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A NOTE ON INEQUALITIES FOR THE RATIO OF ZERO-BALANCED HYPERGEOMETRIC FUNCTIONS

KENDALL C. RICHARDS

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Abstract. Motivated by a question suggested by M. E. H. Ismail in 2017, we present sharp inequalities for the ratio of zero-balanced Gaussian hypergeometric functions. The main theorems generalize known results for complete elliptic integrals of the first kind.

1. Introduction

The Gaussian hypergeometric function is given by

$$ {}_2F_1(a,b;c;r) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} r^n, $$

where

$$ (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} $$

is referred to as the Pochhammer symbol, or rising factorial, and simplifies to $(a)_n = a(a+1)\cdots(a+n-1)$. In the case that the denominator parameter satisfies $c = a+b$, the resulting ${}_2F_1(a,b;a+b;r)$ is said to be "zero-balanced". To provide context, we include a brief history of related estimates involving important special cases of zero-balanced hypergeometric functions. In particular, the complete elliptic integral of the first kind is defined by

$$ \mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2(t)}}dt. \tag{1.1} $$

It is well known that $\mathcal{K}(r)$ can be expressed in terms of ${}_2F_1$ in the following form:

$$ \mathcal{K}(r) = \frac{\pi}{2} {}_2F_1(1/2,1/2;1;r^2), \quad r \in (0,1). $$

Similarly, the generalized complete elliptic integrals of the first kind are defined for $a \in (0,1/2]$ and $r \in (0,1)$ by

$$ \mathcal{K}_a(r) = \frac{\pi}{2} {}_2F_1(a,1-a;1;r^2). $$

For more information on these functions, we refer the reader to [6,9] and, for recently obtained related results, to [7,10,13,14] and the references contained therein.

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The initial thread for this investigation begins with the following elegant inequality obtained by G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen [4] in 1990:

$$\frac{1}{1+r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} \quad \text{for all } r \in (0,1). \tag{1.2}$$

In light of this result, it is natural to ask the following questions:

- What is the best value λ such that

$$\frac{1}{1+\lambda r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} \quad \text{for all } r \in (0,1)?$$

- Can this be extended to the ratio of generalized complete elliptic integrals?

Motivated by these questions, H. Alzer and the author of this paper obtained the following result.

Theorem 1.1 ([1, Theorem 4.1]). *Let $a \in (0,1/2]$. For all $r \in (0,1)$ we have*

$$\frac{1}{1+\lambda_a r} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1}{1+\mu_a r}$$

with the best possible factors $\lambda_a = a(1-a)$ and $\mu_a = 0$.

M. E. H. Ismail [1, p. 1669], [11] asked whether Theorem 1.1 can be extended to the zero-balanced hypergeometric function. It is this question that serves as the catalyst and focal point of this paper, which answers the question in the affirmative.

Before presenting our main results, we provide some additional context and note one important refinement of (1.2) due to Anderson et al. First note that Theorem 1.1, with $a = 1/2$, implies that

$$\frac{1}{1+\lambda r} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} < \frac{1}{1+\mu r} \quad \text{for all } r \in (0,1), \tag{1.3}$$

with the best possible constant factors $\lambda = 1/4$ and $\mu = 0$. While it follows that (1.3) refines (1.2), it is important to note that Anderson, Vamanamurthy, and Vuorinen [5] proved the following inequality in 1992:

$$\frac{1}{\sqrt[4]{1+r}} < \frac{\mathcal{K}(r)}{\mathcal{K}(\sqrt{r})} \quad (0 < r < 1). \tag{1.4}$$

As noted in [1], (1.4) improves the lower bound in (1.3) with $\lambda = 1/4$. Moreover, it is this result by Anderson et al. that suggests a path toward answering the question by Ismail.

2. Main results

Theorem 2.1. *Suppose $a, b > 0$ with $a + b > ab$. For all $r \in (0,1)$ we have*

$$\frac{1}{(1+r)^{\lambda(a,b)}} < \frac{{}_2F_1(a,b;a+b;r^2)}{{}_2F_1(a,b;a+b;r)} < \frac{1}{(1+r)^{\mu(a,b)}}$$

with the best possible exponents $\lambda(a,b) = ab/(a+b)$ and $\mu(a,b) = 0$.

Remarks. The zero-balanced extension of Theorem 1.1 follows from Theorem 2.1 by noting that for $r \in (0,1)$,

$$\frac{1}{1+\lambda r} < \frac{1}{(1+r)^{\lambda}} \quad \text{for } 0 < \lambda < 1,$$

and $\frac{1}{(1+r)^{\mu}} < \frac{1}{1+\mu r}$ for $-1 < \mu < 0$. It is also worth noting that the value of

$$\frac{1}{1+\lambda r} - \frac{{}_2F_1(a,b;a+b;r^2)}{{}_2F_1(a,b;a+b;r)}$$

is not necessarily of constant sign when $\lambda > 1$.

In order to prove Theorem 2.1, we will make use of a result that is an immediate corollary to (the proof of) the following result which was proved in [8]:

Lemma 2.2 ([8, Lemma 1])**.** *Suppose $a, b, c > 0$ and $1 > \lambda > \max\{ab/c, a+b-c\}$. Then*

$$(-\lambda)_n \cdot {}_3F_2(-n, a, b; c, 1+\lambda-n; 1) < 0 \quad \textit{for all } n \in \mathbb{N},$$

where ${}_3F_2$ is the generalized hypergeometric function given by

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; r) := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n}{(b_1)_n (b_2)_n} \frac{r^n}{n!}.$$

The proof of Lemma 2.2 uses a generating function argument applied to the function given by $r \mapsto (1-r)^{\lambda}\, {}_2F_1(a,b;c;r)$ whose series coefficients can be expressed in terms of ${}_3F_2$. Placing our attention on the generating function rather than on its series coefficients, we arrive at the following basic result.

Lemma 2.3. *Suppose $a, b > 0$ and $1 > \lambda \geq ab/(a+b)$. Define*

$$f(r) := (1-r)^{\lambda}\, {}_2F_1(a,b;a+b;r).$$

Then $-f'$ is absolutely monotonic on $(0,1)$.

Remarks. As noted above, the basic result in Lemma 2.3 is a direct corollary to [8, Lemma 1], which appeared in 2001. The monotonicity of the special case $(1-r)^{1/4}\, {}_2F_1(a,b;a+b;r)$ for $4ab/ \leq a+b$ was verified in [2, Theorem 1.7]. Also, the monotonicity and concavity properties of $r'^c \mathcal{K}(r)$ for $c \geq 1/2$ were presented in [6, Theorem 3.21] and generalized in 2000 to $r'^c \mathcal{K}_a(r)$ for $a \in (0, 1/2]$ and $c \geq 2a(1-a)$ in [3, Lemma 5.4 (1)], where $r' = \sqrt{1-r^2}$.

Sketch of proof of Lemma 2.3. Suppose $a, b > 0$ and $1 > \lambda \geq ab/(a+b)$. An argument similar to that used in the proof of Lemma 2.2 as given in [8] yields that

$$-f'(r) = (1-r)^{\lambda-1} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(a+b)_n n!} \left(\lambda - \frac{ab}{a+b+n} \right).$$

Since $1 > \lambda \geq ab/(a+b)$ and

$$(1-r)^{\lambda-1} = \sum_{n=0}^{\infty} \frac{(1-\lambda)_n}{n!} r^n,$$

it follows that $-f'$ is absolutely monotonic since it is the product of two absolutely monotonic functions. □

Proof of Theorem 2.1. Suppose $a, b > 0 \geq \mu$ and $\lambda \geq ab/(a+b)$. We will prove the result that

$$\frac{1}{(1+r)^{\lambda}} < \frac{{}_2F_1(a,b;a+b;r^2)}{{}_2F_1(a,b;a+b;r)} < \frac{1}{(1+r)^{\mu}} \quad \text{for all } r \in (0,1).$$

In this direction, let f be defined as in Lemma 2.3 and, without loss of generality, suppose $\lambda < 1$. For $r \in (0,1)$, it follows that

$$(1-r^2)^\lambda \, {}_2F_1(a,b;a+b;r^2) > (1-r)^\lambda \, {}_2F_1(a,b;a+b;r),$$

which directly implies that

$$\frac{{}_2F_1(a,b;a+b;r^2)}{{}_2F_1(a,b;a+b;r)} > \frac{(1-r)^\lambda}{(1-r^2)^\lambda} = \frac{1}{(1+r)^\lambda}.$$

The sharpness of $\lambda = ab/(a+b)$ follows from the fact that

$$(1+r)^\gamma {}_2F_1(a,b;a+b;r^2) - {}_2F_1(a,b;a+b;r) = (\gamma - ab/(a+b))r + O(r^2).$$

The upper bound follows from the fact that the function $r \mapsto {}_2F_1(a,b;a+b;r)$ is increasing. The sharpness of $\mu = 0$ is obtained by using

$$\lim_{r\to 1} \frac{F(a,b;a+b;r^2)}{F(a,b;a+b;r)} = 1,$$

which follows from the asymptotic relation (see [9, eq. (2), p. 74])

$$F(a,b;a+b;x) \sim -\frac{1}{B(a,b)} \log(1-x) \quad (\text{as } x \to 1).$$

□

The following corollary generalizes the result in (1.4) by Anderson et al. in [5].

Corollary 2.4. *Let $a \in (0,1/2]$. For all $r \in (0,1)$ we have*

$$\frac{1}{(1+r)^{\lambda_a}} < \frac{\mathcal{K}_a(r)}{\mathcal{K}_a(\sqrt{r})} < \frac{1}{(1+r)^{\mu_a}}$$

with the best possible exponents $\lambda_a = a(1-a)$ and $\mu_a = 0$.

3. A further simplification

Before concluding, we will make a simplifying observation that allows us to relax the condition that $a + b > ab$ in Theorem 2.1. The proof will incorporate the following classical results (see [12, 15.5.1] and [12, 15.8.1], respectively). With $F(r) := {}_2F_1(a,b;c;r)$, it follows that

$$F'(r) = \frac{ab}{c} {}_2F_1(a+1,b+1;c+1;r), \tag{3.1}$$

$$F(r) = (1-r)^{c-a-b} {}_2F_1(c-a,c-b;c;r). \tag{3.2}$$

Theorem 3.1. *Suppose $a,b > 0$. For all $r \in (0,1)$ we have*

$$\frac{1}{(1+r)^{\lambda(a,b)}} < \frac{{}_2F_1(a,b;a+b;r^2)}{{}_2F_1(a,b;a+b;r)} < \frac{1}{(1+r)^{\mu(a,b)}}$$

with the best possible exponents $\lambda(a,b) = ab/(a+b)$ and $\mu(a,b) = 0$.

Proof: Let $\lambda \geq \frac{ab}{a+b}$ and define

$$f(r) := (1-r)^{\lambda}\, {}_2F_1(a,b;a+b;r)$$

for $r \in (0,1)$. As a natural extension of [2, Theorem 1.7] and its proof, one can easily show that f is decreasing. In particular, an application of (3.1) followed by (3.2) reveals that

$$\begin{aligned} f'(r) &= -\lambda(1-r)^{\lambda-1}{}_2F_1(a,b;a+b;r) \\ &\quad +(1-r)^{\lambda}\frac{ab}{a+b}\,{}_2F_1(a+1,b+1;a+b+1;r) \\ &= (1-r)^{\lambda-1}\left(\frac{ab}{a+b}\,{}_2F_1(b,a;a+b+1;r)-\lambda{}_2F_1(a,b;a+b;r)\right) \\ &\leq \lambda(1-r)^{\lambda-1}\left({}_2F_1(a,b;a+b+1;r)-{}_2F_1(a,b;a+b;r)\right). \end{aligned}$$

Since $\frac{(a)_n(b)_n}{(a+b+1)_n n!} < \frac{(a)_n(b)_n}{(a+b)_n n!}$, it follows that f is strictly decreasing on $(0,1)$, and the conclusion follows as in the proof of Theorem 2.1. □

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Department of Mathematics and Computer Science, Southwestern University, Georgetown, Texas 78627