THE TWO-SIDED POMPEIU PROBLEM FOR DISCRETE GROUPS

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ABSTRACT. We consider a two-sided Pompeiu type problem for a discrete group \( G \). We give necessary and sufficient conditions for a finite subset \( K \) of \( G \) to have the \( F(G) \)-Pompeiu property. Using group von Neumann algebra techniques, we give necessary and sufficient conditions for \( G \) to be an \( \ell_2(G) \)-Pompeiu group.

1. Introduction

Let \( \mathbb{C} \) be the complex numbers, \( \mathbb{R} \) the real numbers, \( \mathbb{Z} \) the integers and \( \mathbb{N} \) the natural numbers. Let \( 2 \leq n \in \mathbb{N} \) and let \( K \) be a compact subset of \( \mathbb{R}^n \) with positive Lebesgue measure. The Pompeiu problem asks the following: Is \( f = 0 \) the only continuous function on \( \mathbb{R}^n \) that satisfies

\[
\int_{\sigma(K)} f \, dx = 0
\]

for all rigid motions \( \sigma \)? If the answer to the question is yes, then \( K \) is said to have the Pompeiu property. It is known that disks of positive radius do not have the Pompeiu property, see [20, Section 6] and the references therein for the details. The question now becomes: Are disks the only compact subsets of positive measure in \( \mathbb{R}^n \) that do not have the Pompeiu property? This question is still open. It was stated in [19] that all polytopes in \( \mathbb{R}^n, n \geq 2 \), have the Pompeiu property, a proof of this result was recently given in [13, Corollary 1.3]. Since disks are invariant under rotations, and other sets in \( \mathbb{R}^n \) are not, a reasonable question to ask is what would happen if a group of translations replaced the group of rigid motions. Thus it would be interesting to study the Pompeiu problem for groups that are being acted on by translations, see [13,15,17,18,20] and the references therein for more information about various variations of the Pompeiu problem.

In [18] the following version of the Pompeiu problem was studied: Let \( G \) be a unimodular group with Haar measure \( \mu \). Suppose \( K \) is a relatively compact subset of \( G \) with positive measure. Is \( f = 0 \) the only function in \( L^1(G) \) that satisfies

\[
\int_{gK} f(x) \, d\mu = 0
\]

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for all \( g, h \in G \)? This question was studied further in [3]. If \( G \) is abelian, then this problem reduces to a one-sided translation Pompeiu type problem. The purpose of this note is to investigate (1.2) in the discrete group setting and function spaces other than \( \ell^1(G) \).

For the rest of this paper, \( G \) will always be a discrete group and \( \mathcal{C} \) will denote a class of complex-valued functions on \( G \) that contain the zero function. Let \( f \) be a complex-valued function on \( G \). We shall represent \( f \) as a formal sum \( \sum_{g \in G} a_g g \), where \( a_g \in \mathbb{C} \) and \( f(g) = a_g \). Define \( \mathcal{F}(G) \) to be the set of all functions on \( G \), and let \( \ell^1(G) \) be the functions in \( \mathcal{F}(G) \) that satisfy \( \sum_{g \in G} |a_g|^p < \infty \). The group ring \( \mathbb{C}[G] \) consists of those functions where \( a_g = 0 \) for all but finitely many \( g \). The group ring can also be thought of as the functions on \( G \) with compact support. Let \( K \) be a finite subset of \( G \). In this paper we will consider the following discrete version of (1.2):

\[
(1.3) \quad \sum_{x \in gK} f(x) = 0
\]

for all \( g, h \in G \).

The following related Pompeiu type problem for discrete groups was investigated in [16]: When is \( f = 0 \) the only function in \( \mathcal{C} \) that satisfies

\[
(1.4) \quad \sum_{x \in gK} f(x) = 0
\]

for all \( g \in G \)?

It is not difficult to see that \( f = 0 \) is the only function in \( \mathcal{C} \) that satisfies (1.3) if it is the only function in \( \mathcal{C} \) satisfying (1.4). Clearly, (1.3) and (1.4) coincide if \( G \) is abelian.

We shall say that a finite subset \( K \) of \( G \) is a \( \mathcal{C} \)-Pompeiu set if \( f = 0 \) is the only function in \( \mathcal{C} \) that satisfies (1.3). A \( \mathcal{C} \)-Pompeiu group is a group for which every nonempty finite subset is a \( \mathcal{C} \)-Pompeiu set.

The identity element of \( G \) will be denoted by 1. If \( S \subseteq G \), then we will write \( \chi_S \) to indicate the characteristic function on \( S \), \( \chi_S(g) = 1 \) if \( g \in S \) and \( \chi_S(g) = 0 \) if \( g \notin S \). If \( S \) consists of one element \( g \), then \( \chi_g \) will be the usual point mass concentrated at \( g \). For \( g \in G \), the left translation of \( f \) by \( g \) is given by \( L_g f(x) = f(gx) \) and the right translation of \( f \) by \( g \) is denoted by \( R_g f(x) = f(xg^{-1}) \), where \( x \in G \).

One of our main results is:

**Theorem 1.1.** Let \( G \) be a discrete group and suppose \( K \) is a finite subset of \( G \). Let \( I \) be the ideal in \( \mathbb{C}[G] \) that is generated by \( \chi_K \). Then \( K \) is an \( \mathcal{F}(G) \)-Pompeiu set if and only if \( I = \mathbb{C}[G] \).

We will show that as a consequence of this result, algebraically closed groups and universal groups are examples of \( \mathcal{F}(G) \)-Pompeiu groups. This contrasts sharply with the one-sided translation Pompeiu type problem studied in [16], since there are nonzero functions in \( \ell^1(G) \) -in fact \( \mathbb{C}[G] \)-for the above groups that satisfy (1.4).

We also study the case when \( \mathcal{C} = \ell^2(G) \). Suppose \( K \) is a nontrivial subgroup of \( G \). Denote by \( X \) the closed subspace of \( \ell^2(G) \) generated by \( \chi_K \) that is invariant under left and right-translations by elements of \( G \). Let \( Y \) be the closed subspace of \( \ell^2(G) \) generated by \( \chi_K \) that is invariant under left-translations by elements of \( G \). Clearly \( Y \subseteq X \subseteq \ell^2(G) \). We shall see that if \( X = \ell^2(G) \), then \( K \) is an \( \ell^2(G) \)-Pompeiu set. We now assume that the subgroup \( K \) is also finite. It follows from
the paragraph after Proposition 2.3 in [16] that \( Y \neq \ell^2(G) \), but it could still be the case \( X = \ell^2(G) \) which means that \( K \) is an \( \ell^2(G) \)-Pompeiu set. This illuminates the difference between \([1.3]\) and \([1.4]\) in that it is possible there exists a nonzero function in \( \ell^2(G) \) that satisfies \([1.3]\) but the zero function is the only function in \( \ell^2(G) \) that satisfies \([1.4]\). However, this situation changes if \( K \) is also a normal subgroup of \( G \) since \( Y \) will then be invariant under right-translations by elements of \( G \) in addition to being invariant under left-translations. In other words \( Y = X \), which says that \( K \) is not an \( \ell^2(G) \)-Pompeiu set when \( K \) is a nontrivial finite normal subgroup of \( G \). We will prove Theorem 1.2, which is a generalization of [3, Corollary 2.7], that shows the existence of nontrivial finite normal subgroups of \( G \) actually characterizes \( \ell^2(G) \)-Pompeiu groups.

**Theorem 1.2.** Let \( G \) be a discrete group. Then \( G \) is an \( \ell^2(G) \)-Pompeiu group if and only if \( G \) does not contain a nontrivial finite normal subgroup.

Let \( A(G) \) denote the Fourier algebra of \( G \). We shall see that the following result is a consequence of the proof of Theorem 1.2.

**Theorem 1.3.** Let \( G \) be a discrete group. Then \( G \) is an \( A(G) \)-Pompeiu group if and only if \( G \) does not contain a nontrivial finite normal subgroup.

It appears that the first investigation of a Pompeiu type problem in the discrete setting was the paper [21]. For discrete groups the Pompeiu problem with respect to left translations was studied in [16]. Pompeiu type problems for finite subsets of the plane were examined in [7]. An interesting connection between the Fuglede conjecture and the Pompeiu problem for finite abelian groups was established in [8].

In Section 2 we give some preliminary material and preliminary results, including giving an equivalent condition to \([1.3]\) in terms of convolution equations. We prove Theorem 1.1 in Section 3 and we also give examples of groups that are \( \mathcal{F}(G) \)-Pompeiu groups. In Section 4 we discuss group von Neumann algebras and prove Theorem 1.2.

### 2. Preliminaries

In this section we give some necessary background and prove some preliminary results. Let \( f = \sum_{g \in G} a_g g \) and let \( h = \sum_{g \in G} b_g g \) be functions on \( G \). The convolution of \( f \) and \( h \) is given by

\[
  f \ast h = \sum_{g, x \in G} a_g b_x g x = \sum_{g \in G} \left( \sum_{x \in G} a_{g x^{-1}} b_x \right) g.
\]

Sometimes we will write \( f \ast h(g) = \sum_{x \in G} f(g x^{-1}) h(x) \) for when the function \( f \ast h \) is evaluated at \( g \). With respect to pointwise addition and convolution, \( CG \) is a ring. Also if \( f \in \ell^1(G) \) and \( h \in \ell^p(G) \), then \( f \ast h \in \ell^p(G) \). However if both \( f \) and \( h \) are in \( \ell^2(G) \) it might be the case \( f \ast h \) is not in \( \ell^2(G) \).

Recall that \( L_g \) and \( R_g \) denote the left and right translations of a function \( f \) by \( g \in G \). Note that \( L_g f = \chi_{g^{-1}} \ast f \) and \( R_g f = f \ast \chi_g \). For a function \( f \), \( \tilde{f} \) will denote the function \( \tilde{f}(x) = f(x^{-1}) \), where \( x \in G \) and \( \tilde{f} \) will indicate the complex conjugate of \( f \). The following simple lemma gives a useful characterization of \([1.3]\) in terms of convolution equations.
Lemma 2.1. Let $K$ be a finite subset of $G$, and let $f$ be a complex-valued function on $G$. Then $f$ satisfies (1.3) if and only if $\hat{\chi}_K \ast L_g f = 0$ for all $g \in G$.

Proof. Let $g, h \in G$, then
\[
\sum_{x \in gK} f(x) = \sum_{x \in Kh} L_g f(x) = \sum_{k \in K} L_g f(kh) = \sum_{x \in G} \chi_K(x) L_g f(xh) = \hat{\chi}_K \ast L_g f(h).
\]
Thus $\sum_{x \in gK} f(x) = 0$ for all $g, h \in G$ if and only if $\hat{\chi}_K \ast L_g f = 0$ for all $g \in G$. □

A similar calculation shows that (1.3) is also equivalent to $R_g f \ast \hat{\chi}_K = 0$ for all $g \in G$. Observe that $\hat{\chi}_K \ast L_g f = \hat{\chi}_K \ast \chi_g^{-1} \ast f$ and $R_g f \ast \hat{\chi}_K = f \ast \chi_g \ast \hat{\chi}_K$.

Remark 2.2. It was shown in [16, Proposition 2.3] that the single convolution equation $\chi_K \ast \hat{f} = 0$ is equivalent to (1.4).

If $X$ is a set, then $|X|$ will indicate the cardinality of $X$. Let $R$ be a ring and recall that the center of $R$ is the set of all elements in $R$ that commute under multiplication with all elements of $R$. An idempotent in $R$ is an element $e$ that satisfies $e^2 = e$. A central idempotent for $R$ is an idempotent contained in the center of $R$. In [16] it was shown that if $G$ contains a nonidentity element of finite order, then there is a finite subset $K$ of $G$ and a nonzero function in $\mathbb{C}G$ that satisfies (1.4). We shall see that this is not the case for (1.3). What is true though is

Proposition 2.3. Suppose $K$ is a nontrivial finite normal subgroup of $G$, then $K$ is not a $\mathcal{C}$-Pompeiu set for any class of functions that contains $\mathbb{C}G$.

Proof. Write $\chi_K = \sum_{k \in K} k$ for the characteristic function on $K$. Also $\hat{\chi}_K = \chi_K$ since $K$ is a subgroup of $G$. Now,
\[
\chi_K \ast \chi_K = \sum_{k \in K} k|K| = |K| \chi_K,
\]
thus $\chi_K \ast (\chi_K - |K|) = 0$. For $g \in G, \chi_K \ast \chi_g = \chi_g \ast \chi_K$ if and only if $gK = Kg$. Consequently, $\chi_K$ is in the center of $\mathbb{C}G$ since $K$ is normal in $G$. Let $g \in G$, then
\[
\chi_K \ast L_g(\chi_K - |K|) = \chi_K \ast \chi_g^{-1} \ast (\chi_K - |K|) = \chi_g^{-1} \ast \chi_K \ast (\chi_K - |K|) = 0.
\]
Hence, $\hat{\chi}_K \ast L_g(\chi_K - |K|) = 0$ for all $g \in G$. Lemma 2.1 yields that $K$ is not a $\mathcal{C}$-Pompeiu set for any $\mathcal{C}$ containing $\mathbb{C}G$. □

Remark 2.4. In the above proof, $\frac{\chi_K}{|K|}$ is a central idempotent in $\mathbb{C}G$. We shall see that central idempotents play a critical role in the proof of Theorem 1.2.
3. Theorem \[1.1\]

In this section we prove Theorem \[1.1\] and give some examples of groups that are \(\mathcal{F}(G)\)-Pompeiu groups.

Let \( f = \sum_{g \in G} a_g g \in \mathbb{C}G \) and \( h = \sum_{g \in G} b_g g \in \mathcal{F}(G) \). Define a map \( \langle \cdot, \cdot \rangle : \mathbb{C}G \times \mathcal{F}(G) \to \mathbb{C} \) by

\[
\langle f, h \rangle = \sum_{g \in G} a_g \overline{b_g}.
\]

For a fixed \( h \in \mathcal{F}(G) \), \( \langle \cdot, h \rangle \) is a linear functional on \( \mathbb{C}G \). Now suppose \( T \) is a linear functional on \( \mathbb{C}G \). Define \( h(g) = \overline{T(g)} \) for each \( g \in G \). Thus each linear functional on \( \mathbb{C}G \) defines an element of \( \mathcal{F}(G) \). Hence, the vector space dual of \( \mathbb{C}G \) can be identified with \( \mathcal{F}(G) \).

3.1. Proof of Theorem \[1.1\]  We now prove Theorem \[1.1\]. Let \( K \) be a finite subset of \( G \) and let \( I \) be the ideal in \( \mathbb{C}G \) generated by \( \chi_K \). We begin by showing that if \( K \) is an \( \mathcal{F}(G)\)-Pompeiu set, then \( I = \mathbb{C}G \). Now assume that \( K \) is an \( \mathcal{F}(G)\)-Pompeiu set and \( I \neq \mathbb{C}G \). Because \( I \) is a subspace of \( \mathbb{C}G \), there is a nonzero \( f \in \mathcal{F}(G) \) for which \( \langle \alpha, f \rangle = 0 \) for all \( \alpha \in I \).

Fix \( g \in G \) and let \( h \in G \). Since \( I \) is an ideal, \( R_h L_g \chi_K \in I \), which means \( \langle R_h L_g \chi_K, f \rangle = 0 \).

\[
\langle R_h L_g \chi_K, f \rangle = \sum_{y \in G} R_h L_g \chi_K(y) \overline{f(y)}
= \sum_{y \in G} \chi_K(gh^{-1}) \overline{f(y)}
= \sum_{y \in G} \chi_K(yh^{-1}) \overline{f(g^{-1}y)}
= (\overline{\chi_K} \ast L_{g^{-1}}f)(h).
\]

Thus \( (\overline{\chi_K} \ast L_{g^{-1}}f)(h) = 0 \) for all \( h \in G \). Consequently, \( \overline{\chi_K} \ast L_{g^{-1}}f = 0 \) for all \( g \in G \). Thus \( f \) is an nonzero function that satisfies \[1.3\], contradicting our assumption that \( K \) is an \( \mathcal{F}(G)\)-Pompeiu set. Hence, \( I = \mathbb{C}G \).

Conversely, assume \( I = \mathbb{C}G \). We will finish the proof of the theorem by showing that \( f = 0 \) is the only function that satisfies \[1.3\]. Set \( \tilde{I} = \{ \tilde{\alpha} \mid \alpha \in I \} \). Now \( \tilde{I} \) is generated by \( \overline{\chi_K} \) and \( \tilde{I} = \mathbb{C}G \) since \( I = \mathbb{C}G \). Assume that \( f \in \mathcal{F}(G) \) satisfies \[1.3\]. Then by Lemma \[2.1\], \( \overline{\chi_K} \ast L_g f = R_g f = \chi_K = 0 \) for all \( g \in G \). We now obtain \( f \ast \tilde{I} = 0 = \tilde{I} \ast f \) since \( \overline{\chi_K} \ast L_g f = \overline{\chi_K} \ast \chi_{g^{-1}} \ast f \) and \( R_g f = \overline{\chi_K} = f \ast \chi_g \ast \overline{\chi_K} \).

Now \( \chi_1 \in \tilde{I} \) and \( 0 = \chi_1 \ast f = f \), thus \( K \) is an \( \mathcal{F}(G)\)-Pompeiu set and the theorem is proved.

3.2. Examples. Before we give examples of \( \mathcal{F}(G)\)-Pompeiu groups we need to define the augmentation ideal of a group ring. Define a map from \( \mathbb{C}G \) into \( \mathbb{C} \) by

\[
\varepsilon \left( \sum_{g \in G} a_g g \right) = \sum_{g \in G} a_g.
\]

The map \( \varepsilon \) is a ring homomorphism onto \( \mathbb{C} \). The augmentation ideal of \( \mathbb{C}G \), which we will denote by \( \omega(\mathbb{C}G) \), is the kernel of \( \varepsilon \). For information about \( \omega(\mathbb{C}G) \) see \[14\] Chapter 3. If \( K \) is a nonempty finite subset of \( G \), then \( \chi_K \notin \omega(\mathbb{C}G) \) due
to $\varepsilon(\chi_K) = |K|$. The main result of [2] showed that the only nontrivial ideal in $\mathbb{C}G$ for algebraically closed groups and universal groups is $\omega(\mathbb{C}G)$. Thus the ideal generated by $\chi_K$ in these groups must be all of $\mathbb{C}G$. Therefore, algebraically closed groups and universal groups are $\mathcal{F}(G)$-Pompeiu groups.

Remark 3.1. These groups have elements of finite order, which implies that there exists a nontrivial finite subset $K$ in $G$ and a nonzero function in $\mathbb{C}G$ that satisfies (1.4). See [16] Section 2 for the details.

4. Theorem [1.2]

In this section we will prove Theorem 1.2. We begin by discussing group von Neumann algebras. See Dixmier’s book [5] for a general discussion of von Neumann algebras, and for more detailed explanations of group von Neumann algebras see [11] Section 8 and [12] Section 1.1.1. Recall that algebras, and for more detailed explanations of group von Neumann algebras see [11] Section 8 and [12] Section 1.1.1. Recall that $\ell^2(G)$ is the set of all formal sums $\sum_{g \in G} a_g g$ for which $\sum_{g \in G} |a_g|^2 < \infty$. Furthermore, $\ell^2(G)$ is a Hilbert space with Hilbert bases $\{g \mid g \in G\}$. For $f = \sum_{g \in G} a_g g \in \ell^2(G)$ and $h = \sum_{g \in G} b_g g \in \ell^2(G)$, the inner product $\langle f, h \rangle$ is defined to be $\sum_{g \in G} a_g b_g$. If $f \in \mathbb{C}G$ and $h \in \ell^2(G)$, then $f * h \in \ell^2(G)$. In fact, multiplication on the left by $f$ is a continuous linear operator on $\ell^2(G)$. Thus we can consider $\mathbb{C}G$ to be a subring of $B(\ell^2(G))$, the set of bounded operators on $\ell^2(G)$. Denote by $\mathcal{N}(G)$ the weak closure of $\mathbb{C}G$ in $B(\ell^2(G))$. The space $\mathcal{N}(G)$ is known as the *group von Neumann algebra* of $G$. For $T \in B(\ell^2(G))$ the following are standard facts.

(i) $T \in \mathcal{N}(G)$ if and only if there exists a net $(T_n)$ in $\mathbb{C}G$ such that $\lim_{n \to \infty} \langle T_n u, v \rangle \to \langle T u, v \rangle$ for all $u, v \in \ell^2(G)$.

(ii) $T \in \mathcal{N}(G)$ if and only if $(Tf) \star \chi_g = T(f \star \chi_g)$ for all $g \in G$.

Another way of expressing (ii) is that $T \in \mathcal{N}(G)$ if and only if $T$ is a right $\mathbb{C}G$-map. Using (ii) we can see that if $T \in \mathcal{N}(G)$ and $T \chi_1 = 0$, then $T \chi_g = 0$ for all $g \in G$ and hence $Tf = 0$ for all $f \in \mathbb{C}G$. It follows that $T = 0$ and so the map defined by $T \mapsto T \chi_1$ is injective. Therefore the map $T \mapsto T \chi_1$ allows us to identify $\mathcal{N}(G)$ with a subspace of $\ell^2(G)$. Thus algebraically we have

$$\mathbb{C}G \subseteq \mathcal{N}(G) \subseteq \ell^2(G).$$

It is not difficult to show that if $f \in \ell^2(G)$, then $f \in \mathcal{N}(G)$ if and only if $f \star h \in \ell^2(G)$ for all $h \in \ell^2(G)$. For $f = \sum_{g \in G} a_g g \in \ell^2(G)$, define $f^* = \sum_{g \in G} \overline{a_g} g^{-1} \in \ell^2(G)$. Then for $f \in \mathcal{N}(G)$ we have $\langle f \star u, v \rangle = \langle u, f^* \star v \rangle$ for all $u, v \in \ell^2(G)$; thus $f^*$ is the adjoint operator of $f$.

Two elements $x, y$ in $G$ are said to be *conjugate* in $G$ if there exists a $g \in G$ for which $g^{-1} x g = y$. Recall that the conjugation action of $G$ on itself is an equivalence relation. Suppose $C$ is a finite conjugacy class of $G$. Let $c = \sum_{x \in C} x$, then $c \in \mathbb{C}G$. The group ring elements $c$ are known as *finite class sums*. For $x \in G$, denote by $C_x$ the class containing $x$. We will need the following

**Lemma 4.1.** Let $S$ be the set of all finite class sums of $G$. Each element in the center of $\mathcal{N}(G)$, $Z(\mathcal{N}(G))$, is a formal sum of elements in $S$.

**Proof.** Let $f = \sum_{g \in G} a_g g \in \mathcal{N}(G)$. Then $f \in Z(\mathcal{N}(G))$ if and only if $\chi_{g^{-1}} \star f \star \chi_g = f$ for all $g \in G$. Since

$$\chi_{g^{-1}} \star f \star \chi_g = \sum_{x \in G} a_x (g^{-1} x g) = \sum_{y \in G} a_{gyg^{-1}} y,$$

...
we see immediately that \( a_{g_y}^{-1} = a_y \) because \( (x_{g_y}^{-1} \ast f \ast x_y)(y) = f(y) \). Thus \( f \) is
constant on \( C_y \). If \( f(y) \neq 0 \) on \( C_y \), then \( C_y \) is finite due to \( f \in L^2(G) \). The class sums have disjoints supports, thus if \( f \in Z(N(G)) \) then it is a formal sum of finite class sums in \( S \).

The finite conjugate subgroup of \( G \) is defined by
\[
\Delta(G) = \{ g \in G \mid g \text{ has a finite number of conjugates} \}.
\]
The following immediate consequence of Lemma 4.1 will be crucial in our proof of Theorem 4.3.

**Corollary 4.2.** The center of \( N(G) \) is contained in the center of \( N(\Delta G) \).

**Proof.** Every finite class sum contained in \( CG \) is contained in \( C(\Delta G) \).

We will prove Theorem 1.2 by reducing to the \( N(G) \) case, which we now prove.

**Theorem 4.3.** If \( G \) is a group with no nontrivial finite normal subgroups, then \( G \) is an \( N(G) \)-Pompeiu group.

**Proof.** It follows from [14, Lemma 4.1.5(iii)] that \( \Delta(G) \) is a torsion-free abelian group since \( G \) has no nontrivial finite normal subgroups. Let \( K \) be a finite subset of \( G \) and let \( I \) be the weakly closed ideal in \( N(G) \) generated by \( \chi_K \). Now suppose there exists a nonzero \( f \in N(G) \) that satisfies \( \chi_K \ast L_g f = 0 \) for all \( g \in G \). So \( f \) belongs to the annihilator ideal \( I^\perp \) of \( I \) in \( N(G) \). Thus \( N(G) = I \bigoplus I^\perp \), where \( \bigoplus \) denotes the direct sum. Thus there exists a nonzero central idempotent \( e \) in \( N(G) \) for which \( e \ast N(G) = I^\perp \). Because \( \chi_1 \in N(G), e \in I^\perp \) and it follows that \( I \ast e = 0 \). By Corollary 4.2, \( e \) also belongs to the center of \( N(\Delta G) \). Let \( T \) be a right transversal for \( \Delta(G) \) in \( G \). Write \( \chi_K = \sum_{t \in T} \chi_{Kt} \ast t \), where \( \chi_{Kt} \in C(\Delta G) \). Now
\[
0 = \chi_K \ast e = \left( \sum_{t \in T} \chi_{Kt} \ast t \right) \ast e
\]
\[
= \sum_{t \in T} (\chi_{Kt} \ast e) \ast t.
\]
Thus \( \chi_{Kt} \ast e = 0 \) for each \( t \in T \). Because \( \chi_K \neq 0 \) there exists a \( t' \) in \( T \) for which \( \chi_{Kt'} \neq 0 \). This contradicts the fact that \( \chi_{Kt'} \ast \beta = 0 \) for all \( 0 \neq \beta \in L^2(\Delta G) \), which was proved in [4]. Hence, there is no nonzero \( f \in N(G) \) that satisfies \( \chi_K \ast L_g f = 0 \) for all \( g \in G \). Therefore, every finite subset of \( G \) is an \( N(G) \)-Pompeiu set and \( G \) is an \( N(G) \)-Pompeiu group.

4.1. **Proof of Theorem 1.2** We now prove Theorem 1.2. We start with a definition. A nonzero divisor in a ring \( R \) is an element \( s \) such that \( s r \neq 0 \neq rs \) for all \( r \in R \setminus 0 \).

Proposition 2.3 says that if \( G \) contains a nontrivial finite normal subgroup, then it cannot be an \( L^2(G) \)-Pompeiu group.

Conversely, assume there exists a nonempty finite subset \( K \) of \( G \) and a nonzero \( f \in L^2(G) \) that satisfies \( \chi_K \ast L_g f = 0 \) for all \( g \in G \). By [10, Lemma 7] there exists a nonzero divisor \( \theta \in N(G) \) such that \( f \ast \theta \in N(G) \). Suppose \( f \ast \theta = 0 \). Then \( \theta^* \ast f^* = 0 \). If \( e \in B(L^2(G)) \) is the projection from \( L^2(G) \) onto \( f^* \bigotimes CG \), then \( e \in N(G) \) by [9, Lemma 5], \( e \neq 0 \) because \( f^* \neq 0 \) and \( \theta^* \ast e = 0 \). Hence \( e \ast \theta = 0 \), contradicting the fact \( \theta \) is a nonzero divisor in \( N(G) \), so \( f \ast \theta \neq 0 \). It
follows from Theorem 4.3 that there exists a $g \in G$ such that $	ilde{\chi}_K \ast L_g (f \ast \theta) \neq 0$ since $0 \neq f \ast \theta \in N(G)$. But $	ilde{\chi}_K \ast L_g (f \ast \theta) = (\tilde{\chi}_K \ast g^{-1} \ast f) \ast \theta = 0$ because we are assuming $\tilde{\chi}_K \ast L_g f = 0$ for all $g \in G$, a contradiction. Hence, there does not exist a nonzero $f \in \ell^2(G)$ and a nonempty finite subset $K$ of $G$ for which $\tilde{\chi}_K \ast L_g f = 0$ for all $g \in G$. Therefore it follows from Lemma 2.1 that $G$ is an $\ell^2(G)$-Pompeiu group, as desired.

Remark 4.4. A key ingredient in the proof of Theorem 1.2 was [10, Lemma 7], which basically says that if $f \in \ell^2(G)$ then there exists a $\theta \in N(G)$ for which $f \ast \theta \in N(G)$. This allowed us to reduce from the $\ell^2(G)$-case to the $N(G)$-case. What makes this possible is the existence of a ring $\cal U(G)$ that is the classical right quotient ring for $N(G)$. This means that every element in $N(G)$ is either a zero divisor or invertible in $\cal U(G)$. Another important fact is that algebraically

$$CG \subseteq N(G) \subseteq \ell^2(G) \subseteq \cal U(G),$$

see [11] Section 8 for the details.

4.2. Proof of Theorem 1.3. Let $A(G)$ denote the Fourier algebra of $G$. Then every element of $A(G)$ can be written in the form $f_1 \ast f_2$ with $f_1, f_2 \in \ell^2(G)$ [6, Theorem 2.4.3]. Thus

$$\ell^2(G) \subseteq A(G) \subseteq \cal U(G).$$

Now applying the argument used in the proof of Theorem 1.2 establishes Theorem 1.3.

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