

SARNAK'S CONJECTURE FOR A CLASS OF RANK-ONE SUBSHIFTS

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ABSTRACT. Using techniques developed by Kanigowski, Lemańczyk, and Radziwiłł [Fund. Math. 255 (2021), pp. 309–336], we verify Sarnak's conjecture for two classes of rank-one subshifts with unbounded cutting parameters. The first class of rank-one subshifts we consider is called *almost complete congruency classes (acc)*, the definition of which is motivated by the main result of Foreman, Gao, Hill, Silva, and Weiss [Isr. J. Math., To appear], which implies that when a rank-one subshift carries a unique nonatomic invariant probability measure, it is *acc* if it is measure-theoretically isomorphic to an odometer. The second class we consider consists of Katok's map and its generalizations.

1. INTRODUCTION

The Möbius function, $\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}$, is defined such that: $\mu(n) = 0$ if n is divisible by p^2 for some prime number p ; and $\mu(n) = (-1)^k$ if $n = p_1 p_2 \cdots p_k$ where p_1, p_2, \dots, p_k are distinct prime numbers. The Möbius function is one of the most important functions in Number Theory, and in particular the study of the Möbius function is highly consequential in Analytical Number Theory. For instance, the fact that the respective numbers of 1s and -1 s as values of the Möbius function are almost the same is equivalent to the prime number theorem.

Theorem 1.1 (Landau; see [3, §4.9]). *The statement that $\sum_{n \leq N} \mu(n) = o(N)$ is equivalent to the prime number theorem (PNT).*

Furthermore, the Riemann hypothesis can be restated in terms of the rate of cancellation in $\sum_{n \leq N} \mu(n)$.

Theorem 1.2 (Littlewood, 1912; see [15]). *The Riemann hypothesis is equivalent to the statement that for every $\epsilon > 0$ we have $\sum_{n \leq N} \mu(n) = o(N^{1/2+\epsilon})$.*

In this paper we concentrate on the study of the random behavior of the Möbius function and not necessarily the speed of the cancellation. One of the strongest conjectures on the random nature of the sequence $\{\mu(n)\}_{n \in \mathbb{N}}$ is due to Chowla.

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Conjecture 1.3 (Chowla). *Let $0 \leq i_1, i_2, \dots, i_k \leq 2$ be a sequence of integers with at least one taking value 1. Then*

$$\sum_{n \leq N} \mu(n+1)^{i_1} \mu(n+2)^{i_2} \dots \mu(n+k)^{i_k} = o(N).$$

Chowla's conjecture seems out of reach for the moment and a weaker notion (see [1, Theorem 4.10]; [14]) of pseudorandomness for the Möbius function, Sarnak's conjecture, is the main focus of the present work. In an attempt to formalize the random behavior of Möbius function using tools from Dynamical Systems, Sarnak suggested Conjecture 1.4.

Conjecture 1.4 (Sarnak). *Let X be a compact metric space and $T : X \rightarrow X$ be a homeomorphism. If the topological dynamical system (X, T) is of entropy zero, then we have*

$$\sum_{n \leq N} f(T^n x) \mu(n) = o(N)$$

for every continuous function $f : X \rightarrow \mathbb{R}$ and $x \in X$.

Following [12], we say that (X, T) is *Möbius disjoint* if

$$\sum_{n \leq N} f(T^n x) \mu(n) = o(N)$$

for every continuous function $f : X \rightarrow \mathbb{R}$ and $x \in X$. Furthermore, we say a continuous function $f : X \rightarrow \mathbb{R}$ satisfies *Sarnak's property* if

$$\sum_{n \leq N} f(T^n x) \mu(n) = o(N)$$

for every $x \in X$.

Sarnak's conjecture has been studied extensively in recent years (see, for example, [2], [4], [7], [8], [11], [12], and [13]). In particular, the study of the conjecture for symbolic systems corresponding to the class of rank-one transformations is of interest.

Given sequences of positive integers $r_n > 1$ for $n \in \mathbb{N}$ and nonnegative integers $s_{n,i}$ for $n \in \mathbb{N}$ and $0 < i \leq r_n$, define a *generating sequence* v_n of finite words recursively by setting $v_0 = 0$ and

$$(1.1) \quad v_{n+1} = v_n 1^{s_{n,1}} v_n 1^{s_{n,2}} \dots v_n 1^{s_{n,r_n}}$$

for $n \in \mathbb{N}$. An *infinite rank-one word* $V \in 2^{\mathbb{N}}$ is then defined as $V = \lim_{n \rightarrow \infty} v_n$ and the *rank-one subshift* (X_V, T) is given by

$$X_V = \{x \in 2^{\mathbb{Z}} : \text{every finite subword of } x \text{ is a subword of } V\}$$

and $T(x)(a) = x(a+1)$ for all $x \in X_V$ and $a \in \mathbb{Z}$. The sequences $(r_n)_{n \in \mathbb{N}}$ and $(s_{n,i})_{n \in \mathbb{N}, 0 < i \leq r_n}$ are known as, respectively, the *cutting parameter* and the *spacer parameter* of the rank-one subshift. A rank-one subshift (X_V, T) is *nontrivial* if X_V is infinite, or equivalently, V is aperiodic. In this paper we only consider nontrivial rank-one subshifts. Note that a rank-one subshift is always of topological entropy zero. (X_V, T) is *bounded* if there is $M > 0$ such that $r_n < M$ and $s_{n,i} < M$ for all $n \in \mathbb{N}$ and $0 < i \leq r_n$.

Bourgain [4] proved Sarnak's conjecture for bounded rank-one subshifts for the special case that $s_{n,r_n} = 0$ for all $n \in \mathbb{N}$. This was extended to all bounded rank-one subshifts by El Abdalaoui-Lemańczyk-de la Rue [2].

Theorem 1.5 (Bourgain [4]; El Abdalaoui–Lemańczyk–de la Rue [2]). *Let (X, T) be a bounded rank-one subshift. Then (X, T) is Möbius disjoint.*

In this paper we consider two classes of rank-one subshifts with unbounded cutting parameters. The consideration of the first class is motivated by the main result of [9]. In that context the authors assumed that the generating sequence satisfies the condition

$$(1.2) \quad \sum_{n=0}^{\infty} \frac{|v_{n+1}| - r_n|v_n|}{|v_{n+1}|} < +\infty,$$

which guarantees that (X_V, T) admits a unique nonatomic invariant probability measure μ . The main result of [9] is a characterization of when the measure-preserving transformation (X_V, μ, T) is measure-theoretically isomorphic to an odometer. To state this characterization we need to make the following definition. For $n \geq m$, apply (1.1) inductively to write v_n uniquely in the form

$$v_n = v_m 1^{b_1} v_m 1^{b_2} \dots v_m 1^{b_t}$$

and let $I_{m,n}$ be the set of indices for the starting positions of the copies of v_m (starting at index 0 for the starting position of the first copy). Note that $I_{0,n}$ is the set of positions of all 0s in v_m .

Theorem 1.6 (Foreman–Gao–Hill–Silva–Weiss [9]). *The rank-one measure-preserving transformation (X_V, μ, T) is measure-theoretically isomorphic to an odometer if and only if for all $l \in \mathbb{N}$ and all $\epsilon > 0$, there is some $k \in \mathbb{N}$ such that for all $\eta > 0$ there exists an $N \in \mathbb{N}$ such that for all $n > m \geq N$,*

(a) *there is some $j \in \mathbb{Z}/k\mathbb{Z}$ such that*

$$\frac{|\{i \in I_{m,n} : [i]_k \neq j\}|}{|I_{m,n}|} < \eta, \text{ and}$$

(b) *there is some $D \subseteq \mathbb{Z}/k\mathbb{Z}$ such that*

$$\frac{|\{i \leq |v_m| : [i]_k \in D\} \Delta I_{l,m}|}{|I_{l,m}|} < \epsilon,$$

where $[i]_k$ denotes the congruency class of $i \pmod k$.

Note that if (X_V, μ, T) is isomorphic to an odometer and (X_V, T) is bounded, then V is periodic and (X_V, T) is trivial. Motivated by Clause (b) with $l = 0$, we introduce the following notion.

Definition 1.7. Let $M \subseteq \mathbb{Z}$ be nonempty and $A \subseteq \mathbb{N}$ be finite. We say A is a *building block* of M if $0 \in A$ and there exists a nondecreasing sequence $\{a_{i,A}\}_{i \in \mathbb{Z}}$ of integers such that $M = \bigcup_{i \in \mathbb{Z}} (A + a_{i,A})$ and for every $i \in \mathbb{Z}$ we have $a_{i+1,A} - a_{i,A} > \max(A)$ or $a_{i+1,A} = a_{i,A}$.

Definition 1.8. We say $M \subseteq \mathbb{Z}$ is an *almost complete congruency class (acc)* if $M = \emptyset$ or for every $\epsilon > 0$ there exist $k \in \mathbb{N}$ with the following property which we denote as $P(M, \epsilon, k)$:

for every $N \in \mathbb{N}$ there exist a building block A of M and $D_A \subseteq \mathbb{Z}/k\mathbb{Z}$ such that $\max(A) \geq N$ and

$$\frac{|\{0 \leq n \leq \max(A) : [n]_k \in D_A\} \Delta A|}{\max(A)} < \epsilon.$$

Thus Theorem 1.6 implies that if (X_V, μ, T) is isomorphic to an odometer then the set of positions for 0s in V is an *acc*, and in this case it also follows that for every $x \in X_V$, the set of positions for 0s in x is an *acc*. Motivated by this observation, we call a rank-one subshift (X, T) an *acc rank-one subshift* if

$$M_x = \{n \in \mathbb{Z} : x(n) = 0\}$$

is an *acc* for every $x \in X$.

Our first main result of the paper is the following.

Theorem 1.9. *Let (X, T) be an acc rank-one subshift. Then (X, T) is Möbius disjoint.*

Theorem 1.9 will be proved in Section 2. In Sections 3 and 4 we will consider another class of rank-one subshifts which are generalizations of Katok's map studied in [2]. Katok's map is a rank-one subshift where for all $n \in \mathbb{N}$, r_n is even and

$$s_{n,i} = \begin{cases} 0, & \text{for } 0 < i \leq r_n/2, \\ 1, & \text{for } r_n/2 < i \leq r_n. \end{cases}$$

In [2] Sarnak's conjecture for Katok's map was verified under the condition

$$\lim_{n \rightarrow \infty} \frac{r_n}{|v_n|} = +\infty.$$

Here we prove Sarnak's conjecture for a class of generalized Katok's maps under a weaker condition.

The key technique used in all of our proofs is an estimate of the Möbius function on short intervals along arithmetic progressions developed by Kanigowski–Lemańczyk–Radziwiłł [12].

Theorem 1.10 (Kanigowski–Lemańczyk–Radziwiłł [12]). *For each $\epsilon \in (0, \frac{1}{100})$ there exists L_0 such that for each $L \geq L_0$ and $q \geq 1$ with*

$$(1.3) \quad \sum_{\substack{p|q \\ p \text{ prime}}} \frac{1}{p} \leq (1 - \epsilon) \sum_{\substack{p \leq L \\ p \text{ prime}}} \frac{1}{p},$$

we can find $N_0 = N_0(q, L)$ such that for all $N \geq N_0$, we have

$$(1.4) \quad \sum_{j=0}^{N/Lq} \sum_{a=0}^{q-1} \left| \sum_{\substack{m \in [z+jLq, z+(j+1)Lq] \\ m \equiv a \pmod q}} \mu(m) \right| \leq \epsilon N$$

for some $0 \leq z < Lq$.

2. ACCC RANK-ONE SUBSHIFTS

Definition 2.1. Let $M \subseteq \mathbb{Z}$. We say M is *orthogonal to the Möbius function* if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{n \in M \\ 1 \leq n \leq N}} \mu(n) = 0.$$

Note that by Theorem 1.1, if $\mathbb{N} \subseteq M \subseteq \mathbb{Z}$ then M is orthogonal to the Möbius function. Trivially the empty set is orthogonal to the Möbius function.

Theorem 2.2. *Let $M \subseteq \mathbb{Z}$ be an accc and n_1, n_2, \dots, n_l be integers. Then*

$$M' = \{n \in \mathbb{Z} : n + n_1, n + n_2, \dots, n + n_l \in M\}$$

is orthogonal to the Möbius function.

Proof. Fix $0 < \epsilon < \frac{1}{100}$. Assume M is nonempty. Since M is an accc, there exists $k \in \mathbb{N}$ such that property $P(M, \epsilon, k)$ holds. Applying Theorem 1.10 with $q = k$, we obtain L_0 and $L \geq L_0$ satisfying (1.3), and there exists $N_0 = N_0(k, L) > 0$ such that for every $N \geq N_0$ we have

$$\left| \sum_{j=0}^{N/Lk} \sum_{a=0}^{k-1} \sum_{\substack{m \in [z+jLk, z+(j+1)Lk) \\ m \equiv a \pmod k}} \mu(m) \right| \leq \epsilon N$$

for some $0 \leq z < Lk$.

From property $P(M, \epsilon, k)$ we obtain a building block A of M with

$$\max(A) \geq 2N_0Lk/\epsilon, |2n_1/\epsilon|, \dots, |2n_l/\epsilon|,$$

$D_A \subseteq \mathbb{Z}/k\mathbb{Z}$, and a nondecreasing sequence $\{a_{i,A}\}_{i \in \mathbb{Z}}$ of integers such that for every $i \in \mathbb{Z}$ we have $a_{i+1,A} - a_{i,A} > \max(A)$ or $a_{i+1,A} = a_{i,A}$, $M = \bigcup_{i \in \mathbb{Z}} (A + a_{i,A})$, and

$$\frac{|\{0 \leq n \leq \max(A) : [n]_k \in D_A\} \Delta A|}{\max(A)} < \epsilon.$$

Let

$$D'_A = \bigcap_{\lambda=1}^l (-n_\lambda + D_A) \subseteq \mathbb{Z}/k\mathbb{Z}.$$

Fix an arbitrary $i \in \mathbb{Z}$. Let $B = \{0 \leq n \leq \max(A) : [n]_k \in D'_A\}$ and $B_i = \{0 \leq n \leq \max(A) : n + a_{i,A} \in M'\}$. We claim that

$$\frac{|B \Delta B_i|}{\max(A)} \leq (l + 1)\epsilon.$$

To see this, let $\delta = \max(|n_1|, \dots, |n_l|)$, $C = \{\delta \leq n \leq \max(A) - \delta : [n]_k \in D'_A\}$, and $C_i = \{\delta \leq n \leq \max(A) - \delta : n + a_{i,A} \in M'\}$. Note that

$$|(C \Delta C_i) \Delta (B \Delta B_i)| \leq 2\delta \leq \epsilon \max(A).$$

It is therefore sufficient to verify that $|C \Delta C_i| \leq l\epsilon \max(A)$. Observe that if $\delta \leq n \leq \max(A) - \delta$ and $1 \leq \lambda \leq l$, then $n + a_{i,A} + n_\lambda \in M$ iff $n + n_\lambda \in A$. Fix any n with $\delta \leq n \leq \max(A) - \delta$. If $n + a_{i,A} \in M'$ and $[n]_k \notin D'_A$, then there exists $1 \leq \lambda \leq l$ such that $n + n_\lambda \in A$ and $[n + n_\lambda]_k \notin D_A$. Similarly, if $n + a_{i,A} \notin M'$ and $[n]_k \in D'_A$, then there exists $1 \leq \lambda \leq l$ such that $n + n_\lambda \notin A$ and $[n + n_\lambda]_k \in D_A$. In either case, we have

$$n + n_\lambda \in \{0 \leq m \leq \max(A) : [m]_k \in D_A\} \Delta A.$$

Now $|C \Delta C_i| \leq l\epsilon \max(A)$ follows from the fact that we have l -many different possibilities for λ and

$$|\{0 \leq m \leq \max(A) : [m]_k \in D_A\} \Delta A| \leq \epsilon \max(A).$$

This proves the claim.

Let $M'' = \bigcup_{i \in \mathbb{Z}} (B + a_{i,A})$. We next claim that for every $N \geq 2 \max(A)/\epsilon$, we have

$$\frac{|\{n \in M' : 1 \leq n \leq N\} \Delta \{n \in M'' : 1 \leq n \leq N\}|}{N} \leq (l + 1)\epsilon + \epsilon.$$

To see this, let $s = \min\{i : 1 \leq a_{i,A} \leq N\}$ and $r = \max\{i : 1 \leq a_{i,A} \leq N\}$. Then $(r - s) \max(A) \leq N$. Note that

$$M' \cap [1, N] = \bigcup_{i=s}^{r-1} (B_i + a_{i,A}) \cup D_0$$

and

$$M'' \cap [1, N] = \bigcup_{i=s}^{r-1} (B + a_{i,A}) \cup D_1$$

for some $D_0, D_1 \subseteq [1, a_{s,A}] \cup [a_{r,A}, N]$. Thus by the preceding claim we have

$$\begin{aligned} |(M' \cap [1, N]) \Delta (M'' \cap [1, N])| &\leq 2 \max(A) + \sum_{i=s}^{r-1} |B \Delta B_i| \\ &\leq 2 \max(A) + (r - s)(l + 1)\epsilon \max(A) \\ &\leq \epsilon N + (l + 1)\epsilon N = (l + 2)\epsilon N. \end{aligned}$$

It now follows that

$$\left| \frac{1}{N} \sum_{\substack{n \in M' \\ 1 \leq n \leq N}} \mu(n) \right| \leq (l + 2)\epsilon + \left| \frac{1}{N} \sum_{\substack{n \in M'' \\ 1 \leq n \leq N}} \mu(n) \right|.$$

We conclude the proof by showing that, for $N > 3 \max(A)/\epsilon$,

$$\left| \sum_{\substack{n \in M'' \\ 1 \leq n \leq N}} \mu(n) \right| \leq \epsilon N + \epsilon N + \sum_{j=0}^{N/Lk} \sum_{a=0}^{k-1} \left| \sum_{\substack{m \in [z+jLk, z+(j+1)Lk) \\ m \equiv a \pmod k}} \mu(m) \right| \leq 3\epsilon N.$$

To see this, for each $s \leq i \leq r$, let $t_i = \max\{j \in \mathbb{Z} : z + jLk \leq a_{i,A}\}$ and $u_i = \max\{j \in \mathbb{Z} : z + jLk \leq a_{i,A} + \max(A)\}$. Then in the above inequality the first error term of ϵN allows us to consider, instead of $M'' \cap [1, N]$, the set $M'' \cap [z + t_s Lk, z + t_r Lk)$, since the difference is bounded by $2 \max(A) + Lk \leq 3 \max(A) \leq \epsilon N$. Since $N > 3 \max(A)/\epsilon > N_0$, we may apply Theorem 1.10 to get the third term of the above inequality, which is an over-estimate of the sum

$$\left| \sum_{n \in M'' \cap [z+t_s Lk, z+t_r Lk)} \mu(n) \right|$$

except over the intervals $[z + t_i Lk, z + (t_i + 1)Lk)$ and $[z + u_i Lk, z + (u_i + 1)Lk)$ for $s \leq i < r$. Finally, the total error on these intervals is bounded by $(r - s)2Lk \leq [N/\max(A)]2Lk \leq \epsilon N$, which gives the second error term of ϵN . \square

Corollary 2.3. *Let $M \subseteq \mathbb{Z}$ be an accc. Then M is orthogonal to the Möbius function.*

Proof. This is a direct consequence of Theorem 2.2 with $l = 1$ and $n_1 = 0$. \square

Theorem 2.4. *Let (X, T) be an accc rank-one subshift. Then (X, T) is Möbius disjoint.*

Proof. Let

$$F = \{f_{n_1} f_{n_2} \cdots f_{n_l} : l \in \mathbb{N} \text{ and } n_1, n_2, \dots, n_l \in \mathbb{Z}\} \cup \{f_{\text{const}}\},$$

where $f_n : X \rightarrow \mathbb{R}$ is the projection onto the n -th coordinate and $f_{\text{const}} : X \rightarrow \mathbb{R}$ is the constant function $f_{\text{const}}(x) = 1$. Note that F separates points since for every $x, y \in X$ with $x \neq y$ there exists $n \in \mathbb{Z}$ such that $f_n(x) = x(n) \neq y(n) = f_n(y)$. Since F contains a nonzero constant function, by the Stone–Weierstrass Theorem the algebra generated by functions in F is dense in the space of all continuous functions on X with the uniform convergence topology. Furthermore, since Sarnak’s property is closed under taking the limit with uniform convergence topology, it is enough to show Sarnak’s property for every continuous function in the algebra (closed under taking linear combinations and multiplication) generated by F . Note that since F is closed under multiplication, the algebra generated by F is equal to

$$\{c_1g_1 + \dots + c_lg_l : l \in \mathbb{N}, c_1, c_2, \dots, c_l \in \mathbb{R}, \text{ and } g_1, g_2, \dots, g_l \in F\}.$$

We next show Sarnak’s property for $f = f_{n_1}f_{n_2} \dots f_{n_l} \in F$. Let $x \in X$. Then $M_x = \{n \in \mathbb{Z} : x(n) = 0\}$ is an acc. For any subset $I \subseteq \{1, \dots, l\}$, say $I = \{i_1, \dots, i_p\}$, and for $N \in \mathbb{N}$, we have

$$\frac{1}{N} \sum_{\substack{n+n_{i_1}, \dots, n+n_{i_p} \in M_x \\ 1 \leq n \leq N}} \mu(n) = \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)(1 - x(n + n_{i_1})) \dots (1 - x(n + n_{i_p})),$$

which approaches 0 as $N \rightarrow \infty$ by Theorem 2.2. Now observe

$$\begin{aligned} & \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)f_{n_1}(T^n x)f_{n_2}(T^n x) \dots f_{n_l}(T^n x) \\ &= \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)x(n + n_1)x(n + n_2) \dots x(n + n_l) \\ &= \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)(1 - (1 - x(n + n_1))) \dots (1 - (1 - x(n + n_l))) \\ &= \sum_{\substack{I \subseteq \{1, \dots, l\} \\ I = \{i_1, \dots, i_p\}}} (-1)^p \frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)(1 - x(n + n_{i_1})) \dots (1 - x(n + n_{i_p})). \end{aligned}$$

Thus, by applying Theorem 2.2 as above 2^l -many times, we get

$$\frac{1}{N} \sum_{1 \leq n \leq N} \mu(n)f_{n_1}(T^n x)f_{n_2}(T^n x) \dots f_{n_l}(T^n x) \rightarrow 0.$$

Finally, we show Sarnak’s property for $f = c_1g_1 + \dots + c_lg_l$ assuming that each g_i satisfies Sarnak’s property. Here, for every $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for every $N \geq N_0$ we have

$$\left| \frac{1}{N} \sum_{n \leq N} \mu(n)f(T^n x) \right| = \left| \sum_{i=1}^l c_i \frac{1}{N} \sum_{n \leq N} \mu(n)g_i(T^n x) \right| \leq \left(\sum_{i=1}^l |c_i| \right) \epsilon.$$

□

Corollary 2.5. *Let (X, μ, T) be a symbolic rank-one transformation that is measure theoretically isomorphic to an odometer. Then the rank-one subshift (X, T) is Möbius disjoint.*

Proof. This is a direct consequence of Theorem 1.6(b) and Theorem 2.4. □

We note that Corollary 2.5 also follows from Theorem 1.2 in [10]. In fact, Theorem 1.2 in [10] shows Sarnak's conjecture for all topological dynamical systems (X, T) such that every invariant Borel probability measure on X has discrete spectrum. Our method is different from the approach in [10] and allows us to verify Sarnak's conjecture for accc rank-one subshifts.

Also, we note that Sarnak's conjecture is preserved under topological isomorphisms but not necessarily under measure-theoretical isomorphisms. In [5], the authors show Sarnak's conjecture for specific strictly ergodic topological dynamical systems that are models of (measure theoretically isomorphic to) an odometer.

3. GENERALIZED KATOK'S MAPS

In this section we verify Sarnak's conjecture for a class of rank-one subshifts which generalize Katok's map studied in [2]. We first define this class. Recall that a generating sequence $\{v_n\}_{n \in \mathbb{N}}$ of a rank-one subshift is defined recursively from the cutting parameter $\{r_n\}_{n \in \mathbb{N}}$ and the spacer parameter $\{s_{n,i}\}_{n \in \mathbb{N}, 0 < i \leq r_n}$ by $v_0 = 0$ and

$$v_{n+1} = v_n 1^{s_{n,1}} v_n 1^{s_{n,2}} \dots v_n 1^{s_{n,r_n}}$$

for $n \in \mathbb{N}$. For each integer $m \geq 2$, let \mathcal{K}_m be the set of all infinite rank-one words $V \in 2^{\mathbb{N}}$ with generating sequences $\{v_n\}_{n \in \mathbb{N}}$ such that there are natural numbers $0 \leq t_{n,1}, t_{n,2}, \dots, t_{n,m} \leq m - 1$ for each $n \in \mathbb{N}$, satisfying

- (1) r_n is divisible by m .
- (2) $s_{n,i} = t_{n, \lceil \frac{m}{r_n} i \rceil}$ for $0 < i \leq r_n$.
- (3) $\liminf_{n \rightarrow \infty} \frac{\log \log(r_n)}{\log \log \log(|v_n|)} \geq 2$.

Note that the original Katok's map is a special case in \mathcal{K}_2 , and Condition (3) is weaker than the condition in [2] which requires $\lim_{n \rightarrow \infty} r_n/|v_n| = +\infty$. Let $\mathcal{K} = \bigcup_{m \geq 2} \mathcal{K}_m$. We show Sarnak's conjecture for (X_V, T) for all $V \in \mathcal{K}$.

Theorem 3.1. *Let $V \in \mathcal{K}$, $x \in X_V$, and n_1, n_2, \dots, n_l be integers. Let $M_x = \{n \in \mathbb{Z} : x(n) = 0\}$ and*

$$M'_x = \{n \in \mathbb{Z} : n + n_1, n + n_2, \dots, n + n_l \in M_x\}.$$

Then M'_x is orthogonal to the Möbius function.

Proof. Fix $m \geq 2$ such that $V \in \mathcal{K}_m$. We may assume M_x is nonempty since otherwise M'_x is the empty set and therefore orthogonal to the Möbius function. By Condition (3) of the definition of \mathcal{K}_m , there exists $N_0 \in \mathbb{N}$ such that for every $n \geq N_0$ we have $\log \log(r_n) \geq 2 \log \log \log(|v_n|)$. For each $n \in \mathbb{N}$, let $A_n = \{0 \leq i < |v_n| : v_n(i) = 0\}$. Then $0 \in A_n$, A_n is a building block of M_x and we have

$$\max(A_n) = |v_n| - t_{n-1,m} - t_{n-2,m} - \dots - t_{0,m} - 1 = |v_n| - O_m(1).$$

For every $n \in \mathbb{N}$ fix a nondecreasing sequence $\{a_{j,A_n}\}_{j \in \mathbb{Z}}$ of integers such that $M_x = \bigcup_{j \in \mathbb{Z}} (A_n + a_{j,A_n})$ and that $a_{j+1,A_n} - a_{j,A_n} > \max(A_n)$ or $a_{j+1,A_n} = a_{j,A_n}$, for every $j \in \mathbb{Z}$.

Let $\delta = \max(|n_1|, \dots, |n_l|)$. For a moment, fix $j \in \mathbb{Z}$ and $n \in \mathbb{N}$, and consider the set $A_{n+1} + a_{j,A_{n+1}}$, which is one of the translations of the building block A_{n+1} in M_x . By the definition of \mathcal{K}_m , we can write

$$v_{n+1} = (v_n 1^{t_{n,1}})^{\frac{r_n}{m}} (v_n 1^{t_{n,2}})^{\frac{r_n}{m}} \dots (v_n 1^{t_{n,m}})^{\frac{r_n}{m}} = u_{n,1} u_{n,2} \dots u_{n,m},$$

where $u_{n,k} = (v_n 1^{t_{n,k}})^{\frac{r_n}{m}}$ for $1 \leq k \leq m$. For each $0 \leq \ell < m$, let $\sigma_{n,\ell}$ be the starting position of $u_{n,\ell+1}$ in v_{n+1} , that is,

$$\sigma_{n,\ell} = (|v_n| + t_{n,1} + \dots + |v_n| + t_{n,\ell}) \frac{r_n}{m},$$

and let $\sigma_{n,m} = |v_{n+1}|$. We say $B \subseteq \mathbb{Z}$ is a *complete congruency class mod q* (ccc mod q) on an interval $[a, b)$ if for every $n \in [a, b - q)$, $n \in B$ iff $n + p \in B$. Then for each $0 \leq \ell < m$, A_{n+1} is a ccc mod $|v_n| + t_{n,\ell+1}$ on $[\sigma_{n,\ell}, \sigma_{n,\ell+1})$, and $A_{n+1} + a_{j,A_{n+1}}$ is a ccc mod $|v_n| + t_{n,\ell+1}$ on $[a_{j,A_{n+1}} + \sigma_{n,\ell}, a_{j,A_{n+1}} + \sigma_{n,\ell+1})$. Since $M_x \cap [a_{j,A_{n+1}}, a_{j+1,A_{n+1}}) = A_{n+1} + a_{j,A_{n+1}}$, it follows that M_x is a ccc mod $|v_n| + t_{n,\ell+1}$ on $[a_{j,A_{n+1}} + \sigma_{n,\ell}, a_{j,A_{n+1}} + \sigma_{n,\ell+1})$, and that M'_x is a ccc mod $|v_n| + t_{n,\ell+1}$ on $[a_{j,A_{n+1}} + \sigma_{n,\ell} + \delta, a_{j,A_{n+1}} + \sigma_{n,\ell+1} - \delta)$.

For any $n \in \mathbb{N}$ and $0 \leq \ell < m$, let

$$C_{n,\ell} = \bigcup_{j \in \mathbb{Z}} (M'_x \cap [a_{j,A_{n+1}} + \sigma_{n,\ell} + \delta, a_{j,A_{n+1}} + \sigma_{n,\ell+1} - \delta)),$$

$q_{n,\ell} = |v_n| + t_{n,\ell+1}$, and $\lambda_{n,\ell} = q_{n,\ell} \frac{r_n}{m} - 2\delta$. Then each $C_{n,\ell}$ is a union of arithmetic progressions each of which has approximately r_n/m -many terms of common difference $q_{n,\ell}$, contained within an interval of length no longer than $\lambda_{n,\ell}$. Let $C_n = \bigcup_{\ell=0}^{m-1} C_{n,\ell}$. We will find large enough n to apply Theorem 1.10 to each $C_{n,\ell}$.

Fix $0 < \epsilon < \frac{1}{100}$ and a bound L_0 corresponding to ϵ given by Theorem 1.10. For each $n \geq N_0$, we first find $L_n \geq L_0$ such that for all $0 \leq \ell < m$,

$$\sum_{\substack{p|q_{n,\ell} \\ p \text{ prime}}} \frac{1}{p} \leq (1 - \epsilon) \sum_{\substack{p \leq L_n \\ p \text{ prime}}} \frac{1}{p}.$$

By [12, Lemma 3.2] we have

$$\sum_{\substack{p|q \\ p \text{ prime}}} \frac{1}{p} \leq \log \log \log(q) + O(1)$$

and from [3, Theorem 4.12] we have

$$\sum_{\substack{p \leq L \\ p \text{ prime}}} \frac{1}{p} = \log \log(L) + O(1).$$

Since $\log \log \log(q_{n,\ell}) = \log \log \log(|v_n|) + O(1)$ and $\log \log(r_n) \geq 2 \log \log \log(|v_n|)$, there exists $N_1 \geq N_0$ such that, for every $n \geq N_1$, we have $|v_n| \geq m$ and, setting $L_n = \epsilon r_n / 4m$, $L_n \geq L_0$. Now $\log \log(L_n) = \log \log(r_n) + O_\epsilon(1)$ and

$$\sum_{\substack{p|q_{n,\ell} \\ p \text{ prime}}} \frac{1}{p} \leq (1 - \epsilon) \sum_{\substack{p \leq L_n \\ p \text{ prime}}} \frac{1}{p}$$

for all $0 \leq \ell < m$. Note that $L_n \geq L_0$ but

$$L_n q_{n,\ell} \leq \epsilon \frac{r_n}{4m} (|v_n| + m) \leq \epsilon \frac{r_n |v_n|}{2m} \leq \epsilon \frac{|v_{n+1}|}{2m}$$

for all $0 \leq \ell < m$.

Finally let $L = L_{N_1}$ and let

$$N_2 = \max\{N_1, N_0(q_{N_1,\ell}, L) : 0 \leq \ell < m\},$$

where $N_0(\cdot, \cdot)$ is given by Theorem 1.10. Denote $H = |v_{N_1+1}|$ and let $N \geq 2N_2H/\epsilon > N_2$. By Theorem 1.10, for each $p = 0, \dots, m - 1$ and $q = |v_{N_1}| + p$ we have

$$\sum_{j=0}^{N/Lq} \sum_{a=0}^{q-1} \left| \sum_{\substack{i \in [z_q + jLq, z_q + (j+1)Lq] \\ i \equiv a \pmod q}} \mu(i) \right| \leq \epsilon N.$$

On the other hand, let $s = \min\{j \in \mathbb{Z} : 1 \leq a_{j, A_{N_1+1}} \leq N\}$, $r = \max\{j \in \mathbb{Z} : 1 \leq a_{j, A_{N_1+1}} \leq N\}$, $E_\ell = C_{N_1, \ell} \cap [a_{s, A_{N_1+1}}, a_{r, A_{N_1+1}})$, and $E = \bigcup_{\ell=0}^{m-1} E_\ell$. Then the sum

$$\sum_{q=|v_{N_1}|}^{|v_{N_1}|+m-1} \sum_{j=0}^{N/Lq} \sum_{a=0}^{q-1} \left| \sum_{\substack{i \in [z_q + jLq, z_q + (j+1)Lq] \\ i \equiv a \pmod q}} \mu(i) \right|$$

is an over-estimate of

$$\left| \sum_{\substack{i \in E \\ 1 \leq i \leq N}} \mu(i) \right|$$

with an error no bigger than

$$m(r - s)2L(|v_{N_1}| + m) \leq m \frac{N}{H} 2\epsilon \frac{H}{2m} = \epsilon N.$$

To justify this error estimate, note that each application of Theorem 1.10 with $q = |v_{N_1}| + p$ where $p = q_{N_1, \ell}$ gives an over-estimate of

$$\left| \sum_{\substack{i \in E_p \\ 1 \leq i \leq N}} \mu(i) \right|$$

with an error occurring near each end of the interval

$$[a_{j, A_{N_1+1}} + \sigma_{N_1, \ell} + \delta, a_{j, A_{N_1+1}} + \sigma_{N_1, \ell+1} - \delta)$$

within an interval of length $Lq_{N_1, \ell}$.

Finally, since $|(M'_x \cap [1, N]) \Delta E| \leq 2H \leq \epsilon N$, we obtain

$$\left| \sum_{\substack{i \in M'_x \\ 1 \leq i \leq N}} \mu(i) \right| \leq \epsilon N + \left| \sum_{\substack{i \in E \\ 1 \leq i \leq N}} \mu(i) \right| \leq \epsilon N + m\epsilon N + \epsilon N = (m + 2)\epsilon N.$$

□

With an argument similar to the proof of Theorem 2.4, we obtain Corollary 3.2 of Theorem 3.1.

Corollary 3.2. *Let $V \in \mathcal{K}$. Then (X_V, T) is Möbius disjoint.*

4. FURTHER GENERALIZATIONS

In this last short section we note that the results in Section 3 can be generalized further. We use the same notation from previous sections for rank-one subshifts. In particular, let $A_n = \{0 \leq i \leq |v_n| - 1 : v_n(i) = 0\}$. In general, our techniques can only be applied in case there exists $m \in \mathbb{N}$ such that A_n for arbitrarily large n can be approximated with a union of long arithmetic progressions with at most m -many common differences. In order to apply Theorem 1.10, these common differences and the lengths of the arithmetic progressions need to be constrained by (1.3), which

usually results in some growth conditions on the cutting parameter and moderation (or bounded) conditions on the spacer parameter of the rank-one subshift.

Here we specify one concrete class of rank-one subshifts that is broader than \mathcal{K} and satisfies Sarnak’s conjecture. Define

$$C_n = \{1, r_n\} \cup \{2 \leq i \leq r_n - 1 : s_{n,i-1} \neq s_{n,i}\}$$

and enumerate the members of C_n in increasing order as $c_{n,1}, c_{n,2}, \dots, c_{n,p_n}$. The arguments in Section 3 can be repeated to show Sarnak’s conjecture for (X_V, T) under the following conditions:

- (i) $\lim_{n \rightarrow \infty} \frac{\log \log(r_n/p_n)}{\log \log(r_n)} = 1,$
- (ii) $K = \limsup_{n \rightarrow \infty} \frac{\sum_{i=1}^{r_n} s_{n,i}}{r_n |v_n|} < +\infty,$
- (iii) $\limsup_{n \rightarrow \infty} \frac{\log \log(r_n)}{\log \log \log(|v_n|)} > 1,$ and
- (iv) there exists $m \in \mathbb{N}$ such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geq N$ there exists $A \subseteq [1, r_n]$ with $|A| \geq (1-\epsilon)r_n$ and $|\{s_{n,a} : a \in A\}| \leq m.$

We sketch the proof to illustrate how the conditions are applied. Fix $0 < \epsilon < \frac{1}{100}$. The strategy is to approximate A_{n+1} (with a suitably defined, large enough n) with a union of arithmetic progressions with common difference $q_i = |v_n| + s_{n,c_{n,i}}$ and length $L_i = c_{n,i+1} - c_{n,i}$ such that (1.3) is satisfied. Note that condition (i) lets us guarantee that $L_i \geq \sqrt{\frac{r_n}{p_n}}$ by removing at most $\epsilon|v_{n+1}|$ many points, that is, we may assume that $\log \log(r_n) - \log \log(L_i) = O(1)$. Condition (ii) lets us guarantee $q_i \leq |v_n|^2$ by removing at most $\epsilon|v_{n+1}|$ many points, that is, we may assume $\log \log \log(|v_n|) - \log \log \log(q_i) = O(1)$. In light of conditions (i) and (ii), condition (iii) lets us choose n large enough such that (1.3) holds for every i with $q = q_i$ and $L = L_i$. Finally, condition (iv) guarantees that there are at most m -many different values of q_i (and therefore we only need to apply Theorem 1.10 m -many times) by removing at most $\epsilon|v_{n+1}|$ many points.

Conditions (i)–(iv) define a class of rank-one subshifts that is more general than \mathcal{K} . In addition, they also include rank-one subshifts correspondent to certain flat stacks. A flat stack is a rank-one transformation T on a probability measure space (X, μ) (see [6, Definition 2]) with the extra condition that for every $\epsilon > 0$ we can choose F in Definition 2 in [6] such that $\mu(T^h F \Delta F) \leq \epsilon \mu(F)$. In particular, if (X_V, T) is a rank-one subshift correspondent to a flat stack then

$$\lim_{n \rightarrow \infty} \frac{|\{1 \leq i \leq r_n : s_{n,i} \neq 0\}|}{r_n} = 0.$$

Thus, it satisfies conditions (ii) and (iv).

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