PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 5, Pages 1–5 (January 22, 2018)
http://dx.doi.org/10.1090/bproc/31

EXTENSION PROBLEM OF SUBSET-CONTROLLED QUASIMORPHISMS

MORIMICHI KAWASAKI

(Communicated by Ken Bromberg)

Abstract. Let (G, H) be $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞, B_n). We conjecture that any semi-homogeneous subset-controlled quasimorphism on $[G, G]$ can be extended to a homogeneous subset-controlled quasimorphism on G and also give a theorem supporting this conjecture by using a Bavard-type duality theorem on conjugation invariant norms.

1. Problems and results

To state our conjecture, we introduce the notion of subset-controlled quasimorphism (partial quasimorphism, pre-quasimorphism) which is a generalization of quasimorphism.

Definition 1.1. Let G be a group and let H be a subset of G. We define the fragmentation norm q_H with respect to H for an element f of G,

$$q_H(f) = \min\{k; \exists g_1 \dots, g_k \in G, \exists h_1, \dots h_k \in H \textit{ such that } f = g_1^{-1} h_1 g_1 \cdots g_k^{-1} h_k g_k\}.$$

If there is no such decomposition of f as above, we put $q_H(f) = \infty$. H *c-generates* G if such decomposition as above exists for any $f \in G$.

Definition 1.2. Let H, G' be subgroups of a group G. A function $\mu\colon G' \to \mathbb{R}$ is called an *H-quasimorphism on G'* if there exists a positive number C such that for any elements f, g of G',

$$|\mu(fg) - \mu(f) - \mu(g)| < C \cdot \min\{q_H(f), q_H(g)\}.$$

μ is called *homogeneous* if $\mu(f^n) = n\mu(f)$ holds for any element f of G' and any $n \in \mathbb{Z}$. μ is called *semi-homogeneous* if $\mu(f^n) = n\mu(f)$ holds for any element f of G' and any $n \in \mathbb{Z}_{\geq 0}$.

Such generalization as above of quasimorphism appeared first in [EP06]. For a symplectic manifold (M, ω), let $\mathrm{Ham}(M)$ denote the group of Hamiltonian diffeomorphisms with compact support and $(\mathbb{R}^{2n}, \omega_0)$, $(\mathbb{B}^{2n}, \omega_0)$ denote the $2n$-dimensional Euclidean space, ball with the standard symplectic form, respectively. Let B_n denote the n-braid group and B_∞ denote the infinite braid group $\bigcup_n B_n$.

We pose the following conjecture. For a group G, let $[G, G]$ denote the commutator subgroup (the subgroup generated by the set $\{[a, b] = aba^{-1}b^{-1} | a, b \in G\}$ of commutators).

Received by the editors December 18, 2016, and, in revised form, April 19, 2017 and September 1, 2017.
2010 *Mathematics Subject Classification.* Primary 20J06, 53D22; Secondary 57M27.
This work was supported by IBS-R003-D1.

Conjecture 1.3. *Let (G,H) be $(\mathrm{Ham}(\mathbb{R}^{2n}),\mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞,B_n). For a semi-homogeneous H-quasimorphism μ on $[G,G]$, there exists a homogeneous H-quasimorphism $\hat{\mu}$ on G such that $\hat{\mu}|_{[G,G]}=\mu$. In particular, any semi-homogeneous H-quasimorphism on $[G,G]$ is a homogeneous H-quasimorphism.*

The author ([Ka1]) and Kimura ([Ki1]) constructed a non-trivial H-quasimorphism on G when $(G,H)=(\mathrm{Ham}(\mathbb{R}^{2n}),\mathrm{Ham}(\mathbb{B}^{2n}))$, $(G,H)=(B_\infty,B_n)$, respectively. Kimura also proved that the dimension of the linear space of H-quasimorphisms on G is infinite when $(G,H)=(B_\infty,B_n)$ ([Ki2]).

We give examples of semi-homogeneous subset-controlled quasimorphisms which are not homogeneous. Let $\mathbb{T}^2$ be a 2-torus. By Proposition 3.1 of [Ka3], we see that the asymptotic Oh-Schwarz invariant $\mu\colon \mathrm{Ham}(\mathbb{T}^2)\to\mathbb{R}$ with respect to the fundamental class $[\mathbb{T}^2]$ is a semi-homogeneous $\mathrm{Ham}(U)$-quasimorphism for any open subset U of $\mathbb{T}^2$ whose closure $\bar{U}$ is contractible. Since a meridian curve M in the 2-torus $\mathbb{T}^2$ is heavy but not superheavy in the sense of Entov and Polterovich ([EP09]), we see that μ is not homogeneous. Let $\mathbb{A}$ be an annulus embedded to $\mathbb{T}^2$ such that $M,U\subset\mathbb{A}$. By restricting μ to $\mathrm{Ham}(\mathbb{A})$, we can construct a semi-homogeneous subset-controlled quasimorphism which is not homogeneous on $\mathrm{Ham}(\mathbb{A})$.

However, the author does not know whether there is a semi-homogeneous subset-controlled quasimorphism which is not homogeneous on $\mathrm{Ham}(\mathbb{B}^{2n})$.

Our main theorem is the following one which supports the above conjecture.

Theorem 1.4. *Let (G,H) be $(\mathrm{Ham}(\mathbb{R}^{2n}),\mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞,B_n). For a semi-homogeneous H-quasimorphism μ on $[G,G]$ and an element g of $[G,G]$ such that $\mu(g)\neq 0$, there exists a homogeneous H-quasimorphism $\hat{\mu}_g$ on G such that $\hat{\mu}_g(g)\neq 0$.*

In Section 2, we prepare some notions and statements. We prove Theorem 1.4 when $(G,H)=(B_\infty,B_n),(\mathrm{Ham}(\mathbb{R}^{2n}),\mathrm{Ham}(\mathbb{B}^{2n}))$ in Sections 3 and 4, respectively.

2. Preliminaries

Let G' be a G-invariant subgroup of a group G i.e. $g^{-1}g'g\in G'$ holds for any $g'\in G'$ and any $g\in G$. A function $\nu\colon G'\to\mathbb{R}_{\geq 0}$ is called *a G-invariant norm on G'* if ν is a conjugation-invariant norm on G' (see [BIP]) and $\nu(g^{-1}g'g)=\nu(g')$ holds for any $g'\in G'$ and any $g\in G$. A G-invariant norm ν_0 on G' is called *G-extremal* if for any G-invariant norm ν on G', there exist $a,b\in\mathbb{R}_{\geq 0}$ such that $a\nu(g')-b<\nu_0(g')$ holds for any $g'\in G'$.

Let G be a group and H a subgroup of G and $p,q\in\mathbb{Z}_{>0}\cup\{\infty\}$. We define the *$(H,p,q)$-commutator subgroup* $[G,G]^H_{p,q}$ of G with a subgroup H to be the subgroup generated by commutators $[f,g]$ such that $q_H(f)\leq p, q_H(g)\leq q$. We also define the *$(H,p,q)$-commutator length* $\mathrm{cl}^H_{p,q}\colon [G,G]^H_{p,q}\to\mathbb{R}$ by

$$\mathrm{cl}^H_{p,q}(h)=\min\{k\mid \exists f_1,\cdots,f_k,g_1,\cdots,g_k;\\ q_H(f_i)\leq p, q_H(g_j)\leq q\,(i,j=1,\cdots,k);\ h=[f_1,g_1]\cdots[f_k,g_k]\}.$$

We can easily prove that $[G,G]^H_{p,q}$ is a G-invariant subgroup and $\mathrm{cl}^H_{p,q}$ is a G-invariant norm on $[G,G]^H_{p,q}$. To prove Theorem 1.4, we use the following propositions.

Proposition 2.1 ([Ka1],[Ki1])**.** *Let (G,H) be $(\mathrm{Ham}(\mathbb{R}^{2n}),\mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞,B_n). Then $[G,G]^H_{p,q}=[G,G]$ holds for any $p,q\in\mathbb{Z}_{>0}\cup\{\infty\}$.*

For a conjugation-invariant norm ν on a group G, let $s\nu$ denote the stabilization of ν i.e. $s\nu(g) = \lim_{n\to\infty} \frac{\nu(g^n)}{n}$ (this limit exists by Fekete's Lemma).

Proposition 2.2 ([Ka1])**.** *If there exists a semi-homogeneous H-quasimorphism μ on $[G,G]^H_{p,q}$ with $\mu(g) \neq 0$ for some $g \in [G,G]^H_{p,q}$, then $scl^H_{p,q}(g) > 0$ holds for any $p, q \in \mathbb{Z}_{>0} \cup \{\infty\}$.*

Bavard ([Bav]) gave some duality theorem between stable commutator length and quasimorphisms which generalizes Matsumoto and Morita's famous work ([MM]). We use the following Bavard-type duality theorem.

Theorem 2.3 ([Ka2])**.** *Let (G,H) be $(\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ or (B_∞, B_n) and let ν be a conjugation-invariant norm on G. Then, for any element g of G such that $s\nu(g) > 0$, there exists a homogeneous H-quasimorphism $\mu\colon G \to \mathbb{R}$ such that $\mu(g) > 0$.*

3. Proof on braid group

In the present section, let G, H denote B_∞, B_n, respectively.

Theorem 1.4 when $(G,H) = (B_\infty, B_n)$ immediately follows from Proposition 2.2, Theorem 2.3, and the following proposition. Let σ_1 denote the first standard Artin generator of B_∞. It is known that $\{\sigma_1^{\pm 1}\}$ c-generates G.

Proposition 3.1 ([Ki1])**.** *The restriction of $q_{\{\sigma_1^{\pm 1}\}}$ to $[G,G]$ is G-extremal.*

Proof of Theorem 1.4 *when* $(G,H) = (B_\infty, B_n)$. Let g be an element of $[G,G]$ and let μ be a semi-homogeneous H-quasimorphism on $[G,G]$ with $\mu(g) \neq 0$. Since $\mu(g) \neq 0$, Proposition 2.2 implies $scl^H_{p,q}(g) > 0$. Thus, by Proposition 2.1 and Proposition 3.1, $sq_{\{\sigma_1^{\pm 1}\}}(g) > 0$. Then Theorem 2.3 implies that there exists a homogeneous H-quasimorphism $\hat{\mu}_g$ on G such that $\hat{\mu}_g(g) \neq 0$. □

4. Proof on Hamiltonian diffeomorphism group

In the present section, let G, H denote $\mathrm{Ham}(\mathbb{R}^{2n})$, $\mathrm{Ham}(\mathbb{B}^{2n})$, respectively. We follow the notion of [E] and thus let ϕ^t_F denote the time-t map of the Hamiltonian flow generated by F for a (time-dependent) Hamiltonian function $F\colon \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$.

Definition 4.1 ([C],[Ban])**.** *The Calabi homomorphism* $\mathrm{Cal}\colon \mathrm{Ham}(\mathbb{R}^{2n}) \to \mathbb{R}$ is defined by

$$\mathrm{Cal}(h) = \int_0^1 \int_M H {\omega_0}^n dt \text{ for a Hamiltonian diffeomorphism } h,$$

where $H\colon \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ is a Hamiltonian function which generates h. $\mathrm{Cal}(h)$ does not depend on the choice of generating Hamiltonian function H and thus the functional Cal is a well-defined homomorphism.

For proving Theorem 1.4 when $(G,H) = (\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$, it is important to construct a Hamiltonian analogue of $q_{\{\sigma_1^{\pm 1}\}}$. Let $F\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a (time-independent) Hamiltonian function such that $\phi^1_F \notin \mathrm{Ker}(\mathrm{Cal})$ and let h be an element of $\mathrm{Ker}(\mathrm{Cal})$. Note that $\mathrm{Cal}(\phi^t_F) = t\mathrm{Cal}(\phi^1_F)$. We define the conjugation-invariant norm $\nu_{F,h}$ by $\nu_{F,h} = q_{\{\phi^t_F\}_{t\in\mathbb{R}} \cup \{h^{\pm 1}\}}$. Since $[G,G]$ is a simple group and

$[G,G]=\mathrm{Ker}(\mathrm{Cal})$ ([Ban]), the subset $\{\phi_F^t\}_{t\in\mathbb{R}}\cup\{h^{\pm1}\}$ c-generates G. Thus $\nu_{F,h}$ is a conjugation-invariant norm on G.

We use the following proposition which is a Hamiltonian analogue of Proposition 3.1.

Proposition 4.2. *The restriction of $\nu_{F,h}$ to $[G,G]$ is G-extremal.*

To prove Proposition 4.2, we use the following lemma.

Lemma 4.3. *Let ν be a G-invariant norm on $[G,G]$. There exists a positive constant $C_{F,\nu}$ which depends only on F and ν such that $\nu([g,\phi_F^t])<C_{F,\nu}$ holds for any element g of G.*

Proof. Let R be a sufficient large number such that $\mathrm{Supp}(F)\subset Q_R$ where $Q_R=[-R,R]^{2n}\subset\mathbb{R}^{2n}$. Let h_0 be an element of $[G,G]$ such that $Q_R\cap h_0(Q_R)=\emptyset$. Note that $\nu(h_0)$ depends only on F and ν. Fix an element g of G and take an element h_g of G such that $h_g(Q_R)=Q_R$ and $h_gh_0(Q_R)\cap(Q_R\cup\mathrm{Supp}(g))=\emptyset$. Then $(h_gh_0h_g^{-1})(\phi_F^t)^{-1}(h_gh_0h_g^{-1})^{-1}$ commutes with ϕ_F^t and g and thus $[g,\phi_F^t]=[g,[\phi_F^t,h_gh_0h_g^{-1}]]$. Since ν is a G-invariant norm on $[G,G]$,

$$\begin{aligned}
\nu([g,\phi_F^t]) &\leq \nu(g[\phi_F^t,h_gh_0h_g^{-1}]g^{-1})+\nu([\phi_F^t,h_gh_0h_g^{-1}]^{-1})\\
&=2\nu([\phi_F^t,h_gh_0h_g^{-1}])\\
&\leq 2(\nu(\phi_F^t(h_gh_0h_g^{-1})(\phi_F^t)^{-1})+\nu((h_gh_0h_g^{-1})^{-1}))\\
&=4\nu(h_gh_0h_g^{-1})=4\nu(h_0).
\end{aligned}$$

□

Proof of Proposition 4.2. Let ϕ be an element of $[G,G]$ and m a natural number such that $\nu_{F,h}(\phi)\leq m$. Then, by the definition of $\nu_{F,h}$, there exist $f_1,\ldots,f_m\in\{\phi_F^t\}_{t\in\mathbb{R}}\cup\{h^{\pm1}\}$ and $g_1,\ldots,g_m\in G$ such that $\phi=g_1^{-1}f_1g_1\cdots g_m^{-1}f_mg_m$. We define a function $\tau\colon\{\phi_F^t\}_{t\in\mathbb{R}}\cup\{h^{\pm1}\}\to\mathbb{R}$ by

$$\tau(f)=\begin{cases} t & (\text{if } f=\phi_F^t),\\ 0 & (\text{if } f\in\{h^{\pm1}\}).\end{cases}$$

We define real numbers T_k $(k=1,\ldots,m+1)$ by $T_k=\sum_{i=1}^{k-1}\tau(f_i)$ and set $T_1=0$. Then we define elements α_k $(k=1,\ldots,m)$ of $\mathrm{Ker}(\mathrm{Cal})=[G,G]$ by

$$\alpha_k=\begin{cases}[\phi_F^{T_k}g_k^{-1},\phi_F^{t_k}] & (\text{if } f_k=\phi_F^{t_k}),\\ (\phi_F^{T_k}g_k^{-1})f_k(\phi_F^{T_k}g_k^{-1})^{-1} & (\text{if } f_k\in\{h^{\pm1}\}).\end{cases}$$

Fix a G-invariant norm ν on $[G,G]$. Note that Lemma 4.3 implies $\nu(\alpha_k)\leq\max\{C_{F,\nu},\nu(h)\}$ holds for any k. Since $\phi_F^{T_k}g_k^{-1}f_kg_k=\alpha_k\phi_F^{T_{k+1}}$ holds for any k,

$$\begin{aligned}
\phi&=\phi_F^{T_1}g_1^{-1}f_1g_1\cdots g_m^{-1}f_mg_m=\alpha_1\phi_F^{T_2}g_2^{-1}f_2g_2\cdots g_m^{-1}f_mg_m\\
&=\alpha_1\alpha_2\phi_F^{T_3}g_3^{-1}f_3g_3\cdots g_m^{-1}f_mg_m=\ldots=\alpha_1\cdots\alpha_m\phi_F^{T_{m+1}},
\end{aligned}$$

holds. Since $\phi\in\mathrm{Ker}(\mathrm{Cal})$ and $\alpha_k\in\mathrm{Ker}(\mathrm{Cal})$ for any k, $T_{m+1}=0$ and thus $\phi=\alpha_1\cdots\alpha_m$ holds. Since $\nu(\alpha_k)\leq\max\{C_{F,\nu},\nu(h)\}$ holds for any k, $\nu(\phi)\leq\max\{C_{F,\nu},\nu(h)\}\cdot m$ holds. Hence $\nu(\phi)\leq\max\{C_{F,\nu},\nu(h)\}\cdot\nu_{F,h}(\phi)$ holds for any element ϕ of $[G,G]$. □

The proof of Theorem 1.4 when $(G, H) = (\mathrm{Ham}(\mathbb{R}^{2n}), \mathrm{Ham}(\mathbb{B}^{2n}))$ is completely similar to the one when $(G, H) = (B_\infty, B_n)$ if we replace Proposition 3.1 by Proposition 4.2.

Acknowledgment

The author would like to thank Mitsuaki Kimura for some comments.

References

[Ban] Augustin Banyaga, *Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique* (French), Comment. Math. Helv. **53** (1978), no. 2, 174–227. MR490874

[Bav] Christophe Bavard, *Longueur stable des commutateurs* (French), Enseign. Math. (2) **37** (1991), no. 1-2, 109–150. MR1115747

[BIP] Dmitri Burago, Sergei Ivanov, and Leonid Polterovich, *Conjugation-invariant norms on groups of geometric origin*, Groups of diffeomorphisms, Adv. Stud. Pure Math., vol. 52, Math. Soc. Japan, Tokyo, 2008, pp. 221–250. MR2509711

[C] Eugenio Calabi, *On the group of automorphisms of a symplectic manifold*, Problems in analysis (Lectures at the Sympos. in honor of Salomon Bochner, Princeton Univ., Princeton, N.J., 1969), Princeton Univ. Press, Princeton, N.J., 1970, pp. 1–26. MR0350776

[E] M. Entov, *Quasi-morphisms and quasi-states in symplectic topology*, the Proceedings of the International Congress of Mathematicians (Seoul, 2014).

[EP06] Michael Entov and Leonid Polterovich, *Quasi-states and symplectic intersections*, Comment. Math. Helv. **81** (2006), no. 1, 75–99. MR2208798

[EP09] Michael Entov and Leonid Polterovich, *Rigid subsets of symplectic manifolds*, Compos. Math. **145** (2009), no. 3, 773–826. MR2507748

[Ka1] Morimichi Kawasaki, *Relative quasimorphisms and stably unbounded norms on the group of symplectomorphisms of the Euclidean spaces*, J. Symplectic Geom. **14** (2016), no. 1, 297–304. MR3523258

[Ka2] Morimichi Kawasaki, *Bavard's duality theorem on conjugation-invariant norms*, Pacific J. Math. **288** (2017), no. 1, 157–170. MR3667767

[Ka3] M. Kawasaki, *Superheavy Lagrangian immersions in surfaces*, to appear in J. Symplectic Geom..

[Ki1] M. Kimura, *Conjugation-invariant norms on the commutator subgroup of infinite braid group*, to appear in Journal of Topology and Analysis.

[Ki2] M. Kimura, private communication.

[MM] Shigenori Matsumoto and Shigeyuki Morita, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. **94** (1985), no. 3, 539–544. MR787909

Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Republic of Korea

Email address: kawasaki@ibs.re.kr