EFFECTIVE CONTRACTION OF SKINNING MAPS

TOMMASO CREMASCHI AND LORENZO DELLO SCHIAVO

Abstract. Using elementary hyperbolic geometry, we give an explicit formula for the contraction constant of the skinning map over moduli spaces of relatively acylindrical hyperbolic manifolds.

1. Introduction

Let $M_1, M_2$ be hyperbolic manifolds of finite-type, i.e. the interior of compact 3-manifolds, with incompressible boundary, and homeomorphic geometrically finite ends $E_1 \subset M_1$ and $E_2 \subset M_2$. From a topological point of view, since $M_1$ and $M_2$ are tame, the surfaces $S_i$ corresponding to the boundary of the ends $E_i$ are naturally homeomorphic. We can thus glue the two manifolds via an orientation-reversing homeomorphism $\tau$, and obtain a new topological 3-manifold $M = M_1 \cup \tau M_2$. Usually, one seeks sufficient conditions for $M$ to admit a complete hyperbolic metric, which is relevant, for example, in the proof of geometrization for hyperbolic manifolds. We call this the glueing problem for $M$. The skinning map, described below, was first introduced by W. P. Thurston, exactly to study this glueing problem.

The moduli space $GF(M, \mathcal{P})$ of all hyperbolic metrics on $M$ with geometrically finite ends and parabolic locus $\mathcal{P}$ is parameterised by the Teichmüller space $T(\partial_0 M)$ with $\partial_0 M$ the closure in $\partial M$ of the complement $\mathcal{P}^c$ of $\mathcal{P}$, viz. $GF(M, \mathcal{P}) = T(\partial_0 M)$. For simplicity, let us here assume that $\mathcal{P}$ only contains toroidal boundary components of $M$. Now, let $N \in GF(M, \mathcal{P})$ be a uniformization, and $S \in \pi_0(\partial_0 M)$ be a (non-toroidal) boundary component. The cover of $N$ associated to $\pi_1(S)$ is a quasi-Fuchsian manifold $N_S$. The manifold $N_S$ has two ends, $A$ and $B$, of which $A$ is isometric to the end of $M$ corresponding to $S$. One defines the skinning map $\sigma_M$ at $N$ as the conformal structure of the new end $B$. As it turns out, the skinning map is an analytic map $\sigma_M : T(\partial_0 M) \to T(\overline{\partial_0 M})$, where the bar denotes opposite orientation. The glueing instruction determines an isometry $\tau^* : T(\partial_0 M) \to T(\overline{\partial_0 M})$, and any fixed point of $\tau^* \circ \sigma_M$ gives a solution to the glueing problem by the Maskit Combination Theorem, e.g. [11].

Received by the editors November 14, 2021, and, in revised form, November 28, 2021, and May 10, 2022.

2020 Mathematics Subject Classification. Primary 57K32.

Key words and phrases. Skinning map, Poincaré series, deformations of hyperbolic manifolds, Kleinian groups.

The first author was partially supported by the National Science Foundation under Grant No. DMS-1928930 while participating in a program hosted by the Mathematical Sciences Research Institute in Berkeley, California, during the Fall 2020 semester. The second author gratefully acknowledges funding by the Austrian Science Fund (FWF) through grants F65 and ESPRIT 208, by the European Research Council (ERC, grant No. 716117, awarded to Prof. Dr. Jan Maas), and by the Deutsche Forschungsgemeinschaft through the SPP 2265.

©2022 by the author(s) under Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 License (CC BY NC ND 4.0)
Given a covering map between Riemann surfaces \( \pi : Y \to X \) the Poincaré series operator is a push-forward operator \( \Theta_{Y/X} : Q(Y) \to Q(X) \), similar to the push-forward of measures, pushing quadrating differentials on \( Y \) to quadratic differentials on \( X \).

In [12], C. McMullen showed that the skinning map of an acylindrical manifold \( N \) is contracting, with contraction constant only depending on the topology of \( \partial_{0}M \).

Furthermore, he related the skinning map to the Poincaré series operator \( \Theta \) by the following formula:

\[
(1.1) \quad d\sigma_{M}^{*}(\varphi) = \sum_{U \in BN} \Theta_{U/X}(\varphi|_{U}),
\]

where \( BN \) is a collection of sub-surfaces of \( \text{im}(\sigma) \). When \( M \) is acylindrical and \( P = \emptyset \), we have that \( BN \) is just a collection of disks, the leopard spots of [12]. If \( P \neq \emptyset \) and \( M \) is relatively acylindrical, then we can also have punctured disks coming from peripheral cylinders of \( M \).

As a consequence of (1.1), one can estimate the operator norm of the co-derivative \( d\sigma_{M}^{*} \) of the skinning map by bounding the Poincaré series operator of the corresponding surfaces. Using such estimate, we provide here effective bounds, in terms of the topology of \( \partial_{0}M \), on the contraction of the skinning map in the acylindrical case. This builds on previous work [2] of D. E. Barret and J. Diller, who gave an alternative proof of McMullen's estimates on the norm of the Poincaré operator, [12].

Improving on the main result of [2] (Theorem 3.1), we show:

**Theorem 1.1.** Suppose \( X \) is a Riemann surface of finite-type and let \( Y \) be a disk or a punctured disk. Further let \( \pi : Y \to X \) be a holomorphic covering map. Then, the norm of the corresponding Poincaré series operator satisfies:

\[
\|\Theta\|_{\text{op}} < \frac{1}{1 + C_{g,n,\ell}} < 1
\]

for some constant \( C_{g,n,\ell} > 0 \) depending only on the topology of \( X \cong S_{g,n} \) and the injectivity radius \( \ell \) of \( X \).

In contrast with [2], we compute the contraction constant \( C_{g,n,\ell} \) in a completely explicit way and in the case under examination without any extra assumptions on \( \|\Theta\|_{\text{op}} \). The constant \( C_{g,n,\ell} \) only depends on: the genus \( g \) of \( X \), the number of punctures \( n \) of \( X \), the length \( \ell \) of the shortest closed geodesic in \( X \). So, we obtain an explicit bound over the moduli space of geometrically finite hyperbolic manifolds.

Furthermore, \( C_{g,n,\ell} \) is continuous and decreasing as a function of \( \ell \), in fact it is linear in \( \ell \), and satisfies the following asymptotic expansion for \( g,n \gg 1 \). Let \( \chi := 2g - 2 + n \) be the Euler characteristic, and \( \kappa := 3g - 3 + n \) be the complexity of \( X \). Then,

\[
\log \log \left( \frac{\ell}{C_{g,n,\ell}} \right) \asymp \frac{4}{\arcsinh(1)} \chi^{2} + \coth \left( \frac{\pi}{12} \right) \chi + \pi \sinh \left( \frac{1}{2} \arcsinh \left( \tanh(\pi/12) \right) \right) \kappa.
\]

An application to infinite-type 3-manifolds. In [6] the first author studied the class \( \mathcal{M}^{B} \) of infinite-type 3-manifolds \( M \) admitting an exhaustion \( M = \bigcup_{i} M_{i} \) by hyperbolizable 3-manifolds \( M_{i} \) with incompressible boundary and with uniformly bounded genus.
One can use skinning maps to study the space of hyperbolic metrics on the
manifolds in $\mathcal{M}^B$ that admit hyperbolic structures. Indeed, consider all mani-

folds $M \in \mathcal{M}^B$ such that for all $i \in \mathbb{N}$ every component $U_i := M_i \setminus M_{i-1}$ is acylindrical. By the main results of [6] this guarantees that $M$ is in fact hyperbolic, which is in general not the case, see [5,7], or [8,9] for other examples of infinite-type hyperbolic 3-manifolds. We can thus think of a (hyperbolic) metric $g$ on $M$ as a gluing of (hyperbolic) metrics $g_i$ on the $U_i$’s and so it makes sense to investigate

the glueing of pairs $U_i, U_{i+1}$ via skinning maps.

In order to approach the construction of $g$ in this way, it is helpful to know that

the contraction factor of the skinning maps over the Teichmüller spaces relative

to $U_i$ stays well below 1 uniformly in $i$. The latter fact follows from Theorem 1.1,
in view of the uniform bound on the genus of the $M_i$’s.

2. Notation

Throughout the work, $X$ is a hyperbolic Riemann surface of finite-type. Let $\overline{X}$
be the compact Riemannian surface obtained by adding a single point to each end
of $X$. We indicate by

- $g$ the genus of $X$;
- $n$ the cardinality of the set of punctures $P := \overline{X} \setminus X$.

We may thus regard $X$ as an element of the moduli space $\mathcal{M}(S_{g,n})$ of the $n$-
punctured Riemann surface of genus $g$. Further let

- $\chi := 2g - 2 + n$ be the Euler characteristic of $X$;
- $\kappa := 3g - 3 + n$ be the complexity of $X$, with the exception of the surface $S_{0,2}$
  for which $\kappa := 0$.

We say that a curve in $X$ is a short geodesic if it is a closed geodesic of length less
than $2 \text{arcsinh}(1)$, and we define

- $\Gamma$ the set of short geodesics on $X$;
- $\ell := \min_{\gamma \in \Gamma} \ell(\gamma)$ (twice) the injectivity radius of $X$.

For any $A \subset X$, denote by $|A|$ the number of connected components of $A$, and
indicate by $U \in \pi_0(A)$ any of such connected components. Let $d$ be the intrinsic
distance of $X$ and further set

$$(A)_s := \{x \in X : \text{dist}(x, A) \leq s\}, \quad s > 0.$$  

Regions. Denote by $D$ the Poincaré disk, and set $D^* := D \setminus \{0\}$. The cusp $C_p$
about $p \in P$ is the image of the punctured disk $\{0 < |z| < e^{-\pi}\}$ under the holo-
morphic cover $\pi_p : D^* \to X$ about $p$.

We start by recalling the following well-known fact.

**Lemma 2.1** (Thm. 4.1.1)). Let $\gamma$ be a short closed geodesic in $X$ of length $\ell(\gamma)$, and set $w := \text{arcsinh} \left( \frac{1}{\sinh(\ell(\gamma)/2)} \right)$. The collar $C_\gamma$ around $\gamma$ is isometric to $[-w, w] \times S^1$ with the metric $d\rho^2 + \ell(\gamma)^2 \cosh^2(\rho) \, dt^2$.

Note that in the previous statement the local metric, in Fermi coordinates, is parametrised with $\ell$ speed hence the $\ell^2$ factor.

We define:

- the cusp part $X_{\text{cusps}}$ of $X$ as $X_{\text{cusps}} := \cup_{p \in P} C_p$;
- the core $X_{\text{core}}$ of $X$ as $X_{\text{core}} := X \setminus X_{\text{cusps}}$;
• the thick part $X_{\text{thick}}$ of $X$ as $X_{\text{thick}} := X_{\text{core}} \setminus \bigcup_{\gamma \in \Gamma} C_{\gamma}$;
• the thin part $X_{\text{thin}}$ of $X$ as $X_{\text{thin}} := X \setminus X_{\text{thick}}$.

Quadratic differentials. Let $T_{1,0}^* X$ be the holomorphic cotangent bundle of $X$. A quadratic differential on $X$ is any section $\psi$ of $T_{1,0}^* X \otimes T_{1,0}^* X$, satisfying, in local coordinates, $\psi = \psi(z) \, dz^2$. A quadratic differential $\psi$ is holomorphic if its local trivializations $\psi(z)$ are holomorphic. To each holomorphic quadratic differential $\psi$ we can associate a measure $|\psi|$ on $X$ defined by $|\psi| := |\psi(z)| \cdot |dz|^2$. We denote by $\langle \psi(\cdot) \rangle$ the density of the measure $|\psi|$ with respect to the Riemannian volume of $X$.

We say that any $\psi$ as above is integrable if $\|\psi\| := |\psi|(X)$ is finite, and we denote by $Q(X)$ the space of all integrable holomorphic quadratic differentials on $X$, endowed with the norm $\|\cdot\|$. When $X$ has finite topological type, $Q(X)$ is finite-dimensional, its dimension depending only on $g$ and $n$.

Constants. Everywhere in this work, $r, s, t, w$ and $\varepsilon$ are free parameters. We shall make use of the following universal constants:

- $e_0 := \arcsinh(1) \approx 0.8813$ the two-dimensional Margulis constant;
- $c_1 := \coth(\pi/12) \approx 3.9065$;
- $c_2 := \arcsinh\left(\tanh(\pi/12)\right) \approx 0.2532$;
- $c_3 := \frac{\pi \sinh\left(\frac{1}{2} \arcsinh\left(\tanh(\pi/12)\right)\right)}{\arcsinh(\tanh(\pi/12))} \approx 1.5750$;
- $c_4 := (1 - \tanh^2(1/2))^2 \approx 0.6185$;
- $c_5 := 4\pi\left(1 + \sinh(1)\right) \approx 27.3343$;
- $c_6 := (ec_4)^{2c_3+2} \approx 76.5904$;
- $c_7 := \max_x x \cdot \arcsinh\left(\csc(x/2)\right) \approx 1.5536$.

Finally, for simplicity of notation, we shall make use of the following auxiliary constants, also depending on $X$:

- $a_1 := 4 |x|^2 / \varepsilon + 2\kappa \log c_1 + 2c_2 c_3$;
- $a_2 := \log(e c_4)^{a_1+2(1+c_3)}$.

We denote by $a \wedge b$ the minimum between two quantities $a, b \in \mathbb{R}$.

3. OUTLINE

We start by recalling the results of D. E. Barret and J. Diller [2] that we make explicit using classic hyperbolic geometry. The main result of [2] is:

**Theorem 3.1** ([2] Thm. 1.1]). Suppose $X, Y$ are Riemann surfaces of finite-type and let $\pi : Y \to X$ be a holomorphic covering map. Then, the norm of the corresponding Poincaré operator satisfies:

$$\|\Theta\|_{\text{op}} := \sup_{\varphi \in Q(Y) \atop \|\varphi\| = 1} \|\Theta \varphi\| < 1 - k < 1.$$  

Furthermore, $k > 0$ may be taken to depend only on the topology of $X, Y$, and the length $\ell$ of the shortest closed geodesic on $X$. As a function of $\ell$, the number $k$ may be taken to be continuous and increasing.

In order to prove the above theorem, consider a unit-norm quadratic differential $\varphi \in Q(Y)$ such that $\Theta \varphi \neq 0$. In [2], the authors estimate

$$1 - \|\Theta \varphi\|$$
as follows. Let $K \subset \overline{X}$ be any compact set containing the set $Z$ of zeroes of $\Theta \varphi$ and the punctures of $X$, viz. $Z \cup P \subset K$, and such that $\partial K$ is smooth. Further let
\begin{equation}
m(r) := \min_{p \in \partial(K)_r} \langle \Theta \varphi \rangle.
\end{equation}
Then, for every $t > 1$ and every $r_0 > 0$, \cite[Lem. 3.2]{2} proves the following estimate
\begin{equation}
1 - \| \Theta \varphi \| \geq \int_{r_0}^{r_1} m(r) \left[ t^{-1} \text{area}(X \setminus (K)_r) - \text{length}(\partial (K)_r) \right] dr.
\end{equation}
In general the $t$ in the above estimate will depend on the geometry and topology of the covering surface $Y$. In the case at hand however, $Y$ is either the Poincaré disk or a punctured disk and, by work of J. Diller \cite{10}, we can assume that $t = 1$. It is likely that the constants of Diller can be made explicit as well and so that one could have a version of Theorem \ref{theo:3.1} with the constants are explicit in the topology of $X$, $Y$ and their injectivity radii.

In the following sections, we give effective estimates for $m(r)$, $\text{area}(X \setminus (K)_r)$, and $\text{length}(\partial (K)_r)$. In order to estimate $m(r)$ we will need the following result from \cite{2}.

**Theorem 3.2** (\cite[Thm. 4.4]{2}). Let $\psi \in Q(X)$ with zero set $Z$. Suppose $W \subset X \setminus Z$ is a domain such that $\langle \psi(p) \rangle \leq L$ for all $p \in W$, and set $\rho(p) := \min \{1, \text{dist}(p, \partial W)\}$.

Then, if $\gamma \subset W$ is a path connecting $p_1$ and $p_2$ we have:
\begin{equation}
\frac{\langle \psi(p_1) \rangle}{\langle \psi(p_2) \rangle} \geq \left( \frac{\langle \psi(p_2) \rangle}{c_4 L} \right)^{-1 + \exp \left( \frac{\rho}{\text{area}(\rho/2)} \right)}.
\end{equation}

4. Effective Computations

The following is an easy lemma bounding the diameter of components of $(X_{\text{thick}})_\varepsilon$ or $(X_{\text{core}})_\varepsilon$.

**Lemma 4.1.** Let $X \in \mathcal{M}(S_{g,n})$. Then,
\begin{itemize}
  \item[(i)] any pair of points in the same connected component of $(X_{\text{thick}})_\varepsilon$ is joined by a path of length at most $4 |\chi|/\varepsilon$;
  \item[(ii)] any pair of points in $(X_{\text{core}})_\varepsilon$ is joined by a path $\gamma$ in $(X_{\text{core}})_\varepsilon$ satisfying
\end{itemize}
\begin{equation}
\ell(\gamma) \leq 4 |\chi|^2/\varepsilon + 2\kappa \text{arcsinh} \left( \text{csch}(\ell/2) \right).
\end{equation}

**Proof.** Assertion \textbf{(i)} is a consequence of the Bounded Diameter Lemma \cite{13}.

\textbf{Claim.} \textbf{(ii)} Using the fact that each component of $(X_{\text{thick}})_\varepsilon$ contains an essential pair of pants and that the maximal number of pairwise disjoint short curves is $\kappa$ we have:

\begin{itemize}
  \item[(i)] $|X_{\text{thick}}| \leq |\chi|$ and $|(X_{\text{thin}})_\varepsilon| \leq \kappa$.
\end{itemize}

By short-cutting in the region we obtain:

\begin{itemize}
  \item[(i)] A length-minimizing $\gamma$ enters each $U \in \pi_0((X_{\text{core}})_\varepsilon)$, resp. $U \in \pi_0((X_{\text{thin}})_\varepsilon)$ at most once.
\end{itemize}

Let $\gamma$ be length-minimizing. By \textbf{(i)} we have $\text{length}(\gamma \cap U) \leq 4 |\chi|/\varepsilon$. By the Collar Lemma \cite{3},
\begin{equation}
\text{length}(\gamma \cap U) \leq \text{diam}(U) \leq 2 \text{arcsinh} \left( \text{csch}(\ell/2) \right).
\end{equation}

The conclusion follows combining the previous estimates with the two claims. \square

The next lemma is \cite[Lem. 4.6]{2}. We just work out the constant explicitly.
Lemma 4.2. Let $L(s) := \max_{p \in (X_{thick})_s} \langle \psi(p) \rangle$. Then,

(i) $L(0) \geq \frac{\ell(A)}{16|\chi|} \|\psi\|$;
(ii) for all $0 \leq s \leq t$, we have $L(s) \geq e^{s-t}L(t)$.

Proof. (i) Firstly assume that at most half the mass of $\psi$ is concentrated inside the collars of short geodesics. As in [2, Lem. 4.6(i)], it follows that

\begin{equation}
\langle \psi \rangle \geq \frac{\|\psi\|}{2 \text{area}(X)} = \frac{\|\psi\|}{4\pi |\chi|} \geq \frac{\|\psi\|}{16 |\chi|}.
\end{equation}

Assume now that at least half the mass of $\psi$ is concentrated inside collars of short geodesics. Let $\gamma$ be any such geodesic and let $C := C_\gamma$ be the collar around $\gamma$. For $r \leq R := \pi^2/\ell(\gamma)$ and $r$ satisfying $\tan(\pi r/2R) = \csc h(\ell(\gamma)/2)$, we have that

\[
\frac{1}{2 \text{area}(X)} \|\psi\| \leq \int_C |\psi| = \int_0^{2\pi} \int_{e^{-r}}^e |f(z)| \frac{dr}{z^2} \frac{d\theta}{\ell(\gamma)/2} \\
\leq \int_0^{2\pi} \int_{e^{-r}}^e Lr^{-1} dr d\theta = 4\pi Lr,
\]

hence that

\[
\frac{\|\psi\|}{2\pi r \text{area}(X)} \leq 4L.
\]

Computing both $r$ and $R$ in terms of $\ell(\gamma)$,

\[
L(0) \geq \max_{\partial C} \langle \psi \rangle = \frac{4LR^2}{\pi^2} \cos^2 \frac{\pi r}{2R} \\
\geq \frac{\|\psi\|}{2 \text{area}(X) 2R \arctan (\csc h (\ell(\gamma)/2)) \cos^2 (\arctan (\csc h (\ell(\gamma)/2))}.\]

Now, since $\cos^2 (\arctan (\csc h(t))) = \tanh^2(t)$, and substituting $R := 2\pi/\ell(\gamma)$,

\[
L(0) \geq \frac{\|\psi\|}{4 \text{area}(X) \arctan (\csc h (\ell(\gamma)/2))} \cdot \ell(\gamma).
\]

Since $t \mapsto \tanh^2(t/2)/\left(t^2 \arctan(\csc h(t/2))\right)$ has global minimum $\frac{1}{2\pi}$ at $t = 0$, we have that

\[
L(0) \geq \frac{\pi \ell(\gamma)}{8 \text{area}(X)} \|\psi\| \geq \frac{\ell}{16 |\chi|} \|\psi\|.
\]

Combining the above inequality with (4.2) yields the assertion. \[\square\]

Let $\log_+(x) := \max \{0, \log(x)\}$. We start with some estimates towards establishing (3.2).

Lemma 4.3. For each connected component $U \in \pi_0((X_{thick})_s)$, letting $s = \log_+(c_1t)$

(i) $\text{area}(U) - t \text{length}(\partial U) \geq \pi/3$;
(ii) for all \( p \in U \): \( \text{inj}_p \geq c_2/t \); 
(iii) given \( p_1, p_2 \in U \) there exists \( \gamma \subset U \) connecting \( p_1 \) and \( p_2 \) such that

\[
\ell(\gamma) \leq \frac{4|\chi|^2}{\varepsilon} + 2\kappa \log t + 2\kappa \log c_1.
\]

Proof. (i) Let \( g_U \) and \( n_U \) respectively denote the genus of \( U \) and the number of boundary components of \( U \). Further let \( A_1, \ldots, A_{n_U} \) denote the embedded annuli bounded by short closed geodesics on one side and by connected components of \( \partial U \) on the other side. We allow for \( A_j \) being part of a cusp, in which case, on one side, it is bounded by a puncture rather than by a short geodesic.

By the Gauss–Bonnet Theorem,

\[
\text{area}(U) = 2\pi (2g_U + n_U - 2) - \sum \text{area}(A_j).
\]

If \( n_U = 0 \) then \( U = X \), which yields \( \text{area}(U) - t \text{length}(\partial U) = 2\pi |\chi| \). Thus, in the following we may assume without loss of generality that \( n_U \geq 1 \). In this case, either \( g_U \geq 1 \) and \( n_U \geq 1 \), or \( g_U = 0 \) and \( n_U \geq 3 \). Thus,

\[
\text{area}(U) \geq 2\pi \frac{n_U}{3} - \sum \text{area}(A_j).
\]

Let \( \ell_j \) denote the length of the geodesic component of \( \partial A_j \) and \( L_j \) denote the length of the other component. Then,

\[
\text{area}(U) - t \text{length}(\partial U) \geq 2\pi \frac{n_U}{3} + \sum_j \left( (t - 1) \text{area}(A_j) - t(\text{area}(A_j) + L_j) \right).
\]

By Lemma 2.1 setting

\[
w_j := \arcsinh \left( \frac{1}{\sinh(\ell_j/2)} \right),
\]

we have that

\[
\text{area}(A_j) = \int_0^{w_j-s} \int_0^1 \ell_j \cosh(\rho) \, d\rho \, dt = \ell_j \sinh(w_j - s)
\]

and

\[
L_j = \ell_j \cosh(w_j - s).
\]

We see that

\[
\text{area}(A_j) + L_j = \ell_j (\sinh(w_j - s) + \cosh(w_j)) = \frac{e^{-s}\ell_j}{\tanh(\ell_j/4)}
\]

is monotone increasing in \( \ell_j \) (e.g. by differentiating w.r.t. \( \ell_j \)). Thus it achieves its minimum when the two boundary components of \( A_j \) coincide, in which case \( \ell_j = A_j \) and \( \text{area}(A_j) = 0 \). In this case, \( s \) measures the distance from the geodesic to the edge of the collar containing \( A_j \). Therefore, by the Collar Lemma, \( \sin(\ell_j/2) = \text{csch}(s) \), hence

\[
\text{area}(U) - t \text{length}(\partial U) \geq 2n_U \left( \frac{\pi}{3} - t \arcsinh \left( \text{csch}(s) \right) \right)
\]

\[
\geq 2 \left( \frac{\pi}{3} - t \arcsinh \left( \text{csch}(s) \right) \right).
\]

Letting the right-hand side above be larger than \( \pi/3 \) we get

\[
s \geq \arcsinh \left( \text{csch}(\pi/(6t)) \right), \quad t > 1, \quad s = \log \left( \coth \left( \frac{\pi}{12t} \right) t \right).
\]
(ii) Let $C$ be a short collar in $X$. For $p \in (X_{\text{thick}})_\varepsilon + s \cap C$, by the Collar Lemma we have that
$$\inj_p \geq \arcsinh \left( e^{\text{dist}(p, \partial C)} \right) = \arcsinh (e^{-s}) = \arcsinh \left( \frac{1}{c_1 t} \right) \geq \frac{c_2}{t}$$
with $c_2 := \arcsinh(1/c_1)$, and where the last inequality is sharp by a direct computation.

(iii) Let $p_1, p_2 \in U$. Then we can find a rectifiable curve $\gamma$, connecting $p_1$ to $p_2$, and enjoying the following properties:

(a) if $\gamma \cap C \neq \emptyset$, then $\gamma \cap \partial C$ consists of two points belonging to distinct connected components of $\partial C$, and $\text{length}(\gamma|_C) \leq 2s$;

(b) in each connected component of $(X_{\text{thick}})_\varepsilon$, the curve $\gamma$ is a shortest path between its endpoints.

See Figure 1.

We can decompose $\gamma$ into its components in

$$X_1 := (X_{\text{thick}})_\varepsilon \quad \text{and} \quad X_2 := (X_{\text{thick}})_{\varepsilon + s} \setminus (X_{\text{thick}})_\varepsilon \subset (X_{\text{thin}})_\varepsilon.$$

By the Bounded Diameter Lemma [13], the length of each component of $\gamma$ in $X_1$ is bounded by $4 |\chi| / \varepsilon$, and we have at most $|\chi|$ such components. In each connected component of $X_2 \subset (X_{\text{thin}})_\varepsilon$ the length of $\gamma$ is at most $2s$, and there are at most $\kappa$ such components. Thus, for $s = \log(c_1 t)$ we get

$$\ell(\gamma) \leq 4 |\chi|^2 / \varepsilon + 2s\kappa = \frac{4 |\chi|^2}{\varepsilon} + 2\kappa \log(c_1 t).$$

We now show how to estimate the quantities related to $(K)_r$ in Equation (3.2). Let $Z$ be the zeroes of a given quadratic differential $\psi$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The piecewise geodesic curve $\gamma$ connecting $p_1$ to $p_2$ and $C$, shaded, are collars around short geodesics.}
\end{figure}
Lemma 4.4. Let $U \in \pi_0((X_{\text{thick}})_s)$ and $K := X \setminus U \cup Z$. Then, for $r \in (0,1)$, 
$t > 1$, and $s = \log_+ (c_1 t)$

\[
\text{area} \left( X \setminus (K)_r \right) - t \text{length} (\partial (K)_r) \geq \frac{\pi}{3} - \kappa r t \left[ c_7 - s + 4 \pi (1 + \sinh(1)) \frac{|\chi|}{\kappa} \right] \\
\geq \frac{\pi}{3} - \kappa r t \left[ 4 \pi (1 + \sinh(1)) + c_7 \right] \\
= \frac{\pi}{3} - \kappa r t (c_5 + c_7). 
\]

Proof. Since $|Z| \leq 2 |\chi|$ and $r < 1$, we have that

\[
\text{length} (\partial (K)_r) \leq \text{length}(\partial U) + \text{length} (\partial (Z)_r) \leq \text{length}(\partial U) + 2\pi |Z| \sinh(r) 
\]

(4.4)

\[
\leq \text{length}(\partial U) + 4\pi |\chi| \sinh(1) r. 
\]

Furthermore,

\[
\text{area} \left( X \setminus (K)_r \right) = \text{area}(X) - \text{area} \left( (K)_r \right) \\
\geq \text{area}(X) - \left( \text{area} \left( X \setminus U \right) + \text{area} \left( (\partial U^+)_r \right) + \text{area} \left( (Z)_r \right) \right) \\
\geq \text{area}(U) - \text{area} \left( (\partial U^+)_r \right) - 4\pi |\chi| (\cosh(r) - 1) \\
\geq \text{area}(U) - \text{area} \left( (\partial U^+)_r \right) - 4\pi |\chi| r \\
\geq \text{area}(U) - t \text{area} \left( (\partial U^+)_r \right) - 4\pi |\chi| tr 
\]

since $t > 1$. We can estimate $\text{area} \left( (\partial U^+)_r \right)$ by assuming that $(\partial U^+)_r$ is isometrically embedded, so that, by Lemma 2.1,

\[
\text{area} \left( (\partial U^+)_r \right) = \sum_j \ell(\gamma_j) (\sinh(w_j - s + r) - \sinh(w_j - s)). 
\]

Repeat the construction of the annuli $A_j$ in Lemma 4.3 and let $w_j$ be defined as in (4.3). By Taylor expansion of sinh around $w_j - s > 0$, we have that

\[
\text{area}(U) - t \text{area} \left( (\partial U^+)_r \right) \geq \text{area}(U) - rt \sum_j \ell(\gamma_j)(w_j - s) \\
\geq \text{area}(U) - rt \sum_j \ell(\gamma_j) (\text{arc sinh} (\text{csch}(\ell(\gamma_j)/2))) \\
+ rt \log(c_1 t) \sum_j \ell(\gamma_j) \\
\geq \text{area}(U) - c_7 r t + rt \log(c_1 t) \sum_j \ell(\gamma_j). 
\]

As a function of the metric, the summation $\sum_j \ell(\gamma_j)$ attains its maximum over the moduli space $\mathcal{M}(S_{g,n})$ when $\ell(\gamma_j) = \varepsilon_0$ for each $j$, thus its maximum is $\kappa \varepsilon_0$. Therefore,

(4.5)

\[
\text{area} \left( X \setminus (K)_r \right) \geq \text{area}(U) - rt \kappa c_7 + rt \kappa \varepsilon_0 \log(c_1 t) - 4\pi |\chi| tr. 
\]

Multiplying (4.4) by $-t$ and adding (4.5), together with Lemma 4.3(i), yields the conclusion. 

Let $U$ be the component of $(X_{\text{thick}})_{s+}$ containing $p_{\max}(s)$, where $s = \log(c_1 t)$ and $p_{\max}$ satisfies Lemma 4.2. Set $K' := X \setminus U$ and let $K := K' \cup Z$. This is a slight refinement of the previous $K$, in which we chose a specific component $U$
and a slightly larger neighbourhood of $U$. The next lemma will deal with paths in $X \setminus (K)_r$. When $r = 0$, the set $X \setminus K = U \setminus Z$ looks as Figure 2.

**Figure 2.** The set $X \setminus K = \text{int}(U) \setminus Z$ is greyed out and the white points are zeroes of the quadratic differential.

**Lemma 4.5.** Fix $t > 1$. If $r < c_2/(|\chi| t)$, then any two points in $X \setminus (K)_r$ can be joined by a rectifiable curve in $X \setminus (K)_{r/2}$.

**Proof.** We start with the following claim.

**Claim.** Let $V \in \pi_0((K)_{r/2})$. If $V \cap (K')_{r/2} \neq \emptyset$, then $V \subset (K')_r$.

Indeed, for $c > 0$ to be fixed later, let $V \in \pi_0((K)_{cr})$ with $V \cap (K')_{cr} \neq \emptyset$. We need to show that if $V$ is such component it does not separate $X \setminus (K)_r$. Fix $p \in V \setminus (K')_{cr}$. Since $V$ is connected and contained in $(K)_{cr}$, then $p$ is joined to $(K')_{cr}$ by a chain of disks of radius $cr$ centered at points in $Z$. Therefore dist$(p, (K')_{cr}) \leq 2c |Z| r$. Choosing $c < (2 |Z| + 1)^{-1}$, e.g. $c := \frac{1}{2}(2 |Z| + 1)^{-1}$, proves that dist$(p, (K')_{cr}) \leq r/2$ and so that:

$$\text{dist}(p, K') \leq \text{dist}(p, (K')_{cr}) + cr = 2c |Z| r + cr \leq (2 |Z| + 1)cr \leq r/2,$$

proving that $V \subset (K')_r$. This concludes the proof of the claim.

Thus, we need to show that for $r < c_2/t$ and for all $p_0, p_1 \in X \setminus (K)_r \subset (U)_s$ there exists a rectifiable curve $\gamma \subset X \setminus (K)_{c_2 r}$ connecting $p_0$ to $p_1$. By the Collar Lemma,

$$\text{inj}(U)_s := \min_{p \in (U)_s} \text{inj}_p \geq \arcsinh(e^{-s}) = \arcsinh \left( \frac{1}{c_1 t} \right) \geq \frac{c_2}{t},$$

similarly to the proof of Lemma 4.3(ii).

Now, argue by contradiction and assume that there exists no rectifiable curve as in the assertion. Then, there exists a rectifiable loop $\alpha$ in $(Z)_{r/2}$ separating $X \setminus (K)_r \subset (U)_s$ into connected components so that $p_0$ and $p_1$ belong to two distinct such components. See the picture in Figure 3.
Figure 3. The two cases for the loop \( \alpha \) separating \( p_1 \) to \( p_2 \). The shaded regions are part of \( (K)_{r/2} \) and the grey dots are zeroes of the quadratic differential.

For any such \( \alpha \),

\[
\text{length}(\alpha) \leq r |Z| < |Z| \frac{c_2}{|\chi| t} \leq \frac{c_2}{t} \leq \inj(U).
\]

As a consequence, \( \alpha \subset (U) \) is null-homotopic and so we must be on the right side of Figure 3. Therefore, there exists \( L \in \mathbb{R}^+ \) such that \( \alpha \subset B_L(q) \) for \( q \in (U) \), and \( L \leq \ell(\alpha)/2 < r/2 \). Thus, the component \( W \subset X \setminus (K)_r \) containing, say, \( p_1 \), lies in \( B_L(q) \subset B_{r}(q) \) and note that by construction its distance from any zero is at least \( r \). Therefore, \( W \) is at distance \( r/2 + L < r \) from a zero. However, since \( d(W, Z) \geq r \) we have a contradiction. \( \square \)

We now state the main lemma we will use in our estimate of (3.2).

**Lemma 4.6.** Let \( r < c_2/(|\chi| t) \), and set \( a_1 := 4 |\chi|^2 / \varepsilon + 2 \kappa \log c_1 + 2 c_2 c_3 \). Then, any two points in \( X \setminus (K)_r \) are joined by a rectifiable curve \( \gamma \subset X \setminus (K)_{r/2} \) with the following properties:

(i) \( \gamma \) consists of length-minimising geodesic segments and of at most one arc in each of the components of \( \partial (K)_{r/2} \);

(ii) \( \ell(\gamma) \leq a_1 + 2 \kappa \log t \);

(iii) for \( z \in Z \): length \( (\gamma \cap B_w(z)) \leq 2(1 + c_3)w \) for all \( w > 0 \) such that \( B_w(z) \) is embedded.

**Proof.** (i) ![Fix points](image)

By Lemma 4.3 there exists a rectifiable \( \gamma \subset U \) connecting them, with

\[
\ell(\gamma) \leq \frac{4 |\chi|^2}{\varepsilon} + 2 \kappa \log c_1 + 2 \kappa \log t.
\]

The curve \( \gamma \) intersects \( (K)_{r/2} \) in at most \( 2 |\chi| \) components (i.e. balls around zeroes of \( \psi \)). In each such component \( V = B_{r/2}(z) \) (for some \( z \in Z \)) we can replace \( \gamma \big|_V \) by a shortest path on \( \partial V \) as the one in Lemma 4.3(iii).

Since \( V \) is a ball, the length of \( \gamma \big|_V \) is bounded by half the length of the circumference of a great circle on \( V \), i.e.

\[
\pi \sinh(r/2) \leq \pi \sinh(r) \leq c_3 r, \quad r < \frac{c_2}{|\chi| t} < \frac{c_2}{|\chi|}.
\]
By repeating this reasoning on each component $V$ as above, we obtain a path
\[ \gamma': p_0 \to p_1 \text{ satisfying (iii) and such that:} \]
\[
\text{length}(\gamma') \leq \ell(\gamma) + 2|\chi|c_3r
\]
\[
\leq \frac{4|\chi|^2}{\varepsilon} + 2\kappa \log c_1 + 2\kappa \log t + 2 \frac{c_2c_3}{t} \quad (t > 1)
\]
\[
\leq \frac{4|\chi|^2}{\varepsilon} + 2\kappa \log c_1 + 2c_2c_3 + 2\kappa \log t.
\]

(iii) Let $z \in Z$ be a zero of $\psi$ and fix $w > 0$. Each component $\alpha$ of $\gamma$ in $\partial(K)_{c_r'}$ has length at most $c_3r$ and each geodesic arc of $\gamma$ connecting an endpoint of $\alpha$ to $\partial B_w(z)$ has length at most $w$. We now estimate

\[
\left| \pi_0(\gamma \cap B_w(z)) \right| \leq \begin{cases} 0 & w < r/2 \\ 1 & p_1, p_2 \in B_w(z) \\ 1 & p_1 \in B_w(z), p_2 \notin B_w(z) \\ 1 & p_1, p_2 \notin B_w(z). \end{cases}
\]

The first bound holds by definition. The second holds by the convexity of hyperbolic balls: if $p_1, p_2 \in B_w(z)$ then we can choose $\gamma \subset B_w(z)$. The third and fourth one follow from the fact that if $\gamma$ has more than one component in $B_w(z)$, then we can shortcut $\gamma$ inside the ball.

If $w = r/2$, then $\gamma|_{B_w(z)} \subset \partial B_w(z)$, and we may choose $\gamma|_{B_w(z)}$ to be a circumference arc, so that length $\left( \gamma|_{B_{r/2}(z)} \right) \leq \pi \sinh(r/2) \leq c_3r$ by (1.6).

If instead $w > a_1r$, then we may choose $\gamma$ to be either a geodesic segment, or a union $\gamma_1 \cup \gamma_2 \cup \gamma_3$, where $\gamma_1$ and $\gamma_2$ are geodesic segments each connecting $\partial B_w(z)$ to $\partial B_{r/2}(z)$, and $\gamma_3$ is a circumference arc on $\partial B_{r/2}(z)$. In the first case, length $\left( \gamma|_{B_w(z)} \right) \leq 2w$. In the second case,

\[
\text{length } (\gamma|_{B_w(z)}) \leq 2w + \pi \sinh(a_1r) \leq 2w + c_3r \leq 2w + c_3r.
\]

Thus, we obtain that:

\[
\text{length } (\gamma \cap B_w(z)) \leq \begin{cases} 0 & w < r/2 \\ 2c_3w & w = r/2 \\ 2w + c_3w & w \geq r/2 \end{cases} \leq 2(1 + c_3)w,
\]

which concludes the proof. \qed

With $m(r)$ as in (3.1) we can now estimate (3.2) and show our final result.

**Proof of Theorem 1.1** Let $r < c_2/|\chi| \leq 2$. Let $s$ be as in Lemma 4.4 and choose $U$ to be the component of $X_{\text{thick}}(s)$ containing the point $p_{\text{max}}(s)$ as in Lemma 4.2. Let $Z$ be the set of zeroes of $\psi$, $K := X \setminus U \cup Z$, and $K' := X \setminus \overline{U}$. Let $W := (X_{\text{thick}})_{s+1} \setminus Z$, $p_1 \in \partial (K)_r$, and $p_2 = p_{\text{max}}(s) \in (X_{\text{thick}})_s \setminus Z$. Therefore, we have that $\langle \psi(p_2) \rangle = L(s)$. Moreover, let $\gamma \subset W$ be a path from $p_1$ to $p_2$ satisfying the conditions of Lemma 4.6 and note that

\[
\text{dist}(p, \partial W) \geq \min \{1, \text{dist}(p, Z)\}, \quad p \in \gamma.
\]

By Lemma 4.2(ii) we have that:

\[
L(s+1) \leq e \cdot L(s).
\]
By Theorem [2 4.4], we have that:

\[
\langle \psi(p_1) \rangle \geq \langle \psi(p_2) \rangle \left( \frac{\langle \psi(p_2) \rangle}{c_4 \cdot L(s + 1)} \right)^{-1+\exp \left( \int_0^s \frac{d\gamma}{\tanh(1 \wedge \text{dist}(\gamma_s, Z))} \right)}
\]

\[
\geq L(s) \left( \frac{1}{e \cdot c_4} \right)^{-1+\exp \left( \int_0^s \coth(1 \wedge \text{dist}(\gamma_s, Z)) \, ds \right)}
\]

\[
\geq e \cdot c_4 \cdot L(0) \cdot \left( \frac{1}{e \cdot c_4} \right)^{\exp \left( \int_0^s \coth(1 \wedge \text{dist}(\gamma_s, Z)) \, ds \right)},
\]

where we can estimate \( L(0) \) by Lemma [4.2].

\[
\geq \frac{e \cdot c_4}{16 |x|} \| \psi \| \left( \frac{1}{e \cdot c_4} \right)^{\exp \left( \int_0^s \coth(1 \wedge \text{dist}(\gamma_s, Z)) \, ds \right)}
\]

\[
= \frac{e \cdot c_4}{16 |x|} \| \psi \| \exp \left( -\log(e \cdot c_4) \exp \left( \int_0^s \coth(1 \wedge \text{dist}(\gamma_s, Z)) \, ds \right) \right).
\]

We now estimate \( \int_0^s \coth(1 \wedge \text{dist}(\gamma_s, Z)) \, ds \) from above by breaking it into two terms:

\[
\int_0^s \frac{ds}{\tanh(1 \wedge \text{dist}(\gamma_s, Z))} \leq \int_{\gamma \setminus Z(1)} \frac{ds}{\text{dist}(\gamma_s, Z)} + \int_{\gamma \cap Z(1)} \frac{ds}{\text{dist}(\gamma_s, Z)}.
\]

The first term is bounded by \( \ell(\gamma) \) while for the second term we have by Lemma [4.4(i)]

\[
\int_{\gamma \cap Z(1)} \frac{ds}{\text{dist}(\gamma_s, Z)} \leq \int_1^2 \text{length} \left( (\gamma \cap Z)_1 \right) \, du
\]

\[
\leq \int_1^2 \frac{2(1+c_3)}{a^2} \, du = 2(1+c_3)(1-r/2)
\]

since \( r \leq 2 \).

By Lemma [4.4(i)] we have that:

\[
\ell(\gamma) \leq a_1 + 2\kappa \log t.
\]

Thus:

\[
\int_0^s \frac{ds}{\tanh(1 \wedge \text{dist}(\gamma_s, Z))} \leq a_1 + 2\kappa \log t + 2(1+c_3)(1-r/2)
\]

\[
= a_1 + 2(1+c_3) + 2\kappa \log t - (1+c_3)r.
\]

Therefore, since \( \log(e \cdot c_4) > 0 \), for all \( p_1 \in \partial(K)_r \) we get:

\[
\langle \psi(p_1) \rangle \geq \frac{e \cdot c_4}{16 |x|} \| \psi \| \exp \left( -\log(e \cdot c_4) \exp \left( a_1 + 2(1+c_3) + 2\kappa \log t - (1+c_3)r \right) \right).
\]

Thus, by minimizing over \( p_1 \in \partial(K)_r \) we obtain:

\[
m(r) \geq \frac{e \cdot c_4}{16 |x|} \| \psi \| \exp \left( -\log(e \cdot c_4) \exp \left( a_1 + 2(1+c_3) + 2\kappa \log t - (1+c_3)r \right) \right)
\]

which for \( a_2 := \log(e \cdot c_4) e^{a_1 + 2(1+c_3)} > 0 \) can be rewritten as:

\[
m(r) \geq e \cdot c_4 \frac{\ell}{16 |x|} \| \psi \| \exp \left( -a_2 \ell^2 \kappa e^{-(1+c_3)r} \right).
\]

Then, Equation [4.2] with \( K := W \) becomes, for \( r_0 < \frac{1}{4t} \),
1 − ∥ψ∥ ≥ \int_{0}^{r_0} m(r) \left( t^{-1} \text{area}(X \setminus (K)_r) - \text{length}(\partial (K)_r) \right) \, dr.

By Lemma 4.4 we thus have that, for every \( r_0 < \frac{1}{4t} \),

\[
1 − ∥ψ∥ ≥ \frac{e c_4 \ell}{16 |\chi| t} ∥ψ∥ \int_{0}^{r_0} \exp \left( -a_2 t^2 e^{-(1+c_3)r} \right) (\pi/3 - \kappa rt(c_5 + c_7)) \, dr
\geq \frac{e c_4 \ell e^{-a_2 t^2}}{16 |\chi| t} ∥ψ∥ \int_{0}^{r_0} (\pi/3 - \kappa rt(c_5 + c_7)) \, dr.
\]

Maximizing over \( r_0 \in \left( 0, \frac{1}{4t} \right) \) additionally so that the integrand is non-negative, we have therefore that

\[
1 − ∥ψ∥ ≥ \frac{e c_4 \ell e^{-a_2 t^2}}{16 |\chi| t} \int_{\frac{\pi}{12}}^{\pi} \exp \left( -a_2 t^2 \right) (\pi/3 - \kappa rt(c_5 + c_7)) \, dr
\geq \frac{e \pi^2 c_4 \ell}{288 \kappa(c_5 + c_7) |\chi| t^2} ∥ψ∥,
\]

and maximizing the right-hand side over \( t > 1 \), i.e. choosing \( t = 1 \), we conclude that

\[
∥ψ∥ ≤ \frac{1}{1 + \frac{C \ell e^{-a_2}}{\kappa |\chi|} }, \quad C := \frac{e \pi^2 c_4}{288 (c_5 + c_7)}.
\]

**Contraction factors of skinning maps.** We now apply our explicit bounds from Theorem 1.1 to get effective bounds on the contraction factor of the skinning map.

Let \( N \in AH(M, P) \) be a pared acylindrical manifold so that

- \( P \subset \partial M \) is a collection of pairwise disjoint closed annuli and tori;
- \( P \) contains all torus components of \( M \) and \( M \) is acylindrical relative to \( P \).

Let \( \partial_0 M := \partial M \setminus P \). By [12, p. 443] we have that, for every such \( N \),

\[
|d\sigma| = |d\sigma^*|.
\]

By Theorem [12]

\[
d\sigma^*(\varphi) = \sum_{U \in B(N)} \Theta_{U/X} (\varphi|U) \leq \max_{X \in \partial_0 M} \frac{1}{1 + C_{g,n,\ell}} ∥\varphi∥,
\]

where \( \ell \) is the injectivity radius of the conformal boundary \( \partial_\infty N \), and \( C_{g,n,\ell} \). Thus, we obtain Corollary 4.7.

**Corollary 4.7.** Let \( (M, P) \) be a pared acylindrical hyperbolic manifold. Then, the skinning map at \( N \in AH(M, P) \) has contraction factor bounded by

\[
|d\sigma| \leq \max_{X \in \partial_0 M} \frac{1}{1 + C_{g,n,\ell}}.
\]

---

1 For \( AH(M, P) \) the set of hyperbolic 3-manifolds homotopy equivalent to \( M \) with \( P \) parabolic.
REFERENCES


Department of Mathematics, Belval, Maison du Nombre 6, avenue de la Fonte L-4364 Esch-sur-Alzette Luxembourg

Email address: tommaso.cremaschi@uni.lu

Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria

Email address: lorenzo.delloschiavo@ist.ac.at