VOLUME OF THE MINKOWSKI SUMS OF STAR-SHAPED SETS

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Abstract. For a compact set $A \subset \mathbb{R}^d$ and an integer $k \geq 1$, let us denote by

$$A[k] = \{a_1 + \cdots + a_k : a_1, \ldots, a_k \in A\} = \sum_{i=1}^{k} A$$

the Minkowski sum of $k$ copies of $A$. A theorem of Shapley, Folkman and Starr (1969) states that $\frac{1}{k}A[k]$ converges to the convex hull of $A$ in Hausdorff distance as $k$ tends to infinity. Bobkov, Madiman and Wang [Concentration, functional inequalities and isoperimetry, Amer. Math. Soc., Providence, RI, 2011] conjectured that the volume of $\frac{1}{k}A[k]$ is nondecreasing in $k$, or in other words, in terms of the volume deficit between the convex hull of $A$ and $\frac{1}{k}A[k]$, this convergence is monotone. It was proved by Fradelizi, Madiman, Marsiglietti and Zvavitch [C. R. Math. Acad. Sci. Paris 354 (2016), pp. 185–189] that this conjecture holds true if $d = 1$ but fails for any $d \geq 12$. In this paper we show that the conjecture is true for any star-shaped set $A \subset \mathbb{R}^d$ for $d = 2$ and $d = 3$ and also for arbitrary dimensions $d \geq 4$ under the condition $k \geq (d-1)(d-2)$. In addition, we investigate the conjecture for connected sets and present a counterexample to a generalization of the conjecture to the Minkowski sum of possibly distinct sets in $\mathbb{R}^d$, for any $d \geq 7$.

1. Introduction

The Minkowski sum of two sets $K, L \subset \mathbb{R}^d$ is defined as $K + L = \{x + y : x \in K, y \in L\}$, where, for brevity, we set $A[k] = \sum_{i=1}^{k} A$, for any $k \in \mathbb{N}$ and any compact set $A \subset \mathbb{R}^d$. Since Minkowski sum preserves the convexity of the summands and the set $\frac{1}{k}A[k]$ consists in some particular convex combinations of elements of $A$, the containment $\frac{1}{k}A[k] \subseteq \text{conv} A$, and, for the special case of convex sets, the equality $\frac{1}{k}A[k] = \text{conv} A$ trivially holds; here $\text{conv} A$ denotes the convex hull of $A$. These observations suggest that for any compact set $A$, the set $\frac{1}{k}A[k]$ looks “more convex” for larger values of $k$. This intuition was formalized by Starr [St1,St2], crediting also Shapley and Folkman, and independently by Emerson and Greenleaf [EG], by proving that the set $\frac{1}{k}A[k]$ approaches $\text{conv} A$ in Hausdorff distance as $k$ approaches
infinity and by giving bounds on the speed of this convergence (we refer to [FMMZ2]
for more discussion of this fact).

A further step in the investigation of the sequence \( \{ \frac{1}{k} A[k] \} \) is to examine the
monotonicity of this convergence. Whereas this sequence is clearly not monotonous
in terms of containment, the main object of this paper is Conjecture \( \text{II} \) of Bobkov,
Madiman, Wang [BMW], relating the volumes of the elements of the sequence,
and in which \( \text{vol}(K) \) denotes the Lebesgue measure (volume) of the measurable set
\( K \subset \mathbb{R}^d \).

**Conjecture 1** (Bobkov-Madiman-Wang). Let \( A \) be a compact set in \( \mathbb{R}^d \) for some
\( d \in \mathbb{N} \). Then the sequence
\[
\left\{ \text{vol}\left( \frac{1}{k} A[k] \right) \right\}_{k \geq 1}
\]
is nondecreasing in \( k \).

Equivalently, Conjecture \( \text{I} \) asks whether for any integer \( k \geq 1 \) and compact set
\( A \subset \mathbb{R}^d \), the following inequality holds
\[
(1) \quad \text{vol}\left( \frac{1}{k} A[k] \right) \leq \text{vol}\left( \frac{1}{k+1} A[k+1] \right).
\]
This inequality trivially holds for any compact set \( A \) if \( k = 1 \) since \( A \subset \frac{1}{2} A[2] \). In the
same way, it is easy to find monotone subsequences of the sequence \( \{\text{vol}(\frac{1}{k} A[k])\}_{k \geq 1} \)
by the same argument; one such example is \( \{\text{vol}(\frac{1}{2^m} A[2^m])\}_{m \geq 0} \). On the other
hand, even the first nontrivial case; that is, the inequality \( \text{vol}\left( \frac{1}{2} A[2] \right) \leq \text{vol}\left( \frac{1}{3} A[3] \right) \)
seems to require new methods to approach. Conjecture \( \text{II} \) was partially resolved in
[FMMZ1,FMMZ2], where, following the approach of [GMR], the authors proved it
for any 1-dimensional compact set \( A \), but constructed counterexamples in \( \mathbb{R}^d \) for
any \( d \geq 12 \). More precisely, they showed that for every \( k \geq 2 \), there is \( d_k \in \mathbb{N} \)
such that for every \( d \geq d_k \) there is a compact set \( A \subset \mathbb{R}^d \) such that \( \text{vol}\left( \frac{1}{k} A[k] \right) > \text{vol}\left( \frac{1}{k+1} A[k+1] \right) \). In particular, one has \( d_2 = 12 \), whence Conjecture \( \text{II} \) fails for
\( \mathbb{R}^d \) if \( d \geq 12 \).

Our goal is to find additional conditions on \( A \) and \( k \) under which the statement
in Conjecture \( \text{II} \) or more precisely when the inequality \( (1) \), is satisfied.

In the paper, for any set \( A \subset \mathbb{R}^d \) we denote by \( \dim A \) the dimension of the
smallest affine subspace containing \( A \), and for any \( p, q \in \mathbb{R}^d \), we denote the closed
segment with endpoints \( p, q \) by \( [p, q] \). To state our main result, let us recall the
following well-known concept.

**Definition 1.** A nonempty set \( S \subset \mathbb{R}^d \) is called **star-shaped** with respect to a point
\( p \) if for any \( q \in S \), we have \( [p, q] \subseteq S \).

Our main result is the following.

**Theorem 1.** Let \( d \geq 2 \) and \( k \geq \max\{2, (d - 1)(d - 2)\} \) be integers. Then for any
compact, star-shaped set \( S \subset \mathbb{R}^d \) we have
\[
\text{vol}\left( \frac{1}{k+1} S[k+1] \right) \geq \text{vol}\left( \frac{1}{k} S[k] \right),
\]
with equality if only if \( \dim(S) < d \) or \( \frac{1}{k} S[k] = \text{conv}(S) \).
We notice that Theorem 1 establishes Conjecture 1 for star-shaped compact sets in dimensions 2 and 3. It is worth to remark that the compact sets $A$ constructed in [FMMZ2] as counterexamples to Conjecture 1 are star-shaped, which makes Theorem 1 fairly unexpected.

We prove Theorem 1 in Section 2. In Section 3 we adapt our techniques to investigate connected sets. Our main result in this section is summarized in Theorem 2. Finally, in Section 4 we collect some additional remarks and questions, and, in particular, we construct low dimensional counterexamples to a generalization of Conjecture 1 which also appeared in [BMW].

2. Conjecture 1 for star-shaped sets: The proof of Theorem 1

We start this section with a couple of Lemmata which are needed for the proof. Throughout this section, we denote $X_d(t) = \{ (x_1, \ldots, x_d) \in \mathbb{N}^d : x_1 + \ldots + x_d = t \}$ and $N_d(t) = \text{card} X_d(t)$ to be the number of elements of $X_d(t)$. Here and in the rest of the paper we will denote by $\mathbb{N}$ the set of nonnegative integers.

Lemma 1. For any integer $t \geq 1$, and $d \geq 2$, we have $N_d(t) = \binom{t + d - 1}{d - 1}$.

Proof. If $d = 2$, then, clearly, $N_2(t) = t + 1 = \binom{t + 2 - 1}{1}$. On the other hand, by induction, we have

$$N_d(t) = \sum_{s=0}^{t} N_{d-1}(s) = \sum_{s=0}^{t} \binom{s + d - 2}{d - 2} = \binom{t + d - 1}{d - 1}.$$

Lemma 2. Let $d \geq 2$ and $o$ be the origin of $\mathbb{R}^d$, $(p_1, \ldots, p_d)$ be a basis of $\mathbb{R}^d$, and let $B = \bigcup_{i=1}^{d} [0, p_i]$. Consider a compact set $M \subset \mathbb{R}^d$ such that $B[k] \subset M \subset k \text{conv}(B)$ for some $k \geq \max\{2, (d-1)(d-2)\}$, then

$$\text{vol} \left( \frac{1}{k+1}(M + B) \right) \geq \text{vol} \left( \frac{1}{k}M \right),$$

where equality holds if and only if $M = k \text{conv}(B)$. Furthermore, if $\text{vol} \left( \frac{1}{k}M \right) \geq \text{vol} \left( \frac{1}{k+1}(M + B) \right) - \delta$ for some $\delta \geq 0$, then $\text{vol}(M) \geq \text{vol}(k \text{conv}(B)) - C(d, k)\delta$, where the constant $C(d, k) = k^d \left( 1 - \frac{k^d}{(k-d+2)^{(k+1)^{d-1}}} \right)^{-1}$ depends only on $d$ and $k$.

Proof. Since the inequality (2) is independent of a nondegenerate linear transformation applied to $B$ and $M$ simultaneously, we may assume that $(p_1, \ldots, p_d)$ is the canonical basis of $\mathbb{R}^d$. Let

$$V(t) = \{ (x_1, \ldots, x_d) \in [0, 1]^d : x_1 + \ldots + x_d \leq t \}.$$ 

Let $C_i = i + [0, 1]^d$, $i \in \mathbb{Z}^d$ be the unit cube cells of the lattice $\mathbb{Z}^d$, and set $\mu_i = \text{vol}(C_i \cap M)$, and $\lambda_i = \text{vol}(C_i \cap (M + B))$.

Note that for any $i \in X_d(t)$, $\text{vol}(C_i \cap k \text{conv}(B))$ is independent of $i$, namely it is equal to 1, if $t \leq k - d$, and to $V(k - t)$ if $t = k - d + 1, \ldots, k - 1$. A similar statement holds for $\text{vol}(C_i \cap (k+1) \text{conv}(B))$. The number of unit cells contained in $k \text{conv}(B)$ is equal to the number of the solutions of the inequality $x_1 + x_2 + \ldots + x_d \leq k$, where each variable is a positive integer, and thus, it is $\binom{k}{d}$. Hence, if $Y_d(k)$ denotes the union of these cells, then we have that

$$\text{vol}(Y_d(k)) = \frac{k^d}{d!} V,$$
where $k^d = k(k - 1) \ldots (k - d + 1)$, and $V = \text{vol}(\text{conv } B) = \frac{1}{d!}$. Thus,

\begin{equation}
\text{vol}(M) = k^d V + \sum_{t=k-d+1}^{k-1} \sum_{i \in X_d(t)} \mu_i,
\end{equation}

and

\begin{equation}
\text{vol}(M + B) = (k + 1)^d V + \sum_{t=k-d+1}^{k} \sum_{i \in X_d(t)} \lambda_i.
\end{equation}

In the following step, we give a lower bound on the $\lambda_i$’s depending on the values of the $\mu_i$’s. We say that $i \in X_d(t)$ and $i' \in X_d(t + 1)$ are adjacent if the corresponding cells $C_i$ and $C_{i'}$ have a common facet, or in other words, if $i' - i$ coincides with one of the standard basis vectors $p_j$. In this case we write $ii' \in I$. Let $i \in X_d(t)$, and let $S = M \cap C_i$. Then, for every $j = 1, 2, \ldots, d$, $S + p_j \subset (M + B) \cap C_{i'}$ with $i' = i + p_j$. Thus, for any $i \in X_d(t + 1)$,

\begin{equation}
\lambda_i \geq \max \{\mu_{i'} : i' \in X_d(t) \text{ is adjacent to } i\}.
\end{equation}

Note that the right-hand side of this inequality is not less than any convex combination of the corresponding $\mu_{i'}$’s. Using a suitable convex combination for each $i \in X_d(t + 1)$, we show that this inequality implies that

\begin{equation}
\sum_{i \in X_d(t+1)} \lambda_i \geq \frac{t + d}{t + 1} \sum_{i \in X_d(t)} \mu_i.
\end{equation}

Consider some $i = (i_1, i_2, \ldots, i_d) \in X_d(t+1)$. Then the indices in $X_d(t)$ adjacent to $i$ are all of the form $i - p_j$ for some $j = 1, 2, \ldots, d$. Furthermore, $i - p_j$ is adjacent to $i$ iff $i_j \geq 1$, or in other words, iff $i_j \neq 0$. Now, for any $i' \in X_d(t)$ adjacent to $i$
we set \( \alpha_{ii'} = \frac{i_j}{i + 1} \), where \( i - i' = p_j \) (cf. Figure 1). Then, since \( i \in X_d(t + 1) \), we clearly have \( 1 = \sum_{j=1}^{d} i_j + 1 = \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \). Thus, by (5), we have

\[
\lambda_i \geq \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{ii'}
\]

for all \( i \in X_d(t+1) \). Now, let \( i' \in X_d(t) \), and \( i' = (i'_1, i'_2, \ldots, i'_d) \). Then the indices in \( X_d(t+1) \) adjacent to \( i' \) are exactly those of the form \( i' + p_j \) for some \( i = 1, 2, \ldots, d \). Hence,

\[
\sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} = \sum_{j=1}^{d} \frac{i'_j + 1}{t + 1} = \frac{t + d}{t + 1}
\]

Finally, by (7) and (8)

\[
\sum_{i \in X_d(t+1)} \lambda_i \geq \sum_{i \in X_d(t+1)} \sum_{i' \in X_d(t), ii' \in I} \alpha_{ii'} \mu_{ii'} = \sum_{i' \in X_d(t)} \left( \sum_{i \in X_d(t+1), ii' \in I} \alpha_{ii'} \right) \mu_{ii'} = \frac{t + d}{t + 1} \sum_{i' \in X_d(t)} \mu_{ii'}.
\]

Using this inequality and the assumption that \( B[k] \subseteq M \subseteq k \text{ conv}(B) \), we obtain

\[
\text{vol}(M + B) \geq (k + 1)^d V + \sum_{t=k-d+1}^{k-1} \frac{t + d}{t + 1} \sum_{i \in X_d(t)} \mu_i.
\]

Note that the sequence \( \left\{ \frac{t + d}{t + 1} \right\} \), where \( t = 0, 1, 2, \ldots \), is strictly decreasing. Hence, using the fact that if \( i \in X_d(t) \), then \( \mu_i \leq V(k-t) \), one has, for \( k-d+1 \leq t \leq k-1 \),

\[
\frac{t + d}{t + 1} \sum_{i \in X_d(t)} \mu_i \geq \frac{k + 1}{k - d + 2} \sum_{i \in X_d(t)} \mu_i + \left( \frac{t + d}{t + 1} - \frac{k + 1}{k - d + 2} \right) V(k-t) N_d(t)
\]

\[
\geq \frac{k + 1}{k - d + 2} \left( \sum_{i \in X_d(t)} \mu_i - V(k-t) N_d(t) \right) + \frac{t + d}{t + 1} V(k-t) N_d(t).
\]

Hence

\[
\text{vol}(M + B) \geq (k + 1)^d V + \frac{k + 1}{k - d + 2} \sum_{t=k-d+1}^{k-1} \left( \sum_{i \in X_d(t)} \mu_i - V(k-t) N_d(t) \right)
\]

\[
+ \sum_{t=k-d+1}^{k-1} \frac{t + d}{t + 1} V(k-t) N_d(t).
\]

Observe that \( \sum_{t=k-d+1}^{k-1} V(k-t) N_d(t) = (k^d - k^d) V \), since it is the volume of the part of \( k \text{ conv}(B) \) belonging to the cells that are not contained in \( k \text{ conv}(B) \), and the equality follows by (3). Similarly, since

\[
\frac{t + d}{t + 1} N_d(t) = \frac{t + d}{t + 1} \left( \frac{t + d}{d - 1} \right) = \left( \frac{t + d}{d - 1} \right) = N_d(t + 1),
\]
we deduce that
\[ \sum_{t=k-d+1}^{k-1} \frac{t+d}{t+1} V(k-t)N_d(t) = \sum_{t'=k-d+2}^{k} V(k+1-t')N_d(t') = ((k+1)^d - (k+1)^d)V, \]
since it is the volume of the part of \((k+1)\text{conv}(B)\) belonging to cells that are not contained in \((k+1)\text{conv}(B)\). Substituting these into (9) and using (4), we obtain
\[ \text{vol}(M + B) \geq (k+1)^d V + k+1 \frac{k+1}{k-d+2} (\text{vol}(M) - k^d V) + ((k+1)^d - (k+1)^d)V \]
\[ \geq \frac{k+1}{k-d+2} \text{vol}(M) + \left((k+1)^d - \frac{k+1}{k-d+2} \frac{k^d}{d+1}\right) V. \]
Thus,
\[ \text{vol}\left(\frac{1}{k+1}(M + B)\right) \geq \frac{k^d}{(k-d+2)(k+1)^{d-1}} \text{vol}\left(\frac{1}{k}M\right) \]
\[ + \left(1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}}\right) V. \]
Since \(\text{vol}(\frac{1}{k}M) \leq V\), to prove the first inequality of the lemma, it is sufficient to show that the right-hand side of (11) is a convex combination of the volumes, namely that the second coefficient is nonnegative. This is clear if \(d = 2\), while for \(d \geq 3\) using the Binomial Theorem, one has
\[ (k-d+2)(k+1)^{d-1} - k^d \geq (k-d+2)(k^{d-1} + (d-1)k^{d-2}) - k^d \]
\[ = k^{d-1} - (d-1)(d-2)k^{d-2}, \]
which is nonnegative for \(k \geq (d-1)(d-2)\).

Now we prove the equality case. By (11), equality in the lemma implies that \(\text{vol}(\frac{1}{k}M) = V\), or equivalently, \(\text{vol}(k\text{conv}(B) \setminus M) = 0\). Note that since \(\text{vol}(k\text{conv}(B)) > 0\), its interior is not empty. Thus, \(k\text{conv}(B)\) is equal to the closure of its interior. On the other hand, \(\text{vol}(k\text{conv}(B) \setminus M) = 0\) implies that \(\text{int}(k\text{conv} B) \subset M\), but as \(M\) is compact, \(M = k\text{conv} B\) follows.

Finally, if \(\text{vol}\left(\frac{1}{k+1}(M + B)\right) - \delta \leq \text{vol}\left(\frac{1}{k}M\right)\), then in the same way (10) yields the inequality \(\text{vol}(M) \geq \text{vol}(k\text{conv}(B)) - C(d,k)\delta\), with
\[ C(d,k) = \frac{k^d}{1 - \frac{k^d}{(k-d+2)(k+1)^{d-1}}}. \]

\[ \square \]

**Proof of Theorem**. Without loss of generality, we may assume that \(S\) is star-shaped with respect to the origin. Let \(\varepsilon > 0\) be an arbitrary positive number. By Carathéodory’s theorem, we may choose a finite point set \(A_0 \subset S\) such that \(\text{vol}(\text{conv}(S)) - \varepsilon \leq \text{vol}(\text{conv}(A_0))\), and without loss of generality, we may assume that the points of \(A_0\) are in convex position. Clearly, the star-shaped set \(A = \bigcup_{a \in A_0} [0,a]\) is a subset of \(S\), satisfying \(\text{vol}(\text{conv}(S)) - \varepsilon \leq \text{vol}(\text{conv}(A))\). Consider a simplicial decomposition \(\mathcal{F}\) of the boundary of \(\text{conv}(A)\) such that all vertices of \(\mathcal{F}\) are vertices of \(\text{conv}(A)\). Let the \((d-1)\)-dimensional faces of \(\mathcal{F}\) be \(F_1, F_2, \ldots, F_m\), and for \(j = 1, 2, \ldots, m\), let \(B_j = \bigcup_{i=1}^{d} [0, p^j_i]\), where \(p^j_1, p^j_2, \ldots, p^j_d\) are the vertices
of $F_j$. Then $B_j \subseteq S$ for all values of $j$, the sets $\text{conv}(B_j)$ are mutually non-overlapping, and $\text{conv}(A) = \bigcup_{j=1}^{m} \text{conv}(B_j)$. Finally, let $M_j = S[k] \cap (k \text{conv}(B_j))$. Then, since $B_j \subseteq S$, we have $B_j[k] \subseteq M_j \subseteq (k \text{conv}(B_j))$. Thus, Lemma 2 implies that $\text{vol}(\frac{1}{k+1}(M_j + B_j)) \geq \text{vol}(\frac{1}{k}M_j)$. Thus, we have

$$\text{vol}\left(\frac{S[k]}{k} \cap \text{conv}(A)\right) = \sum_{j=1}^{m} \text{vol}\left(\frac{S[k]}{k} \cap \text{conv}(B_j)\right) = \sum_{j=1}^{m} \text{vol}\left(M_j\right) \leq \sum_{j=1}^{m} \text{vol}\left(\frac{M_j + B_j}{k+1}\right) \leq \text{vol}\left(\frac{S[k+1]}{k+1}\right).$$

On the other hand, since $0 \leq \text{vol}(\text{conv}(S)) - \text{vol}(\text{conv}(A)) \leq \varepsilon$, we have

$$0 < \text{vol}\left(\frac{S[k]}{k}\right) - \text{vol}(\text{conv}(A)) \leq \varepsilon,$$

implying that

$$\text{vol}\left(\frac{S[k]}{k}\right) - \varepsilon \leq \sum_{j=1}^{m} \text{vol}\left(\frac{M_j}{k}\right) \leq \sum_{j=1}^{m} \text{vol}\left(\frac{M_j + B_j}{k+1}\right) \leq \text{vol}\left(\frac{S[k+1]}{k+1}\right).$$

This inequality is satisfied for all positive $\varepsilon$, and thus, the inequality part of Theorem 1 holds.

Now, assume that

$$\text{vol}\left(\frac{S[k]}{k}\right) = \text{vol}\left(\frac{S[k+1]}{k+1}\right),$$

and that $\text{dim}(S) = d$. Then, from inequality (12) we deduce that

$$\sum_{j=1}^{m} \left(\text{vol}\left(\frac{M_j + B_j}{k+1}\right) - \text{vol}\left(\frac{M_j}{k}\right)\right) \leq \varepsilon.$$

For $j = 1, 2, \ldots, m$, set $\delta_j = \text{vol}\left(\frac{1}{k+1}(M_j + B_j)\right) - \text{vol}(\frac{1}{k}M_j)$. Then, clearly\n
$$\sum_{j=1}^{m} \delta_j \leq \varepsilon.$$ On the other hand, by Lemma 2 for every $j = 1, 2, \ldots, m$, we have $\text{vol}(k\text{conv}B_j) - \text{vol}(M_j) \leq C(k, d)\delta_j$, where $C(k, d)$ is defined in (11). Thus, summing on $j$, it follows that

$$\varepsilon C(k, d) \geq \text{vol}(\text{conv}(kA)) - \text{vol}(S[k] \cap \text{conv}(kA)),$$

implying that $\varepsilon (k^d + C(k, d)) \geq \text{vol}(\text{conv}(kS)) - \text{vol}(S[k])$. This inequality holds for any value $\varepsilon > 0$, and hence, $\text{vol}(\text{conv}(S)) = \text{vol}(\frac{k}{k}S[k])$, or equivalently, $\text{vol}(\text{conv}(S) \setminus \frac{k}{k}S[k]) = 0$. Since conv(S) is a compact, convex set with nonempty interior, and $\frac{k}{k}S[k]$ is compact, to show the equality $\text{conv}(S) = \frac{k}{k}S[k]$, we may apply the argument at the end of the proof of Lemma 2.

3. Conjecture 1 for connected sets

In the first few lemmata we collect some elementary properties of the Minkowski sum of connected sets. Throughout this section, $e_1, e_2$ denote the elements of the standard orthonormal basis of $\mathbb{R}^2$.

**Lemma 3.** Let $A \subseteq \mathbb{R}^d$ be a compact set with a connected boundary and let $\partial A \subseteq B \subseteq A$. Then $B + B = A + A$. 

Proof. We have $\partial A + \partial A \subseteq B + B \subseteq A + A$. Thus it is sufficient to prove that $\partial A + \partial A = A + A$. Clearly, $A + A \supseteq \partial A + \partial A$. We show that $\frac{A + A}{2} \subseteq \frac{\partial A + \partial A}{2}$, which then yields the assertion. Consider a point $p \in \frac{A + A}{2}$. Then $p$ is the midpoint of a segment whose endpoints are points of $A$. Let $\chi_p : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection about $p$ defined by $\chi_p(x) = 2p - x$, for $x \in \mathbb{R}^d$. To prove that $p \in \frac{\partial A + \partial A}{2}$, we need to show that for some $q \in \partial A$, we have $\chi_p(q) \in \partial A$. To do this, let us define $f_p(x)$ ($x \in \mathbb{R}^d$) as the signed distance of $\chi_p(x)$ from the boundary of $A$, where the sign is positive if $\chi_p(x) \notin A$, and not positive if $\chi_p(x) \in A$. Here we remark that since $A$ is compact, $\partial A$ is compact as well. Let $x_1$ be a point of $\partial A$ farthest from $p$. If $\chi_p(x_1) \in A$ then $\chi_p(x_1) \notin A$, implying that $f_p(x_1) > 0$. Now, since $p \in \frac{A + A}{2}$, we have some $y \in A$ such that $\chi_p(y) \in A$. Let $L$ be the line through $y$, $p$ and $\chi_p(y)$. Let $y'$ and $y''$ be points of $L \cap \partial A$ closest to $y$ and $\chi_p(y)$, respectively. Then the segments $[y, y']$ and $[\chi_p(y), y'']$ are included in $A$. If $0 < |y' - y| \leq |y'' - \chi_p(y)|$, then $y' \in \partial A$ and $\chi_p(y') \in [\chi_p(y), y''] \subseteq A$. If $0 < |y'' - \chi_p(y)| \leq |y' - y|$, then the same holds for $y''$ in place of $y'$. Thus, it follows that for some point $x_2 \in \partial A$, $\chi_p(x_2) \in A$. If $\chi_p(x_2) \in \partial A$, then we are done, and so we may assume that $\chi_p(x_2) \in \text{int} A$, which yields that $f_p(x_2) < 0$.

We have shown that $f_p : \partial A \to \mathbb{R}$ attains both a positive and a negative value on its domain. On the other hand, since $f$ is continuous and $\partial A$ is connected, $f_p(q) = 0$ for some $q \in \partial A$, from which the assertion readily follows. \hfill $\square$

Remark 1. Lemma holds also for the boundary of the external connected component of $\mathbb{R}^d \setminus A$ in place of $\partial A$.

Remark 2. We note that the equality $A_1 + A_2 = \partial A_1 + \partial A_2$ does not hold in general for different compact sets $A_1, A_2$ with connected boundaries. To show it, one may consider the sets $A_1 = B_2^2$ and $A_2 = \varepsilon B_2^2$ for some sufficiently small value of $\varepsilon$, where $B_2^2$ be the Euclidean unit ball of dimension $d$ centered at the origin.

Remark 3. Lemma does not hold if we omit the condition that $\partial A$ is connected. To show it, we may choose $A$ as the union of $B_2^2$ and a singleton $\{p\}$ with $|p|$ being sufficiently large.

Corollary 1. If $A$ is a compact set with a connected boundary then $A + A = A + \partial A = \partial A + \partial A$. Thus, for any positive integer $k \geq 2$, we have $A[k] = \partial A[k]$.

Corollary 2. Let $d \geq 2$ and $k \geq \max\{2, (d - 1)(d - 2)\}$. Let $A$ be a compact set such that $\partial S \subseteq A \subseteq S$ for some compact, star-shaped set $S \subseteq \mathbb{R}^d$. Then we have

$$\text{vol} \left( \frac{1}{k} A[k] \right) \leq \text{vol} \left( \frac{1}{k + 1} A[k + 1] \right).$$

Proof. Without loss of generality, we may assume that $S$ is star-shaped with respect to the origin. Set $S' = S + \varepsilon B_2^d$ for some small value $\varepsilon > 0$.

First, we show that $\partial S'$ is path-connected. Let $L$ be a ray starting at $o$. Since $o \in \text{int} S'$, $L \cap \partial S' \neq \emptyset$. Let $p \in L \cap \partial S'$. Then there is a point $q \in S$ such that $|q - p| = \varepsilon$. Now, if $x$ is any relative interior point of $[o, q]$, then the line through $x$ and parallel to $[p, q]$ intersects $[o, q]$ at a point at distance less than $\varepsilon$ from $x$. Since $[o, q] \subseteq S$, from this it follows that $x \in S + \varepsilon \text{int} B_2^d \subseteq \text{int} S'$. In other words, for any $p \in \partial S'$, all points of $[o, p]$ but $p$ lie in $\text{int} S'$. Thus, $L \cap \partial S'$ is a singleton for any ray $L$ starting at $o$. 

\hfill $\square$
Let $0 < r < R$ such that $\partial S' \subset H = RB_2 \setminus (r \text{ int } B_2)$. Let $P : H \to S^{d-1}$ be the central projection to $S^{d-1}$. Note that $P$ is Lipschitz, and thus continuous on $H$, and its restriction $P|_{\partial S'}$ to $\partial S'$ is bijective. On the other hand, since $\partial S'$ (as also $S'$) are compact, this implies that the inverse of $P|_{\partial S'}$ is continuous, that is, $\partial S'$ and $S^{d-1}$ are homeomorphic. Thus, $\partial S'$ is path-connected.

On the other hand, $\partial S \subseteq A \subseteq S$ implies that $A' = A + \varepsilon B_2 \subseteq S'$, and $\partial S' \subseteq \partial S + \varepsilon S^{d-1} \subseteq \partial S + \varepsilon B_2 \subseteq A'$. Now, we may apply Lemma 3 and Corollary 1, and obtain that for any value of $k \geq 2$, $A'[k] = S'[k]$. Thus, by Theorem 1 it follows that

$$\text{vol} \left( \frac{A[k]}{k} + \varepsilon B_2 \right) = \text{vol} \left( \frac{A'[k]}{k} \right) \leq \text{vol} \left( \frac{A'[k+1]}{k+1} \right) = \text{vol} \left( \frac{A[k+1]}{k+1} + \varepsilon B_2 \right).$$

On the other hand, for any compact set $C$ the function $t \mapsto \text{vol}(C + tB_2)$ is continuous on $[0, +\infty)$, see for example [FX], hence $\lim_{t \to 0^+} \text{vol} \left( \frac{1}{m} A[m] + \varepsilon B_2 \right) = \text{vol} \left( \frac{1}{m} A[m] \right)$, for any integer $m$ which implies the corollary. \hfill $\square$

Let us denote the closure of a set $A \subset \mathbb{R}^d$ by $\text{cl}(A)$.

**Proposition 1.** Let $\gamma \subset \mathbb{R}^2$ be a simple continuous curve connecting $o$ and $e_1$ such that its intersection with the $x$-axis is $\{o, e_1\}$. Let $D$ be the interior of the closed Jordan curve $\gamma \cup [o, e_1]$. For $i = 0, 1$, let $\gamma_i = \frac{1}{2} e_1 + \frac{1}{2} \gamma$, and $D_i = \frac{1}{2} e_1 + \frac{1}{2} D$. Then $\text{cl}(D \setminus (D_0 \Delta D_1)) \subseteq \frac{1}{2} \gamma[2]$, where $\Delta$ denotes symmetric difference.

**Proof.** For convenience, we assume that $\gamma$ lies in the half plane $\{y \leq 0\}$. As in the proof of Lemma 3 let $\chi_p : \mathbb{R}^2 \to \mathbb{R}^2$ denote the reflection about $p \in \mathbb{R}^2$ defined by $\chi_p(x) = 2p - x$, and note that $p \in \frac{1}{2} \gamma[2]$ if and only if there is some point $q \in \gamma$ such that $\chi_p(q) \in \gamma$, or in other words, if $\gamma \cap \chi_p(\gamma) \neq \emptyset$. Let $L$ denote the $x$-axis, $L_p = \chi_p(L)$, and let $S$ be the infinite strip between $L$ and $L_p$ (cf. Figure 2).

First, observe that $o, e_1 \in \gamma$ yields that $\gamma_0 \cup \gamma_1 \subset \frac{1}{2} \gamma[2]$, and $\gamma \subset \frac{1}{2} \gamma[2]$ trivially holds. Thus, we need to show that if for some point $p$ we have $p \in D \setminus \text{cl}(D_0 \cup D_1)$ or $p \in D_0 \cap D_1 \cap D$, then $p \in \frac{1}{2} \gamma[2]$. We do it only for the case $p \in D \setminus \text{cl}(D_0 \cup D_1)$ since for the second case a similar argument can be applied.

Consider some point $p \in D \setminus (D_0 \cup D_1)$. Then $p \notin \text{cl}(D_0 \cup D_1)$ yields that $\chi_p(o) = 2p \notin \text{cl} D$, and the relation $\chi_p(e_1) \notin \text{cl} D$ follows similarly.

**Case 1 ($\gamma \subset S$).** Note that in this case $\chi_p(\gamma) \subset S$. Since $p \in D$ and $\chi_p(o) \notin \text{cl} D$, $\partial D = \gamma \cup [o, e_1]$ and $[\chi_p(o), p] \cap [o, e_1] = \emptyset$, it follows by the continuity of $\gamma$ that

![Figure 2](image-url)
consider the case where $p /∈ D_p$. Then $p$ is an exterior point of $D_p$, and there is a connected component $γ^*$ of $γ_p$, with endpoints on $L_p$, that separates $p$ from $L$. Since the reflections of the endpoints of $γ^*$ about $p$ lie on $L$, we may apply the argument in Case I and obtain that $∅ /∈ γ^∗ \cap χ_p(γ^*) \subseteq γ \cap χ_p(γ)$. Thus, we may assume that $p ∈ D_p$. If $χ_p(x_0) ∈ [o, e_1]$, then the continuity of $γ_i$ and $χ_p(o) /∈ ∂D_i$ implies that $∅ /∈ γ \cap χ_p(γ_i) \subseteq γ \cap χ_p(γ)$. If $χ_p(x_1) ∈ [o, e_1]$, then we may apply a similar argument, and thus we may assume that $χ_p(x_0), χ_p(x_1) /∈ [o, e_1]$. This implies that either $[o, p_1] \subseteq [χ_p(x_0), χ_p(x_1)]$ or $[χ_p(x_0), χ_p(x_1)] \cap [o, p_1]$ are disjoint.

The relation $[o, p_1] \subseteq [χ_p(x_0), χ_p(x_1)]$ yields $[χ_p(o), χ_p(p_1)] \subseteq [o, x_1]$, and, by the previous argument, we have $∅ /∈ χ_p(γ_i) \cap γ_i \subseteq γ \cap χ_p(γ)$. Thus, we are left with the case where $[χ_p(x_1), χ_p(x_2)]$ and $[o, p_1]$ are disjoint; without loss of generality we may assume that $χ_p(x_1), χ_p(x_0), o$ and $e_1$ are in this consecutive order on $L$. Let $U$ be the closure of the connected component of $S \setminus γ_i$ containing $γ_i$. Then $χ_p(p) = p ∈ int U \cap χ_p(U)$, implying that $∅ /∈ γ_i \cap χ_p(γ_i) \subseteq γ \cap χ_p(γ)$.}

The proof of Lemma I below is based on the idea of the proof of Proposition I with some necessary modifications.

**Lemma 4.** Let $k ≥ 2$, and let $γ \subseteq \mathbb{R}^2$ be a convex, continuous curve connecting $o$ and $e_1$ such that its intersection with the $x$-axis is $\{o, e_1\}$. Let $D$ be the interior of the closed Jordan curve $γ \cup [o, e_1]$. For $i = 0, 1, \ldots, k - 1$, let $γ_i = \frac{k}{k-1} e_1 + \frac{1}{k} γ$, and $D_i = \frac{1}{k} e_1 + \frac{1}{k} D$. Then $cl \left( D \setminus \bigcup_{i=1}^{k} D_i \right) \subseteq \frac{1}{k} γ[k]$, and for any $i ≠ j$, $D_i \cap D_j \subseteq \frac{1}{k} γ[k]$.

**Proof.** First observe that $D$ is convex, hence $D_i$ is contained in $D$ for all values of $i$. Let us denote the $x$-axis by $L$ and, for any $p ∈ \mathbb{R}^2$, let $χ^k_p : \mathbb{R}^2 \to \mathbb{R}^2$ be the homothety with center $p$ and ratio $\frac{1}{k-1}$ defined by $χ^k_p(x) = \frac{k}{k-1} p - \frac{x}{k-1}$, for $x ∈ \mathbb{R}^2$. Furthermore, we set $L^k_p = χ^k_p(L)$, and denote the infinite strip between $L$ and $L^k_p$ by $S$. The assertion for $k = 2$ is a special case of Proposition I. To prove it for $k ≥ 3$, we apply induction on $k$, and assume that the lemma holds for $γ[k-1]$. Let $p ∈ cl \left( D \setminus \bigcup_{i=1}^{k} D_i \right)$. Clearly, since $(∂D) \setminus \bigcup_{i=1}^{k} D_i = γ \subseteq γ[k]$, we may assume that $p ∈ D$. By the induction hypothesis for $\frac{k-1}{k} γ$, if $p ∈ X_1 = \frac{k-1}{k} cl D$, then $p ∈ \frac{k-1}{k} \cdot \frac{1}{k-1} γ[k-1] = \frac{1}{k} γ[k-1] \subseteq \frac{1}{k} γ[k]$. Similarly, if $p ∈ X_2 = \frac{1}{k} e_1 + \frac{k-1}{k} cl D$, then $p ∈ \frac{1}{k} e_1 + \frac{1}{k} γ[k-1] \subseteq \frac{1}{k} γ[k]$. Thus, assume that $p /∈ X_1 \cup X_2$, which yields that $χ^k_p(o)$ and $χ^k_p(e_1)$ are in the interior of $D$. Let the (unique) intersection point of $[p, χ^k_p(o)]$ and $γ$ be $q_1$ and the (unique) intersection point of $[p, χ^k_p(e_1)]$ and $γ$
be \(q_2\). As \(\chi^k_p(q_1) \in [o,p]\), the convexity of \(D\) implies that \(\chi^k_p(q_1) \in D\), and the containment \(\chi^k_p(q_2) \in D\) follows similarly.

Similarly like in Proposition 1, if \(\gamma \subseteq S\), then by continuity, \(\gamma \cap \chi^k_p(\gamma) \neq \emptyset\), which implies the containment \(p \in \frac{1}{k} \gamma[k]\). Assume that \(\gamma \nsubseteq S\). Then \(S \cap \gamma\) has two connected components \(\gamma_1, \gamma_2\), where we choose the indices such that \(o \in \gamma_1\), and \(e_1 \in \gamma_2\). Clearly, we have either \(q_1 \in \gamma_2, q_2 \in \gamma_1\), or both. If \(q_1 \in \gamma_2\), then the containment relations \(\chi(q_1) \in D, \chi(e_1) \notin \text{cl} D\), and \(\chi^k_p(\gamma_2) \subseteq S\) yield that \(\emptyset \neq \gamma_1 \cap \chi^k_p(\gamma_2) \subseteq \gamma \cap \chi^k_p(\gamma)\). If \(q_2 \in \gamma_1\), then the assertion follows by a similar argument.

Finally, we consider the case that \(p \in D_i \cap D_j\) for some \(i < j\). In this case the convexity of \(D\) implies that \(p \in D_s\) for any \(i \leq s \leq j\). This yields that there are some distinct values \(i, j \leq k - 1\) or \(i, j \geq 2\) such that \(p \in D_i \cap D_j\). Thus, the assertion readily follows from the induction hypothesis. \(\square\)

**Lemma 5.** Let \(k \geq 2\) and \(\gamma\) be a bounded convex curve in \(\mathbb{R}^2\), and let \(\gamma[k] \subseteq M \subseteq k \text{conv} \gamma\). Then

\[
\text{area}\left(\frac{1}{k} M\right) \leq \text{area}\left(\frac{1}{k+1}(M + \gamma)\right).
\]

**Proof.** If \(\gamma\) is closed, then Lemma 3 yields that \(\frac{1}{k} \gamma[k] = \text{conv} \gamma\) for all \(k \geq 2\), which clearly implies the statement. Assume that \(\gamma\) is not closed. Since the inequalities in Lemma 3 do not change under affine transformations, we may assume that the endpoints of \(\gamma\) are \(o\) and \(e_1\), and the \(x\)-axis is a supporting line of \(\text{conv} \gamma\).

Let us define

\[
D = \text{conv} \gamma, \alpha = \text{area}(D \cap (e_1 + D)), \quad \beta = \text{area}(D \cap ((e_1 + D) \cup (-e_1 + D)))
\]

Note that \(0 \leq \alpha \leq \beta \leq 2\alpha\). Let \(D_i = i e_1 + D\) for \(i = 0, 1, \ldots, k\). For \(0 \leq i \leq k - 1\), let \(\mu_i\) be the area of the region of \(M\) in \(D_i\) that do not belong to any \(D_j, j \neq i\), where we note that since \(k \geq 2\), by Lemma 4 we have that all other points of \(D_i\) belong to \(M\). Similarly, for \(0 \leq i \leq k\), let \(\lambda_i\) be the area of the region of \(M + \gamma\) in \(D_i\) that do not belong to any \(D_j, j \neq i\). An elementary computation shows that

\[
\text{area}(M) = k^2 \text{area}(D) - 2(\text{area}(D) - \alpha) - (k - 2)(\text{area}(D) - \beta) + \sum_{i=0}^{k-1} \mu_i
\]

\[
= (k^2 - k) \text{area}(D) + 2\alpha + (k - 2)\beta + \sum_{i=0}^{k-1} \mu_i,
\]

and similarly,

\[
\text{area}(M + \gamma) = (k^2 + k) \text{area}(D) + 2\alpha + (k - 1)\beta + \sum_{i=0}^{k} \lambda_i.
\]

Since \(o, e_1 \in \gamma\), we have \(M, e_1 + M \subseteq M + \gamma\). Thus, \(\lambda_0 \geq \mu_0, \lambda_k \geq \mu_{k-1}, \lambda_1 \geq \max\{\mu_0 - (\beta - \alpha), \mu_1\}, \lambda_{k-1} \geq \max\{\mu_{k-2}, \mu_{k-1} - (\beta - \alpha)\}\), and for \(2 \leq i \leq k - 2, \lambda_i \geq \max\{\mu_{i-1}, \mu_i\}\). Since \(\lambda_i \geq \frac{i}{k} \mu_{i-1} + \frac{k-i}{k} \mu_i\) if \(2 \leq i \leq k - 2\), and \(\lambda_i \geq \frac{i}{k} \mu_{i-1} + \frac{k-i}{k} \mu_i - \frac{2}{k} (\beta - \alpha)\) if \(i = 1\) or \(i = k - 1\), it follows that

\[
\sum_{i=0}^{k} \lambda_i \geq \frac{k}{k} \sum_{i=1}^{k-1} \mu_i - \frac{2}{k} (\beta - \alpha).
\]
Thus, by (13),
\[
\sum_{i=0}^{k} \lambda_i \geq \frac{k+1}{k} (\operatorname{area}(M) - (k^2 - k) \operatorname{area}(D) - 2\alpha - (k - 2)\beta) - \frac{2}{k} (\beta - \alpha).
\]
After substituting this into (14) and simplifying, we obtain
\[
\operatorname{area}(M + \gamma) \geq \frac{k+1}{k} \operatorname{area}(M) + (k+1) \operatorname{area}(D),
\]
which yields
\[
\operatorname{area} \left( \frac{1}{k+1} (M + \gamma) \right) \geq \frac{k}{k+1} \operatorname{area} \left( \frac{1}{k} M \right) + \frac{1}{k+1} \operatorname{area}(D).
\]
Thus, the inequality \( \frac{1}{k} M \leq \operatorname{area}(D) \) yields the assertion. \( \square \)

In Theorem 2, by an open topological disc we mean the bounded connected component defined by a Jordan curve, and recall that a convex body is a compact, convex set with nonempty interior.

**Theorem 2.** Let \( k \geq 2 \). Let \( K \) be a plane convex body, and let \( F = \{ F_i : i \in I \} \) be a family of pairwise disjoint open topological discs such that if \( F_i \cap \partial K \neq \emptyset \) then \( F_i \cap \partial K \) is a connected curve and \( F_i \) is convex. Let \( X = K \setminus \left( \bigcup_{i \in I} F_i \right) \). Then
\[
\operatorname{area} \left( \frac{1}{k} X[k] \right) \leq \operatorname{area} \left( \frac{1}{k+1} X[k+1] \right).
\]

**Proof.** Clearly, we may assume that each \( F_i \) intersects \( K \), and also for each \( F_i \), \( (\partial K) \setminus F_i \) is infinite, since removing the first type discs does not change \( X \), and if there is some \( F_i \) such that \( (\partial K) \setminus F_i \) is finite, then \( X \) is either \( \emptyset \) or a singleton, and in both cases the statement is trivial. Thus, we have that if \( F_i \) intersects \( \partial K \), then the boundary of the convex set \( F_i \cap K \) consists of the two connected, convex curves \( F_i \cap \partial K \) and \( K \cap \partial F_i \).

First, note that since each member of \( F \) has positive area, it has countably many elements; indeed, for any \( \delta > 0 \) there are only finitely many elements \( F_i \) of \( F \) for which \( \operatorname{area}(F_i \cap K) \geq \delta \), and thus, we may list the elements according to area. Furthermore, since \( X \) is compact, \( \operatorname{area}(X) \) exists.

By Lemma 3, we may assume that every member of \( F \) intersects \( \partial K \); indeed, if some \( F_i \) does not intersect \( \partial K \), then \( \partial D \) is a compact, connected set in \( X \), implying that \( F_i \subseteq \frac{1}{k} (\partial F_i)[k] \subseteq \frac{1}{k} X[k] \) for all \( k \geq 2 \). For any \( i \in I \), let \( \gamma_i \) denote the part of \( \partial F_i \) in \( K \). Clearly, \( \gamma_i \) is a convex curve, and the segment connecting its endpoints lies in \( K \) by convexity. As the two endpoints of \( \gamma_i \) are in \( \partial K \), the line through them supports \( K \setminus F_i \). Choose some finite subfamily \( I_\varepsilon \subseteq I \) such that \( \operatorname{area}(X_\varepsilon \setminus X) \leq \varepsilon \), where \( X_\varepsilon = K \setminus \left( \bigcup_{i \in I_\varepsilon} F_i \right) \). This is possible, since for any ordering of the elements, \( \sum_{i \in I} \operatorname{area}(K \cap F_i) \) is a bounded series with positive elements, and hence, it is absolute convergent, and convex sets with small area and bounded diameter are contained in a small neighborhood of their boundary.

For any \( i \in I_\varepsilon \), we set \( D_i = F_i \cap K \), and observe that \( D_i \) is a convex set separated from \( X_\varepsilon \) by the convex curve \( \gamma_i \). Let the endpoints of \( \gamma_i \) be \( q_i^1 \) and \( q_i^2 \), and let \( D_{i1} \) be the homothetic copy of \( D_i \) with ratio \( \frac{1}{k} \) and center \( j^1 \). Furthermore, for \( j = 2, 3, \ldots, k \), let \( D_{ij} = \frac{j-1}{k} (q_i^2 - q_i^1) + D_{i1} \) (cf. Figure 3). Then, by Lemma 4, \( \frac{1}{k} \gamma_i[k] \subseteq \frac{1}{k} X_\varepsilon[k] \) contains all points of \( D_i \) belonging to none of the \( D_{ij} \)s or to at
least two of them. Let $M_i = (X[k] \cap (kD_i))$. Then $M_i \subseteq \text{conv}(kD_i)$, and thus, Lemma 5 yields that

$$\text{area}\left(\frac{1}{k}M_i\right) \leq \text{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right).$$

On the other hand, with the notation $D_\varepsilon = \bigcup_{i \in I_\varepsilon} D_i$, we have

$$\text{area}\left(\frac{1}{k}X[k] \cap D_\varepsilon\right) = \sum_{i \in I_\varepsilon} \text{area}\left(\frac{1}{k}M_i\right),$$

and

$$\text{area}\left(\frac{1}{k+1}X[k+1] \cap D_\varepsilon\right) \geq \sum_{i \in I_\varepsilon} \text{area}\left(\frac{1}{k+1}(M_i + \gamma_i)\right),$$

and thus, we have $\text{area}\left(\frac{1}{k}X[k] \cap D_\varepsilon\right) \leq \text{area}\left(\frac{1}{k+1}X[k+1] \cap D_\varepsilon\right)$. On the other hand, since $\text{area}(X_\varepsilon \setminus X) < \varepsilon$, $X_\varepsilon \cup D_\varepsilon = \text{conv} X$, and $X \subseteq X_\varepsilon$, we have that $\text{area}(\frac{1}{m}X[m] \setminus D_\varepsilon) \leq \varepsilon$ for all $m \geq 1$. This implies that

$$\text{area}\left(\frac{1}{k}X[k]\right) \leq \text{area}\left(\frac{1}{k+1}X[k+1]\right) - \varepsilon.$$

This holds for all $\varepsilon > 0$, which yields the assertion. □

4. ADDITIONAL REMARKS AND QUESTIONS

**Remark 4.** One can ask if the statement of Theorem holds for arbitrary measure instead of volume. The answer to this question is negative. Indeed, consider the measure $\mu(K) = \text{vol}(K \cap C)$, where $C = [-\frac{1}{d}, \frac{1}{d}]^d$ and $S = \bigcup_{i=1}^d [0, e_i]$, where $e_1, e_2, \ldots, e_d$ are the vectors of the standard orthonormal basis. Then, clearly, we have

$$\mu\left(\frac{1}{2k}S[2k]\right) = \frac{1}{2^d} \text{vol}(C) > \mu\left(\frac{1}{2k+1}S[2k+1]\right).$$
Remark 5. The statement of Theorem 1 does not hold for arbitrary measures even for rotationally invariant measures in the plane: for any value of $k$ there is a compact, star-shaped set $S \subset \mathbb{R}^2$ such that $\text{vol}(\frac{1}{k} S[k] \cap B_2^2) > \text{vol}(\frac{1}{k+1} S[k+1] \cap B_2^2)$. To prove this, set $S = [o, e_1] \cup [o, e_2]$, and let $E$ denote the ellipse centered at $o$ and containing the points $(1 - 1/k, 0)$ and $(1 - 2/k, 1/k)$. It is an elementary computation to check that in this case $\text{vol}(\frac{1}{k} S[k] \cap E) = \frac{1}{4} \text{vol}(E)$. On the other hand, the boundary point $(1 - 2/(k+1), 1/(k+1))$ of $\frac{1}{k+1} S[k+1]$ lies in $\text{int}(E)$, which implies that $\text{vol}(\frac{1}{k+1} S[k+1] \cap E) < \frac{1}{4} \text{vol}(E)$. Now, if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as the linear transformation mapping $E$ into $B_2^2$, then $f(S)$ satisfies the required conditions.

One can use star-shaped sets together with ideas from [FMMZ2] to give a negative answer to a more general version of Conjecture 1 also from [BMY].

Conjecture 2 (Bobkov-Madiman-Wang). For any $k \geq 2$, and any compact sets $A_1, A_2, \ldots, A_{k+1}$ in $\mathbb{R}^d$, we have

$$\text{vol}\left(\sum_{i=1}^{k+1} A_i\right)^{1/d} \geq \frac{1}{k} \sum_{i=1}^{k+1} \text{vol}\left(\sum_{j \neq i} A_j\right)^{1/d}.$$ 

In particular, for $k = 2$,

$$\text{vol}(A_1 + A_2 + A_3)^{1/d} \geq \frac{1}{2} \left( \text{vol}(A_1 + A_2)^{1/d} + \text{vol}(A_1 + A_3)^{1/d} + \text{vol}(A_2 + A_3)^{1/d} \right). \tag{15}$$

Conjecture 2 is trivial for convex sets. Moreover, (15) is true when $A_1 = A_2$ and $A_1$ is convex. Indeed, in this case (15) is equivalent to

$$\text{vol}(A_1 + A_1 + A_3)^{1/d} \geq \text{vol}(A_1)^{1/d} + \text{vol}(A_1 + A_3)^{1/d},$$

which follows from the Brunn-Minkowski inequality [Sch].

It was proved in [FMMZ2] that Conjecture 2 is true in $\mathbb{R}$. Since an affirmative answer to Conjecture 2 implies also Conjecture 1 the former is also false for $d \geq 12$ by [FMMZ1,FMMZ2]. Here we show that Conjecture 2 is false in $\mathbb{R}^d$ for $d \geq 7$.

Proposition 2. For any $d \geq 7$, there are compact, star-shaped sets $A_1, A_2, A_3 \subset \mathbb{R}^d$ satisfying

$$\text{vol}(A_1 + A_2 + A_3)^{1/d} < \frac{1}{2} \left( \text{vol}(A_1 + A_2)^{1/d} + \text{vol}(A_1 + A_3)^{1/d} + \text{vol}(A_2 + A_3)^{1/d} \right).$$

Proof. We give the proof for $d = 7$ and the result follows for $d > 7$ by taking direct products with a cube. Consider the sets $A_1 = [0, 1]^4 \times \{0\}^3; A_2 = \{0\}^4 \times [0, 1]^3$ and $A_3 = ([0, a]^4 \times \{0\}^3) \cup (\{0\}^4 \times [0, b]^3)$, where we select $a, b > 0$ later. Since these sets are lower dimensional, one has $\text{vol}(A_1) = \text{vol}(A_2) = \text{vol}(A_3) = 0$. An elementary consideration shows that $\text{vol}(A_1 + A_3) = b^3$, $\text{vol}(A_2 + A_3) = a^4$ and $\text{vol}(A_1 + A_2) = 1$, and

$$\text{vol}(A_1 + A_2 + A_3) = (a + 1)^4 + (b + 1)^3 - 1.$$
The last step is to show that, with \( a = 3 \) and \( b = 6 \), the quantity
\[
((a + 1)^4 + (b + 1)^3 - 1)^{1/7} - \frac{1}{2} \left( a^{4/7} + b^{3/7} + 1 \right)
\]
is negative, which gives a counterexample to (15).

\[\square\]

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References


