PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY, SERIES B
Volume 11, Pages 371–377 (July 11, 2024)
https://doi.org/10.1090/bproc/211

A NEW PROOF OF THE GAGLIARDO–NIRENBERG AND SOBOLEV INEQUALITIES: HEAT SEMIGROUP APPROACH

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(Communicated by Dmitriy Bilyk)

ABSTRACT. We give a new proof of the Gagliardo–Nirenberg and Sobolev inequalities based on the heat semigroup. Concerning the Gagliardo–Nirenberg inequality, we simplify the previous proof by relying only on the L^p-L^q estimate of the heat semigroup. For the Sobolev inequality, we consider another approach by using the heat semigroup and the Hardy inequality.

1. INTRODUCTION

In this paper, we give a *new proof* of the *Gagliardo–Nirenberg* and *Sobolev inequalities*. In what follows, for $1 \le p \le \infty$, let $W^{1,p}(\mathbb{R}^n)$ denote the Sobolev spaces equipped with the norm $\|f\|_{W^{1,p}(\mathbb{R}^n)} := \|f\|_{L^p(\mathbb{R}^n)} + \|\nabla f\|_{L^p(\mathbb{R}^n)}$. In particular, we set $H^1(\mathbb{R}^n) := W^{1,2}(\mathbb{R}^n)$ for the case of $p = 2$. We shall consider the following inequalities:

Theorem 1.1.

(i) *Let $n \ge 1$ and $1 \le q \le p \le \infty$. Suppose that $1 \le r \le \infty$ satisfies $np/(n+p) < r \le p$. Then for every $f \in L^q(\mathbb{R}^n) \cap W^{1,r}(\mathbb{R}^n)$, it holds that $f \in L^p(\mathbb{R}^n)$ with the estimate*

$$\|f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)}^{1-\sigma}\|\nabla f\|_{L^r(\mathbb{R}^n)}^{\sigma}, \quad \sigma := \frac{n(1/q-1/p)}{1+n(1/q-1/r)},$$

where $C > 0$ is a constant independent of f.

(ii) *Let $n \ge 3$. Then for every $f \in H^1(\mathbb{R}^n)$, it holds that $f \in L^{2n/(n-2)}(\mathbb{R}^n)$ with the estimate*

$$\|f\|_{L^{2n/(n-2)}(\mathbb{R}^n)} \le C\|\nabla f\|_{L^2(\mathbb{R}^n)},$$

where $C > 0$ is a constant independent of f.

The inequality in Theorem 1.1(i) was shown by Gagliardo [8] and Nirenberg [13] in 1959. In addition, the inequality in Theorem 1.1(ii) is called the Sobolev inequality, which may be regarded as a *critical case* of the Gagliardo–Nirenberg inequality. In fact, if we assume that $\sigma = 1$, i.e., $n(1/q - 1/p) = 1 + n(1/q - 1/r)$ in the statement of Theorem 1.1(i), then it holds that $p = nr/(n - r)$. Such an

Received by the editors May 16, 2023, and, in revised form, January 9, 2024.
2020 *Mathematics Subject Classification.* Primary 35A23, 46E35.
Key words and phrases. Gagliardo–Nirenberg inequality, Sobolev inequality, heat semigroup.
The research of the first author was partially supported by JSPS KAKENHI Grant Number JP19H00644. The research of the second author was partially supported by JSPS KAKENHI Grant Number JP22J12100.

exponent $p = nr/(n-r)$ is called the critical Sobolev exponent in L^r sense and in this case there holds

$$\|f\|_{L^{nr/(n-r)}(\mathbb{R}^n)} \le C\|\nabla f\|_{L^r(\mathbb{R}^n)}. \tag{1.1}$$

In particular, by setting $r = 2$, we obtain the same estimate as in Theorem 1.1(ii). Here we should remark that although the inequality (1.1) actually holds for all $f \in W^{1,r}(\mathbb{R}^n)$ with $1 \le r < n$, in this paper we have concentrated only on the case $r = 2$ due to the difficulty of the generalization in our proof; its details will be stated in the last part of this section. Both of the inequalities are well known and quite fundamental tools in the fields of the analysis, so such inequalities have supported various kinds of studies essentially. As well as the studies conducted by applying such inequalities, there are a lot of previous works on the precise analysis of the inequalities themselves. For instance, one of the typical studies of the fundamental functional inequalities may be obtaining the *best constant*. We should refer to Aubin [1], Talenti [14], Del Pino and Dolbeault [6], and Cordero-Erausquin, Nazaret, and Villani [5] for such results on the Gagliardo–Nirenberg and Sobolev inequalities.

In this paper, we focus on the *method* of the proof of the functional inequalities. Here we recall the classical methods to show the Gagliardo–Nirenberg and Sobolev inequalities. The original way by Gagliardo [8] and Nirenberg [13] is mainly based on the fundamental theorem of calculus and the Hölder inequality. Note that their methods are valid for the Sobolev inequality as well. We also refer to the result [3] of Caffarelli, Kohn, and Nirenberg, which gives the generalized Gagliardo–Nirenberg inequality by means of considering the weight $|\cdot|^\alpha$. Compared with the classical methods, the aim of this paper is to give a new proof of Theorem 1.1 based on the *heat semigroup*. Here let us define the heat semigroup $\{e^{t\Delta}\}_{t>0}$ by setting $e^{t\Delta} := G_t\, *$ for $0 < t < \infty$, where $G_t(x) := (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$ for $(t,x) \in (0,\infty) \times \mathbb{R}^n$. We should emphasize that using the semigroup to show the functional inequalities is a well-known method: Here, let us recall the *Nash inequality* [12];

$$\|f\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^1(\mathbb{R}^n)}^{2/(n+2)} \|\nabla f\|_{L^2(\mathbb{R}^n)}^{1-2/(n+2)}, \tag{1.2}$$

which may be regarded as a special case of the Gagliardo–Nirenberg inequality like the Sobolev inequality. Indeed, setting $p = 2$, $q = 1$, and $r = 2$ in the statement of Theorem 1.1(i) yields (1.2). Concerning the Nash inequality (1.2), Carlen, Kusuoka, and Stroock [4] showed that (1.2) and the L^p-L^q estimate of the heat semigroup are essentially *equivalent*. Moreover, in the book [15, II.3.3 Remarks (a)] written by Varopoulos, Saloff-Coste, and Coulhon, they gave proof of (1.2) by using the semigroup as well. In the case of the Gagliardo–Nirenberg inequality, a proof based on the heat semigroup is given by, e.g., Maremonti [11, Theorem 2.2] and Giga, Giga, and Saal [9, Theorem, p. 190].

Now we shall explain the *novelty* of our method by comparing that of [9, Theorem, p.190]. In order to show the Gagliardo–Nirenberg inequality under the conditions $q < p$ and $r \le p$, they used the following characterization of the $L^p(\mathbb{R}^n)$-norm for $1 \le p \le \infty$;

$$\|f\|_{L^p(\mathbb{R}^n)} = \sup \left\{ \int_{\mathbb{R}^n} f(x) \left((e^{t\Delta}\varphi)(x) - \int_0^t \Delta(e^{\tau\Delta}\varphi)(x) d\tau \right) dx \,\middle|\, \|\varphi\|_{L^{p'}(\mathbb{R}^n)} \le 1, \varphi \in C_0^\infty(\mathbb{R}^n) \right\},$$

where p' satisfies $1/p + 1/p' = 1$. Indeed, we may confirm that the above characterization appears in [9, The First Step, p.195] and see that such a relation relies on the *duality* of $L^p(\mathbb{R}^n)$. Note that the same method is used in [11, Theorem 2.2] as well. Moreover, the above characterization is similar to that of [7, Lemma 1.5.3] in the book written by Fukushima, Oshima, and Takeda. On the other hand, our method relies *only* on the L^p-L^q estimate of the heat semigroup, so we may give a more *simple discussion*. Once we may simplify such a part, we also see that the condition $r \le p$ in Theorem 1.1(i) may be removed by the same discussion as in [9, The Second Step, p. 196]. Here we mention that Kozono and Wadade [10, Theorem 2.1] also have used a similar method to ours. In this paper, by considering the more simple case of the Gagliardo–Nirenberg inequality, we concentrate only on giving a short and simple proof without any technical method. Furthermore, Giga, Giga, and Saal [9, The Second Step, p. 196] point out that their method has to exclude the *marginal case*, namely, the method *does not work* for the Sobolev inequality. So, they [9, p. 216] have given the proof of the Sobolev inequality by applying the *Hardy-Littlewood-Sobolev inequality* instead of using the heat semigroup. In our case, although we have to give another proof to show the Sobolev inequality as well, we show the Sobolev inequality by using the heat semigroup and the *Hardy inequality*. Hence, our approach gives *new possibilities* having the heat semigroup in the proof of such functional inequalities. However, as stated above, we should note that Theorem 1.1(ii) has considered only the case of $r = 2$, i.e., $f \in H^1(\mathbb{R}^n)$, so our result does not cover the general Sobolev inequality (1.1) whose proof is given in [9, p. 216]. In fact, our proof relies on the Hardy inequality in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$; we see by [2, Theorem 1.72] that

$$\||\cdot|^{-1} f\|_{L^2(\mathbb{R}^n)} \le \frac{2}{n-2} \|\nabla f\|_{L^2(\mathbb{R}^n)} \tag{1.3}$$

for all $f \in \dot{H}^1(\mathbb{R}^n)$ with $n \ge 3$. We usually assume that $f \in H^1(\mathbb{R}^n)$ to obtain (1.3), but thanks to the fact that $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $\dot{H}^1(\mathbb{R}^n)$ for $n \ge 3$ from [2, Theorem 1.70], we have (1.3) even for $f \in \dot{H}^1(\mathbb{R}^n)$. Such a relaxation of the assumption for (1.3) plays a key role in our proof, hence in other words, it seems that the discussion of the general case (1.1) for $f \in W^{1,r}(\mathbb{R}^n)$ with $1 \le r < n$ will become more delicate. In any case, this paper aims to establish a new method due to the heat semigroup and give another proof of the inequalities.

2. A new proof of the respective inequalities

Proof of Theorem 1.1. (i) In the case of $q = p$, the desired estimate trivially holds. So we may assume that $1 \le q < p \le \infty$. Since the condition $np/(n+p) < r$ yields $-n/p < 1 - n/r$, we see by $q < p$ that $0 < n(1/q - 1/p) < 1 + n(1/q - 1/r)$, which implies $0 < \sigma < 1$. Moreover, in the case of $r = p = \infty$, we see by the assumption $f \in W^{1,\infty}(\mathbb{R}^n)$ that f is a bounded uniformly continuous function on $\mathbb{R}^n$. Thus we have $\lim_{t\to+0}\|e^{t\Delta}f-f\|_{L^\infty(\mathbb{R}^n)}=0$ from the strong continuity of the heat semigroup. In the case of $r<\infty$, the set $C_0^\infty(\mathbb{R}^n)$ is dense in $L^q(\mathbb{R}^n)\cap W^{1,r}(\mathbb{R}^n)$. Therefore, to show the desired estimate, it suffices to consider the case where $f\not\equiv 0$ is bounded and uniformly continuous. Then it holds by $\lim_{t\to+0}\|e^{t\Delta}f-f\|_{L^\infty(\mathbb{R}^n)}=0$ that

$$\begin{aligned} f(x) &= (e^{t\Delta}f)(x)-\{(e^{t\Delta}f)(x)-f(x)\}\\ &= (e^{t\Delta}f)(x)-\int_0^t \partial_\tau(e^{\tau\Delta}f)(x)d\tau\\ &= (e^{t\Delta}f)(x)-\int_0^t \Delta(e^{\tau\Delta}f)(x)d\tau \end{aligned}$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$. Since $\Delta e^{\tau\Delta}f=\nabla\cdot e^{\tau\Delta}\nabla f$ and since the L^{p_0}-L^{q_0} estimate of the heat semigroup [9, Theorem, p.8] yields $\|\nabla^j e^{t\Delta}f\|_{L^{p_0}(\mathbb{R}^n)}\le Ct^{-(n/2)(1/q_0-1/p_0)-j/2}\|f\|_{L^{q_0}(\mathbb{R}^n)}$ for $0<t<\infty$, $j=0,1$, and $1\le q_0\le p_0\le\infty$, we observe that

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} &\le Ct^{-(n/2)(1/q-1/p)}\|f\|_{L^q(\mathbb{R}^n)}+C\int_0^t \tau^{-(n/2)(1/r-1/p)-1/2}\|\nabla f\|_{L^r(\mathbb{R}^n)}d\tau\\ &\le Ct^{-(n/2)(1/q-1/p)}\|f\|_{L^q(\mathbb{R}^n)}+Ct^{1/2-(n/2)(1/r-1/p)}\|\nabla f\|_{L^r(\mathbb{R}^n)} \end{aligned}$$

for all $0<t<\infty$. Here we note that the integrability condition $-(n/2)(1/r-1/p)-1/2>-1$ holds from the assumption $np/(n+p)<r$. Hence, noting that $1-\sigma=(1-n(1/r-1/p))(1+n(1/q-1/r))^{-1}$, we obtain

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} \le C\Big(& t^{-\sigma(1/2+(n/2)(1/q-1/r))}\|f\|_{L^q(\mathbb{R}^n)}\\ &+t^{(1-\sigma)(1/2+(n/2)(1/q-1/r))}\|\nabla f\|_{L^r(\mathbb{R}^n)}\Big) \end{aligned}$$

for all $0<t<\infty$. Since the above constant $C>0$ is independent of t, by taking t so that $t^{1/2+(n/2)(1/q-1/r)}=\|f\|_{L^q(\mathbb{R}^n)}\|\nabla f\|_{L^r(\mathbb{R}^n)}^{-1}$, we have

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^n)} &\le C\Big(\|f\|_{L^q(\mathbb{R}^n)}^{-\sigma}\|\nabla f\|_{L^r(\mathbb{R}^n)}^{\sigma}\cdot\|f\|_{L^q(\mathbb{R}^n)}\\ &\qquad+\|f\|_{L^q(\mathbb{R}^n)}^{1-\sigma}\|\nabla f\|_{L^r(\mathbb{R}^n)}^{-(1-\sigma)}\cdot\|\nabla f\|_{L^r(\mathbb{R}^n)}\Big)\\ &\le C\|f\|_{L^q(\mathbb{R}^n)}^{1-\sigma}\|\nabla f\|_{L^r(\mathbb{R}^n)}^{\sigma}, \end{aligned}$$

which yields the desired estimate.

(ii) It suffices to consider the case of $f\in C_0^\infty(\mathbb{R}^n)$. Since $e^{t\Delta}=G_t\,*$ and since $\int_{\mathbb{R}^n}G_t(y)dy=1$, we have

$$(e^{t\Delta}f)(x)-f(x)=\int_{\mathbb{R}^n}G_t(y)f(x-y)dy-f(x)=\int_{\mathbb{R}^n}G_t(y)\{f(x-y)-f(x)\}dy$$

for all $(t,x)\in(0,\infty)\times\mathbb{R}^n$. We set $\eta\coloneqq 2/n$ and $\theta\coloneqq n/(2n-2)$. Then it holds by $n\ge 3$ that $\eta,\theta\in(0,1)$. In addition, we also define

$$\begin{aligned} F_0(y)&\coloneqq |y|^{\eta/\theta}G_t(y),\\ F_1(x,y)&\coloneqq |y|^{-1}|f(x-y)-f(x)|,\\ F_2(x,y)&\coloneqq \{G_t(y)\}^{(1-\theta)/(1-\eta)}|f(x-y)-f(x)| \end{aligned}$$

for all $0 < t < \infty$ and $x, y \in \mathbb{R}^n$. Since

$$G_t(y)|f(x-y) - f(x)| = \{F_0(y)\}^\theta \{F_1(x,y)\}^\eta \{F_2(x,y)\}^{1-\eta},$$

we obtain

$$|(e^{t\Delta} f)(x) - f(x)| \le \int_{\mathbb{R}^n} \{F_0(y)\}^\theta \{F_1(x,y)\}^\eta \{F_2(x,y)\}^{1-\eta} dy$$

for all $(t,x) \in (0,\infty) \times \mathbb{R}^n$. Noting that $\theta(n-1)/n + \eta/2 + (1-\eta)/2 = 1$, we see by the Hölder inequality that

$$|(e^{t\Delta} f)(x) - f(x)| \le \|F_0\|^\theta_{L^{n/(n-1)}(\mathbb{R}^n)} \|F_1(x,\cdot)\|^\eta_{L^2(\mathbb{R}^n)} \|F_2(x,\cdot)\|^{1-\eta}_{L^2(\mathbb{R}^n)}. \tag{2.1}$$

Here, since we have

$$\int_{\mathbb{R}^n} |x|^\alpha \{G_t(x)\}^\beta dx = t^{\alpha/2-(n/2)(\beta-1)} \int_{\mathbb{R}^n} |x|^\alpha \{G_1(x)\}^\beta dx \le Ct^{\alpha/2-(n/2)(\beta-1)} \tag{2.2}$$

for all $0 < t < \infty$, $0 \le \alpha < \infty$, and $0 < \beta < \infty$, by setting $\alpha = \eta/\theta \cdot n/(n-1) = 4/n$ and $\beta = n/(n-1)$ in the estimate (2.2), we observe that

$$\|F_0\|^{n/(n-1)}_{L^{n/(n-1)}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |x|^{\eta/\theta \cdot n/(n-1)} \{G_t(x)\}^{n/(n-1)} dx \le Ct^{2/n - n/(2n-2)},$$

which yields $\|F_0\|^\theta_{L^{n/(n-1)}(\mathbb{R}^n)} \le Ct^{1/n-n/(4n-4)}$ for all $0 < t < \infty$. Moreover, the Hardy inequality (1.3) in the homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n)$ implies that

$$\|F_1(x,\cdot)\|^2_{L^2(\mathbb{R}^n)} \le \left(\frac{2}{n-2}\right)^2 \int_{\mathbb{R}^n} |\nabla_y \{f(x-y) - f(x)\}|^2 dy = \left(\frac{2}{n-2} \|\nabla f\|_{L^2(\mathbb{R}^n)}\right)^2$$

for all $x \in \mathbb{R}^n$, which gives the estimate $\sup_{x\in\mathbb{R}^n} \|F_1(x,\cdot)\|_{L^2(\mathbb{R}^n)} \le C\|\nabla f\|_{L^2(\mathbb{R}^n)}$. Therefore, by taking $L^{2n/(n-2)}(\mathbb{R}^n)$-norm in the estimate (2.1), we have

$$\begin{aligned}
\|e^{t\Delta} f - f\|_{L^{2n/(n-2)}(\mathbb{R}^n)} &\le Ct^{1/n-n/(4n-4)} \|\nabla f\|^\eta_{L^2(\mathbb{R}^n)} \\
&\quad \cdot \left(\int_{\mathbb{R}^n} \|F_2(x,\cdot)\|^{(1-\eta)\cdot 2n/(n-2)}_{L^2(\mathbb{R}^n)} dx \right)^{(n-2)/(2n)} \\
&= Ct^{1/n-n/(4n-4)} \|\nabla f\|^{2/n}_{L^2(\mathbb{R}^n)} \\
&\quad \cdot \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F_2(x,y)|^2 dy dx \right)^{(n-2)/(2n)}
\end{aligned}$$

for all $0 < t < \infty$. Concerning the estimate of $F_2(x,y)$, since the mean value theorem and the Cauchy–Schwarz inequality yield

$$\begin{aligned}
|f(x-y) - f(x)| &= \left| \int_0^1 y \cdot (\nabla f)(x-\lambda y) d\lambda \right| \le |y| \int_0^1 |(\nabla f)(x-\lambda y)| d\lambda \\
&\le |y| \left(\int_0^1 |(\nabla f)(x-\lambda y)|^2 d\lambda \right)^{1/2}
\end{aligned}$$

for all $x, y \in \mathbb{R}^n$, we see by the Fubini theorem that

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |F_2(x,y)|^2 dydx &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \{G_t(y)\}^{(2-2\theta)/(1-\eta)} \\ &\qquad \cdot \left(|y|^2 \int_0^1 |(\nabla f)(x-\lambda y)|^2 d\lambda\right) dydx \\ &= \int_{\mathbb{R}^n} |y|^2 \{G_t(y)\}^{(2-2\theta)/(1-\eta)} \\ &\qquad \cdot \left(\int_0^1 \int_{\mathbb{R}^n} |(\nabla f)(x-\lambda y)|^2 dxd\lambda\right) dy \\ &= \|\nabla f\|_{L^2(\mathbb{R}^n)}^2 \int_{\mathbb{R}^n} |y|^2 \{G_t(y)\}^{(2-2\theta)/(1-\eta)} dy.\end{aligned}$$

By setting $\alpha = 2$ and $\beta = (2-2\theta)/(1-\eta) = n/(n-1)$ in the estimate (2.2), we obtain $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |F_2(x,y)|^2 dydx \le Ct^{1-n/(2n-2)} \|\nabla f\|_{L^2(\mathbb{R}^n)}^2$ for all $0 < t < \infty$. Hence it holds that

$$\begin{aligned}\|e^{t\Delta} f - f\|_{L^{2n/(n-2)}(\mathbb{R}^n)} &\le Ct^{1/n - n/(4n-4)} \|\nabla f\|_{L^2(\mathbb{R}^n)}^{2/n} \\ &\qquad \cdot t^{(n-2)/(2n) - (n-2)/(4n-4)} \|\nabla f\|_{L^2(\mathbb{R}^n)}^{(n-2)/n} \\ &= C\|\nabla f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Since the above constant $C > 0$ is independent of t and since the estimate of the heat semigroup yields

$$\begin{aligned}\|f\|_{L^{2n/(n-2)}(\mathbb{R}^n)} &\le \|e^{t\Delta} f\|_{L^{2n/(n-2)}(\mathbb{R}^n)} + \|e^{t\Delta} f - f\|_{L^{2n/(n-2)}(\mathbb{R}^n)} \\ &\le Ct^{-1/2} \|f\|_{L^2(\mathbb{R}^n)} + C\|\nabla f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

we have the desired estimate by letting $t \to \infty$. This completes the proof of Theorem 1.1. □

References

[1] T. Aubin, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry **11** (1976), no. 4, 573–598, DOI 10.4310/jdg/1214433725.

[2] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 343, Springer, Heidelberg, 2011, DOI 10.1007/978-3-642-16830-7. MR2768550

[3] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math. **53** (1984), no. 3, 259–275.

[4] E. A. Carlen, S. Kusuoka, and D. W. Stroock, *Upper bounds for symmetric Markov transition functions* (English, with French summary), Ann. Inst. H. Poincaré Probab. Statist. **23** (1987), no. 2, suppl., 245–287. MR898496

[5] D. Cordero-Erausquin, B. Nazaret, and C. Villani, *A mass-transportation approach to sharp Sobolev and Gagliardo–Nirenberg inequalities*, Adv. Math. **182** (2004), no. 2, 307–332, DOI 10.1016/S0001-8708(03)00080-X. MR2032031

[6] M. Del Pino and J. Dolbeault, *Best constants for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions* (English, with English and French summaries), J. Math. Pures Appl. (9) **81** (2002), no. 9, 847–875, DOI 10.1016/S0021-7824(02)01266-7. MR1940370

[7] M. Fukushima, Y. Ōshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 1994, DOI 10.1515/9783110889741. MR1303354

[8] E. Gagliardo, *Ulteriori proprietà di alcune classi di funzioni in più variabili* (Italian), Ricerche Mat. **8** (1959), 24–51. MR109295

[9] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear partial differential equations*, Progress in Nonlinear Differential Equations and their Applications, vol. 79, Birkhäuser Boston, Ltd., Boston, MA, 2010. Asymptotic behavior of solutions and self-similar solutions, DOI 10.1007/978-0-8176-4651-6. MR2656972

[10] H. Kozono and H. Wadade, *Remarks on Gagliardo–Nirenberg type inequality with critical Sobolev space and BMO*, Math. Z. **259** (2008), no. 4, 935–950, DOI 10.1007/s00209-007-0258-5. MR2403750

[11] P. Maremonti, *Some interpolation inequalities involving Stokes operator and first order derivatives*, Ann. Mat. Pura Appl. (4) **175** (1998), 59–91, DOI 10.1007/BF01783676. MR1748216

[12] J. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931–954, DOI 10.2307/2372841.

[13] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3) **13** (1959), 115–162. MR109940

[14] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4) **110** (1976), 353–372, DOI 10.1007/BF02418013.

[15] N. Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and geometry on groups*, Cambridge Tracts in Mathematics, vol. 100, Cambridge University Press, Cambridge, 1992. MR1218884

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