

ON CLUSTER ALGEBRAS FROM UNPUNCTURED SURFACES WITH ONE MARKED POINT

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(Communicated by Harm Derksen)

ABSTRACT. We extend the construction of canonical bases for cluster algebras from unpunctured surfaces to the case where the number of marked points on the boundary is one. We show that the cluster algebra is equal to the upper cluster algebra in this case.

1. INTRODUCTION

Cluster algebras were introduced in [FZ1], and further developed in [FZ2, BFZ, FZ4], motivated by combinatorial aspects of canonical bases in Lie theory [L1, L2]. A cluster algebra is a subalgebra of a field of rational functions in several variables, and it is given by constructing a distinguished set of generators, the *cluster variables*. These cluster variables are constructed recursively and their computation is rather complicated in general. By construction, the cluster variables are rational functions, but Fomin and Zelevinsky showed in [FZ1] that they are Laurent polynomials with integer coefficients. Moreover, these coefficients are known to be non-negative [LS].

An important class of cluster algebras is given by cluster algebras of surface type [GSV, FG1, FG2, FST, FT]. From a classification point of view, this class is very important, since it has been shown in [FeShTu] that almost all (skew-symmetric) mutation finite cluster algebras are of surface type. For generalizations to the skew-symmetrizable case see [FeShTu2, FeShTu3]. The closely related surface skein algebras were studied in [M, T].

If \mathcal{A} is a cluster algebra of surface type, then there exists a surface with (possibly empty) boundary and marked points such that the cluster variables of \mathcal{A} are in bijection with certain isotopy classes of curves, called *arcs*, in the surface. Marked points in the interior of the surface are called *punctures*, and the surface is *unpunctured* if all marked points lie on the boundary. Moreover, the relations between the cluster variables are given by the crossing patterns of the arcs in the surface. In [MSW], building on earlier work [S2, ST, S3, MS], the authors gave a combinatorial

Received by the editors August 23, 2014.

2010 *Mathematics Subject Classification*. Primary 13F60.

Key words and phrases. Cluster algebra, upper cluster algebra, unpunctured surface, basis.

The first author was supported by EPSRC grant number EP/K026364/1, UK and the University of Leicester.

The second author was supported by Wayne State University, the Korea Institute for Advanced Study, AMS Centennial Fellowship and NSA grant H98230-14-1-0323.

The third author was supported by NSF grants DMS-1254567, DMS-1101377 and by the University of Connecticut.

formula for the cluster variables in cluster algebras of surface type. In the sequel [MSW2], the formula was the key ingredient for the construction of two bases for the cluster algebra, in the case where the surface has no punctures and has at least two marked points.

1.1. Bases. Our main result, Theorem 2, shows that the basis construction of [MSW2] also applies to surfaces without punctures and with exactly one marked point. The proof of this result consists in showing that the Laurent polynomials associated to the essential loops in the surface are elements of the cluster algebra. This is shown by exhibiting certain identities in the cluster algebra, that allow us to write the Laurent polynomials in question as polynomials in cluster variables. The main ingredients for the proof of these identities are the snake graph calculus developed in [CS, CS2, CS3], and the skein relations proved in [MW].

1.2. Upper cluster algebras. As an application, we study the relationship between cluster algebras and upper cluster algebras. To define a cluster algebra \mathcal{A} , one needs to specify an initial seed $\Sigma = (\mathbf{x}, B)$ consisting of a cluster \mathbf{x} and an exchange matrix B . In [BFZ], the authors introduced the concept of the upper cluster algebra \mathcal{U} associated to the seed Σ . Recall that \mathcal{U} consists of all elements of the ambient field $\mathbb{Q}(x_1, \dots, x_n)$ which are Laurent polynomials over \mathbb{Z} in the cluster variables from any seed in \mathcal{A} . It follows directly from the Laurent phenomenon that the upper cluster algebra contains the cluster algebra as a subalgebra. However, the question of whether the cluster algebra is equal to the upper cluster algebra is subtle and does not have a uniform answer. Already in [BFZ] it is shown that $\mathcal{A} = \mathcal{U}$ for all acyclic types and, on the other hand, $\mathcal{A} \neq \mathcal{U}$ for the rank 3 case given by the once-punctured torus. Muller introduced the notion of locally acyclic cluster algebras in [M2] and showed that $\mathcal{A} = \mathcal{U}$ for all locally acyclic types in [M3]. It is shown in [MSp] that cluster algebras of Grassmannians are locally acyclic, hence $\mathcal{A} = \mathcal{U}$ in this case. Goodearl and Yakimov announced that $\mathcal{A} = \mathcal{U}$ for double Bruhat cells [GY].

For cluster algebras of finite mutation type, the following results are known:

- (1) $\mathcal{A} = \mathcal{U}$ in the following types:
 - surfaces with non-empty boundary and at least two marked points. This has been shown in [MSW2] for unpunctured surfaces, and in [M2] if at least two marked points are on the boundary. The case where only one point is on the boundary can be reduced to the case with two marked points on the boundary using the Louise property of [MSp]; see Proposition 12.
 - the exceptional types $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$, by [BFZ], since these types are acyclic.
 - the exceptional types $\widetilde{\widetilde{\mathbb{E}}}_6, \widetilde{\widetilde{\mathbb{E}}}_7, \widetilde{\widetilde{\mathbb{E}}}_8$ and \mathbb{X}_6 , by [M], since they are locally acyclic, or by [MSp], since they satisfy the Louise property.
- (2) $\mathcal{A} \neq \mathcal{U}$ in the following types:
 - surfaces without boundary and with exactly one puncture. This has been shown for the torus in [BFZ] and for higher genus in [Lad].
 - the exceptional type \mathbb{X}_7 , [M4].

As an application of Theorem 2, we prove in Theorem 5 that

$\mathcal{A} = \mathcal{U}$ for surfaces without punctures and exactly one marked point.

This leaves the question open only for surfaces without boundary and at least two punctures.

Summarizing, we have the following result.

Theorem 1. *For all unpunctured surfaces, the cluster algebra is equal to the upper cluster algebra.*

1.3. Maximal green sequences. Let Q be a quiver and let \bar{Q} be the quiver obtained from Q by adding a vertex i' and an arrow $i \rightarrow i'$, for each vertex i of Q . The original vertices of Q are called *unfrozen* (or *mutable*) and the new vertices are called *frozen*. A *green sequence* is a sequence of mutations starting at \bar{Q} such that at each step the mutation is performed at an unfrozen vertex i , for which there is no arrow $j' \rightarrow i$, from a frozen vertex j' . A green sequence is *maximal* if it produces a quiver which does not contain such a vertex i .

Maximal green sequences have been introduced in [K]. They are related to quantum dilogarithm identities, DT invariants and BPS states; see [BDP] and the references therein.

The question of whether or not there exists a quiver in the mutation class of Q which admits a maximal green sequence seems related to the question of whether or not $\mathcal{A}(Q) = \mathcal{U}(Q)$. Indeed, given a surface (S, M) which is either a surface with boundary, or a sphere with at least four punctures, or a torus with at least two punctures, there exists a triangulation T whose quiver Q_T admits a maximal green sequence. This has been shown in [ACCERV]; see also [BDP] for the sphere. Moreover, for all the exceptional types except for \mathbb{X}_7 there exists a quiver that admits a maximal green sequence [ACCERV], whereas the mutation class \mathbb{X}_7 does not contain a quiver that admits a maximal green sequence [Se].

On the other hand, it is shown in [BDP, Proposition 8.1] that if Q admits a non-degenerate potential such that the Jacobian algebra is infinite dimensional, then Q does not admit maximal green sequences. Combining this result with [GLS, Proposition 9.13], we see that quivers from closed surfaces with exactly one puncture do not admit maximal green sequences. The question seems to be open for closed surfaces of genus at least 2 and with at least two punctures.

Beyond finite mutation type, explicit maximal green sequences are announced by Yakimov for double Bruhat cells.

In all the known cases, maximal green sequences exist if and only if the cluster algebra is equal to the upper cluster algebra. Our Theorem 5 confirms this observation for unpunctured surfaces with exactly one marked point.

2. MAIN RESULT

Let $\mathcal{A} = \mathcal{A}(S, M)$ be a cluster algebra of an unpunctured surface S with marked points M whose coefficient system is such that the initial exchange matrix has maximal rank. The main result of [MSW2] is the construction of two bases for \mathcal{A} under the assumption that the number of marked points is at least two. These are the *bangles basis* \mathcal{B}_0 and the *bracelet basis* \mathcal{B} . The assumption that the number of marked points is at least two was used in [MSW2] only to show that all elements of \mathcal{B}_0 and \mathcal{B} are actually inside the cluster algebra, and not only in the upper cluster algebra. The proof of the spanning property and of linear independence does not rely on the number of marked points.

The following theorem removes the assumption on the number of marked points.

Theorem 2. *Let \mathcal{A} be the cluster algebra of an unpunctured surface with exactly one marked point and with arbitrary coefficients. Then both \mathcal{B} and \mathcal{B}_0 are bases of the cluster algebra \mathcal{A} .*

For the proof of this theorem, we need the following two lemmas. An *essential loop* in the surface (S, M) is a closed curve in S which is disjoint from the boundary of S , which is not contractible and does not have any self-crossings. Since $|M| = 1$, the surface S has exactly one boundary component. In [MSW2], the authors associate to every essential loop a Laurent polynomial given as a sum over perfect matchings of the corresponding band graph.

Lemma 3. *The Laurent polynomial L of the essential loop around the boundary is in the cluster algebra \mathcal{A} .*

Lemma 4. *The Laurent polynomial x_ζ of every essential loop ζ is in the cluster algebra \mathcal{A} .*

Of course the second lemma implies the first, but the first is needed in the proof of the second. We prefer stating them separately, because their proofs use different techniques. The proof of Lemma 3 is given in Section 3 and the proof of Lemma 4 in Section 4.

Proof of Theorem 2. Let T be a triangulation of the surface and let B_T be the associated exchange matrix. Since there are no punctures and the unique boundary component has an odd number of marked points (namely one), it follows from [FST, Theorem 14.3] that the rank of the matrix B_T is maximal for every coefficient system. Therefore the results of [MSW2] show that both \mathcal{B}_0 and \mathcal{B} are linearly independent and that every element of \mathcal{A} is a linear combination of elements of \mathcal{B}_0 and also a linear combination of elements of \mathcal{B} .

It remains to show that \mathcal{B}_0 and \mathcal{B} are subsets of \mathcal{A} .

Lemma 4 implies that every element of \mathcal{B}_0 is in \mathcal{A} . Moreover the Laurent polynomials associated to the bracelets in \mathcal{B} can be written as Chebyshev polynomials in the Laurent polynomials associated to the essential loops in \mathcal{B}_0 ; see [MSW2, Proposition 4.2]. Thus \mathcal{B} is also a subset of \mathcal{A} . \square

The following result follows directly from Theorem 2.

Theorem 5. *Let \mathcal{A} be the cluster algebra of an unpunctured surface with exactly one marked point and with trivial coefficients, and let \mathcal{U} be its upper cluster algebra. Then*

$$\mathcal{A} = \mathcal{U}.$$

Proof. It follows from [FG1, Theorem 12.3(ii)] that \mathcal{B}_0 is a basis of \mathcal{U} . Theorem 2 implies that $\mathcal{B}_0 \subset \mathcal{A}$, and thus $\mathcal{U} \subset \mathcal{A}$. The other inclusion always holds. \square

3. PROOF OF LEMMA 3

We prove the lemma first for genus 1, then genus 2 and then for higher genus. Throughout the section, we denote by \mathcal{A}_\bullet the cluster algebra with principal coefficients in the initial seed corresponding to our triangulation T and by $\hat{\mathcal{A}}$ a cluster algebra with arbitrary coefficient system and initial seed corresponding to T . More precisely, let $\mathbb{P} = \text{Trop}(y_1, \dots, y_n)$ be the tropical semifield and $\mathbf{y} = (y_1, \dots, y_n)$ the initial coefficient tuple consisting of the generators of \mathbb{P} . Then \mathcal{A}_\bullet is the cluster

algebra with initial seed $(\mathbf{x}_T, \mathbf{y}, B_T)$. On the other hand, let $\hat{\mathbb{P}}$ be any semifield and $\hat{\mathbf{y}} = (\hat{y}_1, \dots, \hat{y}_n)$ any coefficient tuple, thus $\hat{y}_i \in \hat{\mathbb{P}}$. Then $\hat{\mathcal{A}}$ is the cluster algebra with initial seed $(\mathbf{x}_T, \hat{\mathbf{y}}, B_T)$.

Recall that every arc or loop in (S, M) corresponds to a snake or band graph and to a Laurent polynomial given by perfect matchings of the snake or band graph. A band graph is obtained from a snake graph by identifying an edge in the first tile with an edge in the last tile. In the figures, this identification is represented by marking the vertices of the identified edges with bullets. For precise definitions, we refer to [MSW2]. For the snake graph calculus used in this section, see [CS2].

3.1. Genus 1. Let (S, M) be a surface of genus 1 with one boundary component and one marked point. This case is known in the literature as ‘the dreaded torus’, because, among cluster algebras of surface type, it was the smallest example where the question of whether the cluster algebra is equal to the upper cluster algebra was open; see [MM, Remark 7.2.3]. Fix the triangulation T shown on the left in Figure 1, and let $\mathbf{x}_T = (x_1, \dots, x_4)$ be the corresponding cluster in \mathcal{A}_\bullet .

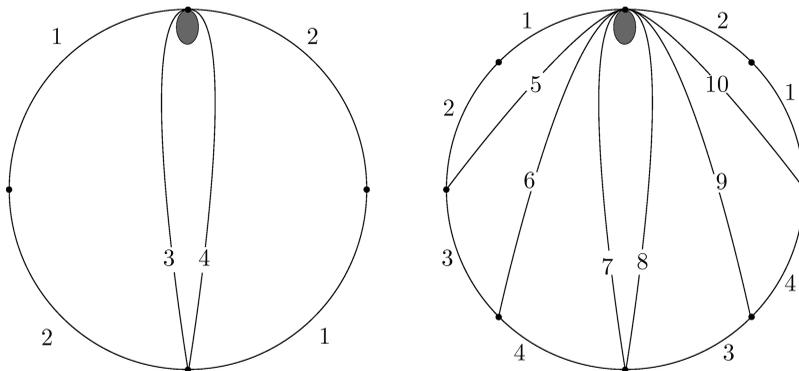


FIGURE 1. A triangulation of (S, M) in genus 1 (left) and a triangulation in genus 2 (right).

Let L be the Laurent polynomial associated to the loop around the boundary. Its band graph is the first graph in the second row in Figure 2.

Lemma 6. $L \in \mathcal{A}_\bullet$.

Proof. Let V_1 be the cluster variable given by the arc that starts at the marked point in the direction between the arc 3 and the boundary and then crosses the arcs 4, 2, 1, 4 in order and ends at the same marked point as it started from in the direction between the arc 3 and the boundary. Let V_2 be the cluster variable given by the symmetric arc, that is, the arc that starts at the marked point in the direction between the arc 4 and the boundary and then crosses the arcs 3, 1, 2, 3 in order and ends at the same marked point as it started from in the direction between the arc 4 and the boundary. The snake graphs of these two cluster variables are the first two snake graphs in the top row of Figure 2. The first two equations in that figure use snake graph calculus of [CS] and [CS2] to compute the product $V_1 V_2$ showing that

$$(1) \quad V_1 V_2 = L + y_3(y_4 X_1 + x_3)(X_1 + y_1 y_2 y_3 x_4),$$

$$\begin{aligned}
& \begin{array}{c} \text{B}^{\min} \\ \begin{array}{cc} & 1 & 4 \\ 4 & 2 & \\ & 3 & 2 \\ & & 1 & 3 \end{array} \\ \text{min} \underbrace{\hspace{1.5cm}}_{V_1} \quad \text{max} \underbrace{\hspace{1.5cm}}_{V_2} \end{array} = \begin{array}{c} & & 1 & 3 \\ & & 3 & 2 \\ & 1 & 4 & \\ 4 & 2 & & \end{array} + y_3 \begin{array}{c} & & 1 & \\ \text{B} & 4 & 2 & \\ \text{min} \underbrace{\hspace{1.5cm}}_3 & & \text{max} \underbrace{\hspace{1.5cm}}_{4^{\max}} & \\ & & 1 & 3 \\ & & 2 & \end{array} \text{B} \\
& = \begin{array}{c} & & 1 & 3 \\ & & 3 & 2 \\ & 1 & 4 & \\ \bullet & 4 & 2 & \\ L & & & \end{array} + y_3 \left(\begin{array}{c} \bullet & 1 \\ \bullet & 2 \\ X_1 & \end{array} + |3 \right) \left(\begin{array}{c} \bullet & 1 \\ \bullet & 2 \\ X_1 & \end{array} + y_1 y_2 y_3 |4 \right)
\end{aligned}$$

$$\begin{aligned}
& \begin{array}{c} \text{max} \\ 2 \underbrace{\hspace{1.5cm}}_{U_1} \begin{array}{cc} 1 & 3 \end{array} \text{B} \text{B} \begin{array}{cc} 4 & 2 \end{array} 1 \\ \text{min} \underbrace{\hspace{1.5cm}}_{U_2} \end{array} = 2 \underbrace{\hspace{1.5cm}}_{\text{min}} \begin{array}{cccc} 1 & 3 & 4 & 2 \end{array} \underbrace{\hspace{1.5cm}}_{\text{min}} 1 + y_1 y_3 |2|1 \\
& = 2 \begin{array}{ccc} \bullet & 1 & 3 & 4 \\ 4 & & 3 & \end{array} |1 + y_4 \begin{array}{cc} & 2 \\ 3 & 1 \\ 4 & \end{array} 1 + y_1 y_3 |2|1 \\
& = y_1 \begin{array}{cc} 3 & 4 \\ W_1 & \end{array} + |3 + y_4 \begin{array}{cc} \bullet & 2 & 1 \\ X_1 & \end{array} + y_1 y_2 y_3 y_4 |4 + y_1 y_3 |2|1
\end{aligned}$$

FIGURE 2. Snake graph calculus with principal coefficients. The edge labels ‘min’, ‘max’ refer to minimal and maximal matchings, which is needed only to determine the y -coefficients. The edge label B refers to the boundary.

where X_1 is the Laurent polynomial defined by the corresponding band graph in the figure. The first equation is the resolution of the grafting operation [CS, section 2.5, case 2] and the second equation is the resolution of three self-grafting operations [CS2, section 3.4]. The graphs of L and X_1 are band graphs.

Thus in order to show that $L \in \mathcal{A}_\bullet$ it suffices to show that $X_1 \in \mathcal{A}_\bullet$. This is done in the second computation in Figure 2, where we compute the product $U_1 U_2$ of the two cluster variables given by the arcs that cross 1, 3 and 4, 2 respectively, showing that

$$(2) \quad U_1 U_2 = y_1 W_1 + x_3 + y_4 X_1 + y_1 y_2 y_3 y_4 x_4 + y_1 y_3 x_1 x_2,$$

where W_1 is the Laurent polynomial defined by the corresponding snake graph in the figure. One can show that U_1, U_2 are cluster variables by checking that the associated arcs in the surface do not have a self-crossing, or by using [CS2] to show that their associated snake graphs do not have any self-crossing overlap. The first equation in this computation is the resolution of a grafting, the second equation is a self-grafting [CS2, section 3.3] and the third is a grafting of a band graph with a single edge [CS3] producing $y_1 W_1 + x_3$, as well as a self-grafting to obtain

$y_4 X_1 + y_1 y_2 y_3 y_4 x_4$. Note that W_1 is a cluster variable corresponding to the arc crossing 3 and 4. This shows that $y_4 X_1 \in \mathcal{A}_\bullet$. Since y_4 is an element of the group \mathbb{P} , its inverse y_4^{-1} is also in \mathbb{P} , and therefore $X_1 \in \mathcal{A}_\bullet$, and thus $L \in \mathcal{A}_\bullet$. \square

Lemma 7. $L \in \hat{\mathcal{A}}$.

Proof. Using Fomin-Zelevinsky's separation of addition formula [FZ4, Theorem 3.7] we see that the elements of $\hat{\mathcal{A}}$ can be computed from the elements in \mathcal{A} by replacing the principal coefficients y_i by the corresponding \hat{y}_i and by dividing by the F -polynomial evaluated over $\hat{\mathbb{P}}$ in \hat{y}_i . For example,

$$\hat{U}_1 = \frac{U_1(x_1, \dots, x_n; \hat{y}_1, \dots, \hat{y}_n)}{\hat{F}_{U_1}},$$

where $\hat{F}_{U_1} = F_{U_1}|_{\hat{\mathbb{P}}}(\hat{y}_1, \dots, \hat{y}_n) \in \hat{\mathbb{P}} \subset \hat{\mathcal{A}}$.

Evaluating the expressions in equations (1) and (2) in $\hat{\mathbf{y}}$ clearly preserves the identities. Then multiplying with the F -polynomials we get

$$(3) \quad \hat{V}_1 \hat{V}_2 \hat{F}_{V_1} \hat{F}_{V_2} = \hat{L} \hat{F}_L + \hat{y}_3 (\hat{y}_4 \hat{X}_1 \hat{F}_{X_1} + x_3) (\hat{X}_1 \hat{F}_{X_1} + \hat{y}_1 \hat{y}_2 \hat{y}_3 x_4),$$

$$(4) \quad \hat{U}_1 \hat{U}_2 \hat{F}_{U_1} \hat{F}_{U_2} = \hat{y}_1 \hat{W}_1 \hat{F}_{W_1} + x_3 + \hat{y}_4 \hat{X}_1 \hat{F}_{X_1} + \hat{y}_1 \hat{y}_2 \hat{y}_3 \hat{y}_4 x_4 + \hat{y}_1 \hat{y}_3 x_1 x_2.$$

From equation (4) we see that $\hat{y}_4 \hat{X}_1 \hat{F}_{X_1} \in \hat{\mathcal{A}}$, and since $\hat{y}_4 \in \hat{\mathbb{P}}$, and thus $\hat{y}_4^{-1} \in \hat{\mathbb{P}}$, this also implies that $\hat{X}_1 \hat{F}_{X_1} \in \hat{\mathcal{A}}$. Now equation (3) implies that $\hat{L} \hat{F}_L \in \hat{\mathcal{A}}$, and, since $\hat{F}_L \in \hat{\mathbb{P}}$, this shows that $\hat{L} \in \hat{\mathcal{A}}$. \square

3.2. Genus 2. Let (S, M) be a surface of genus 2 with one boundary component and one marked point. Fix the triangulation T shown on the right in Figure 1, and let $\mathbf{x}_T = (x_1, \dots, x_{10})$ be the corresponding cluster in \mathcal{A} .

Let L be the Laurent polynomial associated to the loop around the boundary. Its band graph is the first graph in the last row in Figure 3.

Lemma 8. $L \in \mathcal{A}_\bullet$.

Proof. Let V_1 be the cluster variable obtained from \mathbf{x} by the mutation sequence 8, 9, 10, 2, 1, 9, 4, 6, 3, and let V_2 be the cluster variable obtained from \mathbf{x} by the mutation sequence 7, 6, 5, 1, 2, 6, 3, 9, 4. Then V_1 is the cluster variable corresponding to the arc that crosses 8, 9, 10, 2, 1, 10, 4, 6, 3, 8 and V_2 is the cluster variable corresponding to the arc that crosses 7, 4, 9, 3, 5, 2, 1, 5, 6, 7. The corresponding snake graphs \mathcal{G}_1 and \mathcal{G}_2 are illustrated on the left-hand side of the equation in Figure 3.

Let X_i be the Laurent polynomial defined by the band graph obtained from \mathcal{G}_i , for $i = 1, 2$, by deleting the first and the last tile and then glueing. These band graphs are illustrated in the last row of Figure 3.

A simple computation using snake graph calculus shows that

$$V_1 V_2 = L + y_7 (y_8 X_1 + x_7) (X_2 + y_1 y_2 y_3 y_4 y_5^2 y_6 y_7 y_9 x_8).$$

This computation is given in Figure 3. The first equation is the resolution of a grafting, and the second is the resolution of three self-graftings.

In order to show that $L \in \mathcal{A}_\bullet$, it suffices to show that X_1 and X_2 are in the cluster algebra and, by symmetry, it is enough to show that X_1 is.

Let U_1 and U_2 be the Laurent polynomials associated to the snake graphs on the left-hand side of the equation in Figure 4. It follows from [CS2] that none of these two snake graphs has a self-crossing, which implies that U_1 and U_2 are cluster

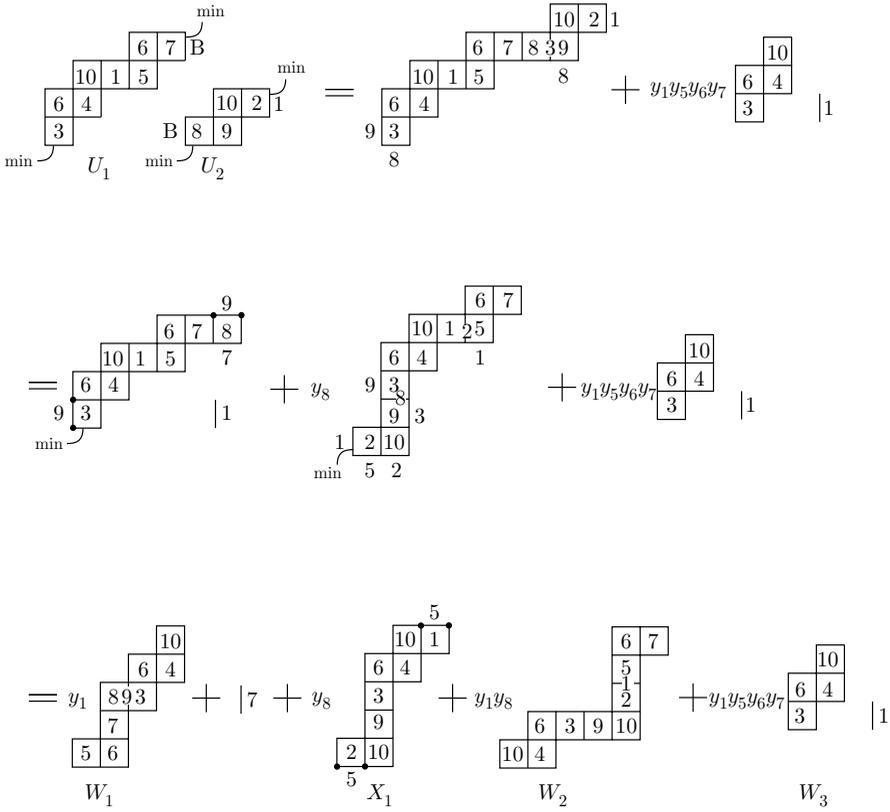


FIGURE 4. Snake graph calculus showing that

$$U_1 U_2 = y_1 W_1 + x_7 + y_8 X_1 + y_1 y_8 W_2 + y_1 y_5 y_6 y_7 W_3 x_1.$$

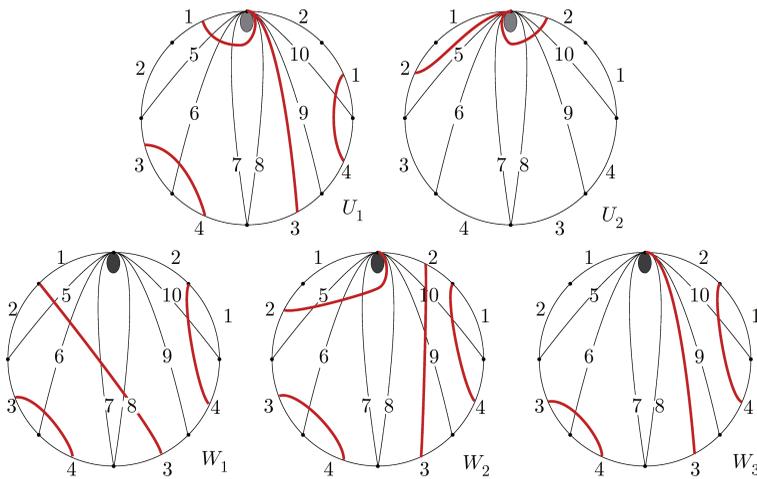


FIGURE 5. The arcs of the cluster variables involved in the computations in genus 2.

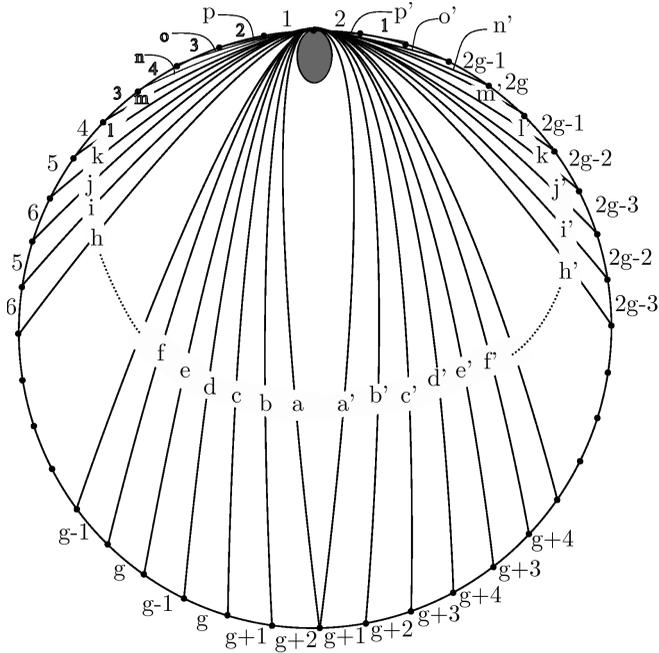


FIGURE 6. Triangulation of a surface of genus g with one boundary component and one marked point.

3.3. Higher genus. In this section, let (S, M) be a surface of genus $g > 2$ with one boundary component and one marked point. Fix the triangulation T shown in Figure 6, and let $\mathbf{x}_T = (x_1, \dots, x_{6g-2})$ be the corresponding cluster in \mathcal{A}_\bullet . Note that the figure is for g an even integer, but the case where g is odd is similar; only the labeling changes slightly.

As in the genus 1 and 2 cases, let L be the Laurent polynomial associated to the loop around the boundary.

Lemma 10. $L \in \mathcal{A}_\bullet$.

Proof. The proof is an adaptation of the genus 2 argument. The cluster variables V_1 and V_2 are now given by the zigzag snake graphs shown in Figure 7. The Laurent polynomials X_1, X_2 are again given by the band graphs obtained from the snake graphs of V_1 and V_2 by deleting the first and the last tile and then gluing. Again snake graph calculus shows that

$$V_1 V_2 = L + y_a(y_{a'} X_1 + x_a)(X_2 + Y x_{a'}),$$

where Y is a monomial in \mathbf{y} .

Finally to show that X_1 , and hence X_2 , is in the cluster algebra, we use the cluster variables U_1, U_2, W_1, W_2, W_3 analogous to the genus 2 case to get

$$U_1 U_2 = y_1 W_1 + x_a + y_{a'} X_1 + y_1 y_{a'} W_2 + y_1 y_p y_o \cdots y_b y_a W_3 x_1.$$

This shows that $X_1 \in \mathcal{A}$ and we are done. \square

Lemma 11. $L \in \hat{\mathcal{A}}$.

Proof. The proof is exactly the same as in the genus 1 case. \square

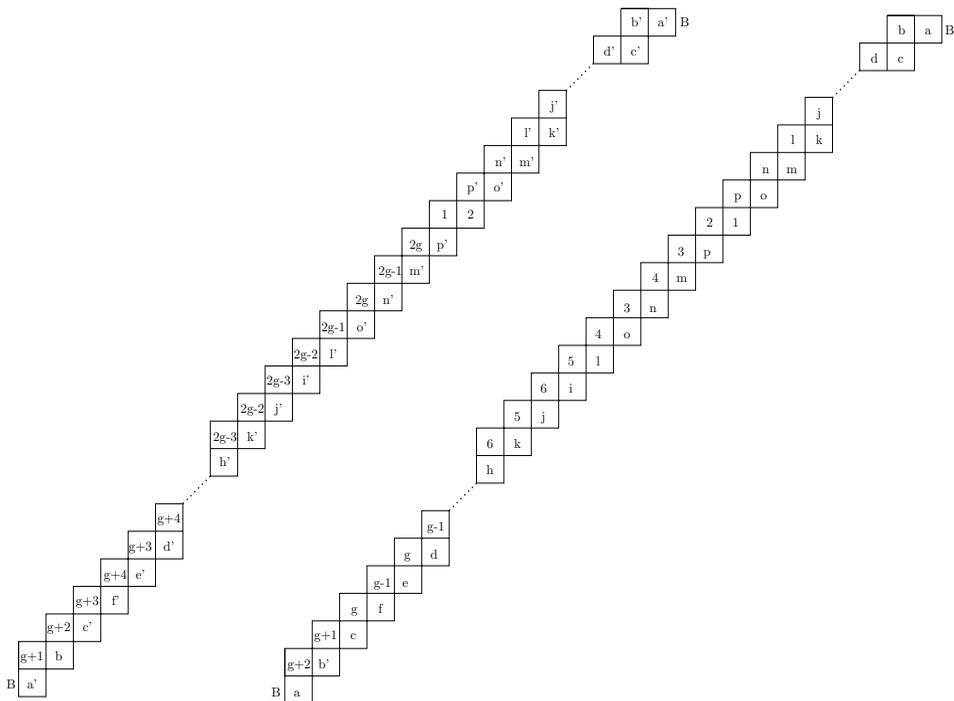


FIGURE 7. Snake graphs of the cluster variables V_1 (left) and V_2 (right).

4. PROOF OF LEMMA 4

Let ζ be an essential loop and let x_ζ be the Laurent polynomial associated to it. In view of Lemma 3 we may suppose that ζ is not the loop around the boundary. The proof uses a geometric argument with skein relations, and the relevant curves are illustrated in Figure 8. Choose a point z on ζ and a simple curve α that goes from the marked point to the point z . Let γ be the arc obtained by the curve $\alpha\zeta\alpha^{-1}$. Let δ be the generalized arc that starts at the marked point, goes around the boundary twice and then ends at the marked point. The curves α, ζ, δ and γ are illustrated in the first row of Figure 8. Let $x_\alpha, x_\zeta, x_\delta$ and x_γ be the corresponding Laurent polynomials.

We compute the product $x_\delta x_\gamma$ using skein relations. Thus each equation is obtained by smoothing a crossing of the curves. This computation is illustrated in the second and third row of Figure 8. The first term on the right-hand side of the equation in the second row is a cluster variable x_ϵ given by the blue arc ϵ (the red arc is the boundary segment B) and the second term on the right-hand side still has a self-crossing. Applying the skein relations to that self-crossing produces the third row in the figure. In this row all red and green curves are boundary segments. Thus we get

$$x_\delta x_\gamma = Y_1 x_\epsilon + Y_2 x_\zeta + Y_3 x_\sigma,$$

where Y_i are some monomials in \mathbf{y} and x_γ, x_ϵ and x_σ are cluster variables. Thus in order to show that $x_\zeta \in \mathcal{A}$, it suffices to show that $x_\delta \in \mathcal{A}$. But it is shown in the

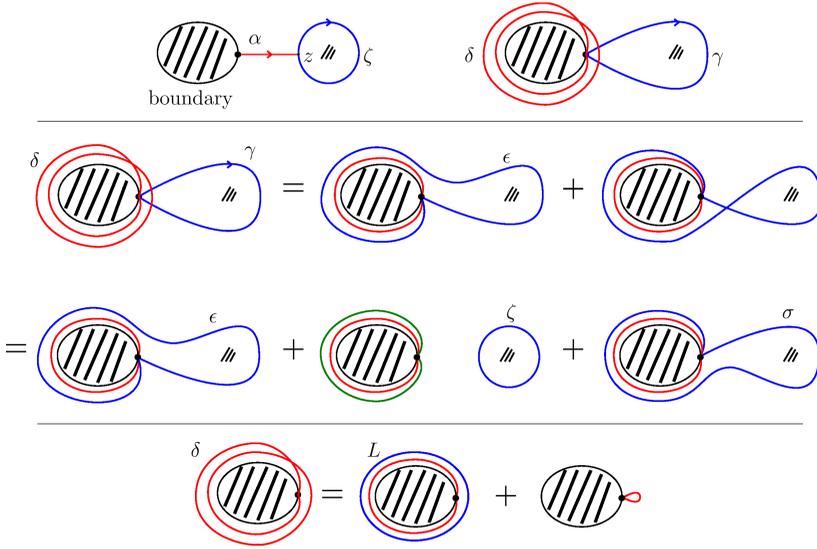


FIGURE 8. Proof of Lemma 4.

last row of Figure 8 that $x_\delta = L + 0$, where L (in blue) is the essential loop around the boundary. Now the result follows from Lemma 3. \square

5. PUNCTURED SURFACES WITH BOUNDARY

In this section, we show how to reduce the case of punctured surfaces with exactly one marked point on the boundary to the case of surfaces with two marked points on the boundary.

Let (S, M) be a punctured surface with boundary and one marked point on the boundary. Let \mathcal{A} be its cluster algebra and \mathcal{U} the corresponding upper cluster algebra.

Proposition 12. $\mathcal{A} = \mathcal{U}$.

Proof. By [MSp] it suffices to show that there is a triangulation whose quiver satisfies the Louise property. Let p_0 be the marked point on the boundary and let p_1 be a puncture. Let α, β be two arcs going from p_0 to p_1 and let γ be the arc from p_0 to p_0 going around the puncture p_1 such that γ cuts the surface into two pieces, one of which contains the boundary, the puncture p_1 and no other punctures; see the picture on the left-hand side of Figure 9. If the genus of S is zero and p_1 is the only puncture, then the cluster algebra is of type \mathbb{A}_1 and $\mathcal{A} = \mathcal{U}$.

Otherwise, γ is not contractible, hence an arc, and we can choose a triangulation T containing the arcs α, β, γ . Then in the corresponding quiver Q_T the vertices α and β are sinks or sources. Now the quiver obtained by removing the vertices α and β from the quiver Q_T is the same as the quiver $Q_{T'}$ of the triangulation $T' = T - \{\alpha, \beta\}$ obtained by deleting the arcs α and β of the surface (S', M') obtained by replacing the puncture p_1 by a second point on the boundary; see the picture on the right-hand side of Figure 9. By induction, $Q_{T'}$ is Louise since (S', M') has two boundary points and one puncture less than (S, M) . \square

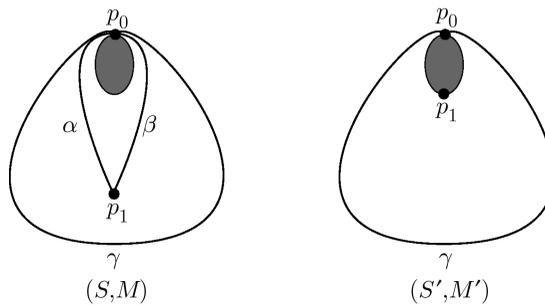


FIGURE 9. Proof of Proposition 12.

ACKNOWLEDGEMENTS

Part of this work was carried out at the Centre de Recherche Mathématiques in Montréal, and the authors thank the CRM for their hospitality. The authors also thank Greg Muller for stimulating discussions.

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