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UNIFORM SIMPLICITY OF GROUPS WITH PROXIMAL ACTION

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WITH AN APPENDIX BY NIR LAZAROVICH

Abstract. We prove that groups acting boundedly and order-primitively on linear orders or acting extremely proximally on a Cantor set (the class including various Higman-Thomson groups; Neretin groups of almost automorphisms of regular trees, also called groups of spheromorphisms; the groups of quasi-isometries and almost-isometries of regular trees) are uniformly simple. Explicit bounds are provided.

Contents

1. Introduction

Let Γ be a group. It is called **N-uniformly simple** if for every nontrivial $f \in \Gamma$ and nontrivial conjugacy class $C \subset \Gamma$ the element f is the product of at most N elements from $C^{\pm 1}$. A group is **uniformly simple** if it is N-uniformly simple for some natural number N. Uniformly simple groups are called sometimes, by other authors, *groups with finite covering number* or *boundedly simple* groups (see, e.g., [15, 19, 22]). We call Γ **boundedly simple** if N is allowed to depend on C. The purpose of this paper is to prove results on uniform simplicity, in particular

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Theorems 1.1, 1.2, and 1.3 below, for a number of naturally occurring infinite permutation groups.

Every uniformly simple group is simple. It is known that many groups with geometric or combinatorial origin are simple. In this paper we prove that, in fact, many of them are uniformly simple.

Below are our main results.

Let $(I, \leq)$ be a linearly ordered set. Let $\mathrm{Aut}(I, \leq)$ denote the group of order-preserving bijections of I. We say that $g \in \mathrm{Aut}(I, \leq)$ is **boundedly supported** if there are $a, b \in I$ such that $g(x) \neq x$ only if $a < x < b$. The subgroup of boundedly supported elements of $\mathrm{Aut}(I, \leq)$ will be denoted by $\mathrm{B}(I, \leq)$.

Theorem 1.1 (Theorem 3.1 below)**.** *Assume that $\Gamma < \mathrm{B}(I, \leq)$ is proximal on a linearly ordered set $(I, \leq)$ (i.e., for every $a < b$ and $c < d$ from I there exists $g \in \Gamma$ such that $g(a) < c < d < g(b)$). Then its commutator group Γ' is six-uniformly simple and the commutator width of this group is at most two.*

For the definition of a **commutator width of a group** see the beginning of Section 2. Observe that every doubly-transitive (i.e., transitive on ordered pairs) action is proximal.

This theorem immediately applies e.g. to $\mathrm{B}(\mathbf{Q}, \leq)$ and the class of Higman-Thomson groups $F_{q,r}$ for $q > r \geq 1$, where the latter are defined as follows. We fix natural numbers $q > r \geq 1$. The **Higman-Thompson group** $F_{q,r}$ is defined as piecewise affine, order-preserving transformations of $((0, r) \cap \mathbf{Z}[1/q], \leq)$ whose breaking points (i.e., singularities) belong to $\mathbf{Z}[1/q]$ and the slopes are q^k for $k \in \mathbf{Z}$ (see [5, Proposition 4.4]). The **Thompson group** F is the group $F_{2,1}$ in the above series. Moreover, $F_{q,r}$ is independent of r (up to isomorphism) [5, 4.1]. The Higman-Thompson groups satisfy the assumptions of Theorem 1.1 due to Lemmata 3.2 and 3.3 from Section 3.

Our result implies that $\Gamma = \mathrm{B}(\mathbf{Q}, \leq)$ is six-uniformly simple. Whereas Droste and Shortt proved in [13, Theorem 1.1(c)] that $\mathrm{B}(\mathbf{Q}, \leq)$ is two-uniformly simple. In fact, they proved that if $\Gamma < \mathrm{B}(I, \leq)$ is proximal (they use the term "feebly 2-transitive" for proximal action) and additionally closed under ω-patching of conjugate elements, then Γ is two-uniformly simple. Thus, our Theorem 1.1 covers a larger class of examples than that from [13] (as we assume only proximality), but with slightly worse bound for uniform simplicity.

The uniform simplicity of the Thompson group $F = F_{2,1}$ was proven implicitly by Bardakov, Tolstykh, and Vershinin [2, Corollary 2.3] and Burago and Ivanov [6]. Although their proofs generalize to the general result given above, we write it down for several reasons. Namely, in the above cited papers some special properties of the linear structure of the real line is used, while the result is true for a general class of proximal actions on linearly ordered sets. The Droste and Shortt argument uses ω-patching, which is not suitable for our case. Furthermore, although in the examples mentioned above the action is doubly-transitive, the right assumption is proximality, which is strictly weaker than double-transitivity. In Theorem 4.2 we construct a bounded and proximal transitive action which is not doubly-transitive. This is discussed in detail in Section 4. The second reason for proving Theorem 1.1 is that a topological analogue of proximality, namely extremal proximality (see the beginning of Section 5), plays a crucial role in the proofs of the subsequent results. Extremal proximality was defined by S. Glasner in [21, p. 96] and [20, p. 328] for a general minimal action of a group on a compact Hausdorff space.

In Section 5 we go away from order-preserving actions, and consider groups acting on a Cantor set, and also groups almost acting on trees. The following theorem is Corollary 6.6(2).

Theorem 1.2. *The commutator subgroup N_q' of the Neretin group N_q of spheromorphisms and the commutator subgroup $G_{q,r}'$ of the Higman-Thomson group $G_{q,r}$ are nine-uniformly simple. The commutator width of each of those groups is at most three.*

The group N_q was introduced by Neretin in [27, 4.5, 3.4] as the group of spheromorphisms (also called almost automorphisms) of a $(q+1)$-regular tree T_q. We will recall the construction in Section 6.

The **Higman-Thompson group** $G_{q,r}$ is defined as the group of automorphisms of the Jónsson-Tarski algebra $V_{q,r}$ [5, 4A]. It can also be described as a certain group of homeomorphisms of a Cantor set [5, p. 57]. Moreover, one can view $G_{q,r}$ as a subgroup of $G_{q,2}$ and the latter as a group acting spheromorphically on the $(q+1)$-regular tree [25, Section 2.2], that is, they are subgroups of N_q. If q is even, then $G_{q,r}' = G_{q,r}$. For odd q, $[G_{q,r} : G_{q,r}'] = 2$ [5, Theorem 4.16], [17, Theorem 5.1]. The group $G_{2,1}$ is also known as the **Thompson group** V. It is known that $F_{q,r} < G_{q,r}$.

Given a group Γ acting on a tree T, in the beginning of Section 5 we will define, following Neretin, the group $[\![\Gamma]\!]$ of partial actions on the boundary of T. Theorem 1.2 is a corollary of a more general theorem about uniform simplicity of partial actions.

Theorem 1.3. *Assume that a group Γ acts on a leafless tree T, whose boundary is a Cantor set, such that Γ does not fix any proper subtree (e.g., $\Gamma\backslash T$ is finite) nor a point in the boundary of T. Then the commutator subgroup $[\![\Gamma]\!]'$ of $[\![\Gamma]\!]$ is nine-uniformly simple.*

This is an immediate corollary of Theorems 5.1 and 6.4. The latter theorem concerns several characterizations of extremal proximality of the group action on the boundary of a tree.

Section 7 is devoted to the proof that the groups of quasi-isometries and almost-isometries of regular trees are five-uniformly simple.

The uniform simplicity of homeomorphism groups of certain spaces has been considered since the beginning of the 1960s, e.g., by Anderson [1]. He proved that the group of all homeomorphisms of $\mathbf{R}^n$ with compact support and the group of all homeomorphisms of a Cantor set are two-uniformly simple (and have commutator width one). His arguments used an infinite iteration arbitrary close to every point, which is not suitable for the study of spheromorphism groups and the Higman-Thompson groups.

N-uniform simplicity is a first-order logic property (for a fixed natural number N). That is, it can be written as a formula in a first-order logic. Therefore, N-uniform simplicity is preserved under elementary equivalence: if G is N-uniform simple, then all other groups elementary equivalent with G are also N-uniformly simple. In particular, all ultraproducts of Neretin groups, and Higman-Thompson groups mentioned above, are nine-uniformly simple.

Another feature of uniformly simple groups comes from [19, Theorem 4.3], where the second author proves the following classification fact about actions of uniformly simple groups (called boundedly simple in [19]) on trees: if a uniformly simple group

acts faithfully on a tree T without invariant proper subtree or invariant end, then essentially T is a bi-regular tree (see Section 6 for definitions).

In Section 4 we discuss *the primitivity* of actions on linearly ordered sets (i.e., lack of proper convex congruences). In fact, we prove that primitivity and proximality are equivalent notions for bounded actions (Theorem 4.1). Our primitivity appears in the literature as *o-primitivity*; see, e.g., [26, Section 7].

Calegari [9] proved that subgroups of the PL^+ homeomorphism of the interval, in particular the Thompson group F, have trivial stable commutator length. Essentially by [7, Lemma 2.4] by Burago, Polterovich, and Ivanov (that we will explain for the completeness of the presentation) we reprove in Lemma 2.2 the commutator width of the commutator subgroup (and other groups covered by Theorems 3.1 and 1.1) of the Thompson group F.

Let us discuss examples and nonexamples of bounded and uniform simplicity. It is known that a simple Chevalley group (that is, the group of points over an arbitrary infinite field K of a quasi-simple quasi-split connected reductive group) is uniformly simple [15, 22]. In fact, there exists a constant d (which is conjecturally 4, at least in the algebraically closed case [23]) such that, any such Chevalley group G is $d \cdot r(G)$-uniformly simple, where $r(G)$ is the relative rank of G [15].

Full automorphism groups of right-angled buildings are simple, but never boundedly simple, because of the existence of nontrivial quasi-morphisms [10], [11, Theorem 1.1] (except if the building is a bi-regular tree [19, Theorem 3.2]).

Compact groups are never uniformly simple. More generally, a topological group Γ is called a SIN *group* if every neighborhood of the identity $e \in \Gamma$ contains a neighborhood of e which is invariant under all inner automorphisms. Every compact group is SIN (as if V is such a neighborhood, then $\bigcap_{\gamma\in\Gamma} \gamma^{-1}V\gamma$ has nonempty interior). Clearly every infinite Hausdorff SIN-group is not uniformly simple. Moreover, many compact linear groups (e.g., $\mathrm{SO}(3, \mathbf{R})$) are boundedly simple, because of the presence of the dimension with good properties. (See also the discussion at the end of this section.)

Let us conclude the introduction by relating simplicity and the notion of central norms on groups. Let Γ be a group. A function $\|\cdot\| : \Gamma \to \mathbf{R}_{\geq 0}$ is called a **seminorm** if

- $\|g\| = \|g^{-1}\|$ for all $g \in \Gamma$, and
- $\|gh\| \leq \|g\| + \|h\|$ for all $g, h \in \Gamma$.

A seminorm is called

- **trivial** if $\|g\| = 0$ for all $g \in \Gamma$,
- **central** if $\|gh\| = \|hg\|$,
- a **norm** if $\|g\| > 0$ for $1 \neq g \in \Gamma$,
- **discrete** if $\inf_{1\neq g\in\Gamma} \|g\| > 0$, and
- **bounded** if $\sup_{g\in\Gamma} \|g\| < \infty$.

A seminorm is discrete if and only if the topology it induces is discrete. A discrete seminorm is a norm. Every central seminorm $\|\cdot\|$ is conjugacy invariant: $\|ghg^{-1}\| = \|h\|$.

A typical example of a nontrivial central and discrete norm is a **word norm** $\|\cdot\|_S$ attached to a subset $S \subseteq \Gamma$ (cf. [16, 2.1]):

$$\|f\|_S = \min\{k \in \mathbf{N} : f = g_1 \cdot \ldots \cdot g_k, \text{ each } g_i \text{ is conjugate with an element from } S \cup S^{-1}\}.$$

For a nontrivial central norm $\|\cdot\|$ we define an invariant $\Delta(\|\cdot\|) = \frac{\sup_{g\in\Gamma}\|g\|}{\inf_{1\neq g\in\Gamma}\|g\|}$. Of course, if $\|\cdot\|$ is either nondiscrete or unbounded, then $\Delta(\|\cdot\|) = \infty$. We define $\Delta(\Gamma)$ to be the supremum of $\Delta(\|\cdot\|)$ for all nontrivial central norms on Γ.

Proposition 1.4. *Let Γ be a group. Then:*

(1) *Γ is simple if and only if any nontrivial central seminorm on Γ is a norm;*
(2) *Γ is boundedly simple if and only if every central seminorm on Γ is a bounded norm;*
(3) *if Γ is uniformly simple, then every central seminorm on Γ is a bounded and discrete norm;*
(4) *Γ is N-uniformly simple if and only if $\Delta(\Gamma) \leq N$.*

Proof. (1) It is obvious that the kernel of a central seminorm is closed under multiplication and conjugacy invariant. Thus, it is a normal subgroup. On the other hand, if $\Gamma_0 \trianglelefteq \Gamma$ is a proper normal subgroup of Γ, then

$$\|g\| = \begin{cases} 0 & \text{if } g \in \Gamma_0, \\ 1 & \text{if } g \notin \Gamma_0 \end{cases}$$

is a nontrivial central seminorm. It is a norm only if $\Gamma_0 = \{1\}$.

(2) Suppose that $\|\cdot\|$ is a central seminorm and assume that Γ is boundedly simple. Choose $1 \neq g \in \Gamma$. There exists $N = N(g)$ such that every element f is a product of at most N conjugates of g and g^{-1}. Thus, by the triangle inequality and conjugacy invariance, $\|f\| \leq N\|g\|$. The number $N\|g\|$ is finite and independent on f. For the converse take $1 \neq g \in \Gamma$ and consider the word norm $\|\cdot\|_S$ attached to $S = {}^{\Gamma}g \cup {}^{\Gamma}g^{-1} = \{x \in \Gamma : x \text{ is conjugated to } g \text{ or } g^{-1}\}$. It is obvious that $\|\cdot\|_S$ is a central seminorm on Γ. Thus Γ is boundedly simple, as $\|\cdot\|_S$ is bounded.

(3 & 4) Suppose Γ is N-uniformly simple, i.e., N is independent on $g \in \Gamma$ and takes nontrivial central norm $\|\cdot\|$. By the triangle inequality we conclude that $\Delta(\|\cdot\|) \leq N$, which proves the necessity of the condition in (4). For the converse, take $1 \neq g \in \Gamma$, and consider the word norm $\|\cdot\|_g$ above. We have that Γ is $\Delta(\|\cdot\|_g) \leq \Delta(\Gamma)$-uniformly simple. This completes the proof of (4), which implies (3). □

In particular, the infinite alternating group $A_\infty = \bigcup_{n\geq 5} A_n$ is simple but not boundedly simple. To see the latter, observe that the cardinality of the support is an unbounded central norm. This norm is maximal up to scaling. Indeed, essentially by [14, Lemma 2.5] every element $\sigma \in A_\infty$ is a product of at most $\left\lfloor \frac{8\#\operatorname{supp}(\sigma)}{\#\operatorname{supp}(\pi)} \right\rfloor + 2 \leq \frac{10\#\operatorname{supp}(\sigma)}{\#\operatorname{supp}(\pi)}$ conjugates of π.

Also, it is easy to see that SO(3) is boundedly simple, but not uniformly simple. The angle of rotation is a full invariant of an element of that group, and this function is a central norm. Clearly it is not discrete. As before, one can observe that if R is a rotation by an angle θ, then every other rotation can be obtained by at most $\lceil \frac{\pi}{\theta} \rceil$ conjugates of R.

Every universal sofic group [14, Section 2] is boundedly simple, but not uniformly simple. Namely, let $(S_n, \|\cdot\|_H)_{n\in\mathbf{N}}$ be the full symmetric group on n letters with the normalized Hamming norm: $\|\sigma\|_H = \frac{1}{n}|\operatorname{supp}(\sigma)|$ for $\sigma \in S_n$. Take

$$(\mathcal{G}, \|\cdot\|) = \prod\nolimits_{\mathcal{U}}^{met} (S_n, \|\cdot\|_H)$$

as the metric ultraproduct of $(S_n, \|\cdot\|_H)_{n\in\mathbf{N}}$ over a nonprincipal ultrafilter $\mathcal{U}$. Then the proof of [14, Proposition 2.3(5)] shows that $\mathcal{G}$ is boundedly simple. Furthermore, $\mathcal{G}$ is not uniformly simple, as $\|\cdot\|$ is a nondiscrete central norm on $\mathcal{G}$ (see Proposition 1.4(4)). As before, this norm is maximal up to scaling due to [14, Lemma 2.5].

2. Burago-Ivanov-Polterovich method

The symbol Γ will always denote a group. For $a, b \in \Gamma$ we use the following notation: ${}^g h := ghg^{-1}$ and $[g,h] := {}^g h.h^{-1} = g.\,{}^h g^{-1}$. By ${}^\Gamma g$ we mean the conjugacy class of $g \in \Gamma$.

Let C be a nontrivial conjugacy class in Γ. By **C-commutator** we mean an element of $[\Gamma, C] = \{[g,h] : g \in \Gamma, h \in C\}$. If $h \in C$ we will use the name h-commutator as a synonym of C-commutator, for short. Of course $[\Gamma, C] = C.C^{-1}$, thus the set of C-commutators is closed under inverses and conjugation.

The **commutator length** of an element $g \in [\Gamma, \Gamma]$ is the minimal number of commutators sufficient to express g as their product. The **commutator width** of Γ is the maximum of the commutator lengths of elements of its derived subgroup $[\Gamma, \Gamma] = \Gamma'$.

We say that f and $g \in \Gamma$ **commute up to conjugation** if there exist $h \in \Gamma$ such that f and ${}^h g$ commute.

Lemma 2.1. *Assume that α and ${}^h\beta$ commute. Then $[\alpha, \beta]$ is a product of two h-commutators. More precisely $[\alpha, \beta]$ can be written as a product of two conjugates of h and two conjugates of h^{-1} by elements from the group generated by α and β.*

Proof. We have $[\alpha, [\beta, h]] = [{}^\alpha\beta, {}^\alpha h]\left[\beta^{-1}, {}^\beta h\right]$. Also, $[\alpha, [\beta, h]] = \left[\alpha, \beta\ {}^h\beta^{-1}\right] = [\alpha, \beta]$, since α^{-1} commute with ${}^h\beta^{-1}$. □

Following Burago, Ivanov, and Polterovich [7, Sec. 2.1] assume that $H < \Gamma$ is a subgroup, $h \in \Gamma$, and $k \in \mathbf{N} \cup \{\infty\}$. We say that an element h **k-displaces** H if

$$\left[f, {}^{h^j} g\right] = e \text{ for all } f, g \in H \text{ and } j = 1, \ldots, k$$

(hence also $\left[{}^{h^i} f, {}^{h^j} g\right] = e$ for $1 \le |i - j| \le k$).

We will say that h **displaces** H if it 1-displaces H. We say that $H < \Gamma$ is **k-displaceable** in Γ if there exists $h \in \Gamma$ such that h k-displaces H (this property is called *strongly k-displaceable* in [7, Sec. 2.1]). In particular, elements of a displaceable subgroup commute up to conjugation.

Lemma 2.2 ([7, Lemma 2.5])**.** *Assume that $h \in \Gamma$ k-displaces $H < \Gamma$. Let $f \in H'$ be a product of at most k commutators ($k \ge 2$). Then there exist α, β, and $\gamma \in \Gamma$ such that $f = [\alpha, \beta][\gamma, h]$.*

Burago, Polterovich, and Ivanov [7, Theorem 2.2(i)] proved that if for every $k \in \mathbf{N}$ some conjugate of g k-displaces H, then every element of H' is a product of seven g-commutators. We get a better result under a stronger assumption.

Proposition 2.3. *Assume that $g \in \Gamma$ is such that for every finitely generated subgroup $H < \Gamma$ and $k \in \mathbf{N}$, there exists a conjugate of g which k-displaces H. Then every element of Γ' is a product of two commutators in Γ and three g-commutators in Γ. Moreover,*

$$\Gamma' \subseteq {}^{\Gamma'}g\,{}^{\Gamma'}g^{-1}\,{}^{\Gamma'}g\,{}^{\Gamma'}g^{-1}\,{}^{\Gamma'}g\,{}^{\Gamma'}g^{-1} = \left({}^{\Gamma'}g\,{}^{\Gamma'}g^{-1}\right)^3.$$

Proof. Every element $f \in \Gamma'$ can be expressed as a product of k commutators of $2k$ elements of Γ for some $k \in \mathbf{N}$. Call the group they generate H. Since some conjugate of g, say h, k-displaces H, by Lemma 2.2, there exist α, β, and $\gamma \in \Gamma$ such that $f = [\alpha, \beta][\gamma, h]$.

Since some conjugate of g displaces the group generated by α and β, by Lemma 2.1, $[\alpha, \beta]$ is a product of two g-commutators. Thus f is a product of three g-commutators.

The "moreover" part follows from the fact that $[\gamma, h] \in {}^{\Gamma}g\,{}^{\Gamma}g^{-1}$ and that ${}^{\Gamma}h$ and ${}^{\Gamma'}h$ are equal. The last claim follows from the fact that if $f \in \Gamma$ commutes up to commutation with ${}^{g}f^{-1}$, then

$$ {}^{f}h = {}^{f}\left({}^{g}f^{-1}h\right) = {}^{[f,g]}h. $$

□

Note that the assumption of the above corollary implies that neither Γ nor Γ' is finitely generated. However, we will use this approach to prove uniform simplicity of the Higman-Thompson groups which are known to be finitely generated.

Lemma 2.4. *Assume that every two elements in Γ commute up to conjugation. Then every commutator in Γ can be expressed as a commutator in Γ'. In particular, $\Gamma' = \Gamma''$ is perfect.*

Proof. Let α and β belong to Γ. Choose h and g such that α and ${}^{h}\beta$ commute and also ${}^{g}\alpha$ and $[\beta, h]$ commute. Then $[[\alpha, g], [\beta, h]] = [\alpha, [\beta, h]] = [\alpha, \beta]$. □

Proposition 2.5. *Let $g \in \Gamma'$ displace $\Gamma_0 < \Gamma$. Assume that, for every $k \in \mathbf{N}$, every finitely generated subgroup $H < \Gamma_0$ is k-displaceable in Γ_0. Then every element of Γ_0' is a product of four g-commutators from Γ_0'. In particular $\Gamma_0' \subseteq \left({}^{\Gamma_0'}g\,{}^{\Gamma_0'}g^{-1}\right)^4$.*

Proof. By Lemma 2.2 every element of Γ_0' is a product of two commutators of Γ_0. By Lemma 2.4 they can be chosen to be commutators of elements of Γ_0'. By Lemma 2.1 each of them is a product of two g-commutators over Γ_0'. □

3. Bounded actions on ordered sets

The purpose of this section is to prove that numerous simple Higman-Thompson groups acting as order-preserving piecewise-linear transformations are, in fact, uniformly simple.

We always assume that a group Γ acts faithfully on the left by order-preserving transformations on a linearly ordered set $(I, \leq)$. Given a map $g\colon I \to I$, we define the support $\operatorname{supp}(g)$ of g to be $\{x \in I : g(x) \neq x\}$. Given a and $b \in I$ we define $(a, b) = \{y \in I : a < y < b\}$. By (a, ∞) we will denote the set $\{x \in I : a < x\}$. The group of all bounded automorphisms of $(I, \leq)$ is denoted by $\mathrm{B}(I, \leq)$.

We call such an action

- **proximal**, if for every $a, b, c, d \in I$ such that $a < b$ and $c < d$ there is $g \in \Gamma$ satisfying $g(a, b) \supseteq (c, d)$;
- **bounded**, if for every $g \in \Gamma$ there are $a, b \in I$ such that $\operatorname{supp}(g) \subseteq (a, b)$.

Note that being proximal implies that $(I, \leq)$ is dense without endpoints.

Theorem 3.1. *Assume that Γ acts faithfully, order-preserving, boundedly, and proximally on a linearly ordered set $(I, \leq)$. Then its commutator group Γ' is six-uniformly simple and the commutator width of Γ' is at most two.*

Proof. We apply Proposition 2.3. Let g be an arbitrary nontrivial element of Γ'. Let $a \in I$ be such that $g(a) \neq a$. Replacing g by g^{-1} we may assume that $a < g(a)$. Choose $b \in I$ such that $a < b < g(a)$. Then $g(a,b) \cap (a,b) = \varnothing$. Let H be an arbitrary finitely generated subgroup of Γ. Then there exists an interval, say (c,d), containing supports of all generators of H, hence also containing supports of all elements of H. By the proximality of the action, we may assume (possibly conjugating g) that $(c,d) \subseteq (a,b)$. It is clear that such a conjugate of g ∞-displaces H. Thus Proposition 2.3 applies. □

Let us apply Theorem 3.1 to the Higman-Thompson groups of order-preserving piecewise-linear maps. We first recall the definitions. Let $q > r \geq 1$ be integers. Recall that $F_{q,r}$ (F_q, respectively) is defined as piecewise affine (we allow only finitely many pieces), order-preserving bijections of $((0,r) \cap \mathbf{Z}[1/q], \leq)$ ($(\mathbf{Z}[1/q], \leq)$, respectively) whose breaking points of the derivatives belong to $\mathbf{Z}[1/q]$ and the slopes are q^k for $k \in \mathbf{Z}$ (see the bottom of page 53 and the top of page 56 in [5]).

Define $\mathrm{B}F_{q,r}$ ($\mathrm{B}F_q$, respectively) to be the subgroup of $F_{q,r}$ (F_q, respectively) consisting of all such transformations γ that are boundedly supported, that is, $\mathrm{supp}(\gamma) \subseteq (x,y)$ for some $x, y \in (0,r) \cap \mathbf{Z}[1/q]$ ($x, y \in \mathbf{Z}[1/q]$, respectively).

We use the following lemma. The first part of it is a known result [3].

Lemma 3.2.

(1) *The groups $\mathrm{B}F_{q,r}$ and $\mathrm{B}F_q$ are isomorphic* [3, Proposition C10.1].
(2) *The commutator subgroups of $F_{q,r}$ and $\mathrm{B}F_{q,r}$ are equal.*

Proof. (2) It is obvious that $\mathrm{B}F'_{q,r} \subseteq F'_{q,r}$. Let us prove $\supseteq$. Note that $F'_{q,r} \subseteq \mathrm{B}F_{q,r}$ (because for $g_1, g_2 \in F_{q,r}$, the element $[g_1, g_2]$ acts as the identity in some small neighborhoods of 0 and r). Thus, if $f \in F'_{q,r}$, then $\mathrm{supp}(f) \subseteq (b_{-j}, b_j)$ for some $j \in \mathbf{Z}$. Therefore $f(b_{-j}, b_j) = (b_{-j}, b_j)$. A slight modification of ψ above gives $\psi_j \colon \mathbf{Z}[1/q] \to (0,r) \cap \mathbf{Z}[1/q]$ which

- sends $(-\infty, b_{-j}] \cap \mathbf{Z}[1/q]$ piecewise affinely onto $(0, b_{-j}] \cap \mathbf{Z}[1/q]$,
- is the identity on $[b_{-j}, b_j] \cap \mathbf{Z}[1/q]$,
- sends $[b_j, +\infty) \cap \mathbf{Z}[1/q]$ piecewise affinely onto $[b_j, r) \cap \mathbf{Z}[1/q]$.

Then $\psi_j^*(x) = \psi_j x \psi_j^{-1}$ is another isomorphism between $\mathrm{B}F_q$ and $\mathrm{B}F_{q,r}$ such that $\psi_j^*(f) = f$ (we regard $F_{q,r}$ as a subgroup of $\mathrm{B}F_q$). Write $f = \prod_{i=1}^m [g_{2i-1}, g_{2i}]$ for $g_i \in F_{q,r} \subset \mathrm{B}F_q$. Then $f = \psi_j^*(f) = \prod_{i=1}^m [\psi_j^*(g_{2i-1}), \psi_j^*(g_{2i})] \in \mathrm{B}F'_{q,r}$. □

We consider the action of $\mathrm{B}F_q$ on $\mathbf{Z}[1/q]$ and its orbits. Let $I \lhd \mathbf{Z}[1/q]$ be the ideal of $\mathbf{Z}[1/q]$ generated by $(q-1)$.

Lemma 3.3 ([3, Theorem A4.1, Corollary A5.1])**.**

(1) *I is $\mathrm{B}F_q$-invariant.*
(2) *$\mathrm{B}F_q$ acts in a doubly-transitive way on I. In particular, the action is proximal.*

As a corollary of the above lemmata we get that groups $F_{q,r}$ satisfy the assumptions of Theorem 1.1.

Corollary 3.4. *$F'_{q,r} \cong \mathrm{B}F'_q$ is six-uniformly simple and the commutator width of it is at most two.*

Remark 3.5. Theorem 3.1 applies to the following groups.

- Bieri and Strebel [3] define a more general class of groups acting boundedly on $\mathbf{R}$. They take a subgroup P in the multiplicative group $\mathbf{R}_{>0}$ and a $\mathbf{Z}[P]$-submodule $A < \mathbf{R}$ and define $\Gamma := B(\mathbf{R}; A, P)$ to be a group of boundedly supported automorphisms of $\mathbf{R}$ consisting of piecewise affine maps with slopes in P and singularities in A. They define an augmentation ideal $I = \langle p - 1 | p \in P\rangle$ of $\mathbf{Z}[P]$ and prove that Γ acts highly transitively on IA. Thus Γ' is six-uniformly simple.
- Another example of doubly-transitive and bounded action on a linear order (thus satisfying the assumptions of Theorem 3.1) was considered by Chehata in [12], who studied partially affine transformations of an ordered field and proved that this group is simple. Theorem 3.1 implies that the Chehata group is six-uniformly simple.

4. Proximality, primitivity, and double-transitivity

In this section we prove (Theorem 4.1) that proximality (from the previous section) and order-primitivity are equivalent properties for bounded group actions. In general, these properties are inequivalent. The action of the group of integers on itself is primitive but neither proximal nor bounded. We also give an example of bounded, transitive, and proximal action, which is not doubly-transitive (Theorem 4.2).

An action of a group Γ on a linearly ordered set $(I, \leq)$ is called **primitive** (or *order-primitive* by some authors), if for any other linearly ordered set $(J, \leq)$ and homomorphism $\Psi\colon \Gamma \to \mathrm{Aut}(J, \leq)$ and order-preserving equivariant map $\psi\colon (I, \leq) \to (J, \leq)$ (that is, $\psi(\gamma x) = \Psi(\gamma)\psi(x)$), the map ψ is injective or $\psi(I)$ is a singleton.

Theorem 4.1. *Every proximal action is primitive. Any bounded and primitive action is proximal.*

Proof. Assume the action is not primitive. Choose a, b, and d such that $a \neq b$ and $\psi(a) = \psi(b) \neq \psi(d)$. Reversing the order if necessary, we may assume $\psi(b) < \psi(d)$. Set $c = a$. This choice contradicts proximality, as if $g(b, a) \subseteq (d, c)$, then

$$\psi(d) \leq \Psi(g)\psi(b) = \Psi(g)\psi(a) \leq \psi(c) = \psi(b) < \psi(d).$$

Assume that action is bounded, but not proximal. Let a, b, c, and d witness the latter. For $x, y \in I$, $x < y$, consider the relation $\sim_{x,y}$ on I defined as

$$s \sim_{x,y} t \text{ if } s \leq t \text{ and there is no } \gamma \in \Gamma \text{ such that } \gamma(s, t) \supseteq (x, y).$$

By the assumption $a \sim_{c,d} b$. Let $\approx_{c,d}$ be the transitive closure of $\sim_{c,d}$. The symmetric closure $\simeq_{c,d}$ of $\approx_{c,d}$ is transitively closed, thus $\simeq_{c,d}$ is an equivalence relation, which has convex classes. Moreover, $\simeq_{c,d}$ is Γ-invariant, that is, $x \simeq_{c,d} y$ implies $\gamma(x) \simeq_{c,d} \gamma(y)$ for all $\gamma \in \Gamma$. It is enough to prove that $\simeq_{c,d}$ is not total, that is, $e \not\simeq_{c,d} f$ for some $e, f \in I$, because then the quotient map

$$\psi\colon I \to I/\simeq_{c,d}$$

proves nonprimitivity of the action ($I/\simeq_{c,d}$ has a natural Γ-action).

First, we claim that there is $\gamma \in \Gamma$ such that $\gamma(c) \geq d$. Indeed, if there is no such group element, define a map $\psi\colon I \to \{0,1\}$ by the formula

$$\psi(x) = \begin{cases} 0 & \text{if there is no } \gamma \in \Gamma \text{ such that } \gamma(x) \geq d, \\ 1 & \text{if there is } \gamma \in \Gamma \text{ such that } \gamma(x) \geq d. \end{cases}$$

This map would contradict primitivity.

Choose e and f from I such that $\operatorname{supp}(\gamma) \subseteq (e,f)$. Then $\{\gamma^t(c,d) : t \in \mathbf{Z}\}$ is a countable family of intervals in (e,f), which are pairwise disjoint. We claim that $e \not\sim_{c,d} f$, as otherwise there are $x, y \in [e,f]$, $x < y$, such that $x \sim_{c,d} y$ and (x,y) contains $\gamma^t(c,d)$ for some $t \in \mathbf{Z}$, which is impossible. □

Clearly, if Γ acts proximally on $(I, \leq)$, then it acts in such a way on any orbit. Thus, we will restrict to transitive actions.

Examples of actions we discuss above are doubly-transitive (cf. Lemma 3.3(2) and Remark 3.5). Thus they are proximal. This property seems to be easier to check than double-transitivity. We construct below an example of bounded, transitive, and proximal action, which is not doubly-transitive. The reader may compare this result with a result of Holland [24, Theorem 4], which says that every bounded, transitive, primitive, and closed under min, max action must be doubly-transitive. Moreover, any group acting boundedly and transitively cannot be finitely generated. Indeed, a finite number of elements have supports in a common bounded interval, thus the whole group is supported in that interval, so does not act transitively.

Theorem 4.2. *There exists a subgroup $\Gamma < \mathrm{B}(\mathbf{Q}, \leq)$ acting transitively and proximally but not doubly-transitively.*

Proof. For each $k \in \mathbf{N}$ we will define a countable linear order $(I_k, \leq)$, a group Γ_k acting on it, and a function $f_k\colon I_k \times I_k \to \mathbf{Z}$ such that:

(1) $\Gamma_k < \Gamma_{k+1}$;
(2) I_k is a Γ_k-equivariant linear bounded suborder of I_{k+1};
(3) for $k > 0$, Γ_k acts transitively and proximally on I_k by order-preserving transformations (but not doubly-transitively);
(4) f_k is Γ_k-invariant: $f_k(\gamma a, \gamma b) = f_k(a,b)$ for $\gamma \in \Gamma_k$, $a, b \in I_k$, and $f_k \subset f_{k+1}$.

Then we take $\Gamma_\infty = \bigcup_{k \in \mathbf{N}} \Gamma_k$, which acts boundedly, transitively, and proximally, but not doubly-transitively on $I_\infty = \bigcup_{k \in \mathbf{N}} I_k$, because of $f_\infty = \bigcup_{k \in \mathbf{N}} f_k$, which is a Γ_∞-invariant map $I_\infty \times I_\infty \to \mathbf{Z}$.

Since $(I_\infty, \leq)$ is a countable and, by proximality, dense linear order without ends, it is isomorphic to $(\mathbf{Q}, \leq)$.

In the following inductive construction we will define three auxiliary points $i_k^- < i_k < i_k^+$ from I_k.

We put $\Gamma_0 = \mathbf{Z}$ and $I_0 = \mathbf{Z}$, where Γ_0 acts on I_0 by translations. Let $f_0(n,m) = n - m$ and $i_0^- = -1$, $i_0 = 0$, $i_0^+ = 1$.

Assume we have constructed I_k, Γ_k, and f_k. Let

$$I_{k+1} = \left\{ a \in {I_k}^{\mathbf{Z}} : \forall^\infty n \in \mathbf{Z},\ a(n) = i_k \right\}$$

and $i_{k+1}(n) = i_k$ for all $n \in \mathbf{Z}$. In plain words, I_{k+1} consists of all functions from $\mathbf{Z}$ to I_k which differ from a constant function (denoted by i_{k+1}) taking the value i_k, only at finitely many places. Define a linear order on I_{k+1} by putting $a < b$ if $\min\{n \in \mathbf{Z} : a(n) < b(n)\} < \min\{n \in \mathbf{Z} : a(n) < b(n)\}$, with the convention that $\min \varnothing > n$ for all $n \in \mathbf{Z}$. Note that I_k embeds into I_{k+1}:

$$I_k \ni a \mapsto \left(n \mapsto \begin{cases} a & \text{if } n = 0, \\ i_k & \text{otherwise} \end{cases} \right) \in I_{k+1}.$$

Consider $\mathrm{Conv}(I_k) = \{a \in I_{k+1} : a(n) = i_k \text{ for all } n < 0\}$, with the following action of Γ_k:

$$(\gamma a)(n) = \begin{cases} \gamma a(0) & \text{if } n = 0, \\ a(n) & \text{otherwise.} \end{cases}$$

Define

$$i^{\pm}_{k+1}(n) = \begin{cases} i^{\pm}_k & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

The interval $(i^-_{k+1}, i^+_{k-1}) \subset I_{k+1}$ contains the embedded copy of I_k.

Extend the action of Γ_k to the whole of I_{k+1} by the identity on the complement $I_{k+1} \smallsetminus \mathrm{Conv}(I_k)$. Thus the action of Γ_k on I_{k+1} is bounded. Define yet another automorphism σ_{k+1} of I_{k+1} by $(\sigma_{k+1}a)(n) = a(n+1)$. Let Γ_{k+1} be the group generated by Γ_k and σ_{k+1}. The action of Γ_{k+1} on I_{k+1} is clearly transitive.

For every pair $a \neq b$ from I_{k+1}, define $m_{a,b} = \min\{n \in \mathbf{Z} : a(n) \neq b(n)\}$.

For $a<b$ and $c<d$ let $\gamma \in \Gamma_k$ be such that $(c(m_{c,d}), d(m_{c,d})) \subseteq \gamma(a(m_{a,b}), b(m_{a,b}))$ (such γ exists by proximality of the action of Γ_k on I_k). Then

$$(c, d) \subseteq \sigma_{k+1}^{-m_{c,d}} \gamma \sigma_{k+1}^{m_{a,b}+1}(a, b),$$

which proves the proximality of the action of Γ_{k+1} on I_{k+1}.

Finally, define $f_{k+1}(a, b) = f_k(a(m_{a,b}), b(m_{a,b}))$. Clearly, f_{k+1} is Γ_{k+1}-invariant, hence the action of Γ_{k+1} on I_{k+1} is not doubly-transitive. □

The element $\sigma_k \in \Gamma_k$ stabilizes i_k and has unbounded orbits on $(i_k, \infty) \subset I_k$. Thus the stabilizer of $i_\infty = \lim i_k$ has unbounded orbits on $(i_\infty, \infty) \subset I_\infty$. This is enough to conclude that the action is proximal.

Question 4.3. Is there any transitive, proximal bounded action without the property that point stabilizers have unbounded orbits?

5. Extremely proximal actions on a Cantor set and uniform simplicity

The main goal of the present section is prove Theorem 5.1, which gives a criterion for a group acting on a Cantor set to be nine-uniformly simple.

Let C be a Cantor set. Assume that a discrete group Γ acts on C by homeomorphisms. By the **topological full group** $[\![\Gamma]\!] < \mathrm{Homeo}(C)$ of Γ we define (see, e.g., [18])

$$[\![\Gamma]\!] = \left\{ g \in \mathrm{Homeo}(C) : \begin{array}{l} \text{for each } x \in C \text{ there exist a neighborhood} \\ U \text{ of } x \text{ and } \gamma \in \Gamma \text{ such that } g|_U = \gamma|_U \end{array} \right\}.$$

Throughout this section we assume that:

- the group Γ acts faithfully by homeomorphisms on a Cantor set C;
- Γ is a topological full group, i.e., $\Gamma = [\![\Gamma]\!]$;
- the action is **extremely proximal**, i.e., for any nonempty and proper clopen sets $V_1, V_2 \subsetneq C$ there exists $g \in \Gamma$ such that $g(V_2) \subsetneq V_1$.

The second assumption is not hard to satisfy as $[\![\Gamma]\!] = [\![[\![\Gamma]\!]]\!]$.

Theorem 5.1. *Assume that Γ satisfies the above assumptions. Then Γ', the commutator subgroup of Γ, is nine-uniformly simple. The commutator width of Γ' is at most three. Therefore, if Γ is perfect (i.e., $\Gamma' = \Gamma$), then Γ is nine-uniformly simple.*

Before proving Theorem 5.1, we need a couple of auxiliary lemmata.

Suppose $x \in C$ and $h \in \Gamma$. By the Hausdorff property of C, if $h(x) \neq x$, then there exists a clopen subset $U \subset C$ containing x such that $h(U) \cap U = \varnothing$. In such a situation we define an element $\tau_{h,U} \in \Gamma$ exchanging U and $h(U)$:

$$\tau_{h,U}(x) = \begin{cases} x & \text{if } x \notin U \cup h(U), \\ h(x) & \text{if } x \in U, \\ h^{-1}(x) & \text{if } x \in h(U). \end{cases}$$

Such an element belongs to Γ, since $\Gamma = [\![\Gamma]\!]$ is a topological full group. Observe that $\tau_{h,U}^2 = \mathrm{id}$ and $f\tau_{h,U}f^{-1} = \tau_{{}^f h, f(U)}$ for $f \in \Gamma$.

Lemma 5.2. *Assume Γ acts extremely proximally on a Cantor set C.*

(1) *Γ' acts extremely proximally on C.*
(2) *For any nontrivial $f \in \Gamma$ and a proper clopen $V \subsetneq C$ there is $h \in \Gamma'$ such that $V \cap {}^h f(V) = \varnothing$.*
(3) *Let $f, g \in \Gamma$ be nontrivial. Then there exists $h \in \Gamma'$ such that ${}^h g.f$ is supported outside a clopen subset.*

Proof. (1) Let U and V be nonempty and proper clopen subsets of C. Shrinking U, if necessary, we may assume that $U \cup V \neq C$ (that is, we may always take $g \in \Gamma$ and $U_1 = g(U)$, $V_1 = g(V)$ such that $U_1 \cup V_1 \neq C$; then $h(U_1) \subsetneq V_1$ implies $h^g(U) \subsetneq V$). By extremal proximality, find elements g_1, g_2, h_1, and h_2 in Γ such that $g_1(U) \subsetneq C \smallsetminus (U \cup V)$, $g_2(U) \subsetneq C \smallsetminus (U \cup V \cup g_1(U))$, $h_1(V) \subsetneq g_1(U)$, and $h_2(U) \subsetneq C \smallsetminus (U \cup V \cup g_1(U))$. Define $g = \tau_{g_2,U}\tau_{g_1,U}$ and $h = \tau_{h_2,U}\tau_{h_1,U}$.

It is straightforward to check that, since U, $g_1(U)$, and $g_2(U)$ are pairwise disjoint, we have $g^3 = 1$ which is equivalent to

$$g = \tau_{g_2,U}\tau_{g_1,U} = [\tau_{g_1,U}\tau_{g_2,U}]\,;$$

and similarly for h. In particular, g and h belong to Γ'. Furthermore, $g^{-1}h(U) = g^{-1}h_1(U) \subsetneq g^{-1}g_1(V) = V$.

(2) Choose U to be a nonempty clopen such that $f(U) \cap U = \varnothing$. Choose, by (1), $h \in \Gamma'$ such that $h^{-1}(V) \subsetneq U$. Then $V \cap {}^h f(V) \subseteq h(U \cap f(U)) = \varnothing$.

(3) We may choose clopens U and V such that $f(U) \cap U = \varnothing = g(V) \cap V$. If $h_1 \in \Gamma'$ satisfies $h_1^{-1}(U) \subsetneq V$, then ${}^{h_1}g(U) \cap U = \varnothing$ (such an h_1 exists by (2)).

If ${}^{h_1}gf$ is the identity on U the proof is finished. Otherwise define $\gamma = {}^{h_1}g$. We may find $W \subset U$ such that $\gamma f(W) \cap W = \varnothing$ and $W \cup f(W) \cup \gamma^{-1}(W) \subsetneq C$. Notice that $\gamma^{-1}(W)$, W, and $f(W)$ are pairwise disjoint.

Choose $\eta \in \Gamma$ such that $\eta(W) \cap \big(W \cup f(W) \cup \gamma^{-1}(W)\big) = \varnothing$. Put $\tau_1 = \tau_{\eta\gamma,\gamma^{-1}W}$, $\tau_2 = \tau_{f\gamma,\gamma^{-1}(W)}$, and $h_2 = [\tau_1, \tau_2]$. As in (1), we have that $h_2 = \tau_1\tau_2 \in \Gamma'$ and if $w \in W$, then $h_2(w) = w$ and $h_2\gamma^{-1}(w) = \tau_1 f^{-1} w = f^{-1}w$.

Hence ${}^{h_2h_1}gf = {}^{h_2}\gamma f$ is the identity on W. Indeed, let $w \in W$. Then $f(w) \in f(W)$. Thus $h_2^{-1}f(w) = \gamma^{-1}(w)$, i.e., $\gamma h_2^{-1}f(w) = w \in W$. Therefore $h_2\gamma h_2^{-1}f(w) = w$. □

For any clopen $U \subset C$, let Γ_U be the subgroup of Γ consisting of elements of Γ supported on U.

Lemma 5.3. *Let $V \subsetneq C$ be a proper clopen set. Then there exists a proper clopen $V \subsetneq U \subsetneq C$ such that $\Gamma' \cap \Gamma_V \subset \Gamma'_U$.*

Proof. Let $\alpha \in \Gamma$ be such that $\alpha(V) \supsetneq V$. Let $U = V \cup \alpha(C \smallsetminus V) \subsetneq C$. Define $\psi\colon U \to C$:

$$\psi(x) = \begin{cases} x & \text{if } x \in V, \\ \alpha^{-1}(x) & \text{if } x \in \alpha(C \smallsetminus V). \end{cases}$$

Then ψ is a homeomorphism, which induces an isomorphism $\Psi\colon \Gamma \to \Gamma_U$ given by

$$\Psi(h)(x) = \begin{cases} x & \text{if } x \in C \smallsetminus U, \\ \psi^{-1}(h(\psi(x))) & \text{if } x \in U, \end{cases}$$

for any $h \in \Gamma$ and $x \in C$. Since Ψ is the identity on Γ_V, $\Psi(f) = f$ for any $f \in \Gamma_V$. Therefore, if $f \in \Gamma'$, then $f \in \Gamma'_U$. □

Lemma 5.4. *Assume that $U \subsetneq V \subseteq C$ are clopens. There exists $h \in \Gamma'_V$ such that for all $k \in \mathbf{Z}$, the sets $h^k(U)$ are pairwise disjoint.*

Proof. Choose clopen W such that $U \subsetneq W \subsetneq V$. By extremal proximality, choose β and $\gamma \in \Gamma$ such that $\beta(W) \subset V \smallsetminus W$ and $\gamma(W) \subset W \smallsetminus U$. Define $\alpha \in \Gamma_V$ by

$$\alpha(x) = \begin{cases} x & \text{if } x \in C \smallsetminus (W \cup \beta(W)), \\ \gamma^{-1}(x) & \text{if } x \in \gamma(W), \\ \beta(x) & \text{if } x \in W \smallsetminus \gamma(W), \\ {}^{\beta}\gamma(x) & \text{if } x \in \beta(W). \end{cases}$$

Then the sets $\alpha^k(U)$ are pairwise disjoint. Indeed, it is sufficient to prove that $\alpha^k(U) \cap U = \varnothing$ for all $k > 0$. Since $U \subset W \smallsetminus \gamma(W)$, we have $\alpha(U) \subset \beta(W)$. As $\alpha\beta(W) \subset \beta(W)$, for $k \geq 1$, $\alpha^k(U) \subset \beta(W)$ which is disjoint from U.

Since $\tau_{\beta,W} \in \Gamma_V$ conjugates α to α^{-1}, the element $h = \alpha^2 = [\alpha, \tau_{\beta,W}]$ satisfies the claim. □

Proof of Theorem 5.1. Let f be an element of Γ' and let A be a nontrivial conjugacy class of Γ'. By Lemmata 5.2(3) and 5.3 we have that $f = g_1^{-1}f_1$ for some $g_1 \in A$ and $f_1 \in \Gamma'_{V_1}$ for some proper clopen $V_1 \subsetneq C$.

We claim that f_1 is a product of four A-commutators in Γ'. Choose $V_1 \subsetneq V_0 \subsetneq C$ and $\omega \in V_0 \smallsetminus V_1$. We apply Proposition 2.5. Namely, let Γ_0 denote the union of groups Γ_V such that V is a clopen contained in $V_0 \smallsetminus \{\omega\}$. Clearly, Γ_0 is a proper subgroup of Γ_{V_0}. By Lemma 5.2(2), we may choose $g \in A$ such that $g(V_0) \cap V_0 = \varnothing$. Thus, g displaces Γ_0. Let H be a finitely generated subgroup of Γ_0. The union of supports of its generators is a clopen U, properly contained in V_0, since $\omega \notin U$. Hence $H < \Gamma_U < \Gamma_0$. Choose $U \subsetneq V \subsetneq V_0$ such that $\omega \notin V$. Let $h \in \Gamma'_V < \Gamma'_0$

be as in Lemma 5.4. Then h ∞-displaces H. Thus Proposition 2.5 applies and $f_1 \in \Gamma'_{V_1} < \Gamma'_0$ is a product of four g-commutators.

By Lemma 2.2, the commutator width of Γ'_0 is at most two. By Lemma 5.2(3), every element decomposes as a product of a conjugate of a given nontrivial element from Γ', say a commutator, and an element conjugate into Γ'_0. Thus every element of Γ' is a product of three commutators. □

6. Groups almost acting on trees

In this section we apply Theorem 5.1 to groups almost acting on trees.

By a **graph** (whose elements are called **vertices**) we mean a set, equipped with a symmetric relation called adjacency. A **path** is a sequence of vertices indexed either by a set $\{1, \dots, n\}$ or $\mathbf{N}$ (in such a case we call the path a **ray**) such that consecutive vertices are adjacent, and no vertices whose indices differ by two coincide (i.e., there are no backtracks). A graph is called a **tree** if it is connected (nonempty) and has no cycles, i.e., paths of positive length starting and ending at the same vertex (in particular, the adjacency relation is irreflexive).

Ends of T are classes of infinite rays in T. Two rays are equivalent if they coincide except for some finite (not necessarily of the same cardinality) subsets. The set of all ends of T is denoted by ∂T and is called the **boundary of** T.

Given a pair of adjacent vertices (called an **oriented edge**) $\vec{e} = (v, w)$, we call the set of terminal vertices of paths starting at $\vec{e}$ a **halftree** of T and we will denote it by $T_{\vec{e}}$. The classes of rays starting at $\vec{e}$ will be called the end of a halftree $T_{\vec{e}}$ and will be denoted by $\partial T_{\vec{e}} \subset \partial T$. By $-\vec{e}$ we denote the pair (w, v).

We endow ∂T with a topology, where the basis of open sets consists of ends of all halftrees.

A **valency** of a vertex v is the cardinality of the set of vertices adjacent to v. A vertex of valency one is called a **leaf**. If every vertex has valency at least three but finite, then the boundary ∂T is easily seen to be compact, totally disconnected, without isolated points, and metrizable. Thus, ∂T is a Cantor set. In such a case, every end $\partial T_{\vec{e}}$ of a halftree is a clopen (open and closed) subset of ∂T.

A **spheromorphism** is a class of permutations of T which preserve all but finitely many adjacency (and nonadjacency) relations. Two such maps are equivalent if they differ on a finite set of vertices (see, e.g., [17, Section 3]). We denote the group of all spheromorphisms of T by $\mathrm{AAut}(T)$. If T is infinite, then the natural map $\mathrm{Aut}(T) \to \mathrm{AAut}(T)$ is an embedding. Every spheromorphism $f \in \mathrm{AAut}(T)$ induces a homeomorphism of its boundary ∂T.

For an integer $q > 1$, by T_q we denote the regular tree whose vertices have degree $(q + 1)$. The group N_q was introduced by Neretin in [27, 4.5, 3.4] as the group $\mathrm{AAut}(T_q)$ of spheromorphisms of the $(q+1)$-regular tree T_q. It is abstractly simple [25].

In what follows, we will be interested in subgroups $\Gamma < \mathrm{Aut}(T)$ acting extremely proximally on the boundary ∂T (see Theorem 6.4 and Corollary 6.7 below). The whole group of automorphisms $\Gamma = \mathrm{Aut}(T_q)$ of T_q is such an example. Another example (cf. Example 6.8) is the automorphism group $\Gamma = \mathrm{Aut}(T_{s,t})$ of a bi-regular tree $T_{s,t}$, $s, t > 2$ (i.e., every vertex of $T_{s,t}$ is black or white, every black vertex is adjacent with s white vertices, every white — with t black vertices). We prove that the group $[\![\Gamma]\!]$ of partial Γ-actions on ∂T is then nine-uniformly simple.

The group $\mathrm{Aut}(T_{s,t})$ itself is virtually 8-uniformly simple [19, Theorem 3.2]. (Bounded simplicity in [19] means uniform simplicity in our context.)

There is a connection between the notion of a spheromorphism and a topological full group acting on a boundary of a tree.

Example 6.1.

(1) Any subdivision of ∂T into clopens can be refined to $\mathcal{U}_1$, a subdivision into ends of halftrees (since any clopen in ∂T is a finite union of boundaries of halftrees). Therefore the Neretin group N_q can be characterized as $N_q = [\![\mathrm{Aut}(T_q)]\!] = \mathrm{AAut}(T_q)$.

(2) Another, well studied, example comes from considering

$$\mathrm{Aut}_0(T_q) = \left\{\begin{matrix}\text{automorphisms of } T_q \text{ preserving chosen cyclic orders} \\ \text{on edges adjacent to any vertex of } T_q\end{matrix}\right\}.$$

One may induce cyclic orders by planar representation of T_q. The group $[\![\mathrm{Aut}_0(T_q)]\!]$ is the Higman-Thompson group $G_{q,2}$ [17, Section 5], [25, 2.2].

(3) The previous two examples can be generalized in the following manner (see [8, Section 3.2]). Let $c\colon E(T_q) \to \{0,\dots,q\}$ be a function from the set $E(T_q)$ of (undirected) edges of the $(q+1)$-regular tree T_q such that for every vertex v, the restriction of c to the set of edges $E(v)$ starting at v gives a bijection with $\{0,\dots,q\}$. We say that such c is a **proper coloring** of T_q. Let $F < S_{q+1}$ be a subgroup of permutations of $\{0,\dots,q\}$. Using proper coloring c and F we define the **universal group** $U(F)$ to be

$$U(F) = \left\{g \in \mathrm{Aut}(T_q) : c \circ g \circ c^{-1}_{|E(v)} \in F \text{ for every vertex } v\right\}.$$

In fact $U(F)$ is independent (up to conjugation in $\mathrm{Aut}(T_q)$) of the choice of proper coloring c. We prove (see Corollary 6.6) that $[\![U(F)]\!]'$ is nine-uniformly simple, provided that F is transitive on $\{0,\dots,q\}$. If F is generated by a $(q+1)$-cycle, then $U(F) = \mathrm{Aut}_0(T_q)$ from (2). If $F = S_{q+1}$, then $U(F) = \mathrm{Aut}(T_q)$.

We call an action for a group Γ on a tree T **minimal** if there is no proper Γ-invariant subtree of T. Given a subset A of a tree, we define its **convex hull** to be the set of all vertices which lie on paths with both ends in the set A. It is a subtree. The action is minimal if and only if the convex hull of any orbit is the whole tree.

Example 6.2. Every action on a leafless tree with a finite quotient is minimal. The converse is not true (see Example 6.8).

Indeed, the distance from a Γ-orbit is a bounded function. Hence the complement of an orbit cannot contain an infinite ray. Thus every vertex lies on a path with endpoints in a given orbit.

Lemma 6.3 ([28, Lemma 4.1])**.** *Assume that a group Γ acts minimally on a leafless tree T. Then for every vertex v and an edge $\vec{e}$ the orbit Γv intersects the halftree $T_{\vec{e}}$.*

Proof. If Γv is all contained in $T_{-\vec{e}}$, so is its convex hull. Thus the claim. □

We call an action for a group Γ on a tree T **parabolic** if Γ has a fixed point in ∂T.

An action of a group by homeomorphisms on a topological space is called **minimal** if there is no proper nonempty closed invariant set (equivalently, if every orbit is dense). This notion should not cause confusion with the notion of minimal actions on trees. (A tree is a set equipped with a relation as opposed to its geometric realization which is a topological space.)

Theorem 6.4. *Assume that T is a leafless tree such that ∂T is a Cantor set. Let Γ act on T. The following are equivalent:*

(1) *The action of Γ on ∂T is extremely proximal (see the beginning of Section 5 for the definitions).*
(2) *The action of $[\![\Gamma]\!]$ on ∂T is extremely proximal.*
(3) *The action of Γ on ∂T is minimal and ∂T does not support any Γ-invariant probability measure.*
(4) *The action of Γ on T is minimal and not parabolic, that is, there is no proper Γ-invariant subtree of T and Γ has no fixed point in ∂T.*

Proof. ($1 \Rightarrow 2$) This is straightforward.

($2 \Rightarrow 3$) Let F be a closed, nonempty, proper, and Γ-invariant subset of ∂T. Choose $x \in F$ and a proper clopen $V \subset \partial T$ containing x. Define $U = \partial T \smallsetminus F$. Then there is no $g \in [\![\Gamma]\!]$ such that $g(V) \subset U$, since $g(x) = \gamma(x) \in F$ for some $\gamma \in \Gamma$; thus a contradiction.

Similarly, let μ be a Γ-invariant measure on ∂T. Decompose $\partial T = U_1 \cup U_2 \cup U_3$, where the U_i's are disjoint nonempty clopens. We may assume that $\mu(U_1) < 1/2$. Then there is no $g \in [\![\Gamma]\!]$ such that $g(U_2 \cup U_3) \subset U_1$. Indeed, for any $g \in [\![\Gamma]\!]$ we may decompose $U_2 \cup U_3$ (by compactness) as a finite disjoint union of clopens $U_2 \cup U_3 = \bigcup_{i=1}^k V_i$ such that $g|_{V_i} = \gamma_i|_{V_i}$ for some $\gamma_i \in \Gamma$ and then

$$1/2 < \mu(U_2 \cup U_3) = \sum_{i=1}^k \mu(V_i) = \sum_{i=1}^k \mu(\gamma_i V_i) < \mu(U_1) < 1/2$$

is a contradiction. Hence, the action is not extremely proximal.

($3 \Rightarrow 4$) If there is an infinite Γ-invariant subtree T' of T or a fixed point $\omega \in \partial T$, then either $\partial T'$ or $\{\omega\}$ is a Γ-invariant closed subset of ∂T.

Suppose that there exists a finite Γ-invariant subtree T' of T. We use the following definition. Given a vertex v of T, we define the **visual measure** associated to v to be the unique measure μ_v on ∂T with the following property: if $\{v_i\}_{i=0}^n$ is any injective path starting at $v_0 = v$, then

$$\mu_v\left(\partial T_{(v_{n-1},v_n)}\right) = \frac{1}{d_0 \prod_{i=1}^{n-1}(d_i - 1)},$$

where d_i is the valence of v_i. The visual metric μ_v is obviously invariant under the action of the stabilizer $\mathrm{Stab}(v)$ of v in $\mathrm{Aut}(T)$.

We can consider the average of the visual measures associated to the vertices of this subtree T'. It will be a Γ-invariant measure on ∂T.

($4 \Rightarrow 1$) By Lemma 6.3 we may assume that, for every pair of edges $\vec{e}$ and $\vec{f}$, there is $\gamma \in \Gamma$ such that either $T_{\gamma\vec{e}}$ or $T_{-\gamma\vec{e}}$ is strictly contained in $T_{\vec{f}}$. It is enough to show that one can find $\gamma \in \Gamma$ such that the latter holds, i.e., $\partial T_{-\gamma(\vec{e})} \subsetneq \partial T_{\vec{f}}$ (indeed, since ends of halftrees constitute a basis, we can find edges $\vec{e}$ and $\vec{f}$ such that $\partial T_{\vec{e}} \subset U$ and $\partial T_{\vec{f}} \subset C \smallsetminus V$ for nonempty proper clopens V and U in ∂T; if there is $\gamma \in \Gamma$ such that $\partial T_{-\gamma(\vec{e})} \subsetneq \partial T_{\vec{f}}$, then $\gamma V \subseteq \partial T_{-\gamma(\vec{e})} \subsetneq \partial T_{\vec{f}} \subseteq U$).

It is enough to prove this claim for $\vec{e} = \vec{f}$. Indeed, if there exists $\gamma_1 \in \Gamma$ such that $T_{\gamma_1\vec{e}} \subsetneq T_{\vec{f}}$ and $T_{-\gamma_2\vec{e}} \subsetneq T_{\vec{e}}$, then $T_{-\gamma_1\gamma_2\vec{e}} \subsetneq T_{\gamma_1\vec{e}} \subsetneq T_{\vec{f}}$.

Assume that there exists $\gamma \in \Gamma$ such that $T_{\gamma\vec{e}} \subsetneq T_{\vec{e}}$. Let $\{v_i\}_{i=0}^n$ be a path such that $\vec{e} = (v_0, v_1)$ and $\gamma\vec{e} = (v_{n-1}, v_n)$. Then $\{v_i\}_{i\in\mathbf{Z}}$, defined as $v_{nq+r} = \gamma^q v_r$, is a bi-infinite path. Let ω be its end as $i \to \infty$. Choose $\eta \in \Gamma$ such that $\eta(\omega) \neq \omega$. Consider the bi-infinite path from ω to $\eta(\omega)$. It coincides with $\{v_i\}_{i<i_-}$ and $\{\eta v_{-i}\}_{i>i_+}$ for some $i_\pm \in \mathbf{Z}$. Therefore $T_{-\eta\gamma^k\vec{e}} \subsetneq T_{\gamma^k\vec{e}}$ for k big enough. Hence, $T_{-\gamma^{-k}\eta\gamma^k\vec{e}} \subsetneq T_{\vec{e}}$. Thus the claim. □

Remark 6.5. Only clause (3) from Theorem 6.4 concerns an action of a group on a tree. The other parts of Theorem 6.4 are about actions on a Cantor set. We do not know if there is a straight argument for proving equivalence of (1) and (4) from Theorem 6.4, without referring to actions on trees.

Below is an application of Theorems 5.1 and 6.4 to the Neretin groups and the Higman-Thompson groups.

Corollary 6.6.

(1) *Suppose $F < S_{q+1}$ is a transitive permutation subgroup and let c be a proper coloring of T_q (see Example* 6.1(3)*). Then $U(F)$ acts transitively on the directed edges of T_q, and thus $[\![U(F)]\!]'$ is nine-uniformly simple.*

(2) *Fix natural numbers $q > r \geq 1$. The commutator subgroup N_q' of the Neretin group N_q and the Higman-Thompson group $G_{q,r}'$, are nine-uniformly simple and have commutator width bounded by three.*

Proof. Let $\Gamma = U(F)$. Then the action of Γ on T_q is not parabolic as there is no $\mathrm{Stab}(v)$-fixed edge adjacent to v, hence no $\mathrm{Stab}(v)$-fixed ray. It is minimal since the action is transitive.

Therefore, in the case of the Neretin group N_q' and the Higman-Thompson group $G_{q,2}'$, Theorem 5.1 applies immediately due to Theorem 6.4.

Suppose $\mathcal{F}$ is a family of pairwise disjoint ends of halftrees $\partial T_{\vec{e}_i} \subset \partial T_q$ for $0 \leq i \leq q-r$. If $\Gamma_{\mathcal{F}}$ is a pointwise stabilizer of $\mathcal{F}$ in $[\![\mathrm{Aut}_0(T_q)]\!]$ (see Example 6.1(2)), then $\Gamma_{\mathcal{F}}$ is isomorphic to $G_{q,r}$ [17, Section 5]. Moreover, $\Gamma_{\mathcal{F}}$ is its own topological full group acting extremely proximally on $C = \partial T_q \smallsetminus \bigcup_{i=0}^{q-r} \partial T_{\vec{e}_i}$. Hence we get the conclusion for $G_{q,r}'$. □

Corollary 6.7. *Suppose $\Gamma = \mathbf{F}_n$ is a free group of rank $n \geq 2$. Then Γ acts on its Cayley graph, which is T_{2n-1}. This action is transitive and clearly not parabolic. Thus the induced action on the boundary is extremely proximal. Therefore $[\![\mathbf{F}_n]\!]'$ is nine-uniformly simple by Theorem* 5.1.

Example 6.8 ([28, Section 5], [19, p. 232])**.** We apply our results to trees constructed by Tits. Any connected graph (G, E) of finite valence, with at least one edge, can appear as a quotient of a (finite valence) tree.

Assume that c is a function from oriented edges of G into the set of positive integers. By a result of Tits, there is a tree T and a group Γ acting on T such that $G = \Gamma\backslash T$ and, for any v and $w \in T$ such that $(\Gamma v, \Gamma w)$ is an edge in G, there are exactly $c(\Gamma v, \Gamma w)$ vertices in Γw adjacent to v (or none if it is not an edge of G).

If c is such that the sum over edges starting at a given vertex is at least three (but finite), then the boundary of T is a Cantor set.

If values of c are at least two, the group action of Γ on T is minimal and not parabolic [28, 5.7], i.e., the action of Γ on ∂T is extremely proximal due to Theorem 6.4, and $[\![\Gamma]\!]$ is nine-uniformly simple due to Theorem 5.1.

Corollary 6.9. *The groups of quasi-isometries and almost-isometries of a regular tree T_q are five-uniformly simple.*

Proof. This follows from Lazarovich's results from the appendix. Let Γ be one of those groups. By Theorem 7.4, $\Gamma = \Gamma'$. Since $\mathrm{Aut}(T_q)$ is a subgroup of Γ, it acts extremely proximally on ∂T_q (see Lemma 7.1) as a topological full group (see Lemma 7.2). This already proves nine-uniform simplicity.

Let $1 \neq g$ and f be two elements of Γ. By Lemma 5.2 there exists g_1, a conjugate of g, such that $f_1 = g_1^{-1} f$ fixes a clopen in ∂T_q. By Lemma 7.3, f_1 is a commutator of two elements fixing an open set in ∂T_q. Thus, by Lemma 2.1, f_1 is a product of two g-commutators. □

7. Appendix by Nir Lazarovich: Simplicity of $\mathrm{AI}(T_q)$ and $\mathrm{QI}(T_q)$

We begin by recalling the following definitions.

For $\lambda \geq 1$ and $K \geq 0$, a (λ, K)**-quasi-isometry** between two metric spaces (X, d_X) and (Y, d_Y) is a map $f\colon X \to Y$ such that for all $x, x' \in X$,

$$\lambda^{-1} d_X(x, x') - K \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + K,$$

and for all $y \in Y$ there exists $x \in X$ such that $d_Y(y, f(x)) \leq K$.

A K**-almost-isometry** is a $(1, K)$-quasi-isometry.

A map f is a **quasi-isometry** (resp., **almost-isometry**) if there exist K and λ (resp., K) for which it is a (λ, K)-quasi-isometry (resp., K-almost-isometry).

Two quasi-isometries $f_1, f_2\colon X \to Y$ are **equivalent** if they are at bounded distance (with respect to the supremum metric).

The group of all quasi-isometries (resp., almost-isometries) from a metric space X to itself, up to equivalence, is denoted by $\mathrm{QI}(X)$ (resp., $\mathrm{AI}(X)$). Thus, for $q \geq 2$, we have the following containments:

$$\mathrm{Aut}(T_q) \subset N_q \subset \mathrm{AI}(T_q) \subset \mathrm{QI}(T_q) \subset \mathrm{Homeo}(\partial T_q),$$

where the last containment follows from the following lemma.

Lemma 7.1. *The group $\mathrm{QI}(T_q)$ acts faithfully on ∂T_q.*

Proof. Let $g \in \mathrm{QI}(T_q)$ be a quasi-isometry. Let $v \in T_q$, and let x_1, x_2, $x_3 \in \partial T_q$ be three distinct points such that v is the median of x_1, x_2, x_3, that is, v is the unique intersection of all three (bi-infinite) geodesics x_1x_2, x_1x_3, x_2x_3. Then, by the stability of quasi-geodesics in Gromov hyperbolic spaces [4, Theorem 1.7], gv is at bounded distance (which does not depend on the vertex v) from the midpoint of gx_1, gx_2, gx_3. This implies that if g induces the identity map at the boundary, then $g \sim \mathrm{id}$. □

In fact, the proof above is valid whenever the space X is a proper geodesic Gromov hyperbolic space X which has a Gromov boundary of cardinality at least three whose convex hull is at bounded distance from X (e.g., any nonelementary hyperbolic group).

For what follows, let Γ be the group $\mathrm{QI}(T_q)$ or $\mathrm{AI}(T_q)$ for $q \geq 2$.

Lemma 7.2. *The group* $\Gamma < \mathrm{Homeo}(\partial T_q)$ *is a topological full group.*

Proof. Fix $g \in [\![\Gamma]\!]$, and let $\{\partial T_{\vec{e}_1}, \dots, \partial T_{\vec{e}_n}\}$ be a disjoint cover of ∂T such that $g|_{\partial T_{\vec{e}_i}} = \gamma_i|_{\partial T_{\vec{e}_i}}$ for some $\gamma_i \in \Gamma$. For each $1 \le i \le n$ let $\vec{e}_{i,1}, \dots, \vec{e}_{i,m}$ be such that $\{\partial T_{\vec{e}_{i,1}}, \dots, \partial T_{\vec{e}_{i,m}}\}$ is a disjoint cover of $gT_{\vec{e}_i}$. We may assume, by changing each γ_i on a bounded set, that $\gamma_i(T_{\vec{e}_i}) = \bigcup_{j=1}^m T_{\vec{e}_{i,j}}$.

Let us define

$$\gamma(v) = \begin{cases} \gamma_i(v) & \text{for } v \in T_{\vec{e}_i}, \\ v & \text{otherwise.} \end{cases}$$

It is clear that if γ is in Γ, then it induces the element g on the boundary.

Let λ, K be the maximal quasi-isometry constants of γ_i, and let M be the diameter of the bounded set $\{\vec{e}_1, \dots, \vec{e}_n, \gamma\vec{e}_1, \dots, \gamma\vec{e}_n\}$.

We claim the following: for all $v, w \in T_q$, $d(\gamma v, \gamma w) \le \lambda d(v, w) + (2K + M)$. Indeed, if v, w are both in some $T_{\vec{e}_i}$ or in $T_q \setminus \bigcup_{i=1}^n T_{\vec{e}_i}$, then the inequality is obvious. If $v \in T_{\vec{e}_i}$ and $w \in T_{\vec{e}_j}$ for some $i \ne j$, then $d(v, w) = d(v, e_i) + d(e_i, e_j) + d(e_j, w)$ and therefore

$$\begin{aligned} d(\gamma x, \gamma y) &= d(\gamma_i x, \gamma_i e_i) + d(\gamma_i e_i, \gamma_j e_j) + d(\gamma_j e_j, \gamma_j w) \\ &\le \lambda d(\gamma v, \gamma e_i) + K + M + \lambda d(e_j, w) + K \\ &\le \lambda d(x, y) + 2K + M. \end{aligned}$$

Similarly, one shows this inequality for $v \in T_{\vec{e}_i}$ and $w \in T_q \setminus \bigcup_{i=1}^n T_{\vec{e}_i}$.

Furthermore, the element γ', defined as

$$\gamma'(v) = \begin{cases} \gamma_i^{-1}(v) & \text{if } v \in \bigcup_{j=1}^m T_{\vec{e}_{i,j}}, \\ v & \text{otherwise,} \end{cases}$$

satisfies that for all $v, w \in T_q$, $d(\gamma' v, \gamma' w) \le \lambda' d(v, w) + (2K' + M')$ for the appropriate λ', K', and M'. Moreover, it is easy to see that $\gamma\gamma' \sim \mathrm{id} \sim \gamma'\gamma$, from which we deduce that γ is a quasi-isometry. □

Lemma 7.3. *Every element* g *in* Γ *that fixes an open set at the boundary is a commutator of two elements fixing a common set at the boundary.*

Proof. Let $\operatorname{supp} g \subset T_{\vec{e}}$. Let $\{x_n\}_{n \in \mathbf{Z}}$ be a bi-infinite line geodesic contained in $T_{-\vec{e}}$ and such that x_0 is the starting point of $\vec{e}$.

Let $t \in \mathrm{Aut}(T_q)$ be a translation along $\{x_n\}_{n \in \mathbf{Z}}$, and let f be the function defined by

$$f(v) = \begin{cases} {}^{t^n}g^{-1}(v) & \text{for } v \in t^n(T_{\vec{e}}) \text{ and } n \ge 0, \\ v & \text{elsewhere.} \end{cases}$$

The function f is in Γ since all the functions ${}^{t^n}g^{-1}$ have the same quasi-isometry constants and $[t, f] = {}^t f f^{-1} = g$.

Let s be a 1-almost-isometry defined as

$$s(v) = \begin{cases} t(v) & \text{for } v \in t^{-2}T_{\vec{e}}, \\ t^{-1}(v) & \text{for } v \in t^{-1}T_{\vec{e}}, \\ v & \text{otherwise.} \end{cases}$$

Then we still have $[ts, f] = g$ as s commutes with f. However both st and f fix $t^{-1}T_{\vec{e}}$. Thus the claim. □

Theorem 7.4. *The group Γ is perfect and has commutator width at most 2.*

Proof. It suffices to show that each element of Γ can be written as a product of two elements of Γ which fix an open set at the boundary, as both of them are single commutators by Lemma 7.3.

Let $1 \neq g \in \Gamma$; there exists $\omega \in \partial T$ such that $g(\omega) \neq \omega$. Let $T_{\vec{e}}$ be a halftree whose boundary contains ω and for which $g\partial T_{\vec{e}}$ and $\partial T_{\vec{e}}$ are disjoint, and do not cover the whole of ∂T. Let $h \in [\![\Gamma]\!] = \Gamma$ be the map defined by

$$h(x) = \begin{cases} g(x) & \text{if } x \in \partial T_{\vec{e}}, \\ g^{-1}(x) & \text{if } x \in g\partial T_{\vec{e}}, \\ x & \text{otherwise.} \end{cases}$$

We see that hg fixes $T_{\vec{e}}$, and thus the claim. □

Remark 7.5. Since, for all $q_1, q_2 \geq 2$, the trees T_{q_1} and T_{q_2} are quasi-isometric, the groups $\mathrm{QI}(T_{q_1})$ and $\mathrm{QI}(T_{q_2})$ are isomorphic.

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