PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 3, Pages 9–17 (November 30, 2016)
http://dx.doi.org/10.1090/bproc/23

UNEXPECTED RELATIONS WHICH CHARACTERIZE OPERATOR MEANS

HIROYUKI OSAKA AND SHUHEI WADA

(Communicated by Stephan Ramon Garcia)

ABSTRACT. We give some characterizations of self-adjointness and symmetricity of operator monotone functions by using the Barbour transform $f \mapsto \frac{t+f}{1+f}$ and show that there are many non-symmetric operator means between the harmonic mean ! and the arithmetic mean ∇. Indeed, we show that there exists a non-symmetric operator mean between any two symmetric operator means.

1. INTRODUCTION

A bounded operator A acting on a Hilbert space H is said to be positive if $(Ax, x) \geq 0$ for all $x \in H$. We denote this by $A \geq 0$. Let $B(H)^+$ be the set of all positive operators on H and let $B(H)^{++}$ be the set of all positive invertible operators on H.

A real-valued function f on $(0, \infty)$ is operator monotone if whenever bounded operators A, B satisfy $0 < A \leq B$, $f(A) \leq f(B)$. Functions $f(t) = t^s \quad (s \in [0, 1])$ and $f(t) = \log t$ are typical operator monotone functions. Let OM_+ be the set of positive operator monotone functions on $(0, \infty)$ and $OM_+^1 = \{f \in OM_+ \mid f(1) = 1\}$.

In [6], Kubo and Ando developed an axiomatic theory for operator connections and operator means for pairs of positive operators. That is, a binary operation σ on the class of positive operators, $(A, B) \mapsto A\sigma B$, is called a connection if the following requirements are fulfilled:

(I) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$.
(II) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$.
(III) If $A_n \searrow A$ and $B_n \searrow B$, then $A_n\sigma B_n \searrow A\sigma B$.

A mean is a connection with normalization condition

(IV) $1\sigma 1 = 1$.

Kubo and Ando showed that there exists an affine order-isomorphism from the class of operator connections onto the class of positive operator monotone functions, $\sigma \mapsto f(t) = 1\sigma(t1)$.

This theory has found a number of applications in operator theory. In particular, Petz [9] connected the theory of monotone metrics with Kubo and Ando's operator

---

Received by the editors February 20, 2016 and, in revised form, June 18, 2016.
2010 *Mathematics Subject Classification.* Primary 47A64, 47A63.
Key words and phrases. Barbour transformation, operator monotone functions, operator connections, symmetric operator means, self-adjoint operator means.
The first author partially supported by the Program for Promotion of International Research (2014) (Ritsumeikan University).

connections. He proved that an operator monotone function $f : (0,\infty) \longrightarrow \mathbf{R}$ satisfying the functional equation

$$f(t) = tf(t^{-1}), \quad t > 0,$$

is related to a Morozova–Chentsov function which gives a monotone metric on the Riemannian manifold of invertible $n \times n$ density matrices.

It is well known that if $f : (0,\infty) \to (0,\infty)$ is operator monotone, the transpose $f'(t) = tf(\frac{1}{t})$, the adjoint $f^*(t) = \frac{1}{f(\frac{1}{t})}$, and the dual $f^\perp = \frac{t}{f(t)}$ are also operator monotone ([6]) and we call f symmetric if $f = f'$ and self-adjoint if $f = f^*$. It is shown in [6] that if f is symmetric with $f(1) = 1$, then the corresponding operator mean exists between the harmonic mean $!$ and the arithmetic mean ∇, that is, $! \leq \sigma_f \leq \nabla$.

In this note, we characterize symmetric functions and self-adjoint functions using Barbour transform $\widehat{\ } : OM_+ \to OM_+^1$ defined by $\hat{f} = \frac{t+f}{1+f}$, and characterize the class of non-symmetric operator functions and the class of operator connections σ which satisfy $\sigma \leq !$ or $\sigma \geq \nabla$. We show the existence of non-symmetric operator means between any pair of symmetric means.

2. Barbour transform

In [7], for any strictly positive continuous functions α, β, γ on $(0,\infty)$, the Barbour path $\phi_{\alpha,\beta,\gamma} : [0,1] \to OM_+^1$ was introduced by

$$\phi_{\alpha,\beta,\gamma}(x) = \frac{x\alpha + (1-x)\beta}{x + (1-x)\gamma}$$

and its basic properties are elucidated. In [1], Barbour examined a function $\phi_{t,\sqrt{t},\sqrt{t}}(x)$ which is an approximation of the power function t^x. We will denote a Barbour path $\phi_{\alpha,\beta,\gamma}(= \phi)$ such that $\phi(0) = f$, $\phi(\frac{1}{2}) = g$, $\phi(1) = h$ by the triple $[f, g, h]$.

Proposition 2.1 ([7]). *For $f \in OM_+$, the Barbour path $\phi_{t,f,f} = [1, \frac{t+f}{1+f}, t]$ exists in OM_+^1.*

The transform $\widehat{\ } : OM_+ \to OM_+^1$ defined by $f \mapsto \phi_{t,f,f}(\frac{1}{2}) = \frac{t+f}{1+f}$ plays an important role in the analysis of OM_+ and we call this transform the Barbour transform.

Proposition 2.2 (cf. [7]).

(1) *For $\lambda \in (0,1)$ the transform $B_\lambda : OM_+ \to OM_+^1$ defined by $f \mapsto \phi_{t,f,f}(\lambda)$ is injective and $B_\lambda(OM_+) = OM_+^1 \backslash \{1, t\}$.*

(2) *For $\lambda \in (0,1)$, $\{f \in OM_+^1 \mid !_\lambda \leq \sigma_f \leq \nabla_\lambda\} = B_\lambda(OM_+^1)$, where $A\nabla_\lambda B = (1-\lambda)A + \lambda B$ and $A!_\lambda B = ((1-\lambda)A^{-1} + \lambda B^{-1})^{-1}$, and notation $!_\lambda \leq \sigma_f \leq \nabla_\lambda$ means that $A!_\lambda B \leq A\sigma_f B \leq A\nabla_\lambda B$ for any positive operators A and B.*

Note that for $f \in OM_+$, $B_{\frac{1}{2}}(f) = \hat{f}$.

For $g \in OM_+^1$, we can define the inverse map $\check{\ }$ of the Barbour transform by

$$\check{g}(t) = \frac{t-g}{g-1};$$

then $\check{g} \in OM_+$. Moreover, for $\lambda \in (0,1)$ and $g \in OM_+^1$, we can define the inverse map $B_\lambda^{-1}(g)$ of the transform $B_\lambda(g)$ by $B_\lambda^{-1}(g) = \frac{\lambda}{1-\lambda} \cdot \check{g}$.

By a simple calculation, we have the following relations.

Lemma 2.3. *For $\lambda \in (0,1)$ and $f \in OM_+$, we have*

$$B_\lambda(f^\perp) = B_{1-\lambda}(f)^\perp, B_\lambda(f^{'}) = B_\lambda(f)^*, B_\lambda(f^*) = B_{1-\lambda}(f)^{'}.$$

In particular, $\widehat{f^\perp} = (\hat{f})^\perp, \widehat{f^*} = (\hat{f})^{'}$, *and* $\widehat{f'} = (\hat{f})^*$.

We next show some properties of the Barbour transform concerning the usual order relation for operator means.

Lemma 2.4. *For $f, g \in OM_+$ and $\lambda \in (0,1)$, the following are equivalent:*

(1) *$f(t) \le g(t)$ for all $t > 0$;*
(2) *$B_\lambda(B_\lambda(f))(t) \le B_\lambda(B_\lambda(g))(t)$ for all $t > 0$;*
(3) *$B_\lambda(f)(t) \le B_\lambda(g)(t)$ for all $t < 1$ and $B_\lambda(f)(t) \ge B_\lambda(g)(t)$ for all $t > 1$.*

Proof. The proof follows from the equation

$$B_\lambda(f) - B_\lambda(g) = \frac{\lambda(1-\lambda)(g-f)(t-1)}{\{\lambda + (1-\lambda)f\}\{\lambda + (1-\lambda)g\}}.$$

□

In the following section, we show that the Barbour transform is a powerful tool for finding examples of non-self-adjoint operator means.

3. Self-adjoint means

In [6], Kubo and Ando asked whether there exist self-adjoint operator means besides the trivial means w_l, w_r, the weighted geometric means corresponding to the operator monotone functions x^p $(0 < p < 1)$. Later, Hansen ([3]) gave an integral form for a strictly positive self-adjoint operator monotone function based on an exponential map.

Using the Barbour transform, we characterize the self-adjointness in OM_+ and give concrete examples in this section.

Proposition 3.1. *Let f be a positive continuous function on $(0,\infty)$ and let $\mathcal{A} = \{c, ct \mid c > 0\}$. The following are equivalent:*

(1) *$f \in OM_+^1 \backslash \{1, t\}$ and $f = f^*$;*
(2) *there exists an operator monotone function $g \in OM_+ \backslash \mathcal{A}$ such that $f = \sqrt{gg^*}$;*
(3) *there exists an operator monotone function $g \in OM_+$ such that*

$$f = \frac{t + g + g'}{1 + g + g'}.$$

Proof. (1) $\leftrightarrow$ (2): Set $g(t) = \sqrt{f}(t)$. Then $g \in OM_+ \backslash \mathcal{A}$, $g = g^*$, and $f = \sqrt{gg^*}$.

Conversely, if $f = \sqrt{gg^*}$ for some $g \in OM_+ \backslash \mathcal{A}$, it is obvious that $f \in OM_+^1$.

(1) $\to$ (3): Since $f \in OM_+^1$, there exists a $g \in OM_+$ such that $f = \hat{g}$. Since $\hat{g^{'}} = (\hat{g})^* = f^* = f = \hat{g}$ and the Barbour transform $\hat{\ }$ is injective, we have $g = g^{'}$; that is, g is symmetric.

Hence,

$$f = \frac{t + \frac{g}{2} + \left(\frac{g}{2}\right)'}{1 + \frac{g}{2} + \left(\frac{g}{2}\right)'}.$$

(3) $\to$ (1): Set $h = g + g'$. Then h is symmetric operator monotone and $\hat{h} = f$. Furthermore,

$$\begin{aligned} f^* &= \left(\hat{h}\right)^* \\ &= \hat{h'} \\ &= \hat{h} = f. \end{aligned}$$

□

The above result shows that we can construct a self-adjoint mean by using a symmetric mean.

Remark 3.2. Using Proposition 3.1, we can construct many examples of self-adjoint means. For example, if $g(t) = \log(t+1)$, then the corresponding operator means of functions $\sqrt{\log(t+1)/\log(t^{-1}+1)}$ and $\dfrac{t + \log(t+1) + t\log(t^{-1}+1)}{1 + \log(t+1) + t\log(t^{-1}+1)}$ are self-adjoint.

4. Symmetric means and non-symmetric means

4.1. **Symmetric means.** Symmetric means have been discussed many times in the literature [4–6, 8]. In contrast to self-adjoint means, many examples of symmetric means are known and appear in the quantum information literature [9].

Proposition 4.1. *Let f be a positive continuous function on $(0,\infty)$. The following are equivalent:*

(1) *$f \in OM_+$ and $f = f'$;*
(2) *there exists an operator monotone function $g \in OM_+$ such that*
$$f = g + g';$$
(3) *there exists an operator monotone function $g \in OM_+ \backslash \mathcal{A}$ such that*
$$f = \frac{t - \sqrt{gg^*}}{\sqrt{gg^*} - 1}.$$

Proof. (1) $\leftrightarrow$ (2): This is obvious.

(1) $\to$ (3): Set $h = \hat{f}$. Then $h \in OM_+ \backslash \{1, t\}$. Since $f = f'$, $h^* = (\hat{f})^* = \widehat{f'} = \hat{f} = h$. Hence by Proposition 3.1 there exists $g \in OM_+ \backslash \mathcal{A}$ such that $h = \sqrt{gg^*}$. Therefore

$$\begin{aligned} f &= \check{\hat{f}} \\ &= \check{h} \\ &= \frac{t - h}{h - 1} \\ &= \frac{t - \sqrt{gg^*}}{\sqrt{gg^*} - 1}. \end{aligned}$$

(3) $\to$ (1): Set $h = \sqrt{gg^*}$. Since $h^* = h$ implies that $(\check{h})' = \check{h}$, $f' = (\check{h})' = \check{h} = f$. □

Proposition 4.2. *Let f be a positive continuous function on $(0,\infty)$. The following are equivalent:*

1. $f \in OM_+^1 \backslash \{1,t\}$ *and* $f = f'$,
2. *there exists an operator monotone function* $g \in OM_+$ *such that*

$$f = \frac{t+\sqrt{gg^*}}{1+\sqrt{gg^*}}.$$

Proof. This follows from the same argument as for Proposition 3.1 using the formula $(\hat{h})' = \widehat{h^*}$ for $h \in OM_+$. □

4.2. **Non-symmetric means between ! and ∇.** It is well known that a symmetric operator mean must be between ! and ∇. To show that the converse is not true, we present an algorithm for constructing a non-symmetric mean σ such that $! \leq \sigma \leq \nabla$.

Lemma 4.3. *Let f be a positive operator monotone function on $(0,\infty)$ with $f(1) = 1$. The following are equivalent:*

1. $\sigma_{\hat{f}}$ *is non-symmetric mean;*
2. f *is non-self-adjoint.*

Proof. (2) → (1): Since $\left(\widehat{f}\right)' = \widehat{f^*}$, if f is non-self-adjoint operator monotone and $f(1) = 1$, $\hat{f}$ is non-symmetric; that is, $\sigma_{\hat{f}}$ is non-symmetric.

(1) → (2): If $\sigma_{\hat{f}}$ is a non-symmetric mean, then $\hat{f} \neq \left(\hat{f}\right)' = \widehat{f^*}$, which implies $f \neq f^*$. □

Lemma 4.4. *If a symmetric operator mean σ is self-adjoint, then $\sigma = \sharp$.*

Proof. Let f be an operator monotone function corresponding to σ. Then

$$f(t) = tf(\frac{1}{t}) = \frac{1}{f(\frac{1}{t})}.$$

Hence, $f(t) = \sqrt{t}$, and $\sigma = \sharp$. □

Remark 4.5. From Lemma 4.4, we know that all the following operator means are non-self-adjoint: arithmetic mean, logarithmic mean, harmonic mean, Heinz mean, and Lehmer mean [8].

Hence, we have the following result.

Proposition 4.6.

$$\begin{aligned}
&\{f \mid f: \text{ non-symmetric}, \ f_! \leq f \leq f_\nabla\} \\
&= \left\{\hat{f} \mid f: \text{ non-self-adjoint, f(1)=1}\right\} \\
&= \left\{\hat{\hat{f}} \mid f: \text{ non-symmetric}\right\} \\
&\supset \left\{\hat{f} \mid f: \text{ symmetric, f(1)=1}\right\} \setminus \{f_\sharp\}.
\end{aligned}$$

Proof. The first equality is clear from Lemma 4.3.

It follows from $\hat{\hat{f'}} = \left(\hat{\hat{f}}\right)'$ and Proposition 2.2 (2) that

$$\{f \mid f: \text{ non-symmetric}, \ f_! \le f \le f_\nabla\} = \left\{\hat{\hat{f}} \mid f: \text{ non-symmetric}\right\}.$$

Thus, we have the second equality.

From Lemma 4.4, we have

$$\{\hat{f} \mid f \neq f^*, f(1) = 1\} \supset \{\hat{f} \mid f = f', f(1) = 1\} \setminus \{f_\sharp\},$$

which means the last inclusion holds. □

Remark 4.7. From Proposition 4.6, a non-self-adjoint positive operator monotone function f with $f(1) = 1$ gives a non-symmetric operator mean $\sigma_{\hat{f}}$ such that $! \le \sigma_{\hat{f}} \le \nabla$. For example, let $p \in [-1, \frac{1}{2}) \cup (\frac{1}{2}, 2]$ and ALG_p be the function corresponding to the power difference mean defined by

$$\mathrm{ALG}_p(t) = \begin{cases} \dfrac{p-1}{p} \dfrac{1-t^p}{1-t^{p-1}} & t \neq 1, \\ 1 & t = 1. \end{cases}$$

Then ALG_p is symmetric and non-self-adjoint by Lemma 4.4. Hence, $\sigma_{\widehat{\mathrm{ALG_p}}}$ is non-symmetric. Moreover, the Petz-Hasegawa function f_p which is defined by

$$f_p(t) = p(p-1)\frac{(t-1)^2}{(t^p - 1)(t^{1-p} - 1)}$$

is non-self-adjoint. Hence, $\sigma_{\widehat{f_p}}$ is a non-symmetric operator mean between ! and ∇.

Later, in Section 5, we shall give an algorithm for constructing non-symmetric operator means between ! and ∇ (see Lemmas 5.1 and 5.2).

4.3. **Non-symmetric means between symmetric means.** In this section, we show the existence of non-symmetric operator means between any pair of symmetric means.

Theorem 4.8. *Let σ_1, σ_2 be symmetric operator means. If $\sigma_1 \le \sigma_2$ and $\sigma_1 \neq \sigma_2$, then there exists a non-symmetric operator mean σ such that $\sigma_1 \le \sigma \le \sigma_2$.*

To prove the above theorem, we need the following lemma.

Lemma 4.9. *Let h_1 and h_2 be self-adjoint positive operator monotone functions on $(0, \infty)$ with $h_1 \neq h_2$ and $h_1(1) = h_2(1) = 1$. If $h_1(t) \le h_2(t)$ for all $t < 1$ and $h_1(t) \ge h_2(t)$ for all $t > 1$, then there exists a non-self-adjoint positive operator monotone function h such that*

$$\begin{aligned} h_1(t) \le h(t) \le h_2(t) \quad & \text{for all } t < 1, \\ h_1(t) \ge h(t) \ge h_2(t) \quad & \text{for all } t > 1. \end{aligned}$$

Proof. Fix a non-self-adjoint mean g. If h is an operator monotone function corresponding to $\sigma_{h_1}(\sigma_g)\sigma_{h_2}$ (see [6, (2.9)]), then h satisfies the above inequalities.

It follows from the assumption $h_1 \neq h_2$ that there exists $\delta > 0$ such that

$$(1-\delta, 1+\delta) \subset \{h_1(t)h_2(t)^{-1} \mid t \in (0, \infty)\},$$

and

$$g(t) \neq g^*(t) \text{ on } (1-\delta, 1+\delta)\setminus\{1\}.$$

So there exists $t_0 \in (0,\infty)\backslash\{1\}$ such that $g\left(h_1(t_0)h_2(t_0)^{-1}\right) \neq g^*\left(h_1(t_0)h_2(t_0)^{-1}\right)$. Thus,

$$\begin{aligned}
h\left(\frac{1}{t_0}\right) &= h_1\left(\frac{1}{t_0}\right) g\left(\frac{h_2\left(\frac{1}{t_0}\right)}{h_1\left(\frac{1}{t_0}\right)}\right) \\
&= \frac{1}{h_1(t_0)} g\left(\frac{h_1(t_0)}{h_2(t_0)}\right) \\
&\neq \frac{1}{h_1(t_0)} g^*\left(\frac{h_1(t_0)}{h_2(t_0)}\right) \\
&= \frac{1}{h_1(t_0)} \frac{1}{g\left(\frac{h_2(t_0)}{h_1(t_0)}\right)} \\
&= \frac{1}{h(t_0)},
\end{aligned}$$

which implies $h \neq h^*$. □

Proof of Theorem 4.8. Let f_1, f_2 be positive operator monotone functions which correspond to σ_1, σ_2, respectively. If we define $h_1 := \check{f}_1, h_2 := \check{f}_2$, then h_1, h_2 satisfy the conditions appearing in Lemma 4.9. Thus, there exists a non-self-adjoint positive operator monotone function h such that

$$h_1(t) \leq h(t) \leq h_2(t) \quad \text{for all } t < 1$$

and

$$h_1(t) \geq h(t) \geq h_2(t) \quad \text{for all } t > 1.$$

By a simple calculation, we have

$$f_1 = \widehat{h_1} \leq \hat{h} \leq \widehat{h_2} = f_2$$

and

$$\hat{h} \neq (\widehat{h^*}) = (\hat{h})',$$

which means that $\sigma_{\hat{h}}$ is a desired mean. □

If we define $g(t) := \frac{1+t}{2}$ in the proof of Lemma 4.9, the function h which is an operator monotone function corresponding to $\sigma_{\check{f}_1}(\sigma_g)\sigma_{\check{f}_2}$ can be written as

$$h = \left(\frac{\check{f}_1 + \check{f}_2}{2}\right).$$

Corollary 4.10. *Let f_1, f_2 be symmetric positive operator monotone functions on $(0,\infty)$ with $f_1(1) = f_2(1) = 1$. If $f_1 \leq f_2$ and $f_1 \neq f_2$, then*

$$\left(\widehat{\frac{\check{f}_1 + \check{f}_2}{2}}\right) = \frac{2t(1-f_1)(1-f_2) + (f_1 - t)(1-f_2) + (f_2 - t)(1-f_1)}{2(1-f_1)(1-f_2) + (f_1 - t)(1-f_2) + (f_2 - t)(1-f_1)}$$

is a non-symmetric positive operator monotone function between f_1 and f_2.

Example 4.11. (1) Since $f_! = \hat{t}$ and $f_\nabla = \hat{1}$, $\widehat{\frac{t+1}{2}} = \frac{3t+1}{t+3}$ is a non-symmetric positive operator monotone function between $f_!$ and f_∇.

(2) Since $f_\sharp = \hat{f}_\sharp$ and $f_\nabla = \hat{1}$, $\widehat{\frac{\sqrt{t}+1}{2}} = \frac{2t+\sqrt{t}+1}{\sqrt{t}+3}$ is a non-symmetric positive operator monotone function between $f_\sharp$ and f_∇.

5. Application

It is well known that a symmetric operator mean exists between ! and ∇. To construct non-symmetric means between ! and ∇, we introduced the Barbour transform in Section 2 to characterize self-adjoint means and symmetric means.

Therefore, we can systematically construct non-symmetric operator means between ! and ∇.

Lemma 5.1. *Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous function. The following are equivalent:*

1. *$f \in OM_+$ and $f \geq f_\nabla$, that is, $f(t) \geq \frac{1+t}{2}$ for $t \in (0,\infty)$;*
2. *there exists an operator monotone function $g \in OM_+ \cup \{0\}$ and non-negative real numbers $a, b \geq \frac{1}{2}$ such that $\lim_{t\to 0} g(t) = 0$, $\lim_{t\to\infty} \frac{g(t)}{t} = 0$, and*
$$f(t) = a + bt + g(t) \ (t \in (0,\infty)).$$

Proof. The implication from (1) to (2): From Löwner's Theorem (for example, see [2, Exercise V.4.9]), there exists a positive Radon measure ρ on $[0,\infty]$ such that

$$f(t) = a + bt + \int_0^\infty \frac{t(1+\lambda)}{t+\lambda} d\rho(\lambda).$$

We have then

$$a = \lim_{t\to 0} f(0) \geq f_\nabla(0) = \frac{1}{2},$$
$$b = \lim_{t\to\infty} \frac{f(t)}{t} \geq \lim_{t\to\infty} \frac{f_\nabla(t)}{t} = \frac{1}{2}.$$

Define a function g on $(0,\infty)$ by $g(t) = \int_0^\infty \frac{t(1+\lambda)}{t+\lambda} d\rho(\lambda)$. Then we have $g \in OM_+$, $\lim_{t\to 0} g(t) = 0$, and $\lim_{t\to\infty} \frac{g(t)}{t} = 0$.

The implication from (2) to (1) is obvious. □

Lemma 5.2. *Let $f\colon (0,\infty) \to (0,\infty)$ be a continuous function. The following are equivalent:*

1. *$f \in OM_+$ and $f \leq f_!$, that is, $f(t) \leq \frac{2t}{1+t}$ $(t \in (0,\infty))$;*
2. *there exists an operator monotone function $g \in OM_+ \cup \{0\}$ and non-negative real numbers $a, b \geq \frac{1}{2}$ such that $\lim_{t\to 0} g(t) = 0$, $\lim_{t\to\infty} \frac{g(t)}{t} = 0$, and*
$$f(t) = \frac{t}{a + bt + g(t)} \ (t \in (0,\infty)).$$

Proof. The lemma follows from the observation that $f \leq f_!$ if and only if $\frac{t}{f} \geq f_\nabla$. □

Remark 5.3. The followings should be well known:

1. If $f \in OM_+^1$ and $f \leq f_!$, then $f = f_!$.
2. If $f \in OM_+^1$ and $f \geq f_\nabla$, then $f = f_\nabla$.

Proposition 5.4. *Suppose that $f \in OM_+$ and $f \leq f_!$. Then $f_! \leq \hat{\hat{\hat{f}}} \leq f_\nabla$ and $\hat{\hat{\hat{f}}}$ is not symmetric.*

Proof. We first consider the case $f(1) < 1$. Suppose that $\hat{f}$ is symmetric. Then $(\hat{f})' = (\widehat{f^*}) = (\hat{f})$. Since $\hat{}$ is injective, $f^* = f$; that is, f is self-adjoint. This is a contradiction and therefore $\hat{f}$ is not symmetric (so is $\hat{\hat{\hat{f}}}$). If $f(1) = 1$, then, by Remark 5.3, we have $f = f_!$ and $\hat{\hat{\hat{f}}}$ is not symmetric. □

Corollary 5.5. *Let a, b be non-negative real numbers greater than or equal to $\frac{1}{2}$ and assume $g \in OM_+ \cup \{0\}$ satisfies condition (2) in Lemma 5.2. Define a function $f : (0, \infty) \to (0, \infty)$ by $f(t) = \frac{t}{a+bt+g(t)}$ $(t \in (0, \infty))$. Then $f \in OM_+$, $f_! \le \hat{\hat{\hat{f}}} \le f_\nabla$ and $\hat{\hat{\hat{f}}}$ is not symmmetric.*

Proof. It is obvious that $f \in OM_+$ and $f \le f_!$. By Proposition 5.4, $f_! \le \hat{\hat{\hat{f}}} \le f_\nabla$ and $\hat{\hat{\hat{f}}}$ is not symmetric. □

References

[1] J. M. Barbour, *A geometrical approximation to the roots of numbers*, Amer. Math. Monthly **64** (1957), 1–9. MR0086098
[2] Rajendra Bhatia, *Matrix analysis*, Graduate Texts in Mathematics, vol. 169, Springer-Verlag, New York, 1997. MR1477662
[3] Frank Hansen, *Selfadjoint means and operator monotone functions*, Math. Ann. **256** (1981), no. 1, 29–35, DOI 10.1007/BF01450940. MR620119
[4] Fumio Hiai and Hideki Kosaki, *Means for matrices and comparison of their norms*, Indiana Univ. Math. J. **48** (1999), no. 3, 899–936, DOI 10.1512/iumj.1999.48.1665. MR1736973
[5] Fumio Hiai and Dénes Petz, *Introduction to matrix analysis and applications*, Universitext, Springer, Cham; Hindustan Book Agency, New Delhi, 2014. MR3184500
[6] Fumio Kubo and Tsuyoshi Ando, *Means of positive linear operators*, Math. Ann. **246** (1979/80), no. 3, 205–224, DOI 10.1007/BF01371042. MR563399
[7] Fumio Kubo, Noboru Nakamura, Koji Ohno, and Shuhei Wada, *Barbour path of operator monotone functions*, Far East J. Math. Sci. (FJMS) **57** (2011), no. 2, 181–192. MR2918503
[8] Yoshihiro Nakamura, *Classes of operator monotone functions and Stieltjes functions*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), Oper. Theory Adv. Appl., vol. 41, Birkhäuser, Basel, 1989, pp. 395–404. MR1038348
[9] Dénes Petz, *Monotone metrics on matrix spaces*, Linear Algebra Appl. **244** (1996), 81–96, DOI 10.1016/0024-3795(94)00211-8. MR1403277

Department of Mathematical Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan

Department of Information and Computer Engineering, Kisarazu National College of Technology, Kisarazu, Chiba 292-0041, Japan
E-mail address: wada@j.kisarazu.ac.jp