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EFFECTIVE CONTRACTION OF SKINNING MAPS

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Abstract. Using elementary hyperbolic geometry, we give an explicit formula for the contraction constant of the skinning map over moduli spaces of relatively acylindrical hyperbolic manifolds.

1. Introduction

Let M_1, M_2 be hyperbolic manifolds of finite-type, i.e. the interior of compact 3-manifolds, with incompressible boundary, and homeomorphic geometrically finite ends $E_1 \subset M_1$ and $E_2 \subset M_2$. From a topological point of view, since M_1 and M_2 are tame, [1, 4], the surfaces S_i corresponding to the boundary of the ends E_i are naturally homeomorphic. We can thus glue the two manifolds via an orientation-reversing homeomorphism τ, and obtain a new topological 3-manifold $M = M_1 \cup_\tau M_2$. Usually, one seeks sufficient conditions for M to admit a complete hyperbolic metric, which is relevant, for example, in the proof of geometrization for hyperbolic manifolds, [11]. We call this the *glueing problem* for M. The *skinning map*, described below, was first introduced by W. P. Thurston, exactly to study this glueing problem, [14].

The moduli space $GF(M, \mathcal{P})$ of all hyperbolic metrics on M with geometrically finite ends and parabolic locus $\mathcal{P}$ is parameterised by the Teichmüller space $\mathcal{T}(\partial_0 M)$ with $\partial_0 M$ the closure in ∂M of the complement $\mathcal{P}^{\mathrm{c}}$ of $\mathcal{P}$, viz. $GF(M, \mathcal{P}) = \mathcal{T}(\partial_0 M)$. For simplicity, let us here assume that $\mathcal{P}$ only contains toroidal boundary components of M. Now, let $N \in GF(M, \mathcal{P})$ be a uniformization, and $S \in \pi_0(\partial_0 M)$ be a (non-toroidal) boundary component. The cover of N associated to $\pi_1(S)$ is a quasi-Fuchsian manifold N_S. The manifold N_S has two ends, A and B, of which A is isometric to the end of M corresponding to S. One defines the skinning map σ_M at N as the conformal structure of the new end B. As it turns out, the skinning map is an analytic map $\sigma_M \colon \mathcal{T}(\partial_0 M) \to \mathcal{T}(\overline{\partial_0 M})$, where the bar denotes opposite orientation. The glueing instruction determines an isometry $\tau^* \colon \mathcal{T}(\partial_0 M) \to \mathcal{T}(\overline{\partial_0 M})$, and any fixed point of $\tau^* \circ \sigma_M$ gives a solution to the glueing problem by the Maskit Combination Theorem, e.g. [11].

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Given a covering map between Riemann surfaces $\pi\colon Y \to X$ the *Poincaré series operator* is a push-forward operator $\Theta_{Y/X} : Q(Y) \to Q(X)$, similar to the push-forward of measures, pushing quadrating differentials on Y to quadratic differentials on X.

In [12], C. McMullen showed that the skinning map of an acylindrical manifold N is contracting, with contraction constant only depending on the topology of $\partial_0 M$. Furthermore, he related the skinning map to the Poincaré series operator Θ by the following formula:

$$\mathrm{d}\sigma_M^*(\varphi) = \sum_{U \in BN} \Theta_{U/X}\left(\varphi|_U\right), \tag{1.1}$$

where BN is a collection of sub-surfaces of $\mathrm{im}(\sigma)$. When M is acylindrical and $\mathcal{P} = \varnothing$, we have that BN is just a collection of disks, the *leopard spots* of [12]. If $\mathcal{P} \neq \varnothing$ and M is relatively acylindrical, then we can also have punctured disks coming from peripheral cylinders of M.

As a consequence of (1.1), one can estimate the operator norm of the co-derivative $\mathrm{d}\sigma_M^*$ of the skinning map by bounding the Poincaré series operator of the corresponding surfaces. Using such estimate, we provide here effective bounds, in terms of the topology of $\partial_0 M$, on the contraction of the skinning map in the acylindrical case. This builds on previous work [2] of D. E. Barret and J. Diller, who gave an alternative proof of McMullen's estimates on the norm of the Poincaré operator, [12].

Improving on the main result of [2] (Theorem 3.1), we show:

Theorem 1.1. *Suppose X is a Riemann surface of finite-type and let Y be a disk or a punctured disk. Further let $\pi\colon Y \to X$ be a holomorphic covering map. Then, the norm of the corresponding Poincaré series operator satisfies:*

$$\|\Theta\|_{\mathrm{op}} < \frac{1}{1 + C_{g,n,\ell}} < 1$$

for some constant $C_{g,n,\ell} > 0$ depending only on the topology of $X \cong S_{g,n}$ and the injectivity radius ℓ of X.

In contrast with [2], we compute the contraction constant $C_{g,n,\ell}$ in a completely explicit way and in the case under examination without any extra assumptions on $\|\Theta\|_{\mathrm{op}}$. The constant $C_{g,n,\ell}$ only depends on: the genus g of X, the number of punctures n of X, the length ℓ of the shortest closed geodesic in X. So, we obtain an explicit bound over the moduli space of geometrically finite hyperbolic manifolds.

Furthermore, $C_{g,n,\ell}$ is continuous and decreasing as a function of ℓ, in fact it is linear in ℓ, and satisfies the following asymptotic expansion for $g, n \gg 1$. Let $\chi := 2g - 2 + n$ be the Euler characteristic, and $\kappa := 3g - 3 + n$ be the complexity of X. Then,

$$\log\log\left(\tfrac{\ell}{C_{g,n,\ell}}\right) \asymp \tfrac{4}{\operatorname{arcsinh}(1)}\,\chi^2 + \coth\left(\tfrac{\pi}{12}\right)\chi + \pi\sinh\left(\tfrac{1}{2}\operatorname{arcsinh}\left(\tanh(\pi/12)\right)\right)\kappa.$$

An application to infinite-type 3-manifolds. In [6] the first author studied the class $\mathcal{M}^B$ of infinite-type 3-manifolds M admitting an exhaustion $M = \cup_i M_i$ by hyperbolizable 3-manifolds M_i with incompressible boundary and with uniformly bounded genus.

One can use skinning maps to study the space of hyperbolic metrics on the manifolds in $\mathcal{M}^B$ that admit hyperbolic structures. Indeed, consider all manifolds $M \in \mathcal{M}^B$ such that for all $i \in \mathbb{N}$ every component $U_i := \overline{M_i \setminus M_{i-1}}$ is acylindrical. By the main results of [6] this guarantees that M is in fact hyperbolic, which is in general not the case, see [5,7], or [8,9] for other examples of infinite-type hyperbolic 3-manifolds. We can thus think of a (hyperbolic) metric g on M as a gluing of (hyperbolic) metrics g_i on the U_i's and so it makes sense to investigate the glueing of pairs U_i, U_{i+1} via skinning maps.

In order to approach the construction of g in this way, it is helpful to know that the contraction factor of the skinning maps over the Teichmüller spaces relative to U_i stays well below 1 uniformly in i. The latter fact follows from Theorem 1.1, in view of the uniform bound on the genus of the M_i's.

2. Notation

Throughout the work, X is a hyperbolic Riemann surface of finite-type. Let $\overline{X}$ be the compact Riemannian surface obtained by adding a single point to each end of X. We indicate by

- g the genus of X;
- n the cardinality of the set of punctures $P := \overline{X} \setminus X$.

We may thus regard X as an element of the moduli space $\mathcal{M}(S_{g,n})$ of the n-punctured Riemann surface of genus g. Further let

- $\chi := 2g - 2 + n$ be the Euler characteristic of X;
- $\kappa := 3g-3+n$ be the complexity of X, with the exception of the surface $S_{0,2}$ for which $\kappa := 0$.

We say that a curve in X is a *short geodesic* if it is a closed geodesic of length less than $2\operatorname{arcsinh}(1)$, and we define

- Γ the set of short geodesics on X;
- $\ell := \min_{\gamma\in\Gamma} \ell(\gamma)$ (twice) the *injectivity radius* of X.

For any $A \subset X$, denote by $|A|$ the number of connected components of A, and indicate by $U \in \pi_0(A)$ any of such connected components. Let d be the intrinsic distance of X and further set

$$(A)_s := \{x \in X : \operatorname{dist}(x, A) \leq s\}, \qquad s > 0.$$

Regions. Denote by D the Poincaré disk, and set $D^* := D \setminus \{0\}$. The *cusp* $\mathcal{C}_p$ about $p \in P$ is the image of the punctured disk $\{0 < |z| < e^{-\pi}\}$ under the holomorphic cover $\pi_p \colon D^* \to X$ about p.

We start by recalling the following well-known fact.

Lemma 2.1 ([3, Thm. 4.1.1])**.** *Let γ be a short closed geodesic in X of length $\ell(\gamma)$, and set $w := \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell(\gamma)/2)}\right)$. The collar $\mathcal{C}_\gamma$ around γ is isometric to $[-w, w] \times \mathbb{S}^1$ with the metric $\mathrm{d}\rho^2 + \ell(\gamma)^2 \cosh^2(\rho)\,\mathrm{d}t^2$.*

Note that in the previous statement the local metric, in Fermi coordinates, is parametrised with ℓ speed hence the ℓ^2 factor.

We define:

- the *cusp part* X_{cusps} of X as $X_{\text{cusps}} := \cup_{p\in P}\, \mathcal{C}_p$;
- the *core* X_{core} of X as $X_{\text{core}} := X \setminus X_{\text{cusps}}$;

- the *thick part* X_{thick} of X as $X_{\mathrm{thick}} := \overline{X_{\mathrm{core}} \setminus \cup_{\gamma\in\Gamma} \mathcal{C}_\gamma}$;
- the *thin part* X_{thin} of X as $X_{\mathrm{thin}} := \overline{X \setminus X_{\mathrm{thick}}}$.

Quadratic differentials. Let $T^*_{1,0}X$ be the holomorphic cotangent bundle of X. A quadratic differential on X is any section ψ of $T^*_{1,0}X \otimes T^*_{1,0}X$, satisfying, in local coordinates, $\psi = \psi(z)\,\mathrm{d}z^2$. A quadratic differential ψ is *holomorphic* if its local trivializations $\psi(z)$ are holomorphic. To each holomorphic quadratic differential ψ we can associate a measure $|\psi|$ on X defined by $|\psi| := |\psi(z)| \cdot |\mathrm{d}z|^2$. We denote by $\langle\psi(\,\cdot\,)\rangle$ the density of the measure $|\psi|$ with respect to the Riemannian volume of X.

We say that any ψ as above is *integrable* if $\|\psi\| := |\psi|(X)$ is finite, and we denote by $Q(X)$ the space of all integrable holomorphic quadratic differentials on X, endowed with the norm $\|\cdot\|$. When X has finite topological type, $Q(X)$ is finite-dimensional, its dimension depending only on g and n.

Constants. Everywhere in this work, r, s, t, w and ε are free parameters. We shall make use of the following universal constants:

- $\varepsilon_0 := \operatorname{arcsinh}(1) \approx 0.8813$ the two-dimensional Margulis constant;
- $c_1 := \coth(\pi/12) \approx 3.9065$;
- $c_2 := \operatorname{arcsinh}\big(\tanh(\pi/12)\big) \approx 0.2532$;
- $c_3 := \frac{\pi \sinh\left(\frac{1}{2}\operatorname{arcsinh}\left(\tanh(\pi/12)\right)\right)}{\operatorname{arcsinh}(\tanh(\pi/12))} \approx 1.5750$;
- $c_4 := \big(1-\tanh^2(1/2)\big)^2 \approx 0.6185$;
- $c_5 := 4\pi\big(1+\sinh(1)\big) \approx 27.3343$;
- $c_6 := (ec_4)^{e^{2c_3+2}} \approx 76.5904$;
- $c_7 := \max_x x \cdot \operatorname{arcsinh}\big(\operatorname{csch}(x/2)\big) \approx 1.5536$.

Finally, for simplicity of notation, we shall make use of the following auxiliary constants, also depending on X:

- $a_1 := 4\,|\chi|^2/\varepsilon + 2\kappa \log c_1 + 2\,c_2\,c_3$;
- $a_2 := \log(e\,c_4)\,e^{a_1+2(1+c_3)}$.

We denote by $a \wedge b$ the minimum between two quantities $a, b \in \mathbb{R}$.

3. Outline

We start by recalling the results of D. E. Barret and J. Diller [2] that we make explicit using classic hyperbolic geometry. The main result of [2] is:

Theorem 3.1 ([2, Thm. 1.1]). *Suppose X, Y are Riemman surfaces of finite-type and let $\pi\colon Y \to X$ be a holomorphic covering map. Then, the norm of the corresponding Poincaré operator satisfies:*

$$\|\Theta\|_{\mathrm{op}} := \sup_{\substack{\varphi\in Q(Y)\\ \|\varphi\|=1}} \|\Theta\varphi\| < 1-k < 1.$$

Furthermore, $k>0$ may be taken to depend only on the topology of X, Y, and the length ℓ of the shortest closed geodesic on X. As a function of ℓ, the number k may be taken to be continuous and increasing.

In order to prove the above theorem, consider a unit-norm quadratic differential $\varphi \in Q(Y)$ such that $\Theta\varphi \neq 0$. In [2], the authors estimate

$$1 - \|\Theta\varphi\|$$

as follows. Let $K \subset \overline{X}$ be any compact set containing the set Z of zeroes of $\Theta\varphi$ and the punctures of X, viz. $Z \cup P \subset K$, and such that ∂K is smooth. Further let

$$m(r) := \min_{p \in \partial (K)_r} \langle \Theta\varphi \rangle . \tag{3.1}$$

Then, for every $t > 1$ and every $r_0 > 0$, [2, Lem. 3.2] proves the following estimate

$$1 - \|\Theta\varphi\| \geq \int_0^{r_0} m(r) \left[t^{-1} \operatorname{area}(X \setminus (K)_r) - \operatorname{length}(\partial (K)_r)\right] \mathrm{d}r. \tag{3.2}$$

In general the t in the above estimate will depend on the geometry and topology of the covering surface Y. In the case at hand however, Y is either the Poincaré disk or a punctured disk and, by work of J. Diller [10], we can assume that $t = 1$. It is likely that the constants of Diller can be made explicit as well and so that one could have a version of Theorem 3.1 were the constants are explicit in the topology of X, Y and their injectivity radii.

In the following sections, we give effective estimates for $m(r)$, $\operatorname{area}(X \setminus (K)_r)$, and $\operatorname{length}(\partial (K)_r)$. In order to estimate $m(r)$ we will need the following result from [2].

Theorem 3.2 ([2, Thm. 4.4])**.** *Let $\psi \in Q(X)$ with zero set Z. Suppose $W \subset X \backslash Z$ is a domain such that $\langle \psi(p) \rangle \leq L$ for all $p \in W$, and set $\rho(p) := \min \{1, \operatorname{dist}(p, \partial W)\}$.*

Then, if $\gamma \subset W$ is a path connecting p_1 and p_2 we have:

$$\frac{\langle \psi(p_1) \rangle}{\langle \psi(p_2) \rangle} \geq \left(\frac{\langle \psi(p_2) \rangle}{c_4 L} \right)^{-1+\exp\left(\int_\gamma \frac{\mathrm{d}s}{\tanh(\rho/2)}\right)} .$$

4. Effective computations

The following is an easy lemma bounding the diameter of components of $(X_{\text{thick}})_\varepsilon$ or $(X_{\text{core}})_\varepsilon$.

Lemma 4.1. *Let $X \in \mathcal{M}(S_{g,n})$. Then,*

(i) *any pair of points in the same connected component of $(X_{\text{thick}})_\varepsilon$ is joined by a path of length at most $4\,|\chi|\,/\varepsilon$;*

(ii) *any pair of points in $(X_{\text{core}})_\varepsilon$ is joined by a path γ in $(X_{\text{core}})_\varepsilon$ satisfying*

$$\ell(\gamma) \leq 4\,|\chi|^2\,/\varepsilon + 2\kappa \operatorname{arcsinh}\big(\operatorname{csch}(\ell/2)\big). \tag{4.1}$$

Proof. Assertion (i) is a consequence of the Bounded Diameter Lemma [13].

(ii) Using the fact that each component of $(X_{\text{thick}})_\varepsilon$ contains an essential pair of pants and that the maximal number of pairwise disjoint short curves is κ we have:

Claim. $|(X_{\text{thick}})_\varepsilon| \leq |\chi|$ and $|(X_{\text{thin}})_\varepsilon| \leq \kappa$.

By short-cutting in the region we obtain:

Claim. A length-minimizing γ enters each $U \in \pi_0\big((X_{\text{core}})_\varepsilon\big)$, resp. $U \in \pi_0\big((X_{\text{thin}})_\varepsilon\big)$ at most once.

Let γ be length-minimizing. By (i) we have $\operatorname{length}(\gamma \cap U) \leq 4\,|\chi|\,/\varepsilon$. By the Collar Lemma [3],

$$\operatorname{length}(\gamma \cap U) \leq \operatorname{diam}(U) \leq 2 \operatorname{arcsinh}\big(\operatorname{csch}(\ell/2)\big).$$

The conclusion follows combining the previous estimates with the two claims. □

The next lemma is [2, Lem. 4.6]. We just work out the constant explicitly.

Lemma 4.2. *Let* $L(s) := \max_{p\in(X_{\text{thick}})_s} \langle\psi(p)\rangle$. *Then,*

(i) $L(0) \geq \frac{\ell\wedge 1}{16\,|\chi|}\,\|\psi\|$;
(ii) *for all* $0 \leq s \leq t$, *we have* $L(s) \geq e^{s-t} L(t)$.

Proof. (i) Firstly assume that at most half the mass of ψ is concentrated inside the collars of short geodesics. As in [2, Lem. 4.6(i)], it follows that

$$\langle\psi\rangle \geq \frac{\|\psi\|}{2\ \text{area}(X)} = \frac{\|\psi\|}{4\pi\,|\chi|} \geq \frac{\|\psi\|}{16\ |\chi|}. \tag{4.2}$$

Assume now that at least half the mass of ψ is concentrated inside collars of short geodesics. Let γ be any such geodesic and let $\mathcal{C} := \mathcal{C}_\gamma$ be the collar around γ. For $r \leq R := \pi^2/\ell(\gamma)$ and r satisfying $\tan\big(\pi r/(2R)\big) = \text{csch}\,\big(\ell(\gamma)/2\big)$, we have that

$$\begin{aligned}\frac{1}{2\ \text{area}(X)}\,\|\psi\| \leq \int_{\mathcal{C}} |\psi| &= \int_0^{2\pi}\int_{e^{-r}}^{e^r} \frac{|f(z)|}{|z|^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &\leq \int_0^{2\pi}\int_{e^{-r}}^{e^r} L r^{-1}\,\mathrm{d}r\,\mathrm{d}\theta = 4\pi L r,\end{aligned}$$

hence that

$$\frac{\|\psi\|}{2\pi r\ \text{area}(X)} \leq 4L.$$

Computing both r and R in terms of $\ell(\gamma)$,

$$\begin{aligned} L(0) &\geq \max_{\partial\mathcal{C}} \langle\psi\rangle = \frac{4LR^2}{\pi^2}\cos^2\frac{\pi r}{2R} \\ &\geq \frac{\|\psi\|}{2\ \text{area}(X)}\,\frac{R^2}{2R\arctan\big(\text{csch}\,\big(\ell(\gamma)/2\big)\big)}\cos^2\big(\arctan\big(\text{csch}\,\big(\ell(\gamma)/2\big)\big)\big).\end{aligned}$$

Now, since $\cos^2(\arctan(\text{csch}(t))) = \tanh^2(t)$, and substituting $R := 2\pi/\ell(\gamma)$,

$$\begin{aligned} L(0) &\geq \frac{\|\psi\|}{4\ \text{area}(X)}\,\frac{R\ \tanh^2\big(\ell(\gamma)/2\big)}{\arctan\big(\text{csch}\,\big(\ell(\gamma)/2\big)\big)} \\ &= \frac{\pi^2\,\|\psi\|}{4\ \text{area}(X)}\,\frac{\tanh^2\big(\ell(\gamma)/2\big)}{\ell(\gamma)^2\cdot\arctan\big(\text{csch}(\ell(\gamma)/2)\big)}\cdot\ell(\gamma).\end{aligned}$$

Since $t \mapsto \tanh^2(t/2)/\big(t^2\arctan(\text{csch}(t/2))\big)$ has global minimum $\frac{1}{2\pi}$ at $t = 0$, we have that

$$L(0) \geq \frac{\pi\,\ell(\gamma)}{8\ \text{area}(X)}\,\|\psi\| \geq \frac{\ell}{16\,|\chi|}\,\|\psi\|\,.$$

Combining the above inequality with (4.2) yields the assertion.

(ii) is [2, Lem. 4.6]. □

Let $\log_+(x) := \max\{0, \log(x)\}$. We start with some estimates towards establishing (3.2).

Lemma 4.3. *For each connected component* $U \in \pi_0\big((X_{\text{thick}})_s\big)$, *letting* $s = \log_+(c_1 t)$

(i) $\text{area}(U) - t\ \text{length}(\partial U) \geq \pi/3$;

(ii) *for all* $p \in U$*:* $\operatorname{inj}_p \geq c_2/t$*;*

(iii) *given* $p_1, p_2 \in U$ *there exists* $\gamma \subset U$ *connecting* p_1 *and* p_2 *such that*

$$\ell(\gamma) \leq \frac{4\,|\chi|^2}{\varepsilon} + 2\kappa \log t + 2\kappa \log c_1.$$

Proof. (i) Let g_U and n_U respectively denote the genus of U and the number of boundary components of U. Further let $A_1, \ldots, A_{n_U}$ denote the embedded annuli bounded by short closed geodesics on one side and by connected components of ∂U on the other side. We allow for A_j being part of a cusp, in which case, on one side, it is bounded by a puncture rather than by a short geodesic.

By the Gauss–Bonnet Theorem,

$$\operatorname{area}(U) = 2\pi(2g_U + n_U - 2) - \sum \operatorname{area}(A_j).$$

If $n_U = 0$ then $U = X$, which yields $\operatorname{area}(U) - t \operatorname{length}(\partial U) = 2\pi\,|\chi|$. Thus, in the following we may assume without loss of generality that $n_U \geq 1$. In this case, either $g_U \geq 1$ and $n_U \geq 1$, or $g_U = 0$ and $n_U \geq 3$. Thus,

$$\operatorname{area}(U) \geq 2\pi \frac{n_U}{3} - \sum_j \operatorname{area}(A_j).$$

Let ℓ_j denote the length of the geodesic component of ∂A_j and L_j denote the length of the other component. Then,

$$\operatorname{area}(U) - t \operatorname{length}(\partial U) \geq 2\pi \frac{n_U}{3} + \sum_j \big((t-1)\operatorname{area}(A_j) - t(\operatorname{area}(A_j) + L_j)\big).$$

By Lemma 2.1, setting

$$w_j := \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell_j/2)}\right), \tag{4.3}$$

we have that

$$\operatorname{area}(A_j) = \int_0^{w_j - s} \int_0^1 \ell_j \cosh(\rho)\,\mathrm{d}\rho\,\mathrm{d}t = \ell_j \sinh(w_j - s)$$

and

$$L_j = \ell_j \cosh(w_j - s).$$

We see that

$$\operatorname{area}(A_j) + L_j = \ell_j\big(\sinh(w_j - s) + \cosh(w_j)\big) = \frac{e^{-s}\ell_j}{\tanh(\ell_j/4)}$$

is monotone increasing in ℓ_j (e.g. by differentiating w.r.t. ℓ_j). Thus it achieves its minimum when the two boundary components of A_j coincide, in which case $\ell_j = A_j$ and $\operatorname{area}(A_j) = 0$. In this case, s measures the distance from the geodesic to the edge of the collar containing A_j. Therefore, by the Collar Lemma, $\sin(\ell_j/2) = \operatorname{csch}(s)$, hence

$$\begin{aligned}\operatorname{area}(U) - t \operatorname{length}(\partial U) &\geq 2n_U \left(\pi/3 - t \operatorname{arcsinh}\left(\operatorname{csch}(s)\right)\right) \\ &\geq 2\left(\pi/3 - t \operatorname{arcsinh}\left(\operatorname{csch}(s)\right)\right).\end{aligned}$$

Letting the right-hand side above be larger than $\pi/3$ we get

$$s \geq \operatorname{arcsinh}\big(\operatorname{csch}(\pi/(6t))\big), \qquad t > 1, \qquad s = \log\left(\coth\left(\tfrac{\pi}{12}\right)t\right).$$

(ii) Let $\mathcal{C}$ be a short collar in X. For $p \in (X_{\text{thick}})_{\varepsilon+s} \cap \mathcal{C}$, by the Collar Lemma we have that

$$\text{inj}_p \geq \operatorname{arcsinh}\left(e^{\operatorname{dist}(p,\partial\mathcal{C})}\right) = \operatorname{arcsinh}\left(e^{-s}\right) = \operatorname{arcsinh}\left(\tfrac{1}{c_1 t}\right) \geq \frac{c_2}{t}$$

with $c_2 := \operatorname{arcsinh}(1/c_1)$, and where the last inequality is sharp by a direct computation.

(iii) Let $p_1, p_2 \in U$. Then we can find a rectifiable curve γ, connecting p_1 to p_2, and enjoying the following properties:

(a) if $\gamma \cap \mathcal{C} \neq \varnothing$, then $\gamma \cap \partial\mathcal{C}$ consists of two points belonging to distinct connected components of $\partial\mathcal{C}$, and $\operatorname{length}(\gamma|_{\mathcal{C}}) \leq 2s$;

(b) in each connected component of $(X_{\text{thick}})_{\varepsilon}$, the curve γ is a shortest path between its endpoints.

See Figure 1.

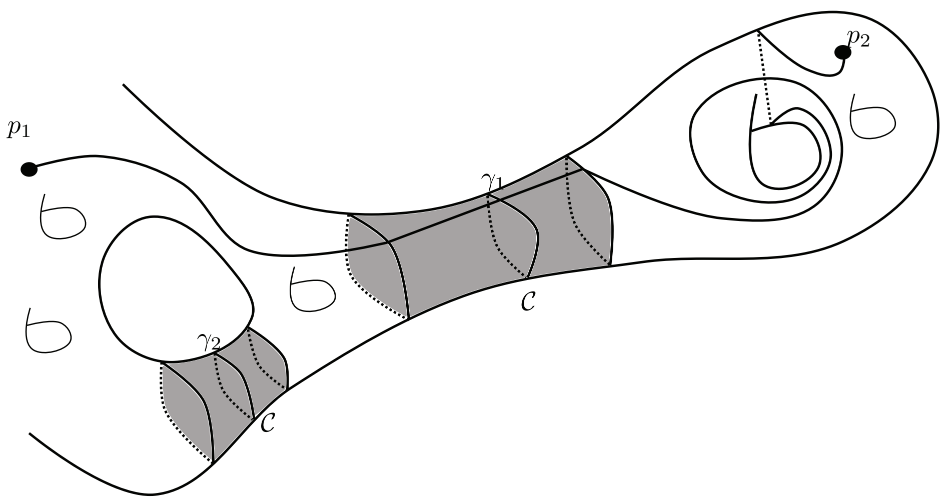

FIGURE 1. The piecewise geodesic curve γ connecting p_1 to p_2 and $\mathcal{C}$, shaded, are collars around short geodesics.

We can decompose γ into its components in

$$X_1 := (X_{\text{thick}})_{\varepsilon} \qquad \text{and} \qquad X_2 := \overline{(X_{\text{thick}})_{\varepsilon+s} \setminus (X_{\text{thick}})_{\varepsilon}} \subset (X_{\text{thin}})_{\varepsilon}\,.$$

By the Bounded Diameter Lemma [13], the length of each component of γ in X_1 is bounded by $4\,|\chi|\,/\varepsilon$, and we have at most $|\chi|$ such components. In each connected component of $X_2 \subset (X_{\text{thin}})_{\varepsilon}$ the length of γ is at most $2s$, and there are at most κ such components. Thus, for $s = \log(c_1 t)$ we get

$$\ell(\gamma) \leq \frac{4\,|\chi|^2}{\varepsilon} + 2s\kappa = \frac{4\,|\chi|^2}{\varepsilon} + 2\kappa\log(c_1 t). \qquad \square$$

We now show how to estimate the quantities related to $(K)_r$ in Equation (3.2). Let Z be the zeroes of a given quadratic differential ψ.

Lemma 4.4. *Let $U \in \pi_0\big((X_{\text{thick}})_s\big)$ and $K := \overline{X \setminus U} \cup Z$. Then, for $r \in (0,1)$, $t > 1$, and $s = \log_+(c_1 t)$*

$$\begin{aligned}\operatorname{area}\big(X \setminus (K)_r\big) - t \operatorname{length}\big(\partial (K)_r\big) &\geq \pi/3 - \kappa r t\left[c_7 - s + 4\pi\big(1+\sinh(1)\big)\frac{|\chi|}{\kappa}\right]\\ &\geq \pi/3 - \kappa r t\left[4\pi\big(1+\sinh(1)\big) + c_7\right]\\ &= \pi/3 - \kappa r t(c_5 + c_7).\end{aligned}$$

Proof. Since $|Z| \leq 2|\chi|$ and $r < 1$, we have that

$$\begin{aligned}\operatorname{length}\big(\partial (K)_r\big) \leq\ & \operatorname{length}(\partial U) + \operatorname{length}\big(\partial (Z)_r\big) \leq \operatorname{length}(\partial U) + 2\pi |Z| \sinh(r)\\ \leq\ & \operatorname{length}(\partial U) + 4\pi|\chi|\sinh(1)\, r.\end{aligned}\tag{4.4}$$

Furthermore,

$$\begin{aligned}\operatorname{area}\big(X \setminus (K)_r\big) =\ & \operatorname{area}(X) - \operatorname{area}\big((K)_r\big)\\ \geq\ & \operatorname{area}(X) - \Big(\operatorname{area}\big(\overline{X\setminus U}\big) + \operatorname{area}\big((\partial U^+)_r\big) + \operatorname{area}\big((Z)_r\big)\Big)\\ \geq\ & \operatorname{area}(U) - \operatorname{area}\big((\partial U^+)_r\big) - 4\pi|\chi|\big(\cosh(r) - 1\big)\\ \geq\ & \operatorname{area}(U) - \operatorname{area}\big((\partial U^+)_r\big) - 4\pi|\chi|\, r\\ \geq\ & \operatorname{area}(U) - t \operatorname{area}\big((\partial U^+)_r\big) - 4\pi|\chi|\, tr\end{aligned}$$

since $t > 1$. We can estimate $\operatorname{area}\big((\partial U^+)_r\big)$ by assuming that $(\partial U^+)_r$ is isometrically embedded, so that, by Lemma 2.1,

$$\operatorname{area}\big((\partial U^+)_r\big) = \sum_j \ell(\gamma_j)\big(\sinh(w_j - s + r) - \sinh(w_j - s)\big).$$

Repeat the construction of the annuli A_j in Lemma 4.3, and let w_j be defined as in (4.3). By Taylor expansion of sinh around $w_j - s > 0$, we have that

$$\begin{aligned}\operatorname{area}(U) - t\operatorname{area}\big((\partial U^+)_r\big) \geq\ & \operatorname{area}(U) - rt\sum_j \ell(\gamma_j)(w_j - s)\\ \geq\ & \operatorname{area}(U) - rt\sum_j \ell(\gamma_j)\ \operatorname{arcsinh}\big(\operatorname{csch}(\ell(\gamma_j)/2)\big)\\ & + rt\ \log(c_1 t)\sum_j \ell(\gamma_j)\\ \geq\ & \operatorname{area}(U) - c_7\kappa r t + rt\ \log(c_1 t)\sum_j \ell(\gamma_j).\end{aligned}$$

As a function of the metric, the summation $\sum_j \ell(\gamma_j)$ attains its maximum over the moduli space $\mathcal{M}(S_{g,n})$ when $\ell(\gamma_j) = \varepsilon_0$ for each j, thus its maximum is $\kappa\varepsilon_0$. Therefore,

$$\operatorname{area}\big(X \setminus (K)_r\big) \geq \operatorname{area}(U) - rt\kappa\, c_7 + rt\kappa\varepsilon_0\ \log(c_1 t) - 4\pi|\chi|\, tr.\tag{4.5}$$

Multiplying (4.4) by $-t$ and adding (4.5), together with Lemma 4.3(i), yields the conclusion. ☐

Let U be the component of $(X_{\text{thick}})_{\varepsilon+s}$ containing $p_{\max}(s)$, where $s = \log(c_1 t)$ and $p_{\max}$ satisfies Lemma 4.2. Set $K' := \overline{X \setminus U}$ and let $K := K' \cup Z$. This is a slight refinement of the previous K, in which we chose a specific component U

and a slightly larger neighbourhood of U. The next lemma will deal with paths in $X \setminus (K)_r$. When $r = 0$, the set $X \setminus K = U \setminus Z$ looks as Figure 2.

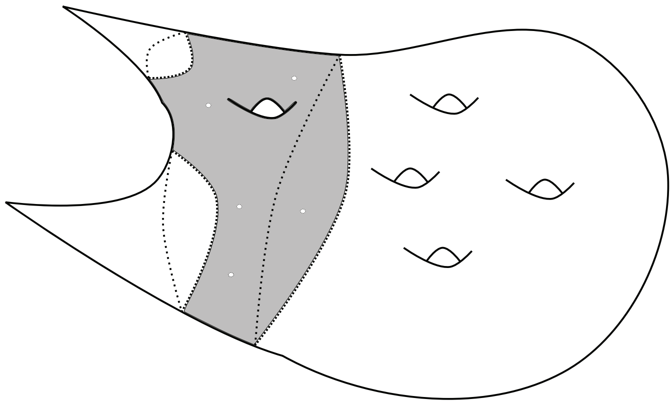

FIGURE 2. The set $X \setminus K = \mathrm{int}(U) \setminus Z$ is greyed out and the white points are zeroes of the quadratic differential.

Lemma 4.5. *Fix $t > 1$. If $r < c_2/(|\chi|\, t)$, then any two points in $X \setminus (K)_r$ can be joined by a rectifiable curve in $X \setminus (K)_{r/2}$.*

Proof. We start with the following claim.

Claim. Let $V \in \pi_0\big((K)_{r/2}\big)$. If $V \cap (K')_{r/2} \neq \varnothing$, then $V \subset (K')_r$.

Indeed, for $c > 0$ to be fixed later, let $V \in \pi_0\big((K)_{cr}\big)$ with $V \cap (K')_{cr} \neq \varnothing$. We need to show that if V is such component it does not separate $X \setminus (K)_r$. Fix $p \in V \setminus (K')_{cr}$. Since V is connected and contained in $(K)_{cr}$, then p is joined to $(K')_{cr}$ by a chain of disks of radius cr centered at points in Z. Therefore $\mathrm{dist}(p, (K')_{cr}) \leq 2c\,|Z|\,r$. Choosing $c < (2\,|Z| + 1)^{-1}$, e.g. $c := \frac{1}{2}(2\,|Z| + 1)^{-1}$, proves that $\mathrm{dist}(p, (K')_{cr}) \leq r/2$ and so that:

$$\mathrm{dist}(p, K') \leq \mathrm{dist}(p, (K')_{cr}) + cr = 2c\,|Z|\,r + cr \leq (2\,|Z| + 1)cr \leq r/2,$$

proving that $V \subset (K')_r$. This concludes the proof of the claim.

Thus, we need to show that for $r < c_2/t$ and for all $p_0, p_1 \in X \setminus (K)_r \subset (U)_s$ there exists a rectifiable curve $\gamma \subset X \setminus (K)_{c_2 r}$ connecting p_0 to p_1. By the Collar Lemma,

$$\mathrm{inj}_{(U)_s} := \min_{p \in (U)_s} \mathrm{inj}_p \geq \mathrm{arcsinh}(e^{-s}) = \mathrm{arcsinh}\left(\frac{1}{c_1 t}\right) \geq \frac{c_2}{t},$$

similarly to the proof of Lemma 4.3(ii).

Now, argue by contradiction and assume that there exists no rectifiable curve as in the assertion. Then, there exists a rectifiable loop α in $(Z)_{r/2}$ separating $X \setminus (K)_r \subset (U)_s$ into connected components so that p_0 and p_1 belong to two distinct such components. See the picture in Figure 3.

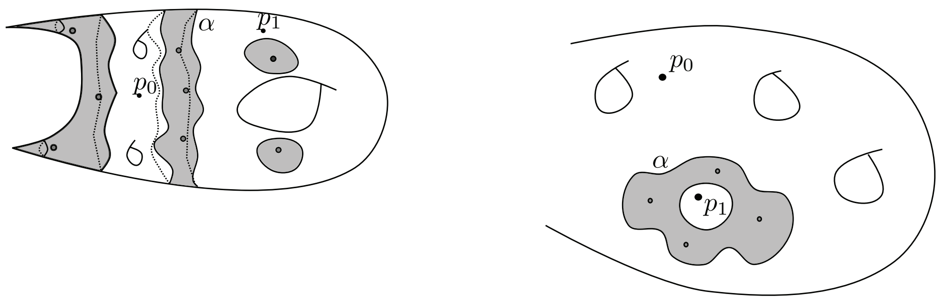

FIGURE 3. The two cases for the loop α separating p_1 to p_2. The shaded regions are part of $(K)_{r/2}$ and the grey dots are zeroes of the quadratic differential.

For any such α,

$$\operatorname{length}(\alpha) \le r\,|Z| < |Z|\,\frac{c_2}{|\chi|\,t} \le \frac{c_2}{t} \le \operatorname{inj}_{(U)_s}.$$

As a consequence, $\alpha \subset (U)_s$ is null-homotopic and so we must be on the right side of Figure 3. Therefore, there exists $L \in \mathbb{R}^+$ such that $\alpha \subset B_L(q)$ for $q \in (U)_s$ and $L \le \ell(\alpha)/2 < r/2$. Thus, the component $W \subset X \setminus (K)_r$ containing, say, p_1, lies in $B_L(q) \subset B_r(q)$ and note that by construction its distance from any zero is at least r. Therefore, W is at distance $r/2 + L < r$ from a zero. However, since $d(W, Z) \ge r$ we have a contradiction. □

We now state the main lemma we will use in our estimate of (3.2).

Lemma 4.6. *Let $r < c_2/(|\chi|\,t)$, and set $a_1 := 4\,|\chi|^2/\varepsilon + 2\kappa\log c_1 + 2\,c_2\,c_3$. Then, any two points in $\overline{X \setminus (K)_r}$ are joined by a rectifiable curve $\gamma \subset \overline{X \setminus (K)_{r/2}}$ with the following properties:*

- (i) *γ consists of length-minimising geodesic segments and of at most one arc in each of the components of $\partial\,(K)_{r/2}$;*
- (ii) *$\ell(\gamma) \le a_1 + 2\kappa\log t$;*
- (iii) *for $z \in Z$: $\operatorname{length}\big(\gamma \cap B_w(z)\big) \le 2(1 + c_3)w$ for all $w > 0$ such that $B_w(z)$ is embedded.*

Proof. (i)–(ii) Fix points $p_0, p_1 \in \overline{X \setminus (K)_r}$. By Lemma 4.3 there exists a rectifiable $\gamma \subset U$ connecting them, with

$$\ell(\gamma) \le \frac{4\,|\chi|^2}{\varepsilon} + 2\kappa\log c_1 + 2\kappa\log t.$$

The curve γ intersects $(K)_{r/2}$ in at most $2\,|\chi|$ components (i.e. balls around zeroes of ψ). In each such component $V = B_{r/2}(z)$ (for some $z \in Z$) we can replace $\gamma\big|_V$ by a shortest path on ∂V as the one in Lemma 4.3 (iii).

Since V is a ball, the length of $\gamma\big|_V$ is bounded by half the length of the circumference of a great circle on V, i.e.

$$\pi\sinh(r/2) \le \pi\sinh(r) \le c_3\,r, \qquad r < \frac{c_2}{|\chi|\,t} < \frac{c_2}{|\chi|}. \tag{4.6}$$

By repeating this reasoning on each component V as above, we obtain a path $\gamma' \colon p_0 \to p_1$ satisfying (i) and such that:

$$\begin{aligned}\text{length}(\gamma') &\leq \ell(\gamma) + 2\,|\chi|\, c_3\, r\\ &\leq \frac{4\,|\chi|^2}{\varepsilon} + 2\kappa \log c_1 + 2\kappa \log t + 2\frac{c_2 c_3}{t} \qquad\qquad (t > 1)\\ &\leq \frac{4\,|\chi|^2}{\varepsilon} + 2\kappa \log c_1 + 2\, c_2\, c_3 + 2\kappa \log t.\end{aligned}$$

(iii) Let $z \in Z$ be a zero of ψ and fix $w > 0$. Each component α of γ in $\partial\,(K)_{c_2 r}$ has length at most $c_3\, r$ and each geodesic arc of γ connecting an endpoint of α to $\partial B_w(z)$ has length at most w. We now estimate

$$\left|\pi_0\big(\gamma \cap \overline{B_w(z)}\big)\right| \leq \begin{cases} 0 & w < r/2\\ 1 & p_1, p_2 \in \overline{B_w(z)}\\ 1 & p_1 \in \overline{B_w(z)}, p_2 \notin \overline{B_w(z)}\\ 1 & p_1, p_2 \notin \overline{B_w(z)}.\end{cases}$$

The first bound holds by definition. The second holds by the convexity of hyperbolic balls: if $p_1, p_2 \in B_w(z)$ then we can choose $\gamma \subset B_w(z)$. The third and fourth one follow from the fact that if γ has more than one component in $B_w(z)$, then we can shortcut γ inside the ball.

If $w = r/2$, then $\gamma\big|_{\overline{B_w(z)}} \subset \partial B_w(z)$, and we may choose $\gamma\big|_{B_w(z)}$ to be a circumference arc, so that length $\big(\gamma\big|_{B_{r/2}(z)}\big) \leq \pi \sinh(r/2) \leq c_3 r$ by (4.6).

If instead $w > a_1 r$, then we may choose γ to be either a geodesic segment, or a union $\gamma_1 \cup \gamma_2 \cup \gamma_3$, where γ_1 and γ_2 are geodesic segments each connecting $\partial B_w(z)$ to $\partial B_{r/2}(z)$, and γ_3 is a circumference arc on $\partial B_{r/2}(z)$. In the first case, length$(\gamma\big|_{B_w(z)}) \leq 2w$. In the second case,

$$\text{length}\,\big(\gamma\big|_{B_w(z)}\big) \leq 2w + \pi \sinh(a_1 r) \leq 2w + c_3 r \leq 2w + c_3 r.$$

Thus, we obtain that:

$$\text{length}\,\big(\gamma \cap \overline{B_w(z)}\big) \leq \begin{cases} 0 & \text{if } w < r/2\\ 2c_3 w & \text{if } w = r/2\\ 2w + c_3 w & \text{if } w \geq r/2\end{cases} \qquad\qquad \leq 2(1 + c_3)w,$$

which concludes the proof. □

With $m(r)$ as in (3.1) we can now estimate (3.2) and show our final result.

Proof of Theorem 1.1. Let $r < c_2/\,|\chi| \leq 2$. Let s be as in Lemma 4.4 and choose U to be the component of $X_{\text{thick}}(s)$ containing the point $p_{\max}(s)$ as in Lemma 4.2. Let Z be the set of zeroes of ψ, $K := \overline{X \setminus U} \cup Z$, and $K' := \overline{X \setminus U}$. Let $W := (X_{\text{thick}})_{s+1} \setminus Z$, $p_1 \in \partial\,(K)_r$, and $p_2 = p_{\max}(s) \in (X_{\text{thick}})_s \setminus Z$. Therefore, we have that $\langle \psi(p_2)\rangle = L(s)$. Moreover, let $\gamma \subset W$ be a path from p_1 to p_2 satisfying the conditions of Lemma 4.6 and note that

$$\text{dist}(p, \partial W) \geq \min\{1, \text{dist}(p, Z)\}\,, \qquad p \in \gamma.$$

By Lemma 4.2(ii) we have that:

$$L(s + 1) \leq e \cdot L(s).$$

By Theorem [2, 4.4], we have that:

$$\begin{aligned}\langle\psi(p_1)\rangle &\geq \langle\psi(p_2)\rangle \left(\frac{\langle\psi(p_2)\rangle}{c_4\cdot L(s+1)}\right)^{-1+\exp\left(\int_\gamma \frac{\mathrm{d}s}{\tanh(1\wedge \operatorname{dist}(\gamma_s,Z))}\right)}\\ &\geq L(s)\left(\frac{1}{e\,c_4}\right)^{-1+\exp\left(\int_\gamma \coth\left(1\wedge \operatorname{dist}(\gamma_s,Z)\right)\mathrm{d}s\right)}\\ &\geq e\,c_4\,L(0)\cdot\left(\frac{1}{e\,c_4}\right)^{\exp\left(\int_\gamma \coth\left(1\wedge \operatorname{dist}(\gamma_s,Z)\right)\mathrm{d}s\right)},\end{aligned}$$

where we can estimate $L(0)$ by Lemma 4.2(i),

$$\begin{aligned}&\geq \frac{e\,c_4\,\ell}{16\,|\chi|}\,\|\psi\|\left(\frac{1}{e\,c_4}\right)^{\exp\left(\int_\gamma \coth\left(1\wedge \operatorname{dist}(\gamma_s,Z)\right)\mathrm{d}s\right)}\\ &= \frac{e\,c_4\,\ell}{16\,|\chi|}\,\|\psi\|\exp\left(-\log(e\,c_4)\exp\left(\int_\gamma \coth\left(1\wedge \operatorname{dist}(\gamma_s,Z)\right)\mathrm{d}s\right)\right).\end{aligned}$$

We now estimate $\int_\gamma \coth(1\wedge \operatorname{dist}(\gamma_s,Z))\,\mathrm{d}s$ from above by breaking it into two terms:

$$\int_\gamma \frac{\mathrm{d}s}{\tanh\left(1\wedge \operatorname{dist}(\gamma_s,Z)\right)} \leq \int_{\gamma\setminus Z(1)} \mathrm{d}s + \int_{\gamma\cap Z(1)} \frac{\mathrm{d}s}{\operatorname{dist}(\gamma_s,Z)}.$$

The first term is bounded by $\ell(\gamma)$ while for the second term we have by Lemma 4.6(i)

$$\begin{aligned}\int_{\gamma\cap Z(1)} \frac{\mathrm{d}s}{\operatorname{dist}(\gamma_s,Z)} &\leq \int_1^{\frac{2}{r}} \operatorname{length}\left(\gamma\cap (Z)_{1/u}\right)\mathrm{d}u\\ &\leq \int_1^{\frac{2}{r}} \frac{2(1+c_3)}{u^2}\,\mathrm{d}u = 2(1+c_3)(1-r/2)\end{aligned}$$

since $r\leq 2$.

By Lemma 4.6(i) we have that:

$$\ell(\gamma)\leq a_1+2\kappa\,\log t.$$

Thus:

$$\begin{aligned}\int_\gamma \frac{\mathrm{d}s}{\tanh\left(1\wedge \operatorname{dist}(\gamma_s,Z)\right)} &\leq a_1+2\kappa\,\log t+2(1+c_3)(1-r/2)\\ &= a_1+2(1+c_3)+2\kappa\,\log t-(1+c_3)r.\end{aligned}$$

Therefore, since $\log(e\,c_4)>0$, for all $p_1\in\partial\,(K)_r$ we get:

$$\langle\psi(p_1)\rangle \geq \frac{e\,c_4\,\ell}{16\,|\chi|}\,\|\psi\|\exp\left(-\log(e\,c_4)\exp\left(a_1+2(1+c_3)+2\kappa\,\log t-(1+c_3)r\right)\right).$$

Thus, by minimizing over $p_1\in\partial\,(K)_r$ we obtain:

$$m(r)\geq \frac{e\,c_4\,\ell}{16\,|\chi|}\,\|\psi\|\exp\left(-\log(e\,c_4)\exp\left(a_1+2(1+c_3)+2\kappa\,\log t-(1+c_3)r\right)\right)$$

which for $a_2:=\log(e\,c_4)\,e^{a_1+2(1+c_3)}>0$ can be rewritten as:

$$m(r)\geq e\,c_4\,\frac{\ell}{16\,|\chi|}\,\|\psi\|\exp\left(-a_2t^{2\kappa}e^{-(1+c_3)r}\right).$$

Then, Equation (3.2) with $K:=W$ becomes, for $r_0<\frac{1}{4t}$,

$$1 - \|\psi\| \geq \int_0^{r_0} m(r) \left(t^{-1} \operatorname{area}\left(X \setminus (K)_r\right) - \operatorname{length}\left(\partial\left(K\right)_r\right)\right) \mathrm{d}r.$$

By Lemma 4.4 we thus have that, for every $r_0 < \frac{1}{4t}$,

$$\begin{aligned}1 - \|\psi\| &\geq \frac{e\, c_4\, \ell}{16\,|\chi|\, t} \|\psi\| \int_0^{r_0} \exp\left(-a_2 t^{2\kappa} e^{-(1+c_3)r}\right) \left(\pi/3 - \kappa r t(c_5 + c_7)\right) \mathrm{d}r \\ &\geq \frac{e\, c_4\, \ell\, e^{-a_2 t^{2\kappa}}}{16\,|\chi|\, t} \|\psi\| \int_0^{r_0} \left(\pi/3 - \kappa r t(c_5 + c_7)\right) \mathrm{d}r.\end{aligned}$$

Maximizing over $r_0 \in \left(0, \frac{1}{4t}\right)$ additionally so that the integrand is non-negative, we have therefore that

$$\begin{aligned}1 - \|\psi\| &\geq \frac{e\, c_4\, \ell\, e^{-a_2 t^{2\kappa}}}{16\,|\chi|\, t} \|\psi\| \int_0^{\frac{1}{4t} \wedge \frac{\pi}{3\kappa t(c_5+c_7)}} \left(\pi/3 - \kappa r t(c_5 + c_7)\right) \mathrm{d}r \\ &= \frac{e\, \pi^2\, c_4}{288\, \kappa(c_5 + c_7)} \frac{\ell\, e^{-a_2 t^{2\kappa}}}{|\chi|\, t^2} \|\psi\|,\end{aligned}$$

and maximizing the right-hand side over $t > 1$, i.e. choosing $t = 1$, we conclude that

$$\|\psi\| \leq \frac{1}{1 + \dfrac{C\, \ell\, e^{-a_2}}{\kappa\, |\chi|}}, \qquad C := \frac{e\, \pi^2\, c_4}{288\,(c_5 + c_7)}. \qquad \square$$

Contraction factors of skinning maps. We now apply our explicit bounds from Theorem 1.1 to get effective bounds on the contraction factor of the skinning map.

Let $N \in AH(M, \mathcal{P})$ [1] be a pared acylindrical manifold so that

- $\mathcal{P} \subset \partial M$ is a collection of pairwise disjoint closed annuli and tori;
- $\mathcal{P}$ contains all tori components of M and M is acylindrical relative to $\mathcal{P}$.

Let $\partial_0 M := \partial M \setminus \mathcal{P}$. By [12, p. 443] we have that, for every such N,

$$|\mathrm{d}\sigma| = |\mathrm{d}\sigma^*|.$$

By Theorem 1.1,

$$\mathrm{d}\sigma^*(\varphi) = \sum_{U \in BN} \Theta_{U/X}\left(\varphi|_U\right) \leq \max_{X \in \partial_0 M} \frac{1}{1 + C_{g,n,\ell}} \|\varphi\|,$$

where ℓ is the injectivity radius of the conformal boundary $\partial_\infty N$, and $C_{g,n,\ell}$. Thus, we obtain Corollary 4.7.

Corollary 4.7. *Let $(M, \mathcal{P})$ be a pared acylindrical hyperbolic manifold. Then, the skinning map at $N \in AH(M, \mathcal{P})$ has contraction factor bounded by*

$$|d\sigma| \leq \max_{X \in \partial_0 M} \frac{1}{1 + C_{g,n,\ell}}.$$

[1] For $AH(M, \mathcal{P})$ the set of hyperbolic 3-manifolds homotopy equivalent to M with $\mathcal{P}$ parabolic.

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