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REMARKS ON THE NAVIER-STOKES EQUATIONS IN SPACE DIMENSION $n \geq 3$

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ABSTRACT. In this paper, we prove some new L^p-estimates of the velocity by the technique of L^p-energy method.

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the Navier-Stokes equations:

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi - \Delta u = 0 \text{ in } \mathbb{R}^n \times (0, T), \tag{1.1}$$

$$\operatorname{div} u = 0 \text{ in } \mathbb{R}^n \times (0, T), \tag{1.2}$$

$$u(\cdot, 0) = u_0, \ \operatorname{div} u_0 = 0 \text{ in } \mathbb{R}^n. \tag{1.3}$$

Here u is the velocity field and π is the pressure. In this paper, both are supposed to exist in a suitable class of function spaces on $\mathbb{R}^n \times [0, T)$ with $n \geq 3$.

In [1], Beirão da Veiga showed the well-known estimate:

$$\|u(\cdot, t)\|_{L^p} \leq \|u_0\|_{L^p} \exp\{C\|u\|^q_{L^q(0,t;L^p)}\} \tag{1.4}$$

for $p \in (n, +\infty)$ with $\frac{2}{q} + \frac{n}{p} = 1$, where $C > 0$ is a constant depending on n, p, q. The proof of (1.4) depends on the standard L^p-energy method.

The Navier-Stokes equation (1.1) is written in the form of the following nonlinear heat equation

$$\partial_t u - \Delta u = -\operatorname{div}(u \otimes u) - \nabla\pi. \tag{1.5}$$

Using the L^∞-estimate of the heat equation, we have

$$\begin{aligned} \|u\|_{L^\infty(\mathbb{R}^n \times (0,T))} &\lesssim \|u \otimes u\|_{L^s(\mathbb{R}^n \times (0,T))} + \|\pi\|_{L^s(\mathbb{R}^n \times (0,T))} + \|u_0\|^2_{L^\infty} \\ &\lesssim \|u\|^2_{L^{2s}(\mathbb{R}^n \times (0,T))} + \|u_0\|^2_{L^\infty} \end{aligned} \tag{1.6}$$

for $s \in (n + 2, +\infty)$ and $u_0 \in L^2 \cap L^\infty$, where we have used the Hölder inequality, the well-known relation

$$\pi = \sum_{j,k=1}^{n} R_j R_k u_j u_k \tag{1.7}$$

with the Riesz operator R_j, and its boundedness in L^s. For other types of L^∞-estimates of the velocity, we refer to [8, 11].

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The estimate (1.6) shows that $L^{2s}(\mathbb{R}^n\times(0,T))$-control governs the uniform space-time control, while the estimate (1.4) preserves the space integrability on both sides. The first purpose of this paper is to broaden the range of admissible space-time integrability on the RHS of (1.4) in the framework of the Serrin condition. We will prove

Theorem 1.1. *Let $n \geq 3$ and $p \in [4,+\infty)$. Let (q,r) satisfy $\frac{2}{q}+\frac{n}{r}=1$ and $r\in(n,+\infty]$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}\exp\left(C\|u\|^q_{L^q(0,t;L^r)}\right) \tag{1.8}$$

for any $t\in[0,T)$ and $u_0\in L^p$.

Remark 1.1. By (1.6) and (1.8), we give a different proof of the classic Ladyzhenskaya-Prodi-Serrin criterion [7].

Similarly, we have

Theorem 1.2. *Let $n \geq 3$ and $p \in (2,+\infty)$. Let (q,r) satisfy $\frac{2}{q}+\frac{n}{r}=2$ and $r\in(\frac{n}{2},+\infty]$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}\exp\left(C\int_0^t\|\nabla u\|^q_{L^r}\mathrm{d}\tau\right) \tag{1.9}$$

for any $t\in[0,T)$ and $u_0\in L^p$.

Remark 1.2. Beirão da Veiga [2] gave a different proof of Theorem 1.2.

Remark 1.3. In the two dimensional case, the estimate (1.9) with $p=4$ and $q=r=2$ is shown in [5]. The method depends on the div-curl lemma and the Hardy-BMO duality.

The next purpose of this paper is to formulate the L^p-bound of the velocity in terms of the pressure or its gradient with admissible space-time integrability in the Serrin condition as in Theorems 1.1 and 1.2.

Theorem 1.3. *Let $n \geq 3$ and $p \in (2,+\infty)$. Let (q,r) satisfy $\frac{2}{q}+\frac{n}{r}=2$ and $r\in(\frac{n}{2},+\infty]$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}\exp\left(C\int_0^t\|\pi\|^q_{L^r}\mathrm{d}\tau\right), \tag{1.10}$$

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}\exp\left(C\int_0^t\|\pi\|_{\mathrm{BMO}}\mathrm{d}\tau\right) \tag{1.11}$$

for any $t\in[0,T)$ and $u_0\in L^p$.

Theorem 1.4. *Let $n \geq 3$ and $p \in (2,+\infty)$. Let (q,r) satisfy $\frac{2}{q}+\frac{n}{r}=3$ and $r\in(\frac{n}{3},+\infty]$. Then there exists a constant C such that*

$$\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}\exp\left(C\int_0^t\|\nabla\pi\|^q_{L^r}\mathrm{d}\tau\right) \tag{1.12}$$

for any $t\in[0,T)$ and $u_0\in L^p$.

Remark 1.4. Recently, Kanamaru and Yamamoto [9] show (1.12) with $p=2n$.

Remark 1.5. When $r = \infty$ and $q = \frac{2}{3}$, one can prove the regularity criterion

$$\int_0^T \frac{\|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}}^{\frac{2}{3}}}{\log^{\frac{1}{3}}(e + \|\nabla\pi\|_{\dot{B}^0_{\infty,\infty}})} \mathrm{d}\tau < +\infty \tag{1.13}$$

by the method in [3,9]. Details are omitted.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We assume that the solution is smooth and only need to show the a priori estimates. Below we consider the case where r is finite since the case $r = +\infty$ is treated in a similar and simpler way.

First, we take $p = 4$.

Testing (1.1) by $|u|^2 u$ and using (1.2), we see that

$$\begin{aligned} &\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^4 \mathrm{d}x + \int |u|^2|\nabla u|^2 \mathrm{d}x + \frac{1}{2}\int |\nabla |u|^2|^2 \mathrm{d}x \\ &= \int \pi u \cdot \nabla |u|^2 \mathrm{d}x \\ &\lesssim \|u\|_{L^r}\|\pi\|_{L^{\frac{2r}{r-2}}}\|\nabla |u|^2\|_{L^2} \\ &\lesssim \|u\|_{L^r}\||u|^2\|_{L^{\frac{2r}{r-2}}}\|\nabla |u|^2\|_{L^2} \\ &\lesssim \|u\|_{L^r}\||u|^2\|_{L^2}^{1-\frac{n}{r}}\|\nabla |u|^2\|_{L^2}^{1+\frac{n}{r}} \\ &\le \frac{1}{4}\|\nabla |u|^2\|_{L^2}^2 + C\|u\|_{L^r}^q\|u\|_{L^4}^2, \end{aligned} \tag{2.1}$$

where we have used the Hölder inequality, the estimate

$$\|\pi\|_{L^s} \lesssim \||u|^2\|_{L^s} \tag{2.2}$$

with $1 < s < +\infty$ via (1.7), and the Gagliardo-Nirenberg inequality

$$\|v\|_{L^{\frac{2r}{r-2}}} \lesssim \|v\|_{L^2}^{1-\frac{n}{r}}\|\nabla v\|_{L^2}^{\frac{n}{r}} \tag{2.3}$$

with $n < r < +\infty$, namely, $0 < \frac{n}{r} < 1$. The estimate (1.8) with $p = 4$ follows by the Gronwall lemma applied to (2.1).

Second, we assume $p > 4$.

Testing (1.1) by $|u|^{p-2}u$ and using (1.2), (2.2), and (2.3), we find that

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p \mathrm{d}x + \int |u|^{p-2}|\nabla u|^2 \mathrm{d}x + 4\frac{p-2}{p^2}\int |\nabla |u|^{\frac{p}{2}}|^2 \mathrm{d}x \\
&= -\int (u\cdot\nabla\pi)|u|^{p-2}\mathrm{d}x = \int \pi u \nabla |u|^{p-2}\mathrm{d}x \\
&\lesssim \left|\int u\pi |u|^{\frac{p}{2}-2}\nabla |u|^{\frac{p}{2}}\mathrm{d}x\right| \\
&\lesssim \|u\|_{L^r}\|\pi\|_{L^{r_1}}\||u|^{\frac{p}{2}-2}\|_{L^{r_2}}\|\nabla |u|^{\frac{p}{2}}\|_{L^2} \\
&\lesssim \|u\|_{L^r}\||u|^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}r_1}}\||u|^{\frac{p}{2}}\|^{\frac{2}{p}(\frac{p}{2}-2)}_{L^{\frac{2}{p}(\frac{p}{2}-2)r_2}}\|\nabla |u|^{\frac{p}{2}}\|_{L^2} \\
&\lesssim \|u\|_{L^r}\||u|^{\frac{p}{2}}\|_{L^{\frac{2r}{r-2}}}\|\nabla |u|^{\frac{p}{2}}\|_{L^2} \\
&\lesssim \|u\|_{L^r}\||u|^{\frac{p}{2}}\|^{1-\frac{n}{r}}_{L^2}\|\nabla |u|^{\frac{p}{2}}\|^{1+\frac{n}{r}}_{L^2} \\
&\le \frac{p-2}{p^2}\|\nabla |u|^{\frac{p}{2}}\|^2_{L^2} + C\|u\|^q_{L^r}\||u|^{\frac{p}{2}}\|^2_{L^2}, \qquad (2.4)
\end{aligned}$$

provided that

$$\frac{1}{r}+\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{2}, \qquad (2.5)$$

$$\frac{4}{p}r_1 = \frac{2}{p}\Big(\frac{p}{2}-2\Big)r_2 = \frac{2r}{r-2}, \qquad (2.6)$$

where we have used the Hölder inequality with (2.5), (2.2) with $s = r_1$, and (2.3) with $v = |u|^{\frac{p}{2}}$. The system of elementary equations (2.5) and (2.6) is solved by

$$r_1 = \frac{p}{2}\cdot\frac{r}{r-2} \quad \text{and} \quad r_2 = \frac{2p}{p-4}\cdot\frac{r}{r-2} \qquad (2.7)$$

if $p > 4$. The estimate (1.1) with $p > 4$ follows by the Gronwall lemma applied to (2.4).

This completes the proof.

3. Proof of Theorem 1.2

We only need to show (1.9). Below we consider the case where r is finite since the case $r = +\infty$ is treated in a similar and simpler way.

Testing (1.1) by $|u|^{p-2}u$ and using (1.2) and (2.3), we derive

$$
\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p \mathrm{d}x + \int |u|^{p-2}|\nabla u|^2 \mathrm{d}x + 4\frac{p-2}{p^2}\int |\nabla |u|^{\frac{p}{2}}|^2 \mathrm{d}x \\
&= \int \pi u \nabla |u|^{p-2} \mathrm{d}x \\
&\lesssim \|\nabla u\|_{L^r}\|\pi\|_{L^{r_3}}\||u|^{p-2}\|_{L^{r_4}} \\
&\lesssim \|\nabla u\|_{L^r}\||u|^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}r_3}}\||u|^{\frac{p}{2}}\|^{\frac{2}{p}(p-2)}_{L^{\frac{2}{p}(p-2)r_4}} \\
&\lesssim \|\nabla u\|_{L^r}\||u|^{\frac{p}{2}}\|^2_{L^{\frac{2r}{r-1}}} \\
&\lesssim \|\nabla u\|_{L^r}\||u|^{\frac{p}{2}}\|^{2-\frac{n}{r}}_{L^2}\|\nabla |u|^{\frac{p}{2}}\|^{\frac{n}{r}}_{L^2} \\
&\le \frac{p-2}{p^2}\|\nabla |u|^{\frac{p}{2}}\|^2_{L^2} + C\|\nabla u\|^q_{L^r}\||u|^{\frac{p}{2}}\|^2_{L^2}, \qquad (3.1)
\end{aligned}
$$

provided that

$$\frac{1}{r} + \frac{1}{r_3} + \frac{1}{r_4} = 1, \qquad (3.2)$$

$$\frac{4}{p}r_3 = \frac{2}{p}(p-2)r_4 = \frac{2r}{r-1}, \qquad (3.3)$$

where we have used the Hölder inequality with (3.2), (2.2) with $s = r_3$, and the Gagliardo-Nirenberg inequality

$$\|v\|_{L^{\frac{2r}{r-1}}} \lesssim \|v\|^{1-\frac{n}{2r}}_{L^2}\|\nabla v\|^{\frac{n}{2r}}_{L^2} \qquad (3.4)$$

with $\frac{n}{2} < r < +\infty$, namely $0 < \frac{n}{2r} < 1$, and $v = |u|^{\frac{p}{2}}$. The system of elementary equations (3.2) and (3.3) is solved by

$$r_3 = \frac{p}{2}\frac{r}{r-1} \quad \text{and} \quad r_4 = \frac{p}{p-2}\frac{r}{r-1} \qquad (3.5)$$

if $p > 2$. The estimate (1.9) follows by the Gronwall lemma applied to (3.1).

This completes the proof.

4. Proof of Theorem 1.3

We only need to show the estimates (1.10) and (1.11). Below we consider the case where r is finite since the case $r = +\infty$ is treated in a similar and simpler way.

We start with the first two equalities in (3.1) and estimate them as

$$
\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p \mathrm{d}x + \int |u|^{p-2}|\nabla u|^2 \mathrm{d}x + 4\frac{p-2}{p^2}\int |\nabla |u|^{\frac{p}{2}}|^2 \mathrm{d}x \\
&= \int \pi u \nabla |u|^{p-2} \mathrm{d}x \\
&\lesssim \int |\pi||u|^{\frac{p}{2}-1}|\nabla |u|^{\frac{p}{2}}|\mathrm{d}x \\
&\lesssim \epsilon\|\nabla |u|^{\frac{p}{2}}\|^2_{L^2} + \frac{1}{\epsilon}\int \pi^2|u|^{p-2}\mathrm{d}x \qquad (4.1)
\end{aligned}
$$

for any $0 < \epsilon < 1$.

We use (2.2) and (3.5) with $v = |u|^{\frac{p}{2}}$ to bound the last integral in (4.1) as

$$\begin{aligned}\int \pi^2|u|^{p-2}dx &\lesssim \|\pi\|_{L^r}\|\pi\|_{L^{r_3}}\||u|^{p-2}\|_{L^{r_4}}\\ &\lesssim \|\pi\|_{L^r}\||u|^{\frac{p}{2}}\|^{\frac{4}{p}}_{L^{\frac{4}{p}r_3}}\||u|^{\frac{p}{2}}\|^{\frac{2}{p}(p-2)}_{L^{\frac{2}{p}(p-2)r_4}}\\ &\lesssim \|\pi\|_{L^r}\||u|^{\frac{p}{2}}\|^2_{L^{\frac{2r}{r-1}}}\\ &\lesssim \|\pi\|_{L^r}\||u|^{\frac{p}{2}}\|^{2-\frac{n}{r}}_{L^2}\|\nabla|u|^{\frac{p}{2}}\|^{\frac{n}{r}}_{L^2}\\ &\lesssim \epsilon^2\|\nabla|u|^{\frac{p}{2}}\|^2_{L^2} + \epsilon^{-2(q-1)}\|\pi\|^q_{L^r}\||u|^{\frac{p}{2}}\|^2_{L^2}, \end{aligned} \tag{4.2}$$

where r_3 and r_4 are given by (3.5). The estimate (1.10) follows by taking ϵ small enough in (4.1) and (4.2) and using the Gronwall lemma on (4.1).

To prove (1.11), we use the interpolation inequality [4, 6, 10]:

$$\|\pi^2\|_{L^{\frac{p}{2}}} \lesssim \|\pi\|_{L^{\frac{p}{2}}}\|\pi\|_{\mathrm{BMO}}. \tag{4.3}$$

We use (2.2) with $s = \frac{p}{2}$ and (4.3) to bound the last integral in (4.1) as

$$\begin{aligned}\int \pi^2|u|^{p-2}dx &\lesssim \|\pi^2\|_{L^{\frac{p}{2}}}\||u|^{p-2}\|_{L^{\frac{p}{p-2}}}\\ &\lesssim \|\pi\|_{L^{\frac{p}{2}}}\|\pi\|_{\mathrm{BMO}}\|u\|^{p-2}_{L^p}\\ &\lesssim \|u\|^2_{L^p}\|\pi\|_{\mathrm{BMO}}\|u\|^{p-2}_{L^p}\\ &\lesssim \|\pi\|_{\mathrm{BMO}}\|u\|^p_{L^p}. \end{aligned} \tag{4.4}$$

Inserting (4.4) into (4.1), taking ϵ small enough, and using the Gronwall lemma, we arrive at (1.11).

This completes the proof.

5. Proof of Theorem 1.4

We only need to show the estimate (1.12). Below we consider the case where r is finite since the case $r = +\infty$ is treated in a similar and simpler way.

Since we have the Sobolev inequality

$$\|\pi\|_{L^{\frac{nr}{n-r}}} \lesssim \|\nabla\pi\|_{L^r} \tag{5.1}$$

with $1 < r < n$ and

$$\|\pi\|_{\mathrm{BMO}} \lesssim \|\nabla\pi\|_{L^n}, \tag{5.2}$$

(1.12) with $r \in (1, n]$ follows from (1.10) and (1.11). Therefore, from now on consider the case $r \in (n, +\infty)$.

We start with the first equality in (3.1) and estimate it with $\Lambda = (-\Delta)^{1/2}$ as

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p \mathrm{d}x + \int |u|^{p-2}|\nabla u|^2 \mathrm{d}x + 4\frac{p-2}{p^2}\int |\nabla |u|^{\frac{p}{2}}|^2 \mathrm{d}x$$
$$= -\int \Lambda^{-\frac{1}{2}-\frac{n}{2r}}\nabla\pi \cdot \Lambda^{\frac{1}{2}+\frac{n}{2r}}(|u|^{p-2}u)\mathrm{d}x$$
$$\lesssim \|\Lambda^{\frac{1}{2}-\frac{n}{2r}}\pi\|_{L^p}\|\Lambda^{\frac{1}{2}+\frac{n}{2r}}(|u|^{p-2}u)\|_{L^{\frac{p}{p-1}}}$$
$$\lesssim \|\pi\|^{\frac{1}{2}}_{L^{\frac{p}{2}}}\|\nabla\pi\|^{\frac{1}{2}}_{L^r}\cdot \||u|^{p-2}u\|^{\frac{1}{2}-\frac{n}{2r}}_{L^{\frac{p}{p-1}}}\|\nabla(|u|^{p-2}u)\|^{\frac{1}{2}+\frac{n}{2r}}_{L^{\frac{p}{p-1}}}$$
$$\lesssim \|u\|_{L^p}\|\nabla\pi\|^{\frac{1}{2}}_{L^r}\cdot\|u\|^{(\frac{1}{2}-\frac{n}{2r})(p-1)}_{L^p}\||\nabla u||u|^{\frac{p}{2}-1}\|^{\frac{1}{2}+\frac{n}{2r}}_{L^2}\||u|^{\frac{p}{2}-1}\|^{\frac{1}{2}+\frac{n}{2r}}_{L^{\frac{2p}{p-2}}}$$
$$\lesssim \|\nabla\pi\|^{\frac{1}{2}}_{L^r}\|u\|^{1+(\frac{1}{2}-\frac{n}{2r})(p-1)+(\frac{1}{2}+\frac{n}{2r})(\frac{p}{2}-1)}_{L^p}\||\nabla u||u|^{\frac{p}{2}-1}\|^{\frac{1}{2}+\frac{n}{2r}}_{L^2}$$
$$\lesssim \epsilon\||\nabla u||u|^{\frac{p}{2}-1}\|^2_{L^2} + \epsilon^{-\frac{r+n}{3r-n}}\|\nabla\pi\|^{\frac{1}{\frac{3}{2}-\frac{n}{2r}}}_{L^r}\|u\|^p_{L^p}, \tag{5.3}$$

which implies (1.12) by taking ϵ small enough and using the Gronwall lemma.

Here we have used (2.2) with $s = \frac{p}{2}$ and the Gagliardo-Nirenberg inequalities

$$\|\Lambda^{\frac{1}{2}-\frac{n}{2r}}\pi\|^2_{L^p} \lesssim \|\pi\|_{L^{\frac{p}{2}}}\|\nabla\pi\|_{L^r}, \tag{5.4}$$

$$\|\Lambda^{\frac{1}{2}+\frac{n}{2r}}(|u|^{p-2}u)\|_{L^{\frac{p}{p-1}}} \lesssim \||u|^{p-2}u\|^{\frac{1}{2}-\frac{n}{2r}}_{L^{\frac{p}{p-1}}}\|\nabla(|u|^{p-2}u)\|^{\frac{1}{2}+\frac{n}{2r}}_{L^{\frac{p}{p-1}}}. \tag{5.5}$$

This completes the proof.

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