorname{Im} w > \|z\|^2$. If $|\alpha(z,w)| < Ae^{-\frac{\lambda}{|w|}}$ for some constants $A > 0$, $\lambda > 0$ and (z,w) belonging to a neighborhood of 0 in M, then we say that α decreases exponentially of order 1. The maximum principle implies that the extension of α satisfies the same kind of estimate. A classical theorem in (1-dimensional) complex analysis known as Watson's Lemma (see Lemma 1) tells us that a function decreasing in

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that way in a half-plane is necessarily 0. Hence $\alpha(0,w)=0$; by induction one can show that $D_z^\beta\alpha(0,w)=0$ for every derivative in the z-directions, and thus we conclude that $\alpha=0$. On the other hand, the functions

$$e^{-\left(\frac{i}{w}\right)^\gamma}, \quad \gamma<1,$$

give examples of functions decaying "of order γ", i.e. like $e^{-1/|w|^\gamma}$ for $\gamma<1$. Our goal in this paper is to generalize this observation to lineally convex hypersurfaces.

Let $M\subset\mathbb{C}^{n+1}$ be a smooth hypersurface, given in coordinates $(z_1,\dots,z_n,w)=(z,w)$ near 0 by a defining function

$$\operatorname{Im} w = h(z,\operatorname{Re} w) = f(z) + (\operatorname{Re} w)g(z,\operatorname{Re} w), \tag{1}$$

where f,g are smooth functions such that $f(0)=0$, $g(0,0)=0$. Recall that M is *lineally convex, of finite order* $k\geq 2$ if $f(z)\geq |z|^k$ for z in a neighborhood of 0. Note that a lineally convex hypersurface is of finite commutator type.

We will need a bit more general notion: If $\mathcal{C}\subset T_0^c(M)=\{w=0\}\cong\mathbb{C}_z^n$ is an open cone, we say that M is *lineally convex along* $\mathcal{C}$ *of finite order* k if $f(cv)\geq|c|^k$ for all $v\in\mathcal{C}$ with $|v|=1$ and c in a neighborhood of 0 in $\mathbb{C}$.

Equivalently, for any $v\in\mathcal{C}$ and $\mathbb{C}_v^2=\operatorname{span}\langle v,\partial/\partial w\rangle$, the manifold $M_v=\mathbb{C}_v^2\cap M$ is a lineally convex hypersurface of $\mathbb{C}^2$ of order at most k.

Let $\psi=-iw_{|_M}$. If M is lineally convex of finite order k, then ψ is a smooth, CR peaking function of finite type at 0 for M, in the sense specified in [1]. If M is lineally convex along the cone $\mathcal{C}$, the restriction of ψ to M_v for any $v\in\mathcal{C}$ is a CR peaking function for M_v. Our aim in this paper is to understand the conditions on the order to which a smooth CR function defined on M can vanish at 0 without vanishing identically. These conditions will be expressed in terms of a comparison with the behavior of ψ: we will show that (in a sense to be made precise below) the function $e^{-1/|\psi|}$ represents the critical rate of decrease for CR functions; that is, we will prove that any CR function that decreases at that speed must vanish, while there exist many non-trivial ones which decrease to a rate "closely" approaching $e^{-1/|\psi|}$.

Let us start by giving a precise meaning to "decreasing like $e^{-1/|\psi|}$":

Definition 1. We say that a function φ, defined in a neighborhood of 0 in M, is *exponentially decreasing of order* 1 at 0 if there exist $A,\lambda>0$ and a neighborhood $\mathcal{V}$ of 0 in M such that

$$|\varphi(p)|\leq Ae^{-\frac{\lambda}{|\psi(p)|}}, \quad p\in\mathcal{V}.$$

Equivalently, we can require that for a certain $\lambda>0$ we have

$$|\varphi(p)|e^{\frac{\lambda}{|\psi(p)|}}\to 0 \text{ as } p\to 0, \quad p\in M.$$

More generally, given an open cone $\mathcal{C}\subset T_0^c(M)$ we say that φ is *exponentially decreasing (of order* 1*) along* $\mathcal{C}$ if the restriction of φ to M_V is exponentially decreasing of order 1 for any $v\in\mathcal{C}$.

Next, we introduce a class of functions $\beta:\mathbb{R}^+\to\mathbb{R}^+$ for which we are able to show the existence of non-trivial CR functions that decrease at least as fast as $e^{-\beta(|\psi|)}$. Essentially, what we ask is that β be integrable in a neighborhood of 0; for technical reasons, we also need some condition controlling the behavior of the first derivative.

Definition 2. Consider a concave function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ of class C^1 having the following properties:

(1) $\int_0^1 \frac{\gamma(t)}{t} dt < +\infty$, and thus, in particular, $\gamma(t) \to 0$ for $t \to 0$;
(2) $t\gamma'(t)$ is monotone increasing.

We call $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ *admissible* if $\beta(t) = \gamma(t)/t$ for such a γ. For example, $\beta(t) = 1/t^a$ (corresponding to $\gamma(t) = t^{1-a}$) is an admissible function for all $0 < a < 1$.

Remark 1. The set of admissible functions thus defined is a convex cone. In the following, with no loss of generality, we will assume (up to, for example, replacing $\beta(t)$ with $\beta(t) + 1/\sqrt{t}$) that $\beta(t) \geq 1/\sqrt{t}$.

If $\beta(t)$ is admissible, then for $t \to 0$ it is diverging at a much slower rate than $1/t$, i.e. $\beta(t)/(1/t) \to 0$. We note, however, that the admissible set contains functions that diverge faster than $1/t^a$ for all $0 < a < 1$. Indeed, if we define

$$\gamma(t) = \sum_{\ell=1}^{\infty} \frac{1}{2^{2\ell}} t^{1/2^\ell},$$

then $\beta(t) = \gamma(t)/t$ has such a property.

Theorem 1. *Let M be lineally convex of finite order k along an open cone $\mathcal{C} \subset T_0^c(M)$, and let φ be a CR function of class C^1, defined on a neighborhood of* 0 *in M, which is exponentially decreasing of order* 1 *along $\mathcal{C}$. Then $\varphi \equiv 0$.*

On the other hand, let M be lineally convex of finite order (i.e. $\mathcal{C} = T_0^c(M)$) and let $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ be an admissible function. Then there exists a non-trivial CR function η of class C^∞, defined on a neighborhood $\mathcal{V}$ of 0 *in M, such that*

$$|\eta(p)| \leq e^{-\beta(|\psi(p)|)}$$

for all $p \in \mathcal{V}$.

For the proof of Theorem 1, we follow the line of argument already used in the introductory example. In order to overcome the additional difficulties from the more general geometry considered here, we will adapt Watson's Lemma to suitable domains (see Corollary 1). In order to prove the second part of the theorem, we will introduce a certain form of enveloping product domain of sufficient smoothness in Lemma 6. Before we start with the preparations, let us give some additional remarks.

Remark 2. Note that in particular, Theorem 1 tells us that the critical rate of decay as measured by the peaking function ψ is 1, as observed in the case of the Lewy hypersurface above.

Remark 3. One might wonder whether the lineal convexity assumption is actually needed in Theorem 1; our techniques do require this assumption at the moment. It is also easy to see that some kind of convexity assumption (or peaking) is needed for the existence of flat CR functions. For the validity of the first part of the theorem, we conjecture that minimality of M is sufficient for the validity of a generalized Watson Lemma.

2. Preparations

We will show that the statement of Theorem 1 is a consequence of certain results in one complex variable, which we review here.

2.1. **Watson's Lemma.**

A sector $S \subset \mathbb{C}$ is a set of the form

$$S = \{z \in \mathbb{C}\colon \alpha < \arg z < \beta\}.$$

We say that f is a germ of a holomorphic function on S if there exists a neighborhood U of 0 such that f is holomorphic on $S \cap U$. A germ f of a holomorphic function on S decreases exponentially of order k if there exist $C, \lambda > 0$,

$$|f(z)| < Ce^{-\frac{\lambda}{|z|^k}},$$

in a neighborhood of 0. In comparing growth rates on sectors, one has a choice of fixing the growth rate and comparing on closed subsectors or fixing the sector and comparing with a strictly greater rate. We shall choose to follow the second path here. Watson's Lemma gives an exact bound of the maximum order of decrease of a non-zero germ on a sector S. This bound is in terms of an exponential rate of decrease whose order k depends on the opening angle of the sector. For simplicity, though, we state the lemma for a half-plane.

Lemma 1 (Watson's Lemma)**.** *Let S be a half-plane of $\mathbb{C}$ and let f be a germ of a holomorphic function on S which is exponentially decreasing of order* 1*. Then $f \equiv 0$.*

For the proof of this version of Watson's Lemma we refer to [5].

Remark 4.

i) There exists a plethora of functions which decrease exponentially of order 1 on any proper subsector of the half-plane S, e.g. the exponentials $e^{-\mu/z}$. In this sense, Lemma 1 is sharp.
ii) More generally, if $S_{\alpha,\beta} = \{z\colon \alpha < \arg z < \beta\}$ is a sector of opening angle $\beta - \alpha$, then any germ of a function decreasing exponentially of order $(\beta - \alpha)/\pi$ near 0 vanishes identically.

2.2. **Smooth extension of Riemann maps.**

We will also need some results about the behavior of holomorphic maps at the boundary, in particular, regarding the extension of the first derivative.

Definition 3. Let φ be a vector-valued, uniformly continuous function defined on some domain $A \subset \mathbb{R}^n$. Its *modulus of continuity* $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as

$$\omega(\delta) = \sup\{|\varphi(x) - \varphi(y)| : x, y \in A, |x - y| \le \delta\}$$

for all $\delta \in \mathbb{R}^+$. The function φ is called *Dini-continuous* if $\int_0^1 \omega(t)/t < +\infty$. In particular, if φ is α-Hölder for some positive α, then it is Dini-continuous.

If $A \subset \mathbb{R}$ and φ is of class C^1, we say that it is *Dini-smooth* if its derivative φ' is Dini-continuous. Accordingly, we call a (1-dimensional) curve in $\mathbb{C}$ Dini-smooth if it admits a Dini-smooth parametrization.

The Riemann map of a simply connected domain whose boundary is Dini-smooth is C^1 up to the boundary. This can be derived from the following result (see [2, Theorem 3.5]):

Theorem 2. *Let $\mathcal{R}$ map the unit disc D conformally onto the inner domain Ω of a Dini-smooth Jordan curve. Then $\mathcal{R}'$ extends continuously to $\overline{D}$. Moreover, the extension of $\mathcal{R}'$ is non-vanishing on bD.*

Corollary 1. *Let Ω be a (simply connected) domain whose boundary is Dini-smooth, $0 \in b\Omega$, and $f \in \mathcal{O}(\Omega)$ be exponentially decreasing of order greater than 1 at 0. Then $f \equiv 0$.*

Proof. Let H be the half-plane and let $\mathcal{R}$ be a Riemann mapping $H \to \Omega$ such that $\mathcal{R}(0) = 0$. By Theorem 2, $\mathcal{R}$ is C^1 up to the boundary and $\mathcal{R}'(0) \neq 0$. In particular, for some constant $C > 0$ we have $|\mathcal{R}(z)| \leq C|z|$ for all z in a neighborhood of 0 in H. Since f is exponentially decreasing of order 1 in Ω, for some $A, \lambda > 0$,

$$|f(\mathcal{R}(z))| \leq Ae^{-\frac{\lambda}{|\mathcal{R}(z)|}} \leq e^{-\frac{\lambda/C}{|z|}}$$

when z is close enough to 0. This implies that $f \circ \mathcal{R}$ is exponentially decreasing of order 1 in H. Applying Lemma 1 we conclude that $f \circ \mathcal{R} \equiv 0$, and hence $f \equiv 0$. ☐

3. Proof of the main theorem

3.1. **Vanishing of CR functions of exponential decay.** We will need the following lemma:

Lemma 2. *Let M be lineally convex, and let (z, w) be coordinates for $\mathbb{C}^{n+1}$ in which M can be expressed as in (1). For a small neighborhood U of 0 in $\mathbb{C}^{n+1}$ let $U_1 = U \cap \{\operatorname{Im} w > h(z, \operatorname{Re} w)\}$, $U_2 = U \setminus \overline{U}_1$. Then for $c \in \mathbb{C}$ close enough to 0 we have that $\{w = c\} \cap U_2 \neq \emptyset$ and $U \cap \{w = c\} \cap M$ is compact.*

Proof. Choose a small $\epsilon > 0$ and let $S_\epsilon = \{z \in \mathbb{C}^n : |z| = \epsilon\}$; moreover, let $M_\epsilon = \max\{|g(z, \operatorname{Re} w)| : z \in S_\epsilon, |\operatorname{Re} w| \leq \epsilon\}$. If we choose $\delta < \epsilon^k/2M_\epsilon$ for all w such that $|\operatorname{Re} w| < \delta$, $\operatorname{Im} w < \epsilon^k/2$ we get by (1)

$$\operatorname{Im} w - h(z, \operatorname{Re} w) = \operatorname{Im} w - f(z) - \operatorname{Re} w \cdot g(z, \operatorname{Re} w) < \epsilon^k/2 - \epsilon^k + \delta M_\epsilon < 0$$

for all $z \in S_\epsilon$, which implies the first claim. For the second one, assume that U is chosen in such a way that $\overline{U} \cap M \cap \{w = 0\} = \{0\}$, and suppose that for a sequence $c_j \to 0$ of complex numbers we have $p_j \in bU \cap M \cap \{w = c_j\} \neq \emptyset$. Up to a subsequence, by continuity we have $p_j \to p \in bU \cap M \cap \{w = 0\}$, a contradiction. Thus, for small enough c the compact subset $\overline{U} \cap \{w = c\} \cap M$ is contained in U, from which follows the second claim. ☐

We turn now to the proof of the first claim of Theorem 1. We first observe that we can reduce to the case of $n = 1$. Indeed, let $v \in \mathcal{C} \subset T_0^c(M)$, and let $\mathbb{C}_v^2 = \operatorname{span}\{\partial/\partial w, v\}$. Then $M_v = M \cap \mathbb{C}_v^2$ is a lineally convex hypersurface of finite type of $\mathbb{C}_v^2$, and $\varphi_v = \varphi_{|M_v}$ is an exponentially decreasing CR function. It is enough to prove that $\varphi_v \equiv 0$ for all $v \in \mathcal{C}$, i.e. that φ vanishes on the set $\bigcup_{v \in \mathcal{C}} M_v$, which contains an open subset of M. Since M is minimal, this implies that φ vanishes identically.

Then let U be the open subset of $\mathbb{C}^2$ defined by $\{\operatorname{Im} w > h(z, \operatorname{Re} w)\}$. By well-known results, φ extends to a holomorphic function $\widetilde{\varphi}$ defined on U and smooth up to the boundary. For any $a \in \mathbb{C}$ lying in a small neighborhood of 0 we let

$\mathcal{U}_a = U \cap \{z = aw\}$; then $\mathcal{U}_a$ can be identified with the domain of $\mathbb{C}_w$ with smooth boundary which is defined by $\{\operatorname{Im} w > h(aw, \operatorname{Re} w)\}$.

We are going to show that the restriction of $\widetilde{\varphi}$ to each $\mathcal{U}_a$ vanishes identically. This is sufficient to conclude that $\widetilde{\varphi} \equiv 0$ (hence $\varphi \equiv 0$) by analytic continuation, because the union of the $\mathcal{U}_a$ has non-empty interior in $\mathbb{C}^2$. So let a be fixed; for any $c \in \mathcal{U}_a \subset \mathbb{C}_w$, we define $\gamma_c = M \cap \{w = c\}$. By Lemma 2, γ_c is a compact, non-empty set if c is small enough. By the maximum principle, for any $w_0 \in \mathcal{U}$ we have

$$|\widetilde{\varphi}(aw_0, w_0)| \leq \max_{z \in \gamma_{w_0}} |\varphi(z, w_0)|.$$

Since the maximum distance of the points of γ_{w_0} to the origin approaches 0 as $w_0 \to 0$, for a suitable $\lambda > 0$ we have

$$|\widetilde{\varphi}(aw_0, w_0)| e^{\frac{\lambda}{|w_0|}} \leq \max_{z \in \gamma_{w_0}} |\varphi(z, w_0)| e^{\frac{\lambda}{|w_0|}} \to 0$$

as $w_0 \to 0$, where we have used the fact that φ is exponentially decreasing of order 1. By Corollary 1, then, it follows that $\widetilde{\varphi}(aw_0, w_0) \equiv 0$.

3.2. **Existence of non-trivial CR functions with admissible decay.** Now, we focus on the second claim of Theorem 1. As before, we are going to derive it from a result in one complex variable, but first we need to establish some properties of β:

Lemma 3. *Let β be an admissible function as defined in Definition 2. Then the following hold:*

(i) *β is of class C^1 and $\beta' \leq 0$;*
(ii) *$t\beta(t) \to 0$ and $t^2\beta'(t) \to 0$ for $t \to 0$;*
(iii) *$\beta(t) \leq C\beta(Ct)$ for all $C > 1$;*
(iv) *for $0 < t_1 < t_2$, the following holds:*

$$|t_2^2\beta'(t_2) - t_1^2\beta'(t_1)| \leq 2(t_2\beta(t_2) - t_1\beta(t_1)) \leq 2(t_2 - t_1)\beta(t_2 - t_1).$$

Proof. Since we are only interested in the behavior of $\beta(t)$ for t close to 0, in the following arguments we implicitly restrict ourselves to a neighborhood of 0 in $\mathbb{R}^+$. Note that the facts that β is of class C^1, $t\beta(t) = \gamma(t) \to 0$ as $t \to 0$ and the last inequality in (iv) (due to the concavity of γ) all follow directly from Definition 2. Computing the derivative of β, we have

$$\beta'(t) = \frac{\gamma'(t)}{t} - \frac{\gamma(t)}{t^2}. \tag{2}$$

The concavity of γ implies that $0 < \gamma'(t) \leq \gamma(t)/t$; hence

$$0 \leq -\beta'(t) \leq \frac{\gamma(t)}{t^2} = \frac{\beta(t)}{t}. \tag{3}$$

This immediately implies assertions (i) and (ii). Now let $C > 1$. Since γ is increasing, we get

$$\beta(t) = \frac{\gamma(t)}{t} = C\frac{\gamma(t)}{Ct} \leq C\frac{\gamma(Ct)}{Ct} = C\beta(Ct),$$

which is the claim at point (iii). To prove (iv), in view of (2) it suffices to estimate $|t_2\gamma'(t_2) - t_1\gamma'(t_1)|$. Let $t_3 > 0$ be defined in such a way that

$$\gamma(t_1) - \gamma'(t_1)(t_1 - t_3) = \gamma(t_2) - \gamma'(t_2)(t_2 - t_3).$$

By the concavity of γ, then, $t_1 < t_3 < t_2$ follows. Let $\tau = t_1 - t_3$. Then $\tau < 0$, and we have

$$\gamma(t_2) - \gamma(t_1) = \gamma'(t_2)(\tau + t_2 - t_1) - \gamma'(t_1)\tau > \gamma'(t_2)t_2 - \gamma'(t_1)t_1.$$

The last inequality is again due to the concavity of γ (since $\gamma'(t_2) < \gamma'(t_1)$, the affine function $x \to \gamma'(t_2)(x + t_2 - t_1) - \gamma'(t_1)x$ is decreasing). Since by assumption $t\gamma'(t)$ is monotone increasing (cf. Definition 2), this is the same as writing $|\gamma'(t_2)t_2 - \gamma'(t_1)t_1| < \gamma(t_2) - \gamma(t_1)$. □

Lemma 4. *Let H be the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, and let β be as in Theorem 1. Then there exists a (simply connected) domain $\Omega_\beta \subset H$ with the following properties:*

- *the boundary of Ω_β is Dini-smooth and $0 \in b\Omega_\beta$;*
- *$\operatorname{Re} \frac{1}{z} \geq \beta(|z|)$ for all z in a neighborhood of 0 in Ω_β.*

Proof. We consider real coordinates (x, y) such that $z = x + iy$, and we define $\Omega_\beta = \{x > \varphi(y)\}$ with $\varphi(y) = 2y^2\beta(|y|)$. The boundary $b\Omega_\beta$ is Dini-smooth if and only if the derivative

$$\frac{d\varphi}{dy}(y) = 4y\beta(|y|) + 2\operatorname{sign}(y)y^2\frac{d\beta}{dy}(|y|)$$

is Dini-continuous. By point (iv) in Lemma 3, the modulus of continuity of $d\varphi/dy$ can be estimated by a multiple of $t\beta(t) = \gamma(t)$. The Dini-smoothness of φ thus follows from Definition 2. Also notice that, by definition, $\varphi(0) = 0$; the first claim is then verified. As for the second one, let $z \in \Omega_\beta$; we have

$$\frac{\left(\operatorname{Re} \frac{1}{z}\right)}{\beta(|z|)} = \frac{x}{\beta\left(\sqrt{x^2+y^2}\right)(x^2+y^2)}.$$

Assume, first, that $x \geq |y|$. Since $\beta(|z|) \leq \beta(x)$, we obtain

$$\frac{\left(\operatorname{Re} \frac{1}{z}\right)}{\beta(|z|)} \geq \frac{1}{2x\beta(x)} \to +\infty$$

as $x \to 0$. If, on the other hand, $\varphi(y) < x \leq |y|$, we estimate as follows:

$$\frac{\left(\operatorname{Re} \frac{1}{z}\right)}{\beta(|z|)} \geq \frac{\varphi(y)}{2y^2\beta(|y|)} = 1$$

(where we used the fact that $\beta(|z|) \leq \beta(|y|)$), which concludes the proof. □

Lemma 5. *Let β, H be as above. There is a non-vanishing holomorphic function α, defined in a neighborhood U of 0 in H and of class C^1 up to the boundary, such that*

$$|\alpha(z)| \leq e^{-\beta(|z|)}$$

for all $z \in U$.

Proof. Let Ω_β be the domain constructed Lemma 4, and let $\mathcal{R}$ be the mapping given by Theorem 2. Since the differential of $\mathcal{R}$ does not vanish, for some $C > 1$, $D > 0$ and z in a neighborhood U of 0 one has $D|z| \leq |\mathcal{R}(z)| \leq C|z|$. By construction, we have

$$\operatorname{Re} \frac{1}{\mathcal{R}(z)} \geq \beta(|\mathcal{R}(z)|) \geq \beta(C|z|) \geq \frac{1}{C}\beta(|z|)$$

for $z \in U$ (here we used claim (iii) of Lemma 3). Hence, letting $\alpha(z) = e^{-C/\mathcal{R}(z)}$, we get

$$|\alpha(z)| = e^{-C \operatorname{Re} \frac{1}{\mathcal{R}(z)}} \leq e^{-\beta(|z|)}$$

for $z \in U$, as desired. As for the smoothness of α, first of all we note that it is clearly of class C^1 outside 0 since this is the case for $1/\mathcal{R}(z)$. Computing the first derivative gives

$$\alpha'(z) = Ce^{-\frac{C}{\mathcal{R}(z)}} \frac{\mathcal{R}'(z)}{\mathcal{R}^2(z)};$$

hence

$$|\alpha'(z)| \leq C' e^{-\beta(|z|)} \frac{1}{|z|^2} \to 0$$

as $z \to 0$ (see Remark 1). This shows that α is of class C^1. □

Corollary 2. *Let α, H be as in the previous lemma, and for $\kappa > 0$ define $H_\kappa = \{z \in \mathbb{C} : \operatorname{Re} z > |z|^\kappa\}$. For any fixed $j \in \mathbb{N}$, there exists $C_{j,\kappa} > 0$ such that*

$$|\alpha^{(j)}(z)| \leq C_{j,\kappa} e^{-\frac{1}{\sqrt{|z|}}}$$

for all $z \in H_\kappa$.

Proof. We note, first, that since $\mathcal{R}$ is bounded the Cauchy estimates give, for any fixed $i \in \mathbb{N}$,

$$|\mathcal{R}^{(i)}(z)| \leq C'_i \frac{1}{(\operatorname{Re} z)^i}, \quad z \in H,$$

for a suitable constant C'_i. It follows that $|\mathcal{R}^{(i)}(z)| \leq C'_i/|z|^{i\kappa}$ for $z \in H_\kappa$.

Now, we have $\alpha^{(j)}(z) = \alpha(z) P_j(1/\mathcal{R}(z), \mathcal{R}'(z), \ldots, \mathcal{R}^{(j)}(z))$, where $P_j \in \mathbb{R}[x_0, x_1, \ldots, x_j]$ is a polynomial whose coefficients are determined by the Faa di Bruno formula. Thus, the polynomial estimates for each $|\mathcal{R}^{(i)}|$ over H_κ imply that $|\alpha^{(j)}(z)| \leq C''_{j,\kappa} e^{-\beta(|z|)}/|z|^\ell$ for a suitable $\ell = \ell(j,\kappa) > 0$ and all $z \in H_\kappa$. The conclusion then follows from Remark 1. □

Lemma 6. *Let M be a smooth lineally convex hypersurface of finite type k, and suppose that coordinates (z, w) for $\mathbb{C}^{n+1}$ are chosen as above. Then there exists a real function r of class $C^{1,\frac{1}{k-1}}$ such that $r(0) = 0$ and, for a neighborhood U of 0 in $\mathbb{C}^{n+1}$,*

$$(M \setminus \{0\}) \cap U \subset \{(z, w) \in U : \operatorname{Im} w > r(\operatorname{Re} w)\}.$$

Proof. We write a local defining equation for M as in (1). By the finite type hypothesis, then, it follows that (for a certain neighborhood U' of 0 in $\mathbb{C}^{n+1}$)

$$(M \setminus \{0\}) \cap U' \subset \{(z, w) \in U' : \operatorname{Im} w > |z|^k - A(|\operatorname{Re} w|\,|z| + |\operatorname{Re} w|^2)\} \tag{4}$$

for a suitable constant $A > 0$. Consider, for a fixed $u > 0$, the function $p_u(x) = x^k - Aux - Au^2$ as defined over $\{x > 0\}$. By looking at the derivative, it is easy to see that p_u assumes a minimum in $x_0 = (Au/k)^{1/(k-1)}$; hence

$$p_u(x) \geq r(u) = A' u^{1+\frac{1}{k-1}} - Au^2 \text{ for } x > 0,$$

where $A' = -(k-1)(A/k)^{k/(k-1)}$ is a (negative) constant. This choice of r, then, satisfies the conditions required by the lemma. □

Remark 5. From the proof of the previous lemma it follows that we can choose r of the form $r(u) = -Du^{1+\frac{1}{k-1}}$ for a large enough $D > 0$.

Now we are in a position to prove the second claim of Theorem 1. Let $\Omega_2 \subset \Omega_1 \subset \mathbb{C}_w$ be the domains defined as $\Omega_1 = \{\operatorname{Im} w > 2r(\operatorname{Re} w)\}$, $\Omega_2 = \{\operatorname{Im} w > r(\operatorname{Re} w)\}$, where r is given by Lemma 6 and by the subsequent Remark 5.

Lemma 7. *For any $w \in \mathbb{C}$, denote by $\delta_1(w)$ the distance between w and $b\Omega_1$. Then there exists $\kappa > 0$ such that $\delta_1(w) \geq |w|^\kappa$ for all $w \in \Omega_2$.*

Proof. Consider the following local diffeomorphism of $\mathbb{C} \cong \mathbb{R}^2$, defined in a neighborhood of 0:

$$\psi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \psi(\operatorname{Re} w, \operatorname{Im} w) = (\operatorname{Re} w, \operatorname{Im} w - 2r(\operatorname{Re} w))$$

where $r(\operatorname{Re} w) = -D(\operatorname{Re} w)^{1+\frac{1}{k-1}}$ as in Remark 5. We note that ψ is of class (at least) C^1, in particular bi-Lipschitz. Therefore there exists a constant $E > 0$ such that $E^{-1}d(w_1, w_2) \leq d(\psi(w_1), \psi(w_2)) \leq Ed(w_1, w_2)$ for all $w_1, w_2 \in \mathbb{C}$ close enough to 0. Furthermore, we have $\psi(\Omega_1) = \Omega_1' = \{\operatorname{Im} w > 0\}$ and $\psi(\Omega_2) = \Omega_2' = \{\operatorname{Im} w > -r(\operatorname{Re} w)\}$. Defining $\delta_1'(w) = \operatorname{dist}(w, b\Omega_1')$, we get $\delta_1'(w) = \operatorname{Im} w$ for all $w \in \Omega_2'$.

Now, for any $w \in \Omega_2'$ close enough to 0 we get

$$\begin{aligned}|w|^2 &= (\operatorname{Re} w)^2 + (\operatorname{Im} w)^2 < ((-r)^{-1}(\operatorname{Im} w))^2 + (\operatorname{Im} w)^2 \\ &= D^{-2}(\operatorname{Im} w)^{\frac{2k-2}{k}} + (\operatorname{Im} w)^2 < D'(\operatorname{Im} w)^{\frac{2k-2}{k}} = D'\delta_1'(w)^{\frac{2k-2}{k}}\end{aligned}$$

for a large enough $D' > 0$. In particular, choosing $\kappa > k/(k-1)$ we have $|w|^\kappa < \delta_1'(w)$ for $w \in \Omega_2'$ around 0.

Since $|w| \leq E|\psi(w)|$ and $\delta_1(w) \geq E^{-1}\delta_1'(\psi(w))$, for $w \in \Omega_2$ we can write $|w|^\kappa \leq E^\kappa|\psi(w)|^\kappa < E^\kappa\delta_1'(\psi(w)) \leq E^{\kappa+1}\delta_1(w)$, which leads to the conclusion up to choosing a slightly larger exponent κ. ☐

Since $b\Omega_1$ is of class $C^{1,\frac{1}{k-1}}$, it is in particular Dini-smooth; let $\mathcal{Q} : \Omega_1 \to H$, $\mathcal{Q}(0) = 0$ be the inverse of the Riemann mapping $\mathcal{R} : H \to \Omega_1$. By Lemma 2 we deduce that $\mathcal{Q}$ is also of class C^1 up to the boundary and $\mathcal{Q}'(0) \neq 0$, so that for some constant $C > 1$ we can write

$$C^{-1}|w_1 - w_2| \leq |\mathcal{Q}(w_1) - \mathcal{Q}(w_2)| \leq C|w_1 - w_2| \tag{5}$$

for w_1, w_2 close enough to 0. We apply Lemma 5 with β replaced by $\beta_1 = C\beta$. Defining $\eta_1(w) = \alpha(\mathcal{Q}(w))$, it follows that η_1 is of class C^1 up to the boundary and that

$$|\eta_1(w)| = |\alpha(\mathcal{Q}(w))| \leq e^{-\beta_1(|\mathcal{Q}(w)|)} \leq e^{-C\beta(C|w|)} \leq e^{-\beta(|w|)}.$$

Now, let $\Omega = \mathbb{C}^n \times \Omega_1 = \{(z, w) \in \mathbb{C}^{n+1} : \operatorname{Im} w > 2r(\operatorname{Re} w)\}$; we define a function $\eta \in \mathcal{O}(\Omega) \cap C^1(\overline{\Omega})$ by $\eta(z, w) = \eta_1(w)$. By Lemma 6, η yields by restriction a non-trivial CR function of class C^1, defined over a neighborhood of 0 in M and clearly satisfying the estimate required by Theorem 1.

We claim that $\eta_{|M}$ is in fact of class C^∞. Fix $j \in \mathbb{N}$. Using the Faa di Bruno formula, we can compute the j-th derivative of η (note that of course only the $(\partial/\partial w)$-derivatives are relevant) as

$$\eta^{(j)}(z, w) = \sum_{i=1}^{j} \alpha^{(i)}(\mathcal{Q}(w))P_i\left(\mathcal{Q}'(w), \ldots, \mathcal{Q}^{(j)}(w)\right)$$

for suitable polynomials $P_i \in \mathbb{Z}[x_1, \ldots, x_j]$. Now, if $p \in M$, $p = (z_p, w_p)$, by Lemma 6 we have that $w_p \in \Omega_2$, and thus by Lemma 7 it follows that $w_p \in \Omega_\kappa := \{w \in \Omega_1 : \delta_1(w) \geq |w|^\kappa\}$ for some $\kappa > 0$. Moreover,

Lemma 8. *For some* $\kappa' > \kappa$ *we have* $\mathcal{Q}(\Omega_\kappa) \subset H_{\kappa'}$*, where* $H_{\kappa'}$ *is defined in Corollary* 2*.*

Proof. If $w \in \Omega_\kappa$, we have by definition $|w - w'| \geq |w|^\kappa$ for all $w' \in b\Omega_1$. Using (5), we deduce $|\mathcal{Q}(w) - \mathcal{Q}(w')| \geq C^{-1}|w - w'| \geq C^{-1}|w|^\kappa \geq C^{-\kappa-1}|\mathcal{Q}(w)|^\kappa$. Since $\mathcal{Q}(b\Omega_1) = bH$, this is the same as writing $|\mathcal{Q}(w) - w''| \geq C^{-\kappa-1}|\mathcal{Q}(w)|^\kappa$ for all $w'' \in bH$; hence we get $\operatorname{Re}\mathcal{Q}(w) \geq C^{-\kappa-1}|\mathcal{Q}(w)|^\kappa$ for all $w \in \Omega_\kappa$. Choosing any $\kappa' > \kappa$ we obtain the claim of the lemma (for w close enough to 0). □

Following, now, the same lines as in the proof of Corollary 2, by the Cauchy estimates it follows that $|\mathcal{Q}^{(i)}(w)| \leq F_i/\delta_1(w)^i$ for $w \in \Omega_1$; thus $|\mathcal{Q}^{(i)}(w_p)| \leq F_i/|w_p|^{i\kappa}$ because $w_p \in \Omega_\kappa$. Hence each term $|P_i\left(\mathcal{Q}'(w_p), \ldots, \mathcal{Q}^{(j)}(w_p)\right)|$ blows up at most polynomially in $1/|w_p|$ as $p \to 0$, $p \in M$. On the other hand, by Corollary 2 we have (since $\mathcal{Q}(w_p) \in H_{\kappa'}$) that $|\alpha^{(i)}(\mathcal{Q}(w_p))| \leq C_i e^{-\frac{1}{\sqrt{|\mathcal{Q}(w_p)|}}} \leq C_i e^{-\frac{1}{C^{1/2}\sqrt{|w_p|}}}$.

From the estimates above it follows that

$$|\alpha^{(i)}(\mathcal{Q}(w_p))| \cdot \left|P_i\left(\mathcal{Q}'(w_p), \ldots, \mathcal{Q}^{(j)}(w_p)\right)\right| \to 0$$

for all $1 \leq i \leq j$ as $p \to 0$, $p \in M$. In conclusion $\eta^{(j)}(p) \to 0$ as $p \to 0$, $p \in M$, and since this holds for any $j \in \mathbb{N}$ we get that $\eta_{|M}$ is of class C^∞.

Remark 6. Since the function η so constructed extends holomorphically to a neighborhood of any $p \in M$ except 0, it is in fact real-analytic on $M \setminus \{0\}$ if M is of class C^ω. Moreover, $\eta(p)$ is only vanishing at $p = 0$.

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Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: giuseppe.dellasala@univie.ac.at

Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria

E-mail address: bernhard.lamel@univie.ac.at