IMPROVEMENT OF THE BERNSTEIN-TYPE THEOREM FOR SPACE-LIKE ZERO MEAN CURVATURE GRAPHS IN LORENTZ-MINKOWSKI SPACE USING FLUID MECHANICAL DUALITY

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ABSTRACT. Calabi’s Bernstein-type theorem asserts that a zero mean curvature entire graph in Lorentz-Minkowski space $L^3$ which admits only space-like points is a space-like plane. Using the fluid mechanical duality between minimal surfaces in Euclidean 3-space $E^3$ and maximal surfaces in Lorentz-Minkowski space $L^3$, we give an improvement of this Bernstein-type theorem. More precisely, we show that a zero mean curvature entire graph in $L^3$ which does not admit time-like points (namely, a graph consists of only space-like and light-like points) is a plane.

1. Introduction

Consider a 2-dimensional barotropic steady flow on a simply connected domain $D$ in the $xy$-plane $R^2$ whose velocity vector field is $v = (u, v)$, with density $\rho$ and pressure $p$. We assume there are no external forces. Then

- the flow is a foliation of the integral curve of $v$,
- $\rho$ is a scalar field on $D$,
- $p: R \to R$ is a monotone function of $\rho$,
- $c := \sqrt{p'(\rho)}$ ($p' := dp/d\rho$) is called the local speed of sound.
- The following Euler’s equation of motion holds:

\[
dp + \frac{\rho}{2} d(|v|^2) = 0.
\]

We also assume the flow is irrotational; that is,

\[
0 = \text{rot}(v) = v_x - u_y
\]
where \( v_x := \partial v/\partial x, u_y := \partial u/\partial y \). Here, ‘the equation of continuity’ is equivalent to the fact that

\[
0 = \text{div}(\rho v) = (\rho u)_x + (\rho v)_y.
\]

By (1.2), there exists a function \( \Phi: D \to \mathbb{R} \), called the potential of the flow, such that

\[
0 = \text{div}(\rho v) = \rho u_x + \rho v_y.
\]

By (1.3), one can easily check that

\[
0 = (c^2 - \Phi^2_x)\Phi_{xx} - 2\Phi_x\Phi_y + (c^2 - \Phi^2_y)\Phi_{yy}.
\]

On the other hand, by (1.3), there exists a function \( \Psi: D \to \mathbb{R} \), called the stream function of the flow, such that

\[
\Psi_x = -\rho v, \quad \Psi_y = \rho u.
\]

If we set \( \xi := \rho u \) and \( \eta := \rho v \), (1.4) can be written as

\[
(\rho^2 c^2 - \xi^2 - \eta^2)(\rho_x, \rho_y) = -\rho(\xi_x + \eta_y, \xi_x + \eta_y).
\]

Since

\[
0 = v_x - u_y = \eta_x - \xi_y - \eta \rho_x - \xi \rho_y,
\]

the identity \( 0 = \rho(\xi^2 + \eta^2 - \rho^2 c^2)(v_x - u_y) \) yields that

\[
0 = (\rho^2 c^2 - \Psi^2_y)\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + (\rho^2 c^2 - \Psi^2_x)\Psi_{yy}.
\]

A flow satisfying

\[
\rho c = 1
\]

is called a Chaplygin gas flow (see [4, p. 24] and also [11, Section 4]). For a given stream function \( \Psi: D \to \mathbb{R} \) of the Chaplygin gas flow, we set

\[
B_\Psi := 1 - \Psi_x^2 - \Psi_y^2.
\]

Let \( D \) be a domain in the \( uv \)-plane \( \mathbb{R}^2 \). Let \( f: D \to \mathbb{L}^3 \) be an immersion into the Lorentz-Minkowski 3-space \( \mathbb{L}^3 \) of signature \((++-)\). We set

\[
P := \begin{pmatrix} f_u \cdot f_u & f_u \cdot f_v \\ f_v \cdot f_u & f_v \cdot f_v \end{pmatrix}
\]

and

\[
B_f := \text{det}(P),
\]

where \( \cdot \) denotes the canonical Lorentzian inner product of \( \mathbb{L}^3 \) and \( \text{det}(P) \) denotes the determinant of the \( 2 \times 2 \) matrix \( P \). A point \( p \in U \) where \( B_f(p) > 0 \) (resp., \( B_f(p) < 0 \), \( B_f(p) = 0 \)) is said to be space-like (resp., time-like, light-like). We set

\[
Q := \begin{pmatrix} f_{uu} \cdot \tilde{v} & f_{uv} \cdot \tilde{v} \\ f_{vu} \cdot \tilde{v} & f_{vv} \cdot \tilde{v} \end{pmatrix},
\]

where \( \tilde{v} := f_u \times_L f_v \) and \( \times_L \) is the canonical Lorentzian vector product of \( \mathbb{L}^3 \). Consider the matrix \( W := P Q \) and set

\[
A_f := \text{trace}(W),
\]
where \( \tilde{P} \) is the cofactor matrix of \( P \). We call \( f \) a zero mean curvature surface if \( A_f \) vanishes identically. In this paper, for the sake of simplicity, we abbreviate ‘zero mean curvature’ by ‘ZMC’. A ZMC-surface consisting only of space-like points is called a maximal surface. On the other hand, a surface in \( \mathbb{L}^3 \) consisting only of light-like points is called a light-like surface. It is known that the identity \( B_f = 0 \) implies that \( A_f = 0 \) (see [21, Proposition 2.1]). In particular, any light-like surfaces are ZMC-surfaces in our sense. Moreover, at a point where \( B_f \neq 0 \), the mean curvature function \( H \) of \( f \) is well-defined, and \( A_f = 0 \) is equivalent to the condition that \( H = 0 \).

We now assume that \( f \) is written in the form \( f(x,y) = (x,y,\Psi(x,y)) \). Then it can be easily checked that \( B_f = B_\Psi \) (cf. (1.9)) and

\[
A_f(x,y) = \left(1 - \Psi_y^2\right)\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + \left(1 - \Psi_x^2\right)\Psi_{yy}.
\]

Under the condition (1.8), the equation (1.7) for the stream function \( \Psi \) reduces to

\[
(1.10) \quad \left(1 - \Psi_y^2\right)\Psi_{xx} + 2\Psi_x\Psi_y\Psi_{xy} + \left(1 - \Psi_x^2\right)\Psi_{yy} = 0,
\]

which implies that \( A_f \) vanishes identically. So we call this the ZMC-equation in \( \mathbb{L}^3 \). If \( \rho_c = 1 \), then \( \frac{1}{\rho^2} = c^2 = \frac{dp}{d\rho} \); that is, \( \frac{dp}{d\rho} = \rho_c^2 \) is obtained. Substituting this into (1.1), we get

\[
(1.11) \quad |v|^2 + \mu = \frac{1}{\rho^2} = c^2.
\]

By (1.6), we can rewrite this as

\[
(1.12) \quad B_\psi = \mu \rho^2.
\]

By (1.11) and (1.12), the sign change of \( B_\psi \) corresponds to the type change of the Chaplygin gas flow from sub-sonic (\( |v| < c \)) to super-sonic (\( |v| > c \)); that is, the sub-sonic part satisfies \( B_\psi > 0 \). If \( \mu = 0 \), then \( B_\psi \) vanishes identically, and the graph of \( \Psi \) gives a light-like surface. Such surfaces are discussed in the appendix, and we now consider the case \( \mu \neq 0 \). Since \( B_\psi \) and \( \mu \) have the same sign (cf. (1.12)), we can write

\[
(1.13) \quad \rho = \frac{1}{\sqrt{|v|^2 + \mu}} = \sqrt{\frac{1 - \Psi_x^2 - \Psi_y^2}{\mu}}.
\]

By (1.11) and the fact that \( |v|^2 = \Phi_x^2 + \Phi_y^2 \), (1.5) can be written as

\[
(1.14) \quad (\mu + \Phi_y^2)\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + (\mu + \Phi_x^2)\Phi_{yy} = 0.
\]

We set

\[
(1.15) \quad \varphi(x,y) := \tilde{\mu}\Phi(\tilde{\mu}x, \tilde{\mu}y) \quad (\tilde{\mu} := 1/\sqrt{|\mu|}).
\]

If \( \mu > 0 \), then (1.14) reduces to

\[
(1.16) \quad (1 + \varphi_y^2)\varphi_{xx} - 2\varphi_x\varphi_y\varphi_{xy} + (1 + \varphi_x^2)\varphi_{yy} = 0,
\]

which is known as the condition that the graph of \( \varphi(x,y) \) gives a minimal surface in the Euclidean 3-space \( \mathbb{E}^3 \). On the other hand, if \( \mu < 0 \), then (1.14) reduces to

\[
(1.17) \quad (1 - \varphi_y^2)\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + (1 - \varphi_x^2)\varphi_{yy} = 0.
\]
which is the ZMC-equation (cf. (1.10)). It can be easily checked that the graph of \( \varphi \) is a time-like ZMC-surface in \( L^3 \). In both of the two cases, it can be easily checked that 
\[
(\psi_x, \psi_y) = \frac{1}{\sqrt{\varphi_x^2 + \varphi_y^2 + \epsilon}} \left( \frac{-\varphi_y}{\varphi_x} \right)
\]
holds, where \( \psi := \Psi(\tilde{\mu}x, \tilde{\mu}y)/\tilde{\mu} \). Note that \( \Psi \) satisfies (1.10) if and only if \( \psi \) satisfies (1.10). Moreover, one can easily check that
\[
(\hat{\rho} := \frac{1}{\sqrt{\varphi_x^2 + \varphi_y^2 + \epsilon}}) = \sqrt{\epsilon} (1 - \psi_x^2 - \psi_y^2)
\]
and
\[
(\varphi_x, \varphi_y) = \frac{1}{\sqrt{\epsilon (1 - \psi_x^2 - \psi_y^2)}} (\psi_y, -\psi_x).
\]
This means that \( \varphi \leftrightarrow \psi \) corresponds to the duality between potentials and stream functions of Chaplygin gas flows such that
\- \( \mu = \pm 1(= \epsilon) \),
\- the density \( \hat{\rho} \) is given as (1.18), and
\- \( p = p_0 - 1/\hat{\rho} \) for some constant \( p_0 \).

When \( \epsilon = 1 \) (resp., \( \epsilon = -1 \)), this gives a correspondence between graphs of minimal surfaces \( (x, y) \rightarrow \varphi(x, y) \) in \( E^3 \) and graphs of maximal surfaces \( (x, y) \rightarrow \psi(x, y) \) in \( L^3 \) (resp., an involution on the set of graphs of time-like ZMC-surfaces in \( L^3 \)) which we call the fluid mechanical duality. 

A part of the above dualities is suggested in the classical book [4]. Calabi [5] also recognized this duality for \( \mu > 0 \) and pointed out the following:

**Fact 1.1** (Calabi’s Bernstein-type theorem). Suppose that the graph of a function \( \psi: \mathbb{R}^2 \rightarrow \mathbb{R} \) gives a maximal surface (that is, a surface consisting only of space-like points whose mean curvature function vanishes identically). Then \( \psi - \psi(0,0) \) is linear.

This is an analogue of the classical Bernstein theorem for minimal surfaces in \( E^3 \). Moreover, Calabi [5] obtained the same conclusion for entire space-like ZMC-graphs in \( L^{n+1} \) \((n \leq 4)\), and Cheng and Yau [6] extended this result for complete maximal hypersurfaces in \( L^{n+1} \) for \( n \geq 5 \). The assumption that the graph consists only of space-like points is crucial. Entire ZMC-graphs which are not planar actually exist. Typical such examples are of the form
\[
(1.19) \quad \psi_0(x, y) := y + g(x),
\]
where \( g: \mathbb{R} \rightarrow \mathbb{R} \) is any \( C^\infty \)-function of one variable. A point \( p = (x_0, y_0) \in \mathbb{R}^2 \) is a light-like point of \( \psi_0 \) if and only if \( g'(x_0) = 0 \). Moreover, if the graph of \( \psi_0 \) does not contain any light-like points, the potential function \( \varphi_0 \) corresponding to \( \psi_0 \) is given by
\[
\varphi_0(x, y) = \pm \left( -y + \int_0^x \frac{du}{g'(u)} \right)
\]
up to a constant, where the sign “\( \pm \)” coincides with that of \( g' \). On the other hand, Osamu Kobayashi [18] pointed out the existence of entire graphs of ZMC-surfaces
with space-like points, light-like points, and time-like points all appearing. Such a surface is called of **mixed type**. Recently, many such examples were constructed in [9].

By definition, any entire ZMC-graph of mixed type has at least one light-like point. So we give the following definition.

**Definition 1.2.** A light-like point $p$ of the function $\psi$ (i.e., $B_\psi(p) = 0$) is said to be **non-degenerate** (resp., **degenerate**) if $\nabla B_\psi$ does not vanish (resp., vanishes) at $p$.

At each non-degenerate light-like point, the graph of $\psi$ changes its causal type from space-like to time-like. This case is now well understood. In fact, under the assumption that the surface is real analytic, it can be reconstructed from a real analytic null regular curve in $L^3$ (cf. Gu [12] and also [11,16,17]).

On the other hand, there are several examples of ZMC-surfaces with degenerate light-like points (cf. [1,2,10,14]). Moreover, a local general existence theorem for maximal surfaces with degenerate light-like points is given in [21]. For such degenerate light-like points, we need a new approach to analyze the behavior of $\psi$ and $\varphi$. The following fact was proved by Klyachin [17] (see also [21]).

**Fact 1.3** (The line theorem for ZMC-surfaces). Let $D$ be a domain of $\mathbb{R}^2$ and let $F: D \to L^3$ be a $C^3$-differentiable ZMC-immersion such that $o \in D$ is a degenerate light-like point. Then, there exists a light-like line segment $\hat{\sigma}$ ($\subset L^3$) passing through $F(o)$ of $L^3$ such that $F(o)$ does not coincide with one of the two end points of $\hat{\sigma}$ and $F(\Sigma)$ contains $\hat{\sigma}$, where $\Sigma$ is the set of degenerate light-like points of $F$.

Recently, Fact 1.3 was generalized to a much wider class of surfaces, including constant mean curvature surfaces in $L^3$; see [21][22]. (In [21], the general local existence theorem of surfaces which changes their causal types along degenerate light-like lines was also shown.) The asymptotic behavior of $\psi$ along the line $l$ consisting of degenerate light-like points is discussed in [21].

The purpose of this paper is to prove the following assertion:

**Theorem A.** An entire $C^3$-differentiable ZMC-graph which is not a plane admits a non-degenerate light-like point if its space-like part is non-empty.

This assertion is proved in Section 2 using the fluid mechanical duality and the half-space theorem for minimal surfaces in $E^3$ given by Hoffman-Meeks [15]. It should be remarked that the half-space theorem does not hold for time-like ZMC-surfaces. In fact, the graph of $\varphi(x,y) := y + \log(\tan x)$ ($x \in (0, \pi/2)$) gives a properly embedded time-like ZMC-surface lying between two parallel vertical planes. In Section 2, we give further examples and provide a few questions related to Theorem A. As an application, we give the following improvement of Calabi’s Bernstein-type theorem:

**Corollary B.** An entire $C^3$-differentiable ZMC-graph which does not admit any time-like points is a plane.

In fact, if the ZMC-graph admits a space-like point, then the assertion immediately follows from Theorem A. So it remains to show the case that the graph consists only of light-like points. However, such a graph must be a plane, as shown in the appendix (see Theorem A.1).
2. Proof of Theorem A

In this section, we prove Theorem A in the introduction. We let \( \psi : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^3 \)-function satisfying the ZMC-equation (1.10). We assume \( \psi \) admits a space-like point \( q_0 \in \mathbb{R}^2 \) but admits no non-degenerate light-like points. By Calabi’s Bernstein-type theorem (cf. Fact 1.1), \( \psi \) has at least one degenerate light-like point. We set

\[
F_\psi(x, y) := (x, y, \psi(x, y)),
\]

which gives the ZMC-graph of \( \psi \). We denote by \( ds^2 \) the positive semi-definite metric which is the pull-back of the canonical Lorentzian metric of \( \mathbb{L}^3 \) by \( F_\psi \). The line theorem (cf. Fact 1.3) yields that the image of \( F_\psi \) contains a light-like line segment \( \sigma \). Then the projection of \( \sigma \) is a line segment \( \sigma \) on the \( xy \)-plane \( \mathbb{R}^2 \). Then \( \sigma \) lies on a line \( l \) on \( \mathbb{R}^2 \). If \( \sigma \neq l \), then there exists an end point \( p \) of \( \sigma \) on \( l \). Since \( p \) is the limit point of degenerate light-like points, \( p \) itself is also a degenerate light-like point. By applying the line theorem again, there exists a light-like line segment \( \sigma' \) containing \( F_\psi(p) \) as its interior point. We denote by \( \sigma'' \) the projection of \( \sigma' \) to the \( xy \)-plane. Since the null direction at \( p \) with respect to the metric \( ds^2 \) is uniquely determined, \( \sigma'' \) also lies on the line \( l \). Thus, the entire graph contains a whole light-like line containing \( \sigma \). In particular, degenerate light-like points on the graph consist of a family of straight lines in \( \mathbb{R}^2 \).

Let \( l \) and \( l' \) be two such straight lines. Then \( l' \) never meets \( l \). In fact, if not, then there is a unique intersection point \( q \in l \cap l' \). By Fact 1.3 two lines \( l, l' \) can be lifted to two light-like lines \( \tilde{l} \) and \( \tilde{l}' \) in \( \mathbb{L}^3 \) passing through \( F_\psi(q) \). The tangential directions of \( \tilde{l} \) and \( \tilde{l}' \) are linearly independent light-like vectors at \( F_\psi(q) \). Then by [19, Lemma 27 in Section 5], \( q \) is a time-like point, a contradiction.

Thus, the set of degenerate light-like points of \( F_\psi \) consists of a family of parallel lines in the \( xy \)-plane. Without loss of generality, we may assume that these lines are vertical and one of them is the \( y \)-axis. Then we can find a domain \( (\Delta \in (0, \infty]) \)

\[
\Omega := \{(x, y) : 0 < x < 2\Delta \}
\]

such that \( q_0 \in \Omega \) and \( F_\psi \) has no light-like points on \( \Omega \) and both of the lines \( l = \{x = 0\} \) and \( l' = \{x = 2\Delta\} \) consist of light-like points unless \( \Delta = \infty \). Since there are no light-like points on \( \Omega \), the potential function \( \varphi : \Omega \to \mathbb{R} \) is induced by \( \psi \) as the fluid mechanical dual. The graph of \( \varphi \) is a minimal surface in \( \mathbb{E}^3 \). In particular, \( \varphi \) is real analytic. If we succeed in proving that the map \( F_\varphi(x, y) := (x, y, \varphi(x, y)) \) is proper, then Theorem A follows. In fact, by the half-space theorem given in [15] the image \( F_\varphi(\Omega) \) lies in a plane in \( \mathbb{E}^3 \). Then the map \( F_\varphi(x, y) \) also lies in a plane \( \Pi \) in \( \mathbb{L}^3 \) on \( \overline{\Omega} \). Since \( F_\psi(l) \) is light-like, the plane \( \Pi \) must be light-like, contradicting the fact that \( q_0 \in \Omega \).

To prove the properness of \( F_\varphi \), it is sufficient to show the following:

**Lemma 2.1.** Let \( \{p_n\}_{n=1}^\infty \) be a sequence of points in \( \Omega \) accumulating to a point on \( l \) or \( l' \). Then \( \{\varphi(p_n)\}_{n=1}^\infty \) diverges.

**Proof.** By switching the roles of \( l \) and \( l' \) if necessary, it is sufficient to consider the case that \( \{p_n\}_{n=1}^\infty \) accumulates to a point on \( l \). Taking a subsequence and using a suitable translation of the \( xy \)-plane, we may assume that \( \{p_n\}_{n=1}^\infty \) converges to the origin \( (0, 0) \in l \) and \( p_n = (x_n, y_n) \) \( (n = 1, 2, 3, \ldots) \) satisfies the following properties:

- there exists \( \epsilon > 0 \) such that \( |y_n| < \epsilon \) for each \( n = 1, 2, \ldots \), and
there exists \((\delta, 0) \in \Omega \ (\delta > 0)\) such that
\[
\delta > x_1 > x_2 > \cdots > x_n > x_{n+1} > \cdots
\]
Since \(l\) consists of degenerate light-like points, there exists a neighborhood \(U\) of \((0, 0)\) such that (see [10] or [21 (6.1)])
\[
\psi(x, y) = y + x^2 h(x, y) \quad ((x, y) \in U),
\]
where \(h(x, y)\) is a \(C^1\)-differentiable function defined on \(U\) (see [21, Appendix A]).

Taking \(\epsilon, \delta\) to be sufficiently small, we may assume that
\[
V := \{(x, y) \in \Omega; |x| \leq \delta, |y| < \epsilon\} \subset U.
\]
Since \(B_{\psi} > 0\), the potential function \(\varphi\) associated to \(\psi\) satisfies (cf. (1.18))
\[
\varphi_x = \frac{\psi_y}{\rho}, \quad \rho = \sqrt{1 - \psi_x^2 - \psi_y^2}.
\]
Since
\[
1 - \psi_x^2 - \psi_y^2 = -x^2 \left( (2h + xh_x)^2 + 2h_y + x^2 h_y^2 \right)
\]
is non-negative on the closure \(\overline{V}\) of \(V\), we can write
(2.1)
\[
\sqrt{\rho} = |x|k(x, y),
\]
where \(k(x, y)\) is a non-negative continuous function defined on \(\overline{V}\) such that \(k\) is positive-valued on \(V\). We set \(p_0 := (\delta, 0)\) and consider the path \(\gamma_n : [0, 1] \to V\) defined by \(\gamma_n(s) := (\delta, 2s y_n)\) if \(0 \leq s \leq 1/2\) and
\[
\gamma_n(s) := (2(x_n - \delta)s - x_n + 2\delta, y_n)
\]
if \(1/2 \leq s \leq 1\), which starts at \(p_0\) and terminates at \(p_n\). This curve \(\gamma_n\) is the union of the vertical subarc \(\gamma_{n,1}\) and the horizontal subarc \(\gamma_{n,2}\). So we can write
\[
\varphi(p_n) - \varphi(p_0) = \int_{\gamma_n} \varphi_x dx + \varphi_y dy
\]
\[
= \int_{\gamma_{n,1}} \varphi_x dx + \int_{\gamma_{n,1}} \varphi_y dy.
\]
Since \([-\epsilon, \epsilon] \ni y \mapsto \varphi_y(\delta, y) \in R\) is a continuous function, we have that
\[
\left| \int_{\gamma_{n,1}} \varphi_y dy \right| \leq \epsilon \max_{|y| \leq \epsilon} \left| \varphi_y(\delta, y) \right| < \infty.
\]
So to prove the lemma, it is sufficient to show that \(\int_{\gamma_{n,2}} \varphi_x dx\) diverges as \(n \to \infty\).
We set
\[
m := \max_{x \in [0, \delta], |y| \leq \epsilon} k(x, y) \ (\geq 0),
\]
where \(k\) is the continuous function given in (2.1). On the other hand, we can take a constant \(m'(> 0)\) such that
\[
\psi_y = 1 + x^2 h_y(x, y) > m' \quad (x \in [0, \delta], |y| \leq \epsilon),
\]
since $\epsilon, \delta$ can be chosen to be sufficiently small. Since $\varphi_x = \psi_y/\rho$, we have
\[
\left| \int_{\gamma_{n,2}} \varphi_x dx \right| = \int_{x_n}^{\delta} \frac{1 + x^2 h_y(x,y)}{x^2k^2(x,y)} dx \geq \frac{m'}{m^2} \int_{x_n}^{\delta} \frac{dx}{x^2} = \frac{m'}{m^2} \left( \frac{1}{x_n} - \frac{1}{\delta} \right) \to \infty,
\]
proving the assertion.

\[\square\]

**Remark 2.2.** In the above proof, we showed that $F_\psi(\Omega)$ lies in a plane using the fluid mechanical duality. We remark here that this can be proved by a different method. In fact, $\psi$ satisfies the assumption of Ecker \[7, Theorem G\] or is a PS-graph on the convex domain $\Omega$ in the sense of Fernandez and Lopez \[8\]. Thus, we can conclude that $\psi(\Omega)$ lies in a light-like plane.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The ZMC-surfaces in Example 2.3 (left) and in Example 2.4 (right), where the white lines indicate light-like points.}
\end{figure}

In \[1\], the first author constructed several ZMC-surfaces foliated by circles and at most countably many straight lines. At the end of this paper, we pick up two important examples of them which contain degenerate light-like points. (In \[1\], these two examples are not precisely indicated. Here we show their explicit parametrization and embeddedness.)

**Example 2.3 (\[1, Figure 5\]).** We set
\[
F(u,v) := (u + a \cos v, a \sin v, u),
\]
where $a > 0$ and $(u,v) \in \mathbb{R} \times [0,2\pi)$. Then the image of $F$ contains two parallel degenerate light-like lines which correspond to the special values $\theta = \pm \pi/2$ (see Figure \[1\] left). The image of $F$ can be characterized by the implicit function $(x-t)^2 + y^2 = a^2$. This ZMC-surface is properly embedded and is not simply connected.

**Example 2.4 (\[1, Figure 2\]).** We set
\[
F(r,\theta) := \left( r + \frac{1}{2a} \log \left( \frac{ar - 1}{ar + 1} \right) + r \cos \theta, r \sin \theta, \frac{1}{2a} \log \left( \frac{ar - 1}{ar + 1} \right) \right),
\]
where $a > 0$ and $\theta \in [0,2\pi)$. This map is defined for $r > 1/a$ or $r < -1/a$, and the closure of the image of $F = (x,y,t)$ can be expressed as
\[
(\Psi :=) a \sinh(at) \left( (x-t)^2 + y^2 \right) + 2(x-t) \cosh(at) = 0.
\]
It can be checked that $(\Psi_x, \Psi_y, \Psi_t)$ never vanishes along $\Psi = 0$. So the closure of $F$ gives a properly embedded ZMC-surface in $L^3$ (see Figure \[1\] right).
Regarding our main result, we state a few open problems:

**Question 1.** Does a properly embedded ZMC-surface which consists only of space-like or light-like points coincide with a plane?

If this question is affirmative, then Corollary 13 follows as a corollary. Suppose that we can find such a non-planar ZMC-surface $S$; it must contain a light-like line. In fact, if $S$ consists only of space-like points, then $S$ is complete, and such a surface must be a plane (see [20, Remark 1.2]). So $S$ has a light-like point $p$. If $p$ is non-degenerate, then $S$ has a time-like point near $p$, so $p$ must be degenerate. By the line theorem (Fact 1.3), $S$ must contain a light-like line consisting of degenerate light-like points.

**Question 2.** Are there entire ZMC-graphs of mixed type containing degenerate light-like points?

This question needs to consider ZMC-graphs of mixed type. In fact, if we choose a function $g(x)$ satisfying $g'(0) = 0$ as in (1.19), then the $y$-axis consists of the degenerate light-like points. If we weaken ‘entire ZMC-graphs’ to ‘properly embedded ZMC-surfaces of mixed type’ the answer is ‘yes’. In fact, Example 2.4 gives a properly embedded ZMC-surface of mixed type which contains a degenerate light-like line $L$. Although the space-like points never accumulate to $L$ in the case of this example, one can show the existence of a function $ψ: U → R$ defined on a domain $U$ in $R^2$ containing the $y$-axis such that

- the $y$-axis corresponds to a degenerate light-like line,
- $ψ$ is of mixed type or consists only of space-like points except along the $y$-axis.

See [3] for details. Also, the following question arises:

**Question 3.** Are there entire ZMC-graphs of mixed type which are not obtained as analytic extensions of Kobayashi surfaces given as in [9]?

In fact, all known examples of entire ZMC-graphs of mixed type are obtained as analytic extensions of Kobayashi surfaces (cf. [9]), and they admit only non-degenerate light-like points.

**Appendix A. A property of light-like surfaces in $L^3$**

It can be easily checked that an embedded surface $S(⊂ L^3)$ is light-like if and only if the restriction of the canonical Lorentzian metric on $L^3$ to the tangent space $T_pS$ of each $p ∈ S$ is positive semi-definite but not positive definite. The purpose of this appendix is to prove the following:

**Theorem A.1.** If an entire $C^2$-differentiable graph of $ψ: R^2 → R$ gives a light-like surface in $L^3$, then $ψ − ψ(0, 0)$ is a linear function.

**Proof.** We set $F(x, y) = (x, y, ψ(x, y))$. Since $F$ is a light-like surface, $ψ_x^2 + ψ_y^2 = 1$ holds on $R^2$. Differentiating this with respect to $x$ and $y$, we get two equations. Since $F$ is light-like, $(ψ_x, ψ_y) ≠ (0, 0)$. By thinking $ψ_x, ψ_y$ are unknown variables of these two equations, the determinant $ψ_{xx}ψ_{yy} − ψ_{xy}^2$ vanishes identically. So the Gaussian curvature of $F$ with respect to the Euclidean metric of $R^3$ vanishes identically. Then, by the Hartman-Nirenberg cylinder theorem, $F$ must be a cylinder. (The proof of the cylinder theorem in [13] needs only $C^2$-differentiability.) That is,
there exist a non-zero vector $a$, a plane $\Pi$ which is not parallel to $a$, and a regular curve $\gamma: \mathbb{R} \to \Pi$ such that $F(u,v) := \gamma(u) + va$ gives a new parametrization of $F$. If $F$ is not a plane, there exists $u_0 \in \mathbb{R}$ such that $\gamma'(u_0)$ and $\gamma''(u_0)$ are linearly independent. Then the point $(u,v) = (u_0,0)$ is not an umbilical point of $F$. Since the asymptotic direction is uniquely determined at each non-umbilical point on a flat surface, the line theorem (cf. Fact 1.3) yields that $a$ is a light-like vector. By a suitable homothetic transformation and an isometric motion in $L^3$, we may set $a := (1,0,1)$. Then it holds that

$$0 = \gamma' \cdot a = x' - t'. \tag{A.1}$$

Since $\gamma' \cdot \gamma' = 0$, we have $y' = 0$. So, without loss of generality, we may assume that $y(u) = 0$. Differentiating (A.1), we have $x'' - t'' = 0$, contradicting the fact that $\gamma'(u_0)$ and $\gamma''(u_0)$ are linearly independent. Thus $F$ is a plane. $\square$

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