HOLOMORPHIC SUPPORT FUNCTIONS FOR UNIFORMLY
PSEUDOCONVEX HYPERSURFACES, WITH AN APPLICATION
TO CR MAPS

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Abstract. We construct holomorphic support functions for smooth weakly
pseudoconvex hypersurfaces with Levi form of constant rank. These are then
applied to show that formal holomorphic curves which are tangential to in-
finitesimal order to such a hypersurface must be formally contained in its Levi
foliation. As a consequence, we obtain a holomorphic deformation theorem
for nowhere smooth CR maps into smooth pseudoconvex hypersurfaces with
one-dimensional Levi foliation, strengthening a very general result of Lamel
and Mir about formal deformations in this particular case.

1. Introduction

In the present paper, we are concerned with smooth real hypersurfaces in complex
Euclidean space, i.e. subsets $M \subset \mathbb{C}^n$ such that for any $p \in M$, there exists an open
neighborhood $O \subset \mathbb{C}^n$ of $p$ and a function $\rho \in C^\infty(O)$ (a defining function) with
$M \cap O = \rho^{-1}(0)$ and $d\rho(p) \neq 0$.

We denote by $T^{1,0}M$ and $T^{0,1}M$ the bundles of holomorphic and antiholomorphic
vectors tangential to $M$, respectively. Given a vector bundle $B$ over $M$, the space
of its sections shall be denoted by $\Gamma(B)$, e.g. $\Gamma(T^{0,1}M)$ for the space of CR vector
fields on $M$. All theorems in this paper are local in nature, and all sections may be
assumed to exist only on a small neighborhood of the point of interest.

Given a smooth real hypersurface $M \subset \mathbb{C}^n$ with local defining function $\rho$, its Levi
form, a smooth hermitian form $\mathcal{L}_\rho: \Gamma(T^{0,1}M) \times \Gamma(T^{0,1}M) \to C^\infty(M)$, is defined by

$$\mathcal{L}_\rho(\bar{V}, \bar{W}) = \sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial \bar{z}^j \partial z^k} \bar{V}^j W^k,$$

for $\bar{V} = \sum_{j=1}^n \bar{V}^j \frac{\partial}{\partial \bar{z}^j}$ and $\bar{W} = \sum_{j=1}^n \bar{W}^j \frac{\partial}{\partial \bar{z}^j}$ sections of $T^{0,1}M$. The Levi form is
tensorial: The value of $\mathcal{L}_\rho(\bar{V}, \bar{W})$ at $p \in M$ depends only on the values of $\bar{V}$
and $\bar{W}$ at $p$. The kernel of the Levi form at $p$, i.e. those $\bar{V} \in T^{0,1}_p M$ such that
$\mathcal{L}_\rho(\bar{V}, \bar{W}) = 0$ for all $\bar{W} \in T^{0,1}_p M$, shall be denoted by $\mathcal{N}_p$. Choosing a different
defining function only multiplies the Levi form by a smooth scalar and therefore,
up to a sign, does not affect its signature. The Levi kernel $\mathcal{N}_p$ likewise does not
depend on the choice of defining function.
If there exists a defining function such that the Levi form of $M$ is positive semi-
definite or definite, we call $M$ pseudoconvex or strongly pseudoconvex, respectively.
If the Levi form has a nontrivial kernel of constant dimension, we call $M$ uniformly
Levi-degenerate, and uniformly pseudoconvex, if $M$ is additionally pseudoconvex.
The key property of uniformly Levi-degenerate hypersurfaces is the existence of the
Levi foliation, a foliation of $M$ by complex manifolds $(\eta_p)_{p \in M}$ such that $T^{0,1}_p \eta_p = N_p$
for all $p \in M$, found by Sommer [9] and later extended to a more general setting
by Freeman [2].

The first main result of this paper is the following construction of a local holo-
morphic support function for uniformly pseudoconvex hypersurfaces. Here, a local
holomorphic support function for a hypersurface $M$ at a point $p \in M$ means a holo-
morphic function $h$, defined on a neighborhood $U \subseteq \mathbb{C}^n$ of $p$, such that $h(p) = 0$
and $\Im(h(q)) \geq 0$ for all $q \in M \cap U$.

**Theorem 1.1.** Let $M \subseteq \mathbb{C}^n$ be a uniformly pseudoconvex hypersurface. For any
point $p \in M$, there exists a coordinate system $(z^j)_{j=1}^n$ on a neighborhood $U \subseteq \mathbb{C}^n$ of
$p$ such that $\Im(z^n(q)) \geq 0$ for all $q \in M \cap U$, with equality if and only if $q \in \eta_p \cap U$.

Uniformly pseudoconvex hypersurfaces bridge two classes of hypersurfaces for
which the existence of holomorphic support functions is well known: strongly pseudo-
convex hypersurfaces, whose Levi kernel is trivial, and Levi-flat hypersurfaces,
whose Levi kernel is as large as possible. This dichotomy was already noted by
Kohn–Nirenberg [4] in a landmark paper which constructs a (not uniformly) pseudo-
convex hypersurface which does not admit a holomorphic support function.

Closely related to holomorphic support functions are holomorphic peak functions,
which are defined on one side of $M$, extend continuously to $M$ and reach their
maximum modulus at $p \in M$. For uniformly pseudoconvex hypersurfaces, local
holomorphic peak functions were constructed by Fu [3, Corollary 3.3], with the
aim of obtaining asymptotics for the Bergman kernel of uniformly pseudoconvex
domains. These peak functions extend continuously to $M$, but not across $M$, since
their construction involves a branch cut on one side of $M$. In our application, it will
be crucial that the support function constructed in Theorem 1.1 is holomorphic in
a whole neighborhood of $p \in M$.

The coordinate system from Theorem 1.1 allows one to carry out the analysis
necessary to prove a smooth deformation theorem akin to the analytic deformation
theorem in [8], something not previously achieved for large classes of merely smooth
(as opposed to real analytic) hypersurfaces.

**Theorem 1.2.** Let $M' \subseteq \mathbb{C}^{n'}$ be a pseudoconvex hypersurface whose Levi form has
a one-dimensional kernel everywhere. Let $M \subseteq \mathbb{C}^n$ be a minimal CR submanifold.
Suppose a CR map $h : M \to M'$ of regularity $C^{n'-1}$ is nowhere smooth. Then there
exists a dense open subset $\Omega \subseteq M$ such that near any $p \in \Omega$, $h$ may be written as
$h(q) = \Phi(q, \psi(q))$ where $\Phi : M \times \mathbb{C} \to M'$ is a smooth CR map defined near $(p, 0)$
and $\psi : M \to \mathbb{C}$ is a nowhere smooth CR function defined near $p$.

Here, minimal means that at no point $p \in M$ does there exist a germ $N_p \subset M$
of a submanifold of strictly smaller dimension such that $T^{0,1}_p M|_{N_p} = T^{0,1}N_p$.
Theorem 1.2 is proven using a powerful formal deformation theorem of Lamel–Mir [6]. The challenge is to show that the formal deformation constructed there
converges, and yields a smooth deformation. The hypothesis in Theorem 1.2 of a
One-dimensional Levi kernel, and therefore a one-dimensional Levi foliation, constrains this formal deformation almost entirely: Using Theorem \[1\] we show each fiber of the deformation formally parametrizes a leaf of the Levi foliation, and thus can be reparametrized to converge.

2. **Uniformly Pseudoconvex Hypersurfaces**

Freeman’s proof in [2] of the existence of the Levi foliation allows one to construct a local parametrization of $M$ adapted to it:

**Lemma 2.1.** Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, and suppose that the kernel of its Levi form is of constant rank $k$ near a point $p \in M$. Then there exists a smooth parametrization $\phi : \mathbb{R}^{2n-2k-1} \times \mathbb{C}^k \supseteq U \times V \to M$ $(r, t) \mapsto \phi(r, t)$ of a neighborhood of $p$ in $M$ such that $\phi(r, t)$ is holomorphic in $t$ for fixed $r$. Furthermore, after a permutation of coordinates, we may assume $t^j = z^{-k+j} \circ \phi(r, t)$ for $j = 1, \ldots, k$.

**Proof.** Freeman’s 1976 construction of the Levi foliation shows that the sections $\Gamma(\mathcal{N} \oplus \overline{\mathcal{N}})$ of the Levi kernel are closed under taking Lie brackets, and hence give rise to a foliation of $M$ by real submanifolds via the real-variable Frobenius Theorem. Furthermore, since their tangent spaces, given by the real part of the Levi kernel, are invariant under the complex structure on $T\mathbb{C}^n$, the leaves of this foliation are in fact complex submanifolds of $\mathbb{C}^n$.

After a permutation of coordinates, we may assume that the coordinate subspace $\Sigma_p$ given by $\{z \in \mathbb{C}^n : z^j = z^j(p), j = n - k + 1, \ldots, n\}$ is transverse to $\eta_p$, the leaf of the Levi foliation through $p$. This means that $z^{-k+1}, \ldots, z^n$ form a coordinate system on $\eta_p$ near $p$. Since transversality is an open condition, the same holds on $\eta_q$ for any $q$ in some open neighborhood $\tilde{U}$ of $p$ in $\Sigma_p \cap M$. We now write $\eta_q, q \in \tilde{U}$, as graphs over $\{z \in \mathbb{C}^n : z^j = z^j(p), j = 1, \ldots, n - k\} \cong \mathbb{C}^k$. We obtain, for each $q \in \tilde{U}$, a parametrization $\tilde{\phi}_q : \mathbb{C}^k \supseteq V \to \eta_q$ such that $t^j = z^{-k+j} \circ \tilde{\phi}_q(t)$, where $V \subseteq \mathbb{C}^k$ is some small open neighborhood of $(z^{-k+1}(p), \ldots, z^n(p))$. Since $\eta_q$ is a complex manifold, $\tilde{\phi}_q(t)$ is holomorphic in $t$.

Next, we parametrize $\tilde{U} \subseteq M \cap \Sigma_p$ by a smooth map $f : \mathbb{R}^{2n-2k-1} \supseteq U \to \tilde{U}$, and define $\phi : \mathbb{R}^{2n-2k-1} \times \mathbb{C}^k \supseteq U \times V \to M$ by $\phi(r, t) = \tilde{\phi}_f(r)(t)$. The map $\phi$ is precisely the parametrization of $M$ adapted to the foliation $\eta$ which is constructed in the standard proof of the Frobenius theorem (see, e.g., [7 Theorem 19.12]), and therefore smooth. By the previous paragraph, it is holomorphic in $t$, and satisfies $t^j = z^{-k+j} \circ \phi(r, t)$ for $j = 1, \ldots, k$. \[\square\]

It may be impossible to holomorphically straighten out more than one leaf of the Levi foliation. However, the following, additional “partial” straightening can be achieved. It is useful to think of this as straightening out not only the Levi foliation’s leaf through a point $p \in M$, but rather the whole Segre variety through $p$, which however only really exists if $M$ is analytic. (Recall that the Segre variety of a real analytic hypersurface $M$ at $p$ is the set $\{z \in \mathbb{C}^n : \rho(z, \bar{z}) = 0\}$, where $\rho(z, \bar{z})$ is an arbitrary real analytic defining function of $M$ near $p$. See [3] Chapter 10.2 for a detailed treatment of Segre varieties.)

This construction actually holds in a more general setting than that of uniformly pseudoconvex hypersurfaces. A CR submanifold $N \subset M$ is called CR-transversal,
if $CN + T^{1,0}M + T^{0,1}M = CTM$. To construct the convenient coordinates of Lemma 2.2, we only require $M$ to contain a Levi-flat CR-transversal CR submanifold. Such a submanifold exists, in particular, if $M$ is uniformly pseudoconvex. To construct it, first take a CR-transversal curve $\gamma : (\varepsilon, \varepsilon) \rightarrow M$ with $\gamma(0) = p$. Then there exists a neighborhood $U \subseteq M$ of $p$ such that the connected components of $\gamma(t) \cap U$ containing $\gamma(t)$ line up to form a Levi-flat CR-transversal CR submanifold.

**Lemma 2.2.** Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface with defining function $\rho$.

(a) Suppose that there exists a $k$-dimensional complex manifold $\eta_p \subset M$ through a point $p \in M$ with $N_q = T_q^{0,1}M$ for all $q \in \eta_p$ near $p$. Then there exist holomorphic coordinates $(z_j)_{j=1}^n$ on a neighborhood $U \subset \mathbb{C}^n$ of $p$, such that

1. $z(p) = 0$, $z(\eta_p) = \{z \in \mathbb{C}^n : z^{k+1} = \cdots = z^n = 0\} \cap z(U)$, and

2. $\partial \rho(z, \bar{z}) = \lambda(z, \bar{z})dz^n$ on $\eta_p$, for a smooth function $\lambda : \eta_p \rightarrow \mathbb{C}$.

(b) Suppose now that $\eta_p$ is contained in a $2k + 1$-dimensional Levi-flat CR-transversal CR submanifold $N \subset M$ of hypersurface type. Then one may choose a possibly different local defining function $\tilde{\rho}$ of $M$ near $p$, together with coordinates as in (a), such that $\partial \tilde{\rho} = dz^n$ on $\eta_p$.

**Proof.** (a) Since $\eta_p$ is a complex manifold, we may choose local holomorphic coordinates straightening out $\eta_p$, such that (1) is satisfied immediately. Consider an arbitrary point $q \in \eta_p$ and an arbitrary vector $V = \sum_{j=1}^n V_j \frac{\partial}{\partial z^j} \in T_q^{1,0}M$. Because $\frac{\partial}{\partial z^j}|_q$ for any $1 \leq l \leq k$ lies in the Levi kernel $N_q$, we have

$$0 = \mathcal{L}_V \left( \frac{\partial}{\partial z^j}|_q, V \right) = \sum_{j=1}^N \frac{\partial^2 \rho}{\partial z^j \partial z^j}(q)V_j.$$  

As $q, V$ were arbitrary, the form $\sum_{j=1}^n \frac{\partial^2 \rho}{\partial z^j \partial z^j}dz^j$ annihilates $T_q^{1,0}M$ on $\eta_p$ and must thus be linearly dependent to $\partial \rho$. Writing $\partial \rho = \sum_{j=1}^n \frac{\partial \rho}{\partial z^j}dz^j = \sum_{j=1}^n A_j(z, \bar{z})dz^j$, and supposing w.l.o.g. that $A_n \neq 0$ on a neighborhood $U$ of $p$, we obtain

$$\frac{\partial}{\partial \bar{z}^j} \frac{\partial}{\partial A_n} A_n = \frac{\partial A_j}{\partial \bar{z}^j} A_n - \frac{\partial A_n}{\partial \bar{z}^j} A_j = 0,$$

for all $1 \leq l \leq k$ on $\eta_p \cap U$, and thus that there are $n - 1$ holomorphic functions $f_1, \ldots, f_{n-1}$ and a smooth function $\lambda$, all defined on $\eta_p$, such that $\partial \rho(z, \bar{z}) = \lambda(z, \bar{z}) \left( dz^n + \sum_{j=1}^{n-1} f_j(z)dz^j \right)$. Since $\eta_p = U \cap z^{-1}(\{\mathbb{C}^k \times \{0\}\}) \subset M$, the coefficients $f_1, \ldots, f_k$ must vanish identically. We extend $f_{k+1}, \ldots, f_{n-1}$ to holomorphic functions on a neighborhood $\tilde{U}$ of $p$ and define new coordinates $(\bar{z})_{j=1}^n$ by

$$\bar{z}^j = z^j, \ 1 \leq j \leq n - 1 \quad \bar{z}^n = z^n + \sum_{j=k+1}^{n-1} f_j z^j.$$  

We calculate $\partial \rho$, on $\eta_p$, in these coordinates.

$$\partial \rho = \lambda \cdot \left( dz^n + \sum_{j=1}^{n-1} f_j dz^j \right) = \lambda \cdot \left( d\bar{z}^n - \sum_{j=k+1}^{n-1} \bar{z}^j df_j \right) = \lambda d\bar{z}^n,$$

as desired, since $\bar{z}^{k+1} = \cdots = \bar{z}^n = 0$ on $\eta_p$.

(b) Using the parametrization $\phi : \mathbb{R} \times \mathbb{C}^k \rightarrow \mathbb{C}^n$, $(r, t) \rightarrow \phi(r, t)$ from Lemma 2.1, applied to the Levi-flat manifold $N$ of hypersurface type, we construct a vector field.
V = \sum_{j=1}^{n} V^j \frac{\partial}{\partial x^j} \text{ such that } \frac{1}{2}(V + \bar{V}) \text{ is tangential to } M, \text{ and such that } V^j \text{ is holomorphic along } \eta_p \text{ for } 1 \leq j \leq n. \text{ In fact, since } \phi_* \frac{\partial}{\partial r} = \sum_{j=1}^{n} \left( \frac{\partial \phi_j}{\partial r} \frac{\partial}{\partial x^j} + \frac{\partial \phi_j}{\partial \bar{z}} \frac{\partial}{\partial \bar{z}} \right) \text{ is tangential to } N \subset M, \text{ and } \frac{\partial}{\partial r} \frac{\partial \phi_j}{\partial r} = \frac{\partial}{\partial \bar{z}} \frac{\partial \phi_j}{\partial \bar{z}} = 0 \text{ for } 1 \leq \ell \leq k, \text{ any vector field } V \text{ with } \frac{1}{2}(V + \bar{V}) \in \Gamma(TM) \text{ that agrees with } \sum_{j=1}^{n} \frac{\partial \phi_j}{\partial r} \frac{\partial}{\partial x^j} \text{ on } \eta_p \text{ has the desired properties.}

In the coordinates from part (a), where \( \partial \rho = \lambda dz^n \) along \( \eta_p \), we have

\[
0 = d\rho(\frac{1}{2}(V + \bar{V})) = \frac{1}{2}(\partial \rho(V) + \bar{\partial} \rho(\bar{V})) = \frac{1}{2}(\lambda V^n + \bar{\lambda} \bar{V}^n) = \Re(\lambda V^n),
\]

hence \( i \lambda V^n \) is a smooth, real valued function on \( \eta_p \). As \( N \) is CR-transversal, and \( \phi_* \frac{\partial}{\partial r} \notin \Gamma(T^{1,0}N) \), the function \( V^n = dz^n (\phi_* \frac{\partial}{\partial r}) \) furthermore does not vanish. Let \( v(z) := V^n(z^1, \ldots, z^k, 0, \ldots, 0) \) and \( a(z) := (i \lambda V^n)(z^1, \ldots, z^k, 0, \ldots, 0) \). By replacing \( z^n \) with \( \bar{z}^n = \frac{i}{a} z^n \) and \( \rho \) by \( \bar{\rho} = \frac{\lambda}{a} d\bar{z}^n = \frac{i}{a} d\bar{z} = d\bar{z}^n \) on \( \eta_p \), as claimed.

By a lemma of Narasimhan, a real hypersurface \( M \subset \mathbb{C}^n \) which is strongly pseudoconvex at a point \( p \in M \) can be “convexified” near \( p \) by a suitable choice of holomorphic coordinate system (see e.g. [5, p. 134]). Concretely, there exists an open neighborhood \( p \subset \mathbb{C}^n \) of \( p \) and a map \( \phi : O \to \mathbb{C}^n \), biholomorphic onto its image, such that \( \phi(M \cap O) \) is given by the graph of a strongly convex function \( f \) of \( 2n - 1 \) real variables:

\[
\phi(M \cap O) = \{ z \in \phi(O) : \Re z^n = f(\Re z^1, \ldots, \Re z^{n-1}, \Re z^n, \Im z^1, \ldots, \Im z^{n-1}) \}.
\]

A merely weakly pseudoconvex \( M \) might not be convexifiable in such a manner [4].

However, if \( M \) is uniformly pseudoconvex, the “straightening out” procedure detailed in Lemma 2.2 yields a holomorphic support function for \( M \) at \( p \) via Narasimhan’s Lemma. To the best present knowledge of the author, this result, although interesting and fairly simple to derive, has not been previously obtained in the literature.

**Proof of Theorem 1.1** Let \( k \) denote the dimension of the Levi kernel of \( M \). As \( M \) is foliated by the Levi foliation, we may apply part (b) of Lemma 2.2 to obtain holomorphic coordinates \((z^j)_{j=1}^n\) and a defining function \( \rho \) such that \( \partial \rho = dz^n \) on \( \eta_p \) near \( p \), and such that \( z(\eta_p) = \{ z \in \mathbb{C}^n : z^{k+1} = \cdots = z^n = 0 \} \cap z(U) \).

Define \( \Sigma_p \subset U \) by \( z(\Sigma_p) = \{ z \in \mathbb{C}^n : z^1 = \cdots = z^k = 0 \} \cap z(U) \). Since \( \Sigma_p \) is a complex submanifold, the set \( S_p = \Sigma_p \cap M \) is a CR submanifold of hypersurface type. Since \( \Sigma_p \) is complementary to the Levi kernel \( \eta_p \), \( S_p \) is strongly pseudoconvex. By Narasimhan’s Lemma, there exist holomorphic coordinates \((z^j)_{j=k+1}^n \) of \( \Sigma_p \), such that \( S_p \) is given by \( \Im (z^n) = \phi(\Re (z^{k+1}), \ldots, \Re (z^n), \Im (z^{k+1}), \ldots, \Im (z^{n-1})) \) for a strongly convex function \( \phi : \mathbb{R}^{2n-2k-1} \to \mathbb{R} \) which satisfies \( \phi(0) = 0 \) and \( d\phi(0) = 0 \).

Coordinates \((w^j)_{j=1}^n\) for a full neighborhood of \( p \in \mathbb{C}^n \) are obtained by setting \( w^j = z^j(z^{k+1}, \ldots, z^n) \) for \( k < j \leq n \), and \( w^j = z^j \) for \( 1 \leq j \leq k \). In these coordinates, \( M \) is given by \( \Im (w^n) = \Phi(\Re (w^1), \ldots, \Re (w^n), \Im (w^1), \ldots, \Im (w^{n-1})) \) for a function \( \Phi : \mathbb{R}^{2n-1} \to \mathbb{R} \) which clearly vanishes on \( \eta_p \). As the unit normal to \( M \) with respect to the \( z \)-coordinates is constant along \( \eta_p \), due to Lemma 2.2(b), and the coordinate change \( z \to w \) is invariant with respect to translations along \( \eta_p \), the unit normal to \( M \) with respect to the new coordinates must also be constant. Hence, \( d\Phi \) vanishes along \( \eta_p \). The real Hessian of \( \Phi \) with respect to the variables
$w^{k+1}, \ldots, w^n$ is strongly positive definite at $p$, and since $M$ is $C^2$, therefore on a whole neighborhood of $p$. To describe the geometry more clearly, introduce for $q \in \eta_p$ the submanifold $\Sigma_q = \{ \xi \in \mathbb{C}^n : z^j(\xi) = z^j(q), k < j \leq n \}$, and let $\Pi = \{ \xi \in \mathbb{C}^n : \Im(w^n(\xi)) = 0 \}$. Then, for each $q \in \eta_p$ near $p$, the restriction $S_q = M \cap \Sigma_q$ is a strongly convex real hypersurface in $\Sigma_q$, and $\Pi \cap \Sigma_q$ is a supporting plane of $S_q$. In particular, $\Im(w^n(S_q)) \geq 0$, with equality only at $q$, which implies the claim. \( \square \)

In the coordinate system constructed in Theorem 1.1, a uniformly pseudoconvex hypersurface does not necessarily become convex, in contrast to strongly pseudoconvex hypersurfaces in Narasimhan’s lemma. To achieve convexity, one would need to control the cross-terms in the Levi form arising from directions along $\Sigma$ parallel to the $(y,z)$-plane are strongly convex, but it is not itself convex.

It is an interesting question whether any uniformly pseudoconvex hypersurface is locally convexifiable by the right choice of coordinates. Nevertheless, for many purposes it is enough to find a holomorphic support function such as the coordinate function $z^n$ in Theorem 1.1 like in the following observation on formal power series.

**Proposition 2.3.** Let $M \subseteq \mathbb{C}^n$ be a uniformly pseudoconvex hypersurface. Suppose a formal holomorphic curve $\Gamma \in \mathbb{C}[[t]]^n$ is tangential to infinite order to $M$ at a point $p \in M$. Then $\Gamma$ is formally contained in the leaf through $p$ of the Levi foliation.

**Proof.** We work with the coordinates $(z_j)_{j=1}^n$ and the defining function $\rho$ constructed in Lemma 2.2 part (b). By definition, $\Gamma$ is tangential to infinite order to $M$ at $p$ if and only if $\rho \circ \Gamma$ vanishes as a formal power series. In the following calculations $\partial^\ell \partial^\rho(z, \bar{z})(\bar{V}_1, \ldots, \bar{V}_k, U_1, \ldots, U_\ell)$ is shorthand for the expression

$$
\sum_{\ell_1=1}^{n} \cdots \sum_{\ell_k=1}^{n} \cdots \sum_{j_1=1}^{n} \cdots \frac{\partial^{k+\ell}\rho(z, \bar{z})}{\partial z_{i_1} \cdots \partial z_{i_k} \partial z_{j_1} \cdots \partial z_{j_\ell}} \bar{V}_{i_1}^{\ell_1} \cdots \bar{V}_{i_k}^{\ell_k} U_1^{j_1} \cdots U_\ell^{j_\ell},
$$

with $\bar{V}_1, \ldots, \bar{V}_k \in T_z^{0,1} \mathbb{C}^n$ and $U_1, \ldots, U_\ell \in T_z^{1,0} \mathbb{C}^n$.

We begin by making three observations:

1. A vector $\bar{V} \in T_p^{0,1} M$ lies in $\mathcal{N}_p$ if and only if $\partial \partial^\rho(p, \bar{p})(\bar{V}, V) = 0$.
2. For $k \geq 1$ vectors $\bar{V}_1, \ldots, \bar{V}_k \in \mathcal{N}_p$ and another vector $U \in T_p^{1,0} M$, we have $\partial^k \partial^\rho(p, \bar{p})(\bar{V}_1, \ldots, \bar{V}_k, U) = 0$.
3. For $k \geq 1$ vectors $\bar{V}_1, \ldots, \bar{V}_k \in \mathcal{N}_p$ and $\ell \geq 0$ vectors $U_1, \ldots, U_\ell \in \bar{\mathcal{N}}_p$,

$$
\partial^k \partial^\rho(p, \bar{p})(\bar{V}_1, \ldots, \bar{V}_k, U_1, \ldots, U_\ell) = 0.
$$

The first claim is immediately seen from the fact that $\partial \partial^\rho(p, \bar{p})(\bar{V}, V) = \mathcal{L}_p(\bar{V}, \bar{V})$ for $\bar{V} \in T_p^{0,1} M$, where $\mathcal{L}_p$ denotes the Levi form derived from the characteristic form $\partial^\rho$, as $\mathcal{L}_p$ is by assumption positive semidefinite with kernel $\mathcal{N}_p$. The second and third observations follow from our choice of coordinates: The function $q \mapsto \partial^\rho(q, \bar{q})(\sum_{j=1}^{n} U_j^1 \partial)_{\partial z^j}$ is constant along $\eta_p$ because $\partial^\rho|_{\eta_p} = dz^n|_{\eta_p}$ is, and
$\bar{\partial}^k \partial \rho(p, \bar{p})(\bar{V}_1, \ldots, \bar{V}_k, U)$ is just a derivative of this constant function along directions $\bar{V}_1, \ldots, \bar{V}_k$ tangential to $\eta_p$. Analogously, $\bar{\partial}^k \partial \rho(p, \bar{p})(V_1, \ldots, V_k, U_1, \ldots, U_l)$ vanishes because only derivatives along $\eta_p$ are taken.

Using these observations, we will show by induction that $\frac{\partial^k \Gamma}{\partial t^k}(0) \in \mathcal{N}_p$ for all $k \in \mathbb{N}$ and therefore $\Gamma$ is formally contained in $\eta_p$ (since, in our coordinate system, $\eta_p$ is straightened out to a linear subspace of $\mathbb{C}^n$ which may be identified with $\mathcal{N}_p$). As $\Gamma$ is tangential to $M$, $\frac{\partial \Gamma}{\partial t}(0) \in T_p^{1,0}M$. By the chain rule for formal power series,

$$\frac{\partial^2}{\partial t \partial \bar{t}} \rho(\Gamma(t), \bar{\Gamma}(t)) = \frac{\partial}{\partial t} \left( \partial \rho(\Gamma(t), \bar{\Gamma}(t)) \left( \frac{\partial \Gamma}{\partial t}(t), \frac{\partial \bar{\Gamma}}{\partial \bar{t}}(t) \right) \right) = \partial \bar{\partial} \rho(\Gamma(t), \bar{\Gamma}(t)) \left( \frac{\partial \Gamma}{\partial t}(t), \frac{\partial \bar{\Gamma}}{\partial \bar{t}}(t) \right),$$

hence $\frac{\partial^k \Gamma}{\partial t^k}(0) \in \mathcal{N}_p$ by observation (1). Inductive application of the chain rule yields

$$\frac{\partial^k}{\partial t^k} \rho(\Gamma(t), \bar{\Gamma}(t)) = \partial \rho(\Gamma(t), \bar{\Gamma}(t)) \left( \frac{\partial^k \Gamma}{\partial t^k} \right)$$

$$+ \sum_{l=2}^{k} \sum_{\alpha \in \mathbb{N}^l} a_{\alpha} \partial^l \rho(\Gamma(t), \bar{\Gamma}(t)) \left( \frac{\partial^{\alpha_1} \Gamma}{\partial t^{\alpha_1}}(t), \ldots, \frac{\partial^{\alpha_l} \Gamma}{\partial t^{\alpha_l}}(t) \right),$$

$$\frac{\partial^2}{\partial t \partial \bar{t}} \rho(\Gamma(t), \bar{\Gamma}(t)) = \partial \bar{\partial} \rho \left( \frac{\partial^k \Gamma}{\partial t^k}(t), \frac{\partial^k \bar{\Gamma}}{\partial \bar{t}^k}(t) \right)$$

$$+ \sum_{l=2}^{k} \sum_{\alpha \in \mathbb{N}^l} b_{\alpha} \partial^l \partial \rho(\Gamma(t), \bar{\Gamma}(t)) \left( \frac{\partial^{\alpha_1} \Gamma}{\partial t^{\alpha_1}}(t), \ldots, \frac{\partial^{\alpha_l} \Gamma}{\partial t^{\alpha_l}}(t) \right)$$

$$+ \sum_{l=2}^{k} \sum_{\alpha \in \mathbb{N}^l} \partial^l \partial \rho \left( \frac{\partial^{\alpha_1} \Gamma}{\partial t^{\alpha_1}}(t), \ldots, \frac{\partial^{\alpha_l} \Gamma}{\partial t^{\alpha_l}}(t) \right)$$

for (irrelevant) universal integer constants $a_{\alpha}, b_{\alpha}$ and $c_{\alpha,\beta}$. Supposing now that $\frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \in \mathcal{N}_p$ for $1 \leq \alpha \leq k - 1$, we see from the first equation, using observation (3), that $\partial \rho(p, \bar{p}) \left( \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \right) = 0$, i.e., that $\frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \in T_p^{1,0}M$. By observation (2), $\partial^l \partial \rho(p, \bar{p}) \left( \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0), \ldots, \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \right) = 0$, and after complex conjugation, $\partial^l \partial \rho(p, \bar{p}) \left( \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0), \ldots, \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \right) = 0$, for any multi-index $\alpha$ of length $l \geq 2$ satisfying $|\alpha| = k$. By (3), for any multi-indices $\alpha$ and $\beta$ of lengths $\ell, m \geq 2$ with $|\alpha| = |\beta| = k$, also $\partial^l \partial^m \rho(p, \bar{p}) \left( \frac{\partial^{\alpha_1} \Gamma}{\partial t^{\alpha_1}}(0), \ldots, \frac{\partial^{\alpha_l} \Gamma}{\partial t^{\alpha_l}}(0), \frac{\partial^{\alpha_1} \Gamma}{\partial t^{\alpha_1}}(0), \ldots, \frac{\partial^{\alpha_m} \Gamma}{\partial t^{\alpha_m}}(0) \right)$ vanishes. Thus all sums in the second equation vanish at $t = 0$, leaving us with the expression $\partial \bar{\partial} \rho(p, \bar{p}) \left( \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0), \frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \right) = 0$, which implies by observation (1) that $\frac{\partial^{\alpha} \Gamma}{\partial t^{\alpha}}(0) \in \mathcal{N}_p$. Together with the fact that $\frac{\partial \Gamma}{\partial t}(0) \in \mathcal{N}_p$, this shows that $\frac{\partial^k \Gamma}{\partial t^k}(0) \in \mathcal{N}_p$ for any $k \in \mathbb{N}$. □
3. The deformation theorem

A remarkable theorem [8, Theorem 1.1] by Nordine Mir shows that, if $M \subset \mathbb{C}^n$ and $M' \subset \mathbb{C}^{n'}$ are real analytic CR manifolds, with $M$ minimal, then any nowhere real-analytic CR map $h : M \to M'$ of regularity $C^{n'-1}$ arises from a holomorphic deformation of a real-analytic CR map, at least almost everywhere.

To be more precise, there exists a dense open set $\Omega \subseteq M$ such that for every point $p \in \Omega$ we can find

- an integer $k$ and a neighborhood $U \times V \subseteq M \times \mathbb{C}^k$ of $(p,0) \in M \times \mathbb{C}^k$;
- a real-analytic CR map $\Phi : U \times V \to M'$ and
- a nowhere real-analytic CR function $\psi : M \supseteq U \to \mathbb{C}^k$,

such that $h(\xi) = \Phi(\xi, \psi(\xi))$.

Lamel and Mir [6, Theorem 2.2] provide an analogue of the previous theorem in the smooth category. The main catch is that the deformation constructed there no longer necessarily converges. What one instead obtains is a formal deformation of the given, nonsmooth, map — in particular, there may not exist a smooth map $h : M \to M'$ at all. For the convenience of the reader, let us restate [6, Thm 2.2] here in a slightly simplified version as it applies to our situation.

Theorem 3.1 (Lamel–Mir). Let $M \subset \mathbb{C}^n$ be a smooth minimal CR submanifold, $M' \subset \mathbb{C}^{n'}$ a smooth CR submanifold and $h : M \to M'$ a CR map of regularity $C^{n'-1}$. Assume that $h$ is nowhere $C^\infty$-smooth. Then there exists a dense open subset $\Omega \subseteq M$ such that for every $p \in \Omega$, there exists a neighborhood $O \subseteq M$ of $p$ and a $C^1$-regular CR family of formal complex curves $(\Gamma_\xi)_{\xi \in O}$ such that $\Gamma_\xi$ is tangential to infinite order to $M'$ at $h(\xi)$, and $\frac{d}{dt}\Gamma_\xi(0) \neq 0$ for all $\xi \in O$.

Here, a CR family of formal complex curves $(\Gamma_\xi)_{\xi \in O}$ associates to each $\xi \in O$ a holomorphic formal power series $\Gamma_\xi \in \mathbb{C}[[t]]^{n'}$, such that all coefficients $\frac{d^k}{dt^k}\Gamma_\xi(0)$, $k \in \mathbb{Z}_{\geq 0}$, are CR functions in $\xi$. Tangency to infinite order to $M'$ at $h(\xi)$ means that $\Gamma_\xi(0) = h(\xi)$ and the composition of $\Gamma_\xi$ with the infinite Taylor series of a defining function $\rho$ of $M'$ vanishes as a formal power series.

The family $(\Gamma_\xi)_{\xi \in O}$ is not unique, because it may be reparametrized freely. Normalizing one of its coordinate components takes care of this issue.

Lemma 3.2. Suppose that, in the setting of Theorem 3.1, we have $\frac{d}{dt}\Gamma_\rho(0)_{n'} \neq 0$. Then, shrinking $O$ if necessary, we may assume $(\Gamma_\xi)_{n'}(t) = h(\xi)_{n'} + t$ for all $\xi \in O$.

Proof. Since $\frac{d}{dt}\Gamma_\xi(0)_{n'} \neq 0$ is an open condition and satisfied at $p$, it is true on a neighborhood of $p$, to which we now restrict $O$. Since its derivative is nonvanishing, the inverse power series to $(\Gamma_\xi)_{n'}(t) - h(\xi)_{n'}$ is defined. Its coefficients are polynomials in $(\frac{d}{dt}\Gamma_\xi(0)_{n'})^{-1}$ and the coefficients of $(\Gamma_\xi - h(\xi))_{n'}$. All these are $C^1$-regular CR functions, so $((\Gamma_\xi - h(\xi))_{n'}^{-1})_{\xi \in O}$ is a $C^1$-regular CR family of formal power series. Now define $(\hat{\Gamma}_\xi)_{\xi \in O}$ by $\hat{\Gamma}_\xi(t) := \Gamma_\xi \circ (\Gamma_\xi - h(\xi))_{n'}^{-1}(t)$. This is a $C^1$-regular CR family of formal complex curves by the same argument as before, and tangential to $M'$ to infinite order as tangency is unaffected by reparametrization. By construction, we have $(\hat{\Gamma}_\xi)_{n'}(t) = h(\xi)_{n'} + t$. □

If a uniformly pseudoconvex hypersurface has one-dimensional Levi kernel, the above trick determines $(\Gamma_\xi)_{\xi \in O}$ uniquely, and thereby forces it to converge. The reason is that, as Proposition 2.3 shows, $\Gamma_\xi$ formally parametrizes $\eta_{h(\xi)}$, the leaf of
the Levi foliation through \( h(\xi) \), since \( \eta_n(\xi) \) is only one-dimensional. By normalizing the \( n' \)-th component of \( \Gamma_\xi \), we pick out the standard parametrization of \( \eta_n(\xi) \) as a graph over the \( z^{n'} \)-axis, which is convergent. With some additional effort, we obtain a smooth deformation result (Theorem 1.2) exactly paralleling Mir’s deformation theorem in the analytic category.

Proof of Theorem 1.2. We apply Theorem 3.1 to obtain a dense open subset of \( M \) such that for any point \( p \) therein, there exists an open neighborhood \( O_p \) of \( p \) and a \( C^1 \) family of formal complex curves \( (\Gamma_\xi)_{\xi \in O_p} \) such that \( \Gamma_\xi \) is tangential to infinite order to \( M' \) at \( h(\xi) \) for every \( \xi \in O_p \). Our objective is now to show that \( (\Gamma_\xi)_{\xi \in O_p} \) actually gives rise to a smooth deformation, at least on a dense open subset of \( O_p \).

To this end, let us take the parametrization \( \phi : \mathbb{R}^{2n-3} \times \mathbb{C} \supseteq U \times V \to M' \subseteq \mathbb{C}^n \) near \( h(p) \), constructed in Lemma 2.1 with \( (\phi(r,\xi))_\xi = \xi \) for all \( (r, \xi) \in U \times V \). As \( \phi \) is an embedding, after shrinking \( O_p \) there exist \( f : O_p \to U \) and \( \psi : O_p \to V \) of regularity \( C^{n'-1} \) such that \( h = \phi \circ (f, \psi) \). Since \( \psi = z^{n'} \circ \phi \circ (f, \psi) = z^{n'} \circ h \) is just a component of \( h \), it is a CR function. By Proposition 2.3, the only formal complex curve \( \Gamma_\xi \) tangential to infinite order to \( M' \) at \( h(\xi) \) satisfying \( (\Gamma_\xi)_\xi (t) = h(\xi)_\xi (t) + t \) is \( \phi(f(\xi), \psi(\xi) + t) \), i.e. the parametrization of the Levi foliation’s leaf through \( h(\xi) \).

We observe that \( \phi(f(\xi), \psi(\xi) + t) \) is holomorphic in \( t \) and converges for all \( t \) such that \( \psi(\xi) + t \in V \). Thus the radius of convergence of \( \Gamma_\xi \) is bounded below near \( p \) by some \( \varepsilon > 0 \). Let us denote \( \frac{1}{k!} \frac{\partial^k \Gamma_\xi}{\partial t^k} (0) \) by \( \Gamma_k(\xi) \). Choosing \( \varepsilon' \in (0, \varepsilon) \), we have that for each \( \xi \in O_p \) there exists a minimal \( N(\xi) \in \mathbb{N} \) such that \( |\Gamma_k(\xi)| \leq \frac{1}{k!} \) for all \( k \geq N(\xi) \). The sets \( O_{p,m} := \{ \xi \in O_p : N(\xi) \leq m \} \) form a chain of nested, relatively closed subsets of \( O_p \) such that \( O_p = \bigcup_{m=1}^\infty O_{p,m} \). By Baire’s theorem, the open set \( \hat{O}_p := \bigcup_{m=1}^\infty O_{p,m} \) is dense in \( O_p \). For any point \( q \in \hat{O}_p \), there exists \( N_q \in \mathbb{N} \) such that \( |\Gamma_k(\xi)| \leq \frac{1}{k!} \) for \( k \geq N_q \) in a neighborhood of \( q \). The coefficients \( \Gamma_k(\xi) \) are \( C^1 \) CR functions in \( \xi \). As \( M \) is minimal, Tumanov’s theorem implies that there exists a wedge \( \mathcal{T}_q \) with edge \( M \) centered at \( q \), such that the coefficients \( \Gamma_k(\xi) \) extend continuously to holomorphic functions defined on \( \mathcal{T}_q \).

By a coordinate translation, we may assume \( z^{n'}(q) = 0 \) (and thus \( \psi(q) = 0 \)). Choose \( \varepsilon'' \in (0, \varepsilon') \) such that \( \mathbb{D}_{\varepsilon''}(0) \subseteq \mathbb{D}_{\varepsilon'}(\psi(\xi)) \) for all \( \xi \in O_q \), after possibly shrinking \( O_q \). Define \( \Phi : O_q \times \mathbb{D}_{\varepsilon''} \to M' \) by \( \Phi(\xi, t) = \phi(f(\xi), t) \). We can rewrite

\footnote{For contradiction, suppose \( \bigcup_{m=1}^\infty O_{p,m} \) was not dense. Then there would exist an open set \( \hat{O} \subseteq O_p \) not intersecting \( O_{p,m} \) for any \( m \). Thus, \( O_{p,m} \cap \hat{O} \) would be relatively closed sets in \( \hat{O} \) and have empty interior. Since \( \hat{O} \) is a Baire space, this would imply \( \bigcup_{m=1}^\infty O_{p,m} \cap \hat{O} \neq \hat{O} \), a contradiction.}
\( \phi(f(\xi), t) = \phi(f(\xi), \psi(\xi) + (t - \psi(\xi))) \) as a composition of CR maps, since \( t - \psi(\xi) \) is a CR function on \( O_q \times \mathbb{D}_{\varepsilon''} \), mapping into \( \mathbb{D}_{\varepsilon'} \), and \( \phi(f(\xi), \psi(\xi) + t) \) is a CR map from \( O_q \times \mathbb{D}_{\varepsilon'} \) to \( M' \), by the preceding paragraph. Thus, \( \Phi \) is a CR map. As \( \phi \) is smooth, \( \Phi(\xi, 0) = \phi(f(\xi), 0) \) is \( C^{n-1} \)-regular. By construction, \( \Phi(\xi, 0) \) maps into the strongly pseudoconvex slice \( M' \cap \{ z^{n'} = 0 \} \), so it is smooth on a dense open subset \( \tilde{O}_q \) of \( O_q \), e.g. by [3] Cor. 2.3. Since \( \phi(f(\xi), 0) \) is smooth on \( \tilde{O}_q \), and \( \phi \) is an immersion, \( f \) is smooth on \( \tilde{O}_q \) as well. Thus \( \Phi(\xi, t) = \phi(f(\xi), t) \), as a composition of smooth maps, is smooth on \( \tilde{O}_q \times \mathbb{D}_{\varepsilon''} \). Since \( h(\xi) = \Phi(\xi, \psi(\xi)) \) as desired, this finishes the proof. □

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