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DIMENSION AND TRACE OF THE KAUFFMAN BRACKET SKEIN ALGEBRA

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ABSTRACT. Let F be a finite type surface and ζ a complex root of unity. The Kauffman bracket skein algebra $K_\zeta(F)$ is an important object in both classical and quantum topology as it has relations to the character variety, the Teichmüller space, the Jones polynomial, and the Witten-Reshetikhin-Turaev Topological Quantum Field Theories. We compute the rank and trace of $K_\zeta(F)$ over its center, and we extend a theorem of the first and second authors in [Math. Z. 289 (2018), pp. 889–920] which says the skein algebra has a splitting coming from two pants decompositions of F.

1. Introduction

Let F be a finite type surface and ζ a complex root of unity. The Kauffman bracket skein algebra $K_\zeta(F)$ is an important object in both classical and quantum topology as it has relations to the character variety, the Teichmüller space, the Jones polynomial, and the Witten-Reshetikhin-Turaev Topological Quantum Field Theories. We recall the definition of $K_\zeta(F)$ in Section 3.

The linear representations of $K_\zeta(F)$ play an important role in hyperbolic Topological Quantum Field Theories. In [13] we prove the Unicity Conjecture of Bonahon and Wong [6] which among other things states that generically all irreducible representations of $K_\zeta(F)$ have the same dimension equal to the square root of the dimension of $K_\zeta(F)$ over its center $Z_\zeta(F)$. Here if A is an algebra whose center is a domain Z then the dimension of A over Z, denoted by $\dim_Z A$, is defined to be the dimension of the vector space $A \otimes_Z \tilde{Z}$ over the field of fractions $\tilde{Z}$ of Z. The calculation of the dimension $\dim_{Z_\zeta(F)} K_\zeta(F)$ is one of the main result of this paper.

Theorem 1 (See Theorem 6.1)**.** *Suppose F is a finite type surface of genus g with p punctures and negative Euler characteristic, and ζ is a root of unity of order* $\mathrm{ord}(\zeta)$. *Let m be the order of ζ^4, then*

$$\dim_{Z_\zeta(F)} K_\zeta(F) = \begin{cases} m^{6g-6+2p} & \text{if } \mathrm{ord}(\zeta) \not\equiv 0 \pmod 4 \\ 2^{2g}\, m^{6g-6+2p} & \text{if } \mathrm{ord}(\zeta) \equiv 0 \pmod 4. \end{cases}$$

We show that $\tilde{K}_\zeta(F) := K_\zeta(F) \otimes_{Z_\zeta(F)} \tilde{Z}_\zeta(F)$, where $\tilde{Z}_\zeta(F)$ is the field of fractions of $Z_\zeta(F)$, is a division algebra having finite dimension over its center $\tilde{Z}_\zeta(F)$. Thus every element $\alpha \in \tilde{K}_\zeta(F)$ lies in a finite field extension of $\tilde{Z}_\zeta(F)$ and hence has a

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reduced trace $\mathrm{tr}_{\tilde{Z}_\zeta(F)}(\alpha) \in \tilde{Z}_\zeta(F)$. We recall the definition of the reduced trace in Section 2.

The second goal of the paper is to compute the reduced trace of elements of $K_\zeta(F)$. To state the theorem, denote by $\mathscr{S}$ the set of all isotopy classes of simple diagrams on surface F, where a simple diagram is the union of disjoint, non-trivial simple closed curves on F. For each $\alpha \in \mathscr{S}$ one can define an element $T(\alpha) \in K_\zeta(F)$, such that the set $\{T(\alpha) \mid \alpha \in \mathscr{S}\}$ is a $\mathbb{C}$-basis of $K_\zeta(F)$ and $T(\alpha)$ is central if and only if α is in a certain subset $\mathscr{S}_\zeta$ of $\mathscr{S}$. See Section 3.3 for details. The definition of $T(\alpha)$ involves Bonahon and Wong's threading map [5]. As the $\mathbb{C}$-vector space $K_\zeta(F)$ has basis $\{T(\alpha) \mid \alpha \in \mathscr{S}\}$, hence it is enough to compute the trace of each $T(\alpha)$.

Theorem 2 (See Theorem 8.1)**.** *Let F be a finite type surface and ζ be a root of 1. For a simple diagram $\alpha \in \mathscr{S}$ one has*

$$\mathrm{tr}_{\tilde{Z}_\zeta(F)}(T(\alpha)) = \begin{cases} T(\alpha) & \text{if } T(\alpha) \text{ is central, i.e., } \alpha \in \mathscr{S}_\zeta \\ 0 & \text{otherwise.} \end{cases}$$

Along the way we develop tools for determining when a collection of skeins forms a basis for $\tilde{K}_\zeta(F)$.

The last goal of the paper is to prove that there exists a splitting of $\tilde{K}_\zeta(F)$ over its center coming from pairs of pants decompositions of the surface.

Theorem 3 (See Theorem 9.1)**.** *Let F be a finite type surface of negative Euler characteristic. There exist two pants decompositions $\mathcal{P}$ and $\mathcal{Q}$ of F such that for any root of unity ζ the $\tilde{Z}_\zeta(F)$-linear map*

$$\psi : \mathcal{C}_\zeta(\mathcal{P}) \otimes_{\tilde{Z}_\zeta(F)} \mathcal{C}_\zeta(\mathcal{Q}) \to \tilde{K}_\zeta(F),$$

defined by the property that $\psi(x \otimes y) = xy$, is a $\tilde{Z}_\zeta(F)$-linear isomorphism of vector spaces. Here $\mathcal{C}_\zeta(\mathcal{P})$ (respectively $\mathcal{C}_\zeta(\mathcal{Q})$) is the $\tilde{Z}_\zeta(F)$-subalgebra of $\tilde{K}_\zeta(F)$ generated by the curves in $\mathcal{P}$ (respectively in $\mathcal{Q}$). Both $\mathcal{C}_\zeta(\mathcal{P})$ and $\mathcal{C}_\zeta(\mathcal{Q})$ are maximal commutative subalgebras of the division algebra $\tilde{K}_\zeta(F)$.

This theorem has an application in defining invariants of links in 3-manifold which will be investigated in a future work.

The paper is organized as follows. In Section 2 we survey results about division algebras that have finite rank over their center, and facts about trace and filtrations of algebras, with the goal of applying these to the Kauffman bracket skein algebra. We follow by introducing the Kauffman bracket skein algebra in Section 3. Its basis is given in terms of simple diagrams, so we describe ways of parametrizing simple diagrams on a surface. We also introduce a residue group. In section 4 we show that after enough twisting the Dehn Thurston coordinates of a simple diagram on a closed surface stabilize to become an affine function of the number of twists. This allows us to define stable Dehn-Thurston coordinates. In Section 5 we introduce a degree map and use it to formulate a criterion for independence of a collection of skeins over its center. Section 6 computes the dimension of the Kauffman bracket skein algebra over its center, proving Theorem 1. In section 7 we find bases for commutative subalgebras of $\tilde{K}_\zeta(F)$ generated by the curves in a primitive non-peripheral diagram on F with coefficients in $\tilde{Z}_\zeta(F)$. In Section 8 we find a formula for computing the trace, proving Theorem 2. The paper concludes in Section 9 which proves the splitting theorem (Theorem 3).

2. Division algebras, trace, filtrations

In this section we survey some well-known facts about division algebras, trace, and filtrations of algebras that will be used in the paper.

2.1. **Notations and conventions.** Throughout the paper $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote respectively the set of natural numbers, integers, rational numbers, real numbers, and complex numbers. Note that $0 \in \mathbb{N}$. Let $\mathbb{Z}_2 = \mathbb{Z}/(2\mathbb{Z})$ be the field with 2 elements.

A complex number ζ is a *root of 1* if there is a positive integer n such that $\zeta^n = 1$, and the smallest such positive integer is called the *order* of ζ, denoted by $\mathrm{ord}(\zeta)$.

All rings are assumed to be associative with unit, and ring homomorphisms preserve 1. A *domain* is a ring A, not necessarily commutative, such that if $xy = 0$ with $x, y \in A$, then $x = 0$ or $y = 0$. For a ring A denote by A^* the set of all non-zero elements in A. For example, $\mathbb{C}^*$ is the set of all non-zero complex number.

2.2. **Algebras finitely generated over their centers.** Recall that a *division algebra* is an associative algebra with unit such that every nonzero element has a multiplicative inverse. Note that a commutative division algebra is a field. The following is well-known, and we present a simple proof for completeness.

Proposition 2.1.

(a) If k is a field and A is k-algebra which is a domain and has finite dimension over k, then A is a division algebra.

(b) Let Z be the center of a domain A and $\tilde{Z}$ be the field of fractions of Z. Assume A is finitely generated as a Z-module. Then $\tilde{A} = A \otimes_Z \tilde{Z}$ is a division algebra.

Proof.

(a) We need to show that any non-zero element of A has an inverse. Suppose $0 \neq a \in A$. The k-subalgebra of A generated by a is a commutative domain, and it is a finite extension of the field k. This implies that the k-subalgebra generated by a is a field (see Proposition 5.7 in [3]), hence a has an inverse.

(b) Every element of $\tilde{A}$ can be presented in the form az^{-1} where $a \in A$ and $0 \neq z \in Z$. From here it is easy to show that $\tilde{A}$ is a domain and the natural map $A \to \tilde{A}$ is an embedding. Since A is Z-finitely generated, $\tilde{A}$ is also $\tilde{Z}$-finitely generated, and (b) follows from (a). □

The above proposition reduces many problems concerning domains which are finitely generated as modules over their centers to the case of division algebras finitely generated over their centers.

2.3. **Trace.** Suppose k is a field and A is a k-algebra which is finite-dimensional as a k-vector space. For $a \in A$, the left multiplication by a is a k-linear operator acting on A, and its trace is denoted by $\mathrm{TR}_{A/k}(a)$. The *reduced trace* is defined by

$$\mathrm{tr}_{A/k}(a) = \frac{1}{\dim_k A} \mathrm{TR}_{A/k}(a) \in k. \tag{1}$$

Again, the following is well-known.

Proposition 2.2. *Suppose that k is a field, and A is a division k-algebra having finite dimension over k. Suppose $0 \neq a \in A$.*

(a) If $P(x) = x^l + c_{l-1}x^{l-1} + \dots c_1 x + c_0$ is the minimal polynomial of a over k, then $\mathrm{tr}_{A/k}(a) = -c_{l-1}/l$.

(b) If B is a division algebra with $a \in B \subset A$, then $\mathrm{tr}_{B/k}(a) = \mathrm{tr}_{A/k}(a)$.

(c) The function $\mathrm{tr}_{A/k} : A \to k$ is non-degenerate in the sense that for $0 \neq a \in A$ there exists $b \in A$ such that $\mathrm{tr}_{A/k}(ab) \neq 0$. In particular, A is a Frobenius algebra.

Proof.

(a) Let C be the k-subalgebra of A generated by a, then C is a field. By the definition of the minimal polynomial,

$$C \cong k[x]/(P(x)). \tag{2}$$

As a k-vector space C has basis $\{1, a, \dots, a^{l-1}\}$. It follows that $\mathrm{TR}_{C/k} = -c_{l-1}$. Let $\{a_1, \dots, a_t\}$ be a basis of A over C so that $A = \bigoplus_{i=1}^t Ca_i$. Each Ca_i is invariant under the left multiplication by a, and the action of a on each Ca_i has trace equal to $\mathrm{TR}_{C/k}(a)$. Hence,

$$\mathrm{TR}_{A/k}(a) = t\mathrm{TR}_{C/k}(a) = -tc_{l-1} = (\dim_k A)\frac{-c_{l-1}}{l}. \tag{3}$$

From here we have $\mathrm{tr}_{A/k}(a) = \frac{-c_{l-1}}{l}$.

(b) follows immediately from (a).

(c) Let $b = a^{-1}$. Then $\mathrm{tr}(ab) = \mathrm{tr}(1) = 1 \neq 0$. □

2.4. **Maximal commutative subalgebras.** Suppose A is a division algebra with center k. If $C \subset A$ is a maximal commutative subalgebra, then C is a field and

$$\dim_k A = (\dim_k C)^2. \tag{4}$$

This follows directly from Theorem 15.8 in [15]. Moreover A splits over C, i.e., $A \otimes_k C$ is isomorphic to the algebra of $d \times d$ matrices with entries in C for some d.

2.5. **Dimension.** Suppose a $\mathbb{C}$-algebra A has center Z which is a commutative domain. Let $\tilde{Z}$ be the field of fractions of Z. The *dimension* $\dim_Z A$ is defined to be the dimension of the $\tilde{Z}$-vector space $A \otimes_Z \tilde{Z}$.

A *filtration compatible with the product* of A is a sequence $\{F_i\}_{i=0}^\infty$ of $\mathbb{C}$-subspaces of A such that $F_i \subset F_{i+1}$, $\bigcup_{i=0}^\infty F_i = A$, and $F_j F_j \subset F_{i+j}$. For any subset $X \subset A$ let $F_i(X) = F_i \cap X$.

Lemma 2.3. *Suppose $\dim_Z A < \infty$ and $\dim_{\mathbb{C}} F_i(A) < \infty$ for every i. There exists a positive integer u such that for all $k \geq u$,*

$$\dim_Z A \leq \frac{\dim_{\mathbb{C}} F_k(A)}{\dim_{\mathbb{C}} F_{k-u}(Z)}. \tag{5}$$

Proof. Assume $a_1, \dots, a_d \in A$ form a basis of $A \otimes_Z \tilde{Z}$ over $\tilde{Z}$. Let u be a number such that all a_j are in $F_u(A)$. Since $a_1, \dots, a_d$ are linearly independent over Z, the sum $\sum_{j=1}^d Za_j$ is a direct sum. We have

$$\bigoplus_{j=1}^{d} F_{k-u}(Z)a_j \subset F_k(\bigoplus_{j=1}^{d} Za_j) \subset F_k(A). \tag{6}$$

The dimension of the first space in (6) is $d \dim_{\mathbb{C}} F_{k-u}(Z)$, while the dimension of the last one is $\dim_{\mathbb{C}} F_k(A)$. Hence, $d \dim_{\mathbb{C}} F_{k-u}(Z) \leq \dim_{\mathbb{C}} F_k(A)$, which is (5). □

2.6. **Lattice points in a polytope.** Suppose $\mathbb{R}^n$ is the standard n-dimensional Euclidean space. A *lattice* $\Lambda \subset \mathbb{R}^n$ is any abelian subgroup of maximal rank n. A *convex polyhedron* Q is the convex hull of a finite number of points in $\mathbb{R}^n$ and its n-dimensional volume is denoted by $\mathrm{vol}(Q)$. Let

$$kQ = \{kx \mid x \in Q\}. \tag{7}$$

Lemma 2.4. *Suppose* $\Gamma \subset \Lambda$ *are lattices in* $\mathbb{R}^n$ *and* $Q \subset \mathbb{R}^n$ *is the union of a finite number of convex polyhedra with* $\mathrm{vol}(Q) > 0$. *Let* u *be a positive integer. One has*

$$\lim_{k\to\infty} \frac{|\Lambda \cap kQ|}{|\Gamma \cap (k-u)Q|} = [\Lambda : \Gamma]. \tag{8}$$

Proof. Define $\mathrm{vol}(\Lambda)$ to be the n-dimensional volume of the parallelepiped spanned by a $\mathbb{Z}$-basis of Λ. One has

$$[\Lambda : \Gamma] = \frac{\mathrm{vol}(\Gamma)}{\mathrm{vol}(\Lambda)}, \qquad \lim_{k\to\infty} \frac{|\Lambda \cap kQ|}{k^n} = \frac{\mathrm{vol}(Q)}{\mathrm{vol}(\Lambda)} \tag{9}$$

from which one easily obtains (8). □

3. Kauffman bracket skein algebra

The Kauffman bracket skein module of a 3-manifold was introduced independently by Przytycki [26] and Turaev [32, 33]. In this section we recall the definition of the Kauffman bracket skein algebra of a finite type surface F, and present some results concerning its center. We also explain how to coordinatize the set of curves on F and use coordinates to define a residue group associated to F and a root of 1.

3.1. **Finite type surface.** An oriented surface F of the form $F = \bar{F} \setminus \mathcal{V}$, where $\bar{F}$ is an oriented closed connected surface and $\mathcal{V}$ is finite (possibly empty), is called a *finite type surface*. A point in $\mathcal{V}$ is called a puncture. The genus $g = g(F)$ and the puncture number $p = |\mathcal{V}|$ totally determine the diffeomorphism class of F, and for this reason we denote $F = F_{g,p}$. The Euler characteristic of F is $2 - 2g - p$, which is non-negative only in 4 cases:

$$(g, p) = (0, 0), (0, 1), (0, 2), \text{ or } (1, 0). \tag{10}$$

Since the analysis of these four surfaces is simple and requires other techniques, very often we consider these cases separately.

Throughout this section we fix a finite type surface $F = F_{g,p}$.

In this paper a *loop* on F is a unoriented submanifold diffeomorphic to the standard circle. A loop is *trivial* if it bounds a disk in F; it is *peripheral* if it bounds a disk in $\bar{F}$ which contains exactly one puncture. A *simple diagram* is the union of several disjoint non-trivial loops. A simple diagram is *peripheral* if all its components are peripheral.

If $a : [0, 1] \to \bar{F}$ is a smooth map such that $a(0), a(1) \in \mathcal{V}$ and a embeds $(0, 1)$ into F then the image of $(0, 1)$ is called an *ideal arc*. Isotopies of ideal arcs are always considered in the class of ideal arcs.

Suppose $\alpha \subset F$ is either an ideal arc or a simple diagram and $\beta \subset F$ is a simple diagram. The *geometric intersection number* $I(\alpha, \beta)$ is the minimum of $|\alpha' \cap \beta'|$, with all possible α' isotopic to α and β' isotopic to β. We say that α is β-*taut*, or α and β are *taut*, if they are transverse and $|\alpha \cap \beta| = I(\alpha, \beta)$.

A simple diagram $\alpha \subset F$ is *even* if $I(\alpha, a)$ is even for every loop $a \subset F$. It is easy to see that α is even if and only if α represents the zero element in the homology group $H_1(\bar{F}, \mathbb{Z}_2)$.

Very often we identify a simple diagram with its isotopy class. Denote by $\mathscr{S} = \mathscr{S}(F)$ the set of all isotopy classes of simple diagrams on F. Let $\mathscr{S}^{\mathrm{ev}} \subset \mathscr{S}$ be the subset of all classes of even simple diagrams, and $\mathscr{S}^{\partial} \subset \mathscr{S}$ be the subset of all peripheral ones. For convenience, we make the convention that the empty set $\emptyset$ is a peripheral simple diagram. Thus

$$\emptyset \in \mathscr{S}^{\partial} \subset \mathscr{S}^{\mathrm{ev}} \subset \mathscr{S}. \tag{11}$$

3.2. **Kauffman bracket skein algebra.** A *framed link* in $F \times [0, 1]$ is an embedding of a disjoint union of oriented annuli in $F \times [0, 1]$. By convention the empty set is considered as a framed link with 0 components and is isotopic only to itself.

For a non-zero complex number ζ, the *Kauffman bracket skein module* of F at ζ, denoted by $K_\zeta(F)$, is the $\mathbb{C}$-vector space freely spanned by all isotopy classes of framed links in $F \times [0, 1]$ subject to the following *skein relations*

$$\diagup\!\!\!\!\diagdown = \zeta \asymp + \zeta^{-1})(, \tag{12}$$

$$\bigcirc \sqcup L = (-\zeta^2 - \zeta^{-2}) L. \tag{13}$$

Here the framed links in each expression are identical outside the balls pictured in the diagrams, and the arcs in the pictures are supposed to have blackboard framing. If the two arcs in the crossing belong to the same component then it is assumed that the same side of the annulus is up.

Theorem 3.1 ([27,29])**.** *The set $\mathscr{S}$ of isotopy classes of simple diagrams is a basis of $K_\zeta(F)$ over $\mathbb{C}$.*

For two framed links L_1 and L_2 in $F \times [0, 1]$, their product, $L_1 L_2$, is defined by first isotoping L_1 into $F \times (1/2, 1)$ and L_2 into $F \times (0, 1/2)$ and then taking the union of the two. This product gives $K_\zeta(F)$ the structure of a $\mathbb{C}$-algebra, which is in most cases non-commutative. Let $Z_\zeta(F)$ be the center of $K_\zeta(F)$.

Theorem 3.2. *Let F be a finite type surface and ζ a root of 1.*
(a) [28]. *The algebra $K_\zeta(F)$ is a domain.*
(b) [13]. *The module $K_\zeta(F)$ is finitely generated as a $Z_\zeta(F)$-module.*

The localized skein algebra of F is defined by

$$\tilde{K}_\zeta(F) = K_\zeta(F) \otimes_{Z_\zeta(F)} \tilde{Z}_\zeta(F), \tag{14}$$

where $\tilde{Z}_\zeta(F)$ is the field of fractions of the center $Z_\zeta(F)$. From Proposition 2.1, we have the following corollary.

Corollary 3.3. *For a finite type surface F and a root of unity ζ, the localized skein algebra $\tilde{K}_\zeta(F)$ is a division algebra.*

Remark 3.4. Corollary 3.3 was proved in [2] for the case when $p \geq 1$ and $\mathrm{ord}(\zeta) \neq 0$ mod 4, using explicit calculation of the trace.

3.3. **Chebyshev basis and center.** The *Chebyshev polynomials of the first kind* are defined recursively by

$$T_0(x) = 2, \quad T_1(x) = x \quad \text{and } T_k(x) = xT_{k-1}(x) - T_{k-2}(x). \tag{15}$$

They satisfy the product to sum formula,

$$T_k(x)T_l(x) = T_{k+l}(x) + T_{|k-l|}(x). \tag{16}$$

Suppose $\alpha \in \mathscr{S}$ is a simple diagram. Some components of α may be isotopic to each other. Let $C_1, \dots, C_k$ be a maximal collection of components of α such that no two of them are isotopic. Then there are positive integers $(l_1, \dots, l_k)$ such that α is the union of l_j parallel copies of C_j with $j = 1, \dots, k$. In other words, $\alpha = \prod_{j=1}^k C_j^{l_j}$ in $K_\zeta(F)$. Let

$$T(\alpha) = \prod_{j=1}^{k} T_{l_j}(C_j) \in K_\zeta(F). \tag{17}$$

If $\alpha, \beta \in \mathscr{S}$ then in general $\alpha\beta \notin \mathscr{S}$. However, if $I(\alpha, \beta) = 0$ where $\alpha, \beta \in \mathscr{S}$, then α and β can be represented by disjoint simple diagrams, and hence $\alpha\beta \in \mathscr{S}$. In particular, if $\alpha \in \mathscr{S}$ and $\beta \in \mathscr{S}^\partial$, then $\alpha\beta = \beta\alpha \in \mathscr{S}$. Further, if $\alpha \in \mathscr{S}$, then $\alpha^k \in \mathscr{S}$.

Recall that $\mathscr{S}^{\text{ev}}$ denotes the set of even simple diagrams. For a root ζ of 1 with $\text{ord}(\zeta^4) = m$ define the following subset $\mathscr{S}_\zeta$ of $\mathscr{S}$:

$$\mathscr{S}_\zeta := \begin{cases} \{\alpha\beta^m \mid \alpha \in \mathscr{S}^\partial, \beta \in \mathscr{S}\} & \text{if } \text{ord}(\zeta) \neq 0 \mod 4 \\ \{\alpha\beta^m \mid \alpha \in \mathscr{S}^\partial, \beta \in \mathscr{S}^{\text{ev}}\} & \text{if } \text{ord}(\zeta) = 0 \mod 4. \end{cases} \tag{18}$$

Theorem 3.5 ([13])**.** *Let F be a finite type surface and ζ be a root of 1. Recall that $Z_\zeta(F)$ is the center of the skein algebra $K_\zeta(F)$. We have the following Chebyshev bases:*

(a) $\{T(\alpha) \mid \alpha \in \mathscr{S}\}$ is a basis of the $\mathbb{C}$-vector space $K_\zeta(F)$.

(b) $\{T(\alpha) \mid \alpha \in \mathscr{S}_\zeta\}$ is a basis of the $\mathbb{C}$-vector space $Z_\zeta(F)$.

The point is while $K_\zeta(F)$ has a $\mathbb{C}$-basis parameterized by $\mathscr{S}$, its center $Z_\zeta(F)$ has a basis parameterized by $\mathscr{S}_\zeta$.

Remark 3.6. The Chebyshev basis is important in the theory of quantum cluster algebras and quantum Teichmüller spaces of surfaces. It was first used, for the case when F is a torus, by Frohman-Gelca [11]. The famous positivity conjecture states that the Chebyshev basis is positive [17, 30].

3.4. **Filtrations on skein algebras.** Suppose $\mathfrak{A} = \{a_1, \dots, a_k\}$, where each a_i is an ideal arc or a loop on F.

For each $n \in \mathbb{N}$ define $F_n = F_n^{\mathfrak{A}}(K_\zeta(F))$ to be the $\mathbb{C}$-subspace of $K_\zeta(F)$ spanned by simple diagrams $\alpha \in \mathscr{S}$ such that $\sum_{i=1}^k I(\alpha, a_i) \leq n$. The following is well-known and its variants were used extensively in the study of skein algebras, see e.g., [13, 16, 18].

Proposition 3.7. *The sequence $\{F_n\}_{n=0}^\infty$ is a filtration of $K_\zeta(F)$ compatible with the product.*

Proof. It is clear that $F_n \subset F_{n+1}$ and $\bigcup F_n = K_\zeta(F)$. It remains to show that $F_n F_l \subset F_{n+l}$. Suppose $\alpha, \beta \in \mathscr{S}$ and $\alpha \in F_n, \beta \in F_l$. The product $\alpha\beta$ is obtained by placing α above β. Using the skein relation (12) we see that $\alpha\beta = \sum c_\gamma \gamma$, where each γ is a diagram obtained by a smoothing of all the crossings in $\alpha\beta$ and hence $I(\gamma, a_i) \le I(\alpha, a_i) + I(\beta, a_i)$. It follows that $\alpha\beta \in F_{n+l}$. □

3.5. **Coordinates and residues, open surface case.** When $F = F_{g,p}$ (a finite type surface of genus g with p punctures) has negative Euler characteristic, one can parameterize the set $\mathscr{S}$ of simple diagrams on F by embedding it into the free abelian group $\mathbb{Z}^r$, where

$$r = 6g - 6 + 3p = -3\chi(F). \tag{19}$$

This embedding depends on an object that we call the *coordinate datum.* In this subsection we describe this embedding for open surfaces.

Suppose $F = F_{g,p}$ with $p \ge 1$ and F has negative Euler characteristic, $\chi(F) = 2 - 2g - p$.

By definition, a *coordinate datum* of F is an *ordered ideal triangulation*, which is any sequence $\{e_1, \dots, e_r\}$ of disjoint ideal arcs on F such that no two are isotopic. Here r is given by (19).

Such ideal triangulations always exist and we fix one of them. The ideal arcs $(e_i)_{i=1}^r$ cut F into triangles.

Recall that $I(\alpha, e_i)$ is the geometric intersection number. Define

$$\nu : \mathscr{S} \to \mathbb{N}^r, \quad \text{by} \quad \nu(\alpha) := (\nu_i(\alpha))_{i=1}^r, \text{ where } \nu_i(\alpha) := I(\alpha, e_i). \tag{20}$$

It is known that ν is injective, and its image $\mathcal{A} := \nu(\mathscr{S})$ consists of all $(n_1, \dots, n_r) \in \mathbb{N}^r$ such that whenever e_i, e_j, e_k are edges of a triangle

$$n_i + n_j + n_k \text{ is even and } n_i \le n_j + n_k. \tag{21}$$

We call $\nu(\alpha)$ the *edge-coordinates* of α with respect to the coordinate datum. Let

$$S : \mathcal{A} \to \mathscr{S} \tag{22}$$

be the inverse of ν. That is, $S(\mathbf{n})$ is the simple diagram whose edge coordinates are $\mathbf{n}$. Note that $0 = \nu(\emptyset) \in \mathcal{A}$, and $\mathcal{A}$ is closed under addition. Hence $\mathcal{A}$ is a submonoid of $\mathbb{Z}^r$. For a submonoid X of $\mathbb{Z}^r$ let $\overline{X}$ be the subgroup of $\mathbb{Z}^r$ generated by X, then

$$\overline{X} = \{x_1 - x_2 \mid x_1, x_2 \in X\}. \tag{23}$$

Lemma 3.8. *Let X be a submonoid of $\mathbb{Z}^k$ and Y a subgroup of $\overline{X}$. If $\overline{X}/Y$ is finite then the monoid homomorphism $\phi : X \to \overline{X}/Y$ is surjective.*

Proof. As X generates $\overline{X}$ as a group, $\phi(X)$ generates $\overline{X}/Y$. The monoid $\phi(X)$, being finite, is a group. Hence $\phi(X) = \overline{X}/Y$. □

Lemma 3.9. *Let $F = F_{g,p}$ be a finite type surface with a fixed ideal triangulation, $r = 6g - 6 + 3p$, and $\mathcal{A} \subset \mathbb{Z}^r$ be the subset of edge-coordinates as defined above.*

(a) $\overline{\mathcal{A}}$ is the subset of $\mathbb{Z}^r$ consisting of $(n_1, \dots, n_r)$ such that whenever e_i, e_j, e_k are edges of a triangle, $n_i + n_j + n_k$ is even.

(b) The index of $\overline{\mathcal{A}}$ in $\mathbb{Z}^r$ is $2^{4g-5+2p} = 2^{-2\chi(F)-1}$.

Proof.

(a) follows from the description (21) of $\mathcal{A}$.

(b) Recall that $\bar{F}$ denotes the oriented closed connected surface obtained from F by adding back all punctures. Consider the triangulation of F as a cellular decomposition of $\bar{F}$ which has p zero-cells and r one-cells. Identify $(\mathbb{Z}_2)^r$ with the set of all maps from one-cells to $\mathbb{Z}_2$, and let $C_1 \subset (\mathbb{Z}_2)^r$ be the set of all one-cocycles. Let $f : \mathbb{Z}^r \to (\mathbb{Z}_2)^r$ be the reduction modulo 2, then $\overline{\mathcal{A}} = f^{-1}(C_1)$. Therefore $\mathbb{Z}^r/\overline{\mathcal{A}} \cong (\mathbb{Z}_2)^r/C_1$ and hence

$$[\mathbb{Z}^r : \overline{\mathcal{A}}] = |(\mathbb{Z}_2)^r|/|C_1|= 2^r/|C_1|. \tag{24}$$

The following sequence is exact

$$0 \to H_0(\bar{F}, \mathbb{Z}_2) \to C_0 \xrightarrow{\delta} C_1 \to H_1(\bar{F}, \mathbb{Z}_2) \to 0, \tag{25}$$

where $C_0 = (\mathbb{Z}_2)^p$ is the set of all 0-cochains. In an exact sequence of finite groups, the alternating product of orders of groups is 1. Hence

$$|C_1|= \frac{|C_0||H_1(\bar{F}, \mathbb{Z}_2)|}{|H_0(\bar{F}, \mathbb{Z}_2)|} = \frac{2^p 2^{2g}}{2} = 2^{2g+p-1}. \tag{26}$$

Plugging this value of $|C_1|$ in (24), we get the result. □

The following follows easily from the definition.

Proposition 3.10. *Let $F = F_{g,p}$ be a finite type surface with fixed coordinate datum. If $\alpha, \beta \in \mathscr{S}$ are two simple diagrams with $I(\alpha, \beta) = 0$, then $\alpha\beta \in \mathscr{S}$ and*

$$\nu(\alpha\beta) = \nu(\alpha) + \nu(\beta). \tag{27}$$

□

Recall that $\mathscr{S}^{\text{ev}} \subset \mathscr{S}$ is the subset of all classes of even simple diagrams and $\mathscr{S}^\partial \subset \mathscr{S}$ is the subset of all peripheral ones. Let $\mathcal{A}^{\text{ev}} := \nu(\mathscr{S}^{\text{ev}}), \mathcal{A}^\partial = \nu(\mathscr{S}^\partial), \mathcal{A}_\zeta = \nu(\mathscr{S}_\zeta)$, where $\mathscr{S}_\zeta$ is defined by (18). Each of $\mathcal{A}, \mathcal{A}^{\text{ev}}, \mathcal{A}^\partial, \mathcal{A}_\zeta$ is a submonoid of $\mathbb{N}^r \subset \mathbb{Z}^r$.

Recall that for a submonoid X of $\mathbb{Z}^r$ we denote by $\overline{X}$ the subgroup of $\mathbb{Z}^r$ generated by X.

Proposition 3.11. *Let $F = F_{g,p}$ be a finite type surface with fixed coordinate datum, and let $\mathcal{A}^\partial$, $\mathcal{A}^{\text{ev}}$ be the submonoids of $\mathbb{Z}^r$ defined above, where $r = 6g-6+3p$.*

(a) The group $\overline{\mathcal{A}^\partial}$ is a direct summand of $\overline{\mathcal{A}}$.

(b) The quotient $\overline{\mathcal{A}}/\overline{\mathcal{A}^{\text{ev}}}$ is isomorphic to $(\mathbb{Z}_2)^{2g}$.

Proof.

(a) The group $\overline{\mathcal{A}^\partial}$ is a direct summand of $\overline{\mathcal{A}}$ if and only if it is *primitive* in the sense that

(*) if $kx \in \overline{\mathcal{A}^\partial}$, where k is a positive integer and $x \in \overline{\mathcal{A}}$,

then $x \in \overline{\mathcal{A}^\partial}$.

Let $\mathscr{S}^\bullet \subset \mathscr{S}$ be the set of all simple diagrams containing no peripheral loops, with the convention that the empty diagram is in $\mathscr{S}^\bullet$, and let $\mathcal{A}^\bullet = \nu(\mathscr{S}^\bullet)$. Every simple diagram can be presented in a unique way as the product of an element in $\mathcal{A}^\partial$ and an element in $\mathcal{A}^\bullet$. By Proposition 3.10 every $x \in \mathcal{A}$ can be presented uniquely as

$$x = x^\bullet + x^\partial, \quad \text{where } x^\bullet \in \mathcal{A}^\bullet, x^\partial \in \mathcal{A}^\partial. \tag{28}$$

Suppose $x \in \overline{\mathcal{A}}$ satisfies (*). There are $x_1, x_2 \in \mathcal{A}$ such that $x = x_1 - x_2$ (see (23)). Since $kx \in \overline{\mathcal{A}^\partial}$, there are $y_1, y_2 \in \mathcal{A}^\partial$ such that $k(x_1 - x_2) = y_2 - y_1$. Using the decomposition (28) for x_1 and x_2, we get

$$kx_1^\bullet + (kx_1^\partial + y_1) = kx_2^\bullet + (kx_2^\partial + y_2). \tag{29}$$

By Proposition 3.10, we have $kx_1^\bullet, kx_2^\bullet \in \mathcal{A}^\bullet$. The uniqueness of (28) shows that $kx_1^\bullet = kx_2^\bullet$, or $x_1^\bullet = x_2^\bullet$. It follows that $x = x_1 - x_2 = x_1^\partial - x_2^\partial \in \overline{\mathcal{A}^\partial}$, proving (a).

(b) The composition

$$\mathcal{A} \xrightarrow{S} \mathscr{S} \xrightarrow{h} H_1(\bar{F}, \mathbb{Z}_2), \tag{30}$$

where $h(\alpha)$ is the homology class of α and S is defined by (22), is a surjective monoid homomorphism and extends to a surjective group homomorphism $\bar{h} : \overline{\mathcal{A}} \to H_1(\bar{F}, \mathbb{Z}_2)$. By definition, $\ker \bar{h} = \overline{\mathcal{A}^{\mathrm{ev}}}$. Hence $\overline{\mathcal{A}}/\overline{\mathcal{A}^{\mathrm{ev}}} \cong H_1(\bar{F}, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$. □

Suppose ζ is a root of 1, and consider the Kauffman bracket skein module of surface F at ζ. From Theorem 3.5 and the bijection $\nu : \mathscr{S} \to \mathcal{A}$, we see that $K_\zeta(F)$ has a $\mathbb{C}$-basis parameterized by $\mathcal{A}$ while the center $Z_\zeta(F)$ has a $\mathbb{C}$-basis parameterized by $\mathcal{A}_\zeta$. Hence we want to study the quotient $\mathcal{A}/\mathcal{A}_\zeta$. Define the ζ-residue group of surface F

$$\mathfrak{R}_\zeta(F) = \overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta}, \tag{31}$$

which depends on a coordinate datum of F. Let $m = \mathrm{ord}(\zeta^4)$. Define the following integer

$$D_\zeta(F) = \begin{cases} m^{6g-6+2p} & \text{if} \quad \mathrm{ord}(\zeta) \neq 0 \mod 4 \\ 2^{2g} m^{6g-6+2p} & \text{if} \quad \mathrm{ord}(\zeta) = 0 \mod 4 \end{cases}. \tag{32}$$

The following proposition gives the size of the ζ-residue group of a surface F with at least one puncture.

Proposition 3.12. *Suppose that the finite type surface $F = F_{g,p}$ has negative Euler characteristic and that $p \geq 1$, and let ζ be a root of 1. For any coordinate datum of F,*

$$|\mathfrak{R}_\zeta(F)| = D_\zeta(F). \tag{33}$$

Proof.

Case 1 ($\mathrm{ord}(\zeta) \neq 0 \mod 4$). Then $\mathcal{A}_\zeta = \mathcal{A}^\partial + m\mathcal{A}$, and

$$\mathfrak{R}_\zeta(F) = \frac{\overline{\mathcal{A}}}{\overline{\mathcal{A}^\partial + m\mathcal{A}}} \cong \frac{\overline{\mathcal{A}}/\overline{\mathcal{A}^\partial}}{(\overline{\mathcal{A}^\partial} + m\overline{\mathcal{A}})/\overline{\mathcal{A}^\partial}} \cong \frac{\overline{\mathcal{A}}/\overline{\mathcal{A}^\partial}}{m(\overline{\mathcal{A}}/\overline{\mathcal{A}^\partial})}. \tag{34}$$

Note that $\mathrm{rk}\overline{\mathcal{A}} = 6g - 6 + 3p$ while $\mathrm{rk}\overline{\mathcal{A}^\partial} = p$. Since $\overline{\mathcal{A}^\partial}$ is a direct summand of $\overline{\mathcal{A}}$ by Proposition 3.11, the group $\overline{\mathcal{A}}/\overline{\mathcal{A}^\partial}$ is free abelian of rank $6g - 6 + 2p$. It follows that $\frac{\overline{\mathcal{A}}/\overline{\mathcal{A}^\partial}}{m(\overline{\mathcal{A}}/\overline{\mathcal{A}^\partial})}$ has cardinality $m^{6g-6+2p}$.

Case 2 ($\mathrm{ord}(\zeta) = 0 \mod 4$). Then $\mathcal{A}_\zeta = \mathcal{A}^\partial + m\mathcal{A}^{\mathrm{ev}}$. Since $\mathcal{A}^\partial \subset \mathcal{A}^{\mathrm{ev}}$, the same argument as in Case 1 with $\mathcal{A}$ replaced by $\mathcal{A}^{\mathrm{ev}}$ gives

$$|\overline{\mathcal{A}^{\mathrm{ev}}}/(\overline{\mathcal{A}^\partial} + m\overline{\mathcal{A}^{\mathrm{ev}}})| = m^{6g-6+2p}. \tag{35}$$

By Proposition 3.11, we have $|\mathcal{A}/\mathcal{A}^{\mathrm{ev}}| = 2^{2g}$. Hence

$$|\mathfrak{R}_\zeta(F)| = \left| \frac{\overline{\mathcal{A}}}{\overline{\mathcal{A}^\partial + m\mathcal{A}^{\mathrm{ev}}}} \right| = \left| \frac{\overline{\mathcal{A}}}{\overline{\mathcal{A}^{\mathrm{ev}}}} \right| \left| \frac{\overline{\mathcal{A}^{\mathrm{ev}}}}{\overline{\mathcal{A}^\partial + m\mathcal{A}^{\mathrm{ev}}}} \right| = 2^{2g} m^{6g-6+2p}. \tag{36}$$

In all cases we have $|\mathfrak{R}_\zeta(F)| = D_\zeta(F)$. □

We use the collection $\mathfrak{A} = \{e_1, \ldots, e_r\}$ to define the filtrations $F_k = F_k^{\mathfrak{A}}(K_\zeta(F))$, as described in Subsection 3.4. That is, F_k is the $\mathbb{C}$-subspace of $K_\zeta(F)$ spanned by

$S(\mathbf{n}), \mathbf{n} \in \mathcal{A}$ such that $|\mathbf{n}| := \sum_{i=1}^{r} n_i \le k$, where $S(\mathbf{n})$ is the simple diagram whose edge coordinates are $\mathbf{n}$.

Proposition 3.13. *Let F be a finite type surface with at least one puncture and a fixed coordinate datum. Let ζ be a non-zero complex number. For $\mathbf{n}, \mathbf{n}' \in \mathcal{A}$ there exists $j(\mathbf{n}, \mathbf{n}') \in \mathbb{Z}$ such that*

$$S(\mathbf{n})S(\mathbf{n}') = \zeta^{j(\mathbf{n},\mathbf{n}')} S(\mathbf{n}+\mathbf{n}') \pmod{F_{|\mathbf{n}|+|\mathbf{n}'|-1}}. \tag{37}$$

The monomial product formula (37), which plays an important role later, was proved in [2, 13] where an explicit formula for $j(\mathbf{n}, \mathbf{n}')$ is given. The monomial product formula (37) can also be obtained from Bonahon-Wong's quantum trace [4] by taking the top degree term.

3.6. **Coordinates and residues, closed surface case.** Now we consider coordinates of simple diagrams for the case when F is a closed oriented surface of genus $g > 1$. A *coordinate datum* of F consists of an ordered pants decomposition $\mathcal{P}$ and a dual graph $\mathcal{D}$ defined as follows.

An ordered pants decomposition of F is a sequence $\mathcal{P} = (P_1, \dots, P_{3g-3})$ of disjoint non-trivial loops such that no two of them are isotopic. The collection $\mathcal{P}$ cuts F into $2g-2$ pairs of pants (i.e., thrice punctured spheres). A *dual graph $\mathcal{D}$ to $\mathcal{P}$* is a trivalent graph embedded into F having exactly $2g-2$ vertices, one in the interior of each pair of pants, and $3g-3$ edges $(e_i)_{i=1}^{3g-3}$ such that e_i intersects P_i transversally in a single point and is disjoint with P_j for $j \ne i$. The pair $(\mathcal{P}, \mathcal{D})$ is called a *coordinate datum* for F. For technical simplicity we assume that $\mathcal{D}$ does not have an edge with endpoints in the same vertex, in other words each pair of pants has 3 different boundary components. Such a coordinate datum always exists, and we fix one.

Given $(\mathcal{P}, \mathcal{D})$ as above, one can define the Dehn-Thurston coordinates function from the set $\mathscr{S}$ of all isotopy classes of simple diagrams into $\mathbb{Z}^r$, where $r = 6g-6$,

$$\nu : \mathscr{S} \hookrightarrow \mathbb{Z}^{6g-6}, \quad \nu(\alpha) = (\nu_i(\alpha))_{i=1}^{6g-g}. \tag{38}$$

Suppose $\alpha \in \mathscr{S}$ is a simple diagram and $i \le 3g-3$. The i-th coordinate $\nu(\alpha)_i$ is the geometric intersection number $I(\alpha, P_i)$, and is called the *i-th pant coordinate*. The $(i+3g-3)$-th coordinate $\nu(\alpha)_{i+3g-3}$ is the twist coordinate at the curve P_i. Its definition is more involved, and we refer the reader to [22] for a precise definition. In [22], a red hexagon in each pair of pants must be fixed in order to define the twist coordiates, and the red hexagon is determined by the dual graph $\mathcal{D}$ as explained in Figure 1.

Again we use $\mathcal{A} \subset \mathbb{Z}^r$ to denote the image of ν in (38). It is known that $\mathcal{A}$ is the set of all $\mathbf{n} = (n_1, \dots, n_{6g-6}) \in \mathbb{Z}^{6g-6}$ such that

$$\begin{cases} \text{If } i \le 3g-3 \text{ then } n_i \ge 0. \\ \text{If } P_i, P_j, P_l \text{ bound a pair of pants then } n_i + n_j + n_l \text{ is even.} \\ \text{If } n_i = 0 \text{ for some } i \le 3g-3 \text{ then } n_{i+3g-3} \ge 0. \end{cases} \tag{39}$$

Since these conditions are linear, $\mathcal{A}$ is a submonoid of $\mathbb{Z}^{6g-6}$. Let $S : \mathcal{A} \to \mathscr{S}$ be the inverse of ν.

Remark 3.14. Note that we use the same terminologies and notations for coordinates of simple diagrams both on open surfaces and closed surfaces, even though the geometric nature of the two cases are different. The reason is we want to have uniform formulations and treatments of results for both types of surfaces in most cases.

Let $N(\mathcal{D})$ be a regular closed neighborhood of the dual graph $\mathcal{D}$. In other words $N(\mathcal{D})$ is a subsurface with boundary of F containing $\mathcal{D}$ in its interior such that $\mathcal{D}$ is a strong deformation retract of $N(\mathcal{D})$. Moreover $N(\mathcal{D})$ is a closed subset of F. Let $\Omega = \partial N(\mathcal{D})$ be the boundary of $N(\mathcal{D})$. We assume that for each pair of pants C the intersection $C \cap N(\mathcal{D})$ is a regular neighborhood (in C) of $\mathcal{D} \cap C$, and $\Omega \cap C$ consists of 3 arcs as in Figure 1.

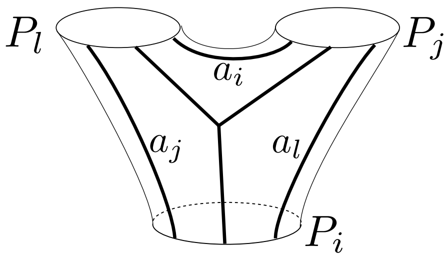

FIGURE 1. The pair of pants C bounded by loops P_i, P_j, P_l. The trivalent graph is $\mathcal{D} \cap C$. The bold arcs a_i, a_j, a_l are $\Omega \cap C$. The hexagon which contains the trivalent graph and is bounded by a_i, a_j, a_l and parts of P_i, P_j, P_l, is the red hexagon of [22].

Lemma 3.15. *Let $F = F_g$ be a closed oriented surface of genus $g > 1$ with fixed coordinate datum $(\mathcal{P}, \mathcal{D})$, and let $\mathcal{A} \subset \mathbb{Z}^{6g-6}$ denote the image of coordinates of simple diagrams on F. Let $\overline{\mathcal{A}}$ be the subgroup of $\mathbb{Z}^{6g-6}$ generated by $\mathcal{A}$. The index of $\overline{\mathcal{A}}$ in $\mathbb{Z}^{6g-6}$ is 2^{2g-3}.*

Proof. Identify $\mathbb{Z}^{6g-6} = \mathbb{Z}^{3g-3} \oplus \mathbb{Z}^{3g-3}$ and let p_1 be the projection from $\mathbb{Z}^{6g-6}$ onto the first summand. For any pants curve $P_j \in \mathcal{P}$ note that $\nu(P_j) = \vec{0} \oplus \boldsymbol{\delta}_j$, where $\vec{0} \in \mathbb{Z}^{3g-3}$ is the zero element and $\boldsymbol{\delta}_j \in \mathbb{Z}^{3g-3}$ is the element all of whose coordinates are zero except the j-th entry which is 1. It follows that $\overline{\mathcal{A}} = p_1(\overline{\mathcal{A}}) \oplus \mathbb{Z}^{3g-3}$. Hence the index of $\overline{\mathcal{A}}$ in $\mathbb{Z}^{6g-6}$ is equal to the index of $p_1(\overline{\mathcal{A}})$ in $\mathbb{Z}^{3g-3}$. By (39), set $p_1(\overline{\mathcal{A}})$ is the subset of $\mathbb{Z}^{3g-3}$ such that

(*) whenever P_i, P_j, P_k bound a pair of pants, $n_i + n_j + n_k$ is even.

Recall that $N(\mathcal{D})$ denotes a regular neighborhood of the graph $\mathcal{D}$ dual to the pants decomposition. The interior $\mathring{N}(\mathcal{D})$ of $N(\mathcal{D})$ is a finite type open surface with Euler characteristic $\chi(\mathring{N}(\mathcal{D})) = 1 - g$. Let $e_i = P_i \cap \mathring{N}(\mathcal{D})$, then $(e_i)_{i=1}^{3g-3}$ is an ideal triangulation, giving rise to edge-coordinates of simple diagrams on $\mathring{N}(\mathcal{D})$, and the set of all such edge-coordinates is denoted by $\mathcal{A}'$. Note that P_i, P_j, P_k bound a pair of pants if and only if e_i, e_j, e_k are edges of an ideal triangle. Condition (*) and Lemma 3.9(a) show that $p_1(\overline{\mathcal{A}}) = \overline{\mathcal{A}'}$. Hence

$$[\mathbb{Z}^{3g-3} : p_1(\overline{\mathcal{A}})] = [\mathbb{Z}^{3g-3} : \overline{\mathcal{A}'}] = 2^{-2\chi(\mathring{N}(\mathcal{D}))-1} = 2^{2g-3}, \tag{40}$$

where the second identity follows from Lemma 3.9(b). □

Suppose ζ is a root of 1. Let $\mathcal{A}^{\text{ev}} = \nu(\mathscr{S}^{\text{ev}})$, where $\mathscr{S}^{\text{ev}} \subset \mathscr{S}$ denotes the subset of all classes of even simple diagrams. If $\mathcal{A}_\zeta = \nu(\mathscr{S}_\zeta)$, where $\mathscr{S}_\zeta$ is defined by (18), then $\mathcal{A}^{\text{ev}}, \mathcal{A}_\zeta$ are submonoids of $\mathcal{A}$.

Let $m = \text{ord}(\zeta^4)$. Define the ζ-residue group $\mathfrak{R}_\zeta(F)$ and the number $D_\zeta(F)$ just like in the case of open surfaces (see (31)), noting that for a closed surface we have $p = 0$. That is,

$$\mathfrak{R}_\zeta(F) = \overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta} \tag{41}$$

which depends on a coordinate datum of F, and $D_\zeta(F)$ is given by

$$D_\zeta(F) = \begin{cases} m^{6g-6} & \text{if} \quad \text{ord}(\zeta) \neq 0 \mod 4 \\ 2^{2g} m^{6g-6} & \text{if} \quad \text{ord}(\zeta) = 0 \mod 4. \end{cases} \tag{42}$$

The following proposition is analogous to Proposition 3.12.

Proposition 3.16. *Suppose $F = F_g$ is a closed finite type surface with genus $g \geq 2$. Let ζ be a root of 1. For any coordinate datum of F,*

$$|\mathfrak{R}_\zeta(F)| = D_\zeta(F). \tag{43}$$

Proof. The proof is almost identical to that in the case of open surfaces.

Case 1 ($\text{ord}(\zeta) \neq 0 \mod 4$). In this case $\mathcal{A}_\zeta = m\mathcal{A}$, and $\mathfrak{R}_\zeta(F) = \overline{\mathcal{A}}/m\overline{\mathcal{A}}$. Since $\text{rk}\overline{\mathcal{A}} = 6g - 6$, we have $|\overline{\mathcal{A}}/m\overline{\mathcal{A}}| = m^{6g-6}$.

Case 2 ($\text{ord}(\zeta) = 0 \mod 4$). In this case $\mathcal{A}_\zeta = m\mathcal{A}^{\text{ev}}$.

First note that $|\overline{\mathcal{A}}/\overline{\mathcal{A}^{\text{ev}}}| = 2^{2g}$. The proof of this fact is identical to that of Proposition 3.11. Now we have

$$|\mathfrak{R}_\zeta(F)| = |\overline{\mathcal{A}}/m\overline{\mathcal{A}^{\text{ev}}}| = |\overline{\mathcal{A}}/\overline{\mathcal{A}^{\text{ev}}}||\overline{\mathcal{A}^{\text{ev}}}/m\overline{\mathcal{A}^{\text{ev}}}| = 2^{2g} m^{6g-6} = D_\zeta(F). \tag{44}$$

□

When F is closed we do not have a monomial product formula similar to (37). However, a monomial product formula still holds for the class of triangular simple diagrams defined as follows.

A simple diagram $\alpha \subset F$ is *triangular* with respect to the pants decomposition $\mathcal{P}$ if after being brought to a taut position with respect to $\mathcal{P}$, each connected component of the intersection of α with a pair of pants C is an arc whose two endpoints are in two different components of ∂C. In particular, α cannot have a component isotopic to any P_i.

Let $\mathscr{S}^\Delta \subset \mathscr{S}$ be the subset consisting of triangular simple diagrams on F, and $\mathcal{A}^\Delta = \nu(\mathscr{S}^\Delta)$. Then $\mathbf{n} = (n_1, \dots, n_{6g-6}) \in \mathcal{A}$ is in $\mathcal{A}^\Delta$ if and only if

- whenever P_i, P_j, P_k bound a pair of pants, $n_i \leq n_j + n_k$, and
- whenever $n_i = 0$ for some $i \leq 3g - 3$, one has $n_{i+3g-3} = 0$.

Note that the first condition above explains the terminology "triangular simple diagram."

Recall that $S(\mathbf{n})$ denotes a simple diagram on surface F with coordinates $\mathbf{n}$. Similarly to the case of an open surface, we use the collection $\mathfrak{A} = \{P_1, \dots, P_{3g-3}\}$ to define the filtrations of the Kauffman bracket skein algebra $F_k = F_k^{\mathfrak{A}}(K_\zeta(F))$, as described in Subsection 3.4. That is, F_k is the $\mathbb{C}$-subspace spanned by $S(\mathbf{n}), \mathbf{n} \in \mathcal{A}$ such that $|\mathbf{n}|_1 := \sum_{i=1}^{3g-3} n_i \leq k$. Unlike the case of open surface, F_k has infinite dimension over $\mathbb{C}$.

We have the following monomial product formula for triangular simple diagrams.

Proposition 3.17 ([13]). *Let F be a closed, oriented connected surface of genus $g \geq 2$ equipped with a coordinate datum, and let ζ be a non-zero complex number.*

For $\mathbf{n}, \mathbf{n}' \in \mathcal{A}^{\Delta}$ there is $j(\mathbf{n}, \mathbf{n}') \in \mathbb{Z}$ such that

$$S(\mathbf{n})S(\mathbf{n}') = \zeta^{j(\mathbf{n},\mathbf{n}')} S(\mathbf{n}+\mathbf{n}') \mod F_{|\mathbf{n}|_1+|\mathbf{n}'|_1-1}. \tag{45}$$

□

In [13] we gave the exact value of $j(\mathbf{n}, \mathbf{n}')$ in (45).

4. Stable Dehn-Thurston coordinates

Throughout this section $F = F_g$ is a closed surface with genus $g \geq 2$, and with a fixed coordinate datum $(\mathcal{P}, \mathcal{D})$ from which one can define the Dehn-Thurston coordinates of simple diagrams. Unlike for surfaces with punctures, the monomial product formula (45) works only for triangular simple diagrams. In this section we describe a stabilization process which transforms any simple diagram to a triangular one. The ordinary Dehn-Thurston coordinates will be replaced by a stable Dehn-Thurston coordinates which satisfy a monomial product formula.

4.1. Stable Dehn-Thurston coordinates.

The main goal of this section is to prove the following theorem.

Theorem 4.1. *Let $F = F_g$ be a closed surface of genus $g \geq 2$ equipped with a coordinate datum $(\mathcal{P}, \mathcal{D})$, where $\mathcal{P}$ is a pants decomposition of F and $\mathcal{D}$ is an embedding of its dual graph. Let $o_1, \dots, o_l$ denote the components of $\Omega = \partial N(\mathcal{D})$, where $N(\mathcal{D})$ is a regular neighborhood of $\mathcal{D}$. Let $h_\Omega : F \to F$ be the product of the Dehn twists about $o_1, \dots, o_l$.*

Let $\alpha \subset F$ be a simple diagram. There exists $\eta(\alpha) \in \mathbb{Z}^{6g-6}$ such that if k is large enough then $h_\Omega^k(\alpha)$ is triangular with respect to $\mathcal{P}$, and the Dehn-Thurston coordinates ν of the twisted diagram satisfy

$$\nu(h_\Omega^k(\alpha)) = k\mu(\alpha) + \eta(\alpha), \tag{46}$$

where

$$\mu(\alpha) = \sum_{j=1}^{l} I(\alpha, o_j)\nu(o_j), \tag{47}$$

and $I(\alpha, o_j)$ denotes geometric intersection number of α with o_j. In particular, the last $3g-3$ coordinates of $\mu(\alpha)$ are equal to 0.

Note that the action of any element of the mapping class of the surface on diagrams extends linearly to yield an automorphism of the skein algebra. In specific, it makes sense to talk about $h_\Omega^k(x)$ for $x \in K_\zeta(F)$.

We present the proof of Theorem 4.1 in Subsection 4.3.

4.2. Piecewise affine functions.

A function $f : \mathbb{R}^k \to \mathbb{R}^l$ is *affine* if there is an $l \times k$ matrix A and a vector B such that

$$f(x) = A \cdot x + B. \tag{48}$$

A function $f : \mathbb{R}^k \to \mathbb{R}^l$ is *piecewise affine* if there is a finite collection of proper affine subspaces of $\mathbb{R}^k$ such that in the closure of any connected component of the complement of these affine spaces f is equal to an affine function. It is easy to see that the class of piecewise affine functions is closed under linear combinations and compositions.

A function $f : X \to \mathbb{Z}^l$, where $X \subset \mathbb{R}^k$, is *piecewise affine* if it is the restriction of a piecewise affine function $\overline{f} : \mathbb{R}^k \to \mathbb{R}^l$.

Lemma 4.2. *If $f : \mathbb{N} \to \mathbb{N}$ is convex and bounded from above by an affine function, then f is piecewise affine.*

Proof. The lemma follows easily from the definition. □

Lemma 4.3. *Let F be a closed finite type surface with a fixed coordinate datum, and let h_Ω be the product of the Dehn twists about components of Ω. For any two simple diagrams $\alpha, \beta \in \mathscr{S}$ and $k > 0$ one has*

$$|I(h_\Omega^k(\alpha), \beta) - k \sum_{j=1}^{l} I(\alpha, o_j) I(o_j, \beta)| \leq I(\alpha, \beta). \tag{49}$$

Proof. This is a special case of Proposition A1 of [9, Section 4]. □

In order to prove the next proposition we need to explore the topology of the coordinate datum $(\mathcal{P}, \mathcal{D})$ of a closed surface F. To define a geometric intersection of a simple diagram S with the graph $\mathcal{D}$ we add the assumption that S misses the vertices of $\mathcal{D}$. That is, $I(S, \mathcal{D})$ is the minimum cardinality of $S' \cap \mathcal{D}$ where S' is isotopic to S, misses the vertices of $\mathcal{D}$ and is transverse to its edges.

Lemma 4.4. *If $(\mathcal{P}, \mathcal{D})$ is a coordinate datum for the closed surface F and S is a simple diagram with Dehn-Thurston coordinates $(\mathbf{n}(S), \mathbf{t}(S))$, then*

$$\sum_j |t_j(S)| \leq 2I(S, \mathcal{D}). \tag{50}$$

Proof. Given coordinate datum for F let A_j denote the annuli which are collars of the pants curves P_j, and let Q_i be the pairs of pants that are the components of the complement of $\cup A_j$ in F. These are the *shrunken pairs of pants*, versus the pairs of pants C_j as defined in section 3.6. By assumption $\mathcal{D}$ is transverse to the pants curves P_j and minimizes its intersection with the boundaries of the annuli A_j. We say that a simple diagram is in *standard position* if

- its intersection with the Q_i is isotopic to standard model curves in the complement of $\mathcal{D}$,
- its intersection with ∂A_j is disjoint from $\partial A_j \cap \mathcal{D}$,
- it minimizes its intersection with $\mathcal{D} \cap A_j$, for each j.

If a simple diagram is in standard position then its twists coordinates are given by its signed intersection numbers with $\mathcal{D} \cap A_i$.

We are most interested in the case that a simple diagram S is triangular. Recall that model curves describe possible ways in which a simple diagram in standard position intersects a pair of pants (see, e.g., [13]). In Figure 2 we show the triangular model curve d_{12} and another curve d'_{12} that will play a role in the rest of the proof. For each pair of boundary components of each shrunken pair of pants Q_i there are two curves like this, the model curve d_{ij} and its mate d'_{ij}.

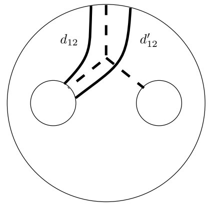

FIGURE 2. The model curve d_{12} and the curve d'_{12}.

We say a triangular diagram S is in *special position* if:

- It realizes $I(S, \partial A_j)$ for all j;
- It realizes $I(S, \mathcal{D})$;
- It does not intersect $\mathcal{D} \cap \partial A_j$ for any j;
- It minimizes the intersection $S \cap \mathcal{D} \cap Q_i$ for all i, among all S satisfying the first condition.

It is easy to see that if S is a triangular diagram in special position then it intersects each Q_j in curves parallel to model curves d_{kl} or to the new curves d'_{kl}.

To move a curve from special position to standard position, each curve of type d'_{kl} needs to be isotoped to a curve of type d_{kl}. In the Figure 3 we show a curve of the form d'_{kl} in the process of being pushed into standard position. Its intersection with $\mathcal{D}$ needs to be pushed inside the annuli A_j. As the result the intersection of S with $\mathcal{D} \cap A_j$ in each of the annuli on either end of d_{kl} is incremented by 1.

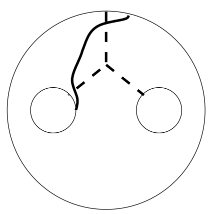

FIGURE 3. Deforming the curve d'_{12} on the way to standard position.

Thus

$$\sum_j |t_j(S)| \leq 2I(S, \mathcal{D}). \tag{51}$$

□

To prove Theorem 4.1 we use the following proposition. Note that part (a) states that the Dehn-Thurston coordinates of a simple diagrams are bounded by a piecewise affine function that depends on the intersection of the diagram with an auxiliary collection of curves. Part (b) asserts that upon twisting a simple diagram its intersection with any other simple diagram is a piecewise-affine function of the number of twists. Part (c) shows that the absolute values of the twist coordinates of any simple diagram do not increase when the diagram is twisted along Ω.

Proposition 4.5. *Let $F = F_g$ be a closed surface of genus g with a coordinate datum $(\mathcal{P}, \mathcal{D})$. The set of isotopy classes of simple diagrams on F is denoted by $\mathscr{S}$. Dehn-Thurston coordinates of $\alpha \in \mathscr{S}$ are denoted by $\nu(\alpha)$, and h_Ω denotes the product of Dehn twists about the components of the boundary Ω of a regular neighborhood of the dual graph $\mathcal{D}$.*

(a) There exists an additional collection of $6g-6$ loops P_j with $j = 3g-2, \dots, 9g-9$, and a piecewise affine function $G : \mathbb{Z}^{9g-9} \to \mathbb{Z}^{6g-6}$ such that for all $\alpha \in \mathscr{S}$,

$$\nu(\alpha) = G\Big(I(\alpha, P_1), \dots, I(\alpha, P_{9g-9})\Big). \tag{52}$$

(b) For any $\alpha, \beta \in \mathscr{S}$, the function $f_{\alpha,\beta} : \mathbb{Z} \to \mathbb{Z}$, defined by $f_{\alpha,\beta}(k) = I(h^k_\Omega(\alpha), \beta))$, is piecewise affine.

(c) For any $\alpha \in \mathscr{S}$, the twist coordinates of $h^k_\Omega(\alpha)$ are bounded, meaning that there is a constant $M > 0$ such that for all k and all $i = 3g-2, \dots, 6g-6$, one has $|(h^k_\Omega(\alpha))_i| < M$.

Proof.

(a) This statement, with G being homogeneous continuous instead of being piecewise affine, is proved in [22, Proposition 4.4]. But the explicit formulas for G in [22] are actually piecewise affine.

(b) By [21, Corollary 3], the function $f_{\alpha,\beta} : \mathbb{N} \to \mathbb{N}$ is convex. By Lemma 4.3, $f_{\alpha,\beta}$ is bounded from above by an affine function. By Lemma 4.2 $f_{\alpha,\beta}$ is piecewise affine.

(c) Notice that applying h^k_Ω to a diagram S does not increase $i(S, \mathcal{D})$. By Lemma 4.4 the twist coordinates of $h^k_\Omega(S)$ are bounded above by $2i(S, \mathcal{D})$. □

4.3. **Proof of Theorem 4.1.** Let $\alpha \in \mathscr{S}$. Parts (a) and (b) of Proposition 4.5 along with Formula (52) imply that the function $k \to \nu(h^k_\Omega(\alpha))$ is piecewise affine. Hence it is affine for big enough k. Thus there exist k_0 and $\mu(\alpha), \eta(\alpha) \in \mathbb{Z}^{6g-6}$ such that if $k \geq k_0$ then

$$\nu(h^k_\Omega(\alpha)) = k\mu(\alpha) + \eta(\alpha). \tag{53}$$

By Proposition 4.5(c), the twist coordinates of $h^k_\Omega(\alpha)$ are bounded. It follows that the last $3g-3$ coordinates of $\mu(\alpha)$ must be 0.

Assume $i \leq 3g-3$. By definition, $\nu_i(h^k_\Omega(\alpha)) = I(h^k_\Omega(\alpha), P_i)$. Comparing the slope of $\nu_i(h^k_\Omega(\alpha))$ in (53) and (49), with $\beta = P_i$, we get

$$\mu_i(\alpha) = \sum_{j=1}^{l} I(\alpha, o_j) I(o_j, P_i) = \sum_{j=1}^{l} I(\alpha, o_j)\nu_i(o_j). \tag{54}$$

As the twists coordinates of o_j, as well as the last $3g-3$ coordinates of $\mu(\alpha)$, are all 0, we have

$$\mu(\alpha) = \sum_{j=1}^{l} I(\alpha, o_j)\nu(o_j). \tag{55}$$

It remains to show that the diagram $h^k_\Omega(\alpha)$ is triangular for large k. Suppose $P_i, P_j, P_l \in \mathcal{P}$ bound a pair of pants C. An arc in C having one endpoint in P_j and one endpoint in P_l is called a (j,l)-arc. An arc having both endpoints in P_i is called an (i,i)-arc.

Lemma 4.6. *Let β be a $\mathcal{P}$-taut simple diagram.*
(a) If $\beta \cap C$ has a (j,l)-arc, then $\beta \cap C$ does not have (i,i)-arcs.
(b) If $\beta \cap C$ does not have (i,i)-arcs, then $\nu_i(\beta) \le \nu_j(\beta) + \nu_l(\beta)$.

Proof. Both statements follow from the well-known facts (see [9, Exposé 4]):
- The number of (j,l)-arcs is $\max(0, (\nu_j(\beta) + \nu_l(\beta) - \nu_i(\beta))/2)$,
- The number of (i,i)-arcs is $\max(0, (\nu_i(\beta) - \nu_j(\beta) - \nu_l(\beta))/2)$. □

Note that $\Omega \cap C$ consists of 3 arcs, a (j,l)-arc a_i, a (k,i)-arc a_j, and a (j,i)-arc a_k, see Figure 1. For each $s = i,j,l$ let m_s be the intersection number of α with the component of Ω containing a_s. By (53) and (55), for $k \ge k_0$ we have

$$\begin{aligned} \nu_i(h_\Omega^k(\alpha)) &= k(m_j + m_l) + \eta_i(\alpha) \\ \nu_j(h_\Omega^k(\alpha)) &= k(m_i + m_l) + \eta_j(\alpha) \\ \nu_l(h_\Omega^k(\alpha)) &= k(m_i + m_j) + \eta_l(\alpha). \end{aligned} \tag{56}$$

It follows that

$$\nu_j(h_\Omega^k(\alpha)) + \nu_l(h_\Omega^k(\alpha)) - \nu_i(h_\Omega^k(\alpha)) = 2km_i + \eta_l(\alpha) + \eta_j(\alpha) - \eta_i(\alpha). \tag{57}$$

Hence if $m_i > 0$ then, for sufficiently large k, we have

$$\nu_j(h_\Omega^k(\alpha)) + \nu_l(h_\Omega^k(\alpha)) - \nu_i(h_\Omega^k(\alpha)) \ge 0. \tag{58}$$

Suppose now $m_i = 0$. Let o_s be the component of Ω containing a_i. Then $I(\alpha, o_s) = m_i = 0$ and we can assume $\alpha \cap o_s = \emptyset$. Since o_s is a component of Ω, for any $k > 0$ we also have $h_\Omega^k(\alpha) \cap o_s = \emptyset$. Let $\beta = h_\Omega^k(\alpha) \cup o_s$, then β has a (j,l)-arc in C, which is a_i. By Lemma 4.3(a), β does not have (i,i)-arcs. As $h_\Omega^k(\alpha)$ is a sub-diagram of β, it does not have an (i,i) arc neither. By Lemma 4.3(b), we have (58). Thus in all cases, the coordinates of the twisted diagram $h_\Omega^k(\alpha)$ satisfy the triangular inequality (58) for large enough k.

It remains to show that for k large, $h_\Omega^k(\alpha)$ does not have a component isotopic to P_i. Suppose $h_\Omega^{k_1}(\alpha)$ has a component isotopic to P_i for some k_1. As P_i has non-trivial intersection with the components of Ω which contain a_j and a_l, we have $m_j, m_l > 0$. From (56) it follows that for large k we have $\nu_i(h_\Omega^k(\alpha)) > 0$, implying $h_\Omega^k(\alpha)$ does not have components isotopic to P_i. This completes the proof of Theorem 4.1. □

4.4. **More on Theorem 4.1.** We call

$$\tilde{\nu}(\alpha) := (\mu(\alpha), \eta(\alpha)) \in \mathbb{Z}^{12g-12} \tag{59}$$

the *stable Dehn-Thurston-coordinate* of a simple diagram $\alpha \in \mathscr{S}$ with respect to the coordinate datum $(\mathcal{P}, \mathcal{D})$ on a closed surface F.

Recall that $\mathcal{A}$ denotes the image of coordinates of simple diagrams on F, and $\overline{\mathcal{A}}$ is the subgroup of $\mathbb{Z}^{6g-6}$ generated by the monoid $\mathcal{A}$. Let $\vec{0} \in \mathbb{Z}^{3g-3}$ be the element having all 0's as entries, and $\boldsymbol{\delta}_i \in \mathbb{Z}^{3g-3}$ be the element having all 0's as entries except a 1 in the i-th entry.

The following Proposition gives properties of stable Dehn-Thurston coordinates of simple diagrams.

Proposition 4.7. *Let F be a finite type surface of genus g without punctures with a fixed Dehn-Thurston coordinate datum $(\mathcal{P},\mathcal{D})$, and associated stable Dehn-Thurston coordinates $\tilde{\nu}$ of simple diagrams on F.*

(a) The map $\tilde{\nu}:\mathscr{S}\to\mathbb{Z}^{12g-12}$ is injective.

(b) For any $\alpha\in\mathscr{S}$ one has $\eta(\alpha)\in\overline{\mathcal{A}}$, where $\eta(\alpha)$ is defined by (53).

(c) If $I(\alpha,\Omega)=0$ then $\eta(\alpha)=\nu(\alpha)$ and $\mu(\alpha)=(\vec{0},\vec{0})$. Here Ω denotes the boundary of a regular neighborhood of the dual graph $\mathcal{D}$.

(d) Let h_Ω denote the composition of Dehn twists along the components of Ω. For all $k\in\mathbb{Z}$ one has

$$\eta(h_\Omega^k(\alpha))=\eta(\alpha)+k\sum_{i=1}^{l}I(\alpha,o_i)\nu(o_i). \tag{60}$$

(e) For any $P_j\in\mathcal{P}$ one has $\eta(P_j)=(\vec{0},-\boldsymbol{\delta}_j)$.

Proof.

(a) If $(\mu(\alpha),\eta(\alpha))=(\mu(\beta),\eta(\beta))$, then (46) shows $h_\Omega^k(\alpha)=h_\Omega^k(\beta)$ for large k. Applying h_Ω^{-k}, we get $\alpha=\beta$.

(b) As $\eta(\alpha)=\nu(h_\Omega^k(\alpha))-k\sum_i I(\alpha,o_i)\nu(i)$ is the difference of two elements in $\mathcal{A}$, we have $\eta(\alpha)\in\overline{\mathcal{A}}$.

(c) If $I(\alpha,\Omega)=0$ then $h_\Omega^k(\alpha)=\alpha$ for all k. The result follows.

(d) Applying h_Ω^j to (46) and noting that $h_\Omega(o_i)=o_i$, we get

$$\eta(h_\Omega^j(\alpha))=\eta(\alpha)+j\sum_{i=1}^{l}I(\alpha,o_i)\nu(o_i), \tag{61}$$

which is (60) with $k=j>0$. Replacing α by h_Ω^{-k} in (60) we get (60) with k replaced by $-k$.

(e) Note that $\nu(P_j)=(\vec{0},\boldsymbol{\delta}_j)$. After twisting once along Ω its twist coordinate becomes $-\boldsymbol{\delta}_j$,

$$\nu(h_\Omega^k(P_j))=(\vec{0},-\boldsymbol{\delta}_j)+k\sum_{i=1}^{l}I(P_j,o_i)\nu(o_i). \tag{62}$$

Since $h_\Omega(P_j)$ makes no bigons with P_j and its intersection with D is contained inside the annulus around P_j, therefore its Dehn-Thurston coordinates change linearly, and the result follows. ☐

Remark 4.8. One can prove that $\mu(\alpha)$ can be expressed through $\eta(\alpha)$, and hence $\eta:\mathscr{S}\hookrightarrow\mathbb{Z}^{6g-6}$ does give a coordinate system for the set $\mathscr{S}$ of simple diagrams.

5. Independence over the center

We formulate a criterion for independence of a collection of elements of the skein algebra of a surface $K_\zeta(F)$ over its center. Throughout this section we fix a finite type surface $F=F_{g,p}$ with negative Euler characteristic equipped with coordinate datum. One can define the coordinates of simple diagrams on the surface, $\nu:\mathscr{S}\hookrightarrow\mathbb{Z}^r$, where $r=6g-6+3p$. The set of possible coordinates $\mathcal{A}=\nu(\mathscr{S})$ is a submonoid of $\mathbb{Z}^r$. Let $\overline{\mathcal{A}}$ denote the subgroup of $\mathbb{Z}^r$ generated by $\mathcal{A}$. For a root of unity ζ we also define the submonoid $\mathcal{A}_\zeta$ and its group $\overline{\mathcal{A}_\zeta}$ as in Section 3.

The formulation of the criterion does not depend on whether the number of punctures is non-zero, however the proofs are different for closed and open surfaces.

Note that, as in previous sections, we use the same notation ν, $\mathcal{A}$, etc., although the coordinate datum and consequently the definition of these objects differs depending on whether $p = 0$ or not.

For a ring R we denote by R^* the set of non-zero elements of R. Since $K_\zeta(F)$ is a domain, $K_\zeta(F)^*$ is a monoid under multiplication.

5.1. **General result.** The degree of polynomials in one variable satisfies the following two properties: for non-zero polynomials x and y,

(i) $\deg(xy) = \deg(x) + \deg(y)$ (monoid homomorphism),
(ii) if $\deg(x_i)$ for $i = 1, \dots, d$ are pairwise distinct, then $\sum_i x_i \neq 0$.

For a domain R, a *degree map* is a map $\deg : R^* \to M$, where M is a monoid, satisfying the above two properties.

Recall that $\mathcal{A}_\zeta$ denotes coordinates of central diagrams as defined by (18), and $\mathfrak{R}_\zeta$ is the residue group as defined by (31) for open surfaces and by (41) for closed surfaces.

Theorem 5.1. *Given a finite type surface $F = F_{g,p}$ with negative Euler characteristic and a fixed coordinate datum, let ζ be a root of 1. Let $\overline{\mathcal{A}}$ denote the subgroup of of $\mathbb{Z}^r$ generated by the set $\mathcal{A}$ of possible coordinates of simple diagrams on F. There exists a degree map*

$$\deg : K_\zeta(F)^* \to \overline{\mathcal{A}} \tag{63}$$

such that $\deg(Z_\zeta(F)^*) \subset \overline{\mathcal{A}_\zeta}$. *Moreover, the composition*

$$\deg_\zeta : K_\zeta(F)^* \xrightarrow{deg} \overline{\mathcal{A}} \to \overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta} = \mathfrak{R}_\zeta \tag{64}$$

is a surjective monoid homomorphism onto the ζ-residue group.

We will construct the map deg in later subsections. We want to mention an important corollary that we will use in the future.

Corollary 5.2. *Assume the hypothesis of Theorem* 5.1.

(a) If $x_1, \dots, x_d \in K_\zeta(F)^*$ such that $\deg_\zeta(x_1), \dots, \deg_\zeta(x_d)$ *are pairwise distinct, then $x_1, \dots, x_d$ are linearly independent over $Z_\zeta(F)$.*

(b) One has $\dim_{Z_\zeta(F)} K_\zeta(F) \geq |\mathfrak{R}_\zeta| = D_\zeta(F)$, *where $D_\zeta(F)$ is given by* (32).

Proof.

(a) Suppose $z_1, \dots z_d \in Z_\zeta(F)^*$. From the assumption, the elements $\deg(z_1x_1), \dots, \deg(z_dx_d)$ are pairwise distinct in $\overline{\mathcal{A}}$. By Property (ii) of degree maps, the sum $\sum z_ix_i \neq 0$.

(b) Since $\deg_\zeta(K_\zeta(F)^*) = \mathfrak{R}_\zeta(F)$, from (a) we have $\dim_{Z_\zeta(F)} K_\zeta(F) \geq |\mathfrak{R}_\zeta|$, which is equal to $D_\zeta(F)$ by Propositions 3.12 and 3.16. □

Recall that $\tilde{K}_\zeta(F) = K_\zeta(F) \otimes_{Z_\zeta(F)} \tilde{Z}_\zeta(F)$, where $\tilde{Z}_\zeta(F)$ is the field of fractions of the center $Z_\zeta(F)$. Since $\deg_\zeta(Z_\zeta(F)) = 0$, the map $\deg_\zeta$ extends to a surjective group homomorphism from the localized skein algebra to the residue group, also denoted by $\deg_\zeta$:

$$\deg_\zeta : \tilde{K}_\zeta(F)^* \to \overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta} = \mathfrak{R}_\zeta. \tag{65}$$

5.2. **Lead term.** Since the set $\mathscr{S}$ of isotopy classes of simple diagrams on a surface F is a $\mathbb{C}$-basis of the Kauffman bracket skein algebra $K_\zeta(F)$, for every $x \in K_\zeta(F)^*$ there is a unique set, $\mathrm{supp}(x) \subset \mathscr{S}$, such that skein x has the presentation

$$x = \sum_{\alpha \in \mathrm{supp}(x)} c_\alpha \alpha, \quad 0 \neq c_\alpha \in \mathbb{C}. \tag{66}$$

If $\leq$ is a total order on $\mathscr{S}$, then there is the largest simple diagram appearing in the expression (66) for the skein x, and $c_\alpha \alpha$, where $\alpha = \max \mathrm{supp}(x)$, is called the ($\leq$)-*lead term* of x. We can write

$$x = c_\alpha \alpha + \sum_{\gamma \in G_<(\alpha) \cap \mathrm{supp}(x)} c_\gamma \gamma \tag{67}$$

where $G_<(\alpha)$ is the $\mathbb{C}$-span of $\{\beta \in \mathscr{S} \mid \beta < \alpha\}$.

5.3. **Proof of Theorem 5.1for open surfaces.** Suppose F is an open finite type surface with negative Euler characteristic and coordinate datum $\{e_1, \dots, e_r\}$. Let $\iota : \mathbb{Z}^r \hookrightarrow \mathbb{Z}^{r+1}$ be the embedding

$$\iota(\mathbf{n}) = (|\mathbf{n}|, \mathbf{n}), \tag{68}$$

where $|\mathbf{n}|= \sum n_i$. Let $\trianglelefteq$ be the total order on $\mathscr{S}$ and $\mathcal{A}$ induced from the lexicographic order on $\mathbb{Z}^{r+1}$ via the embeddings

$$\mathscr{S} \xrightarrow{\nu} \mathcal{A} \hookrightarrow \overline{\mathcal{A}} \hookrightarrow \mathbb{Z}^r \overset{\iota}{\hookrightarrow} \mathbb{Z}^{r+1}. \tag{69}$$

The order $\trianglelefteq$ makes $\overline{\mathcal{A}}$ an ordered group. Define $\deg : K_\zeta(F)^* \to \overline{\mathcal{A}}$ by

$$\deg(x) = \nu(\alpha), \quad \text{where } \alpha = \max \mathrm{supp}(x). \tag{70}$$

Suppose $x_1 \dots, x_n \in K_\zeta(F)^*$. If $\deg(x_1), \dots, \deg(x_d)$ are distinct, then $\sum_i x_i \neq 0$ and $\deg(\sum_i x_i) = \max\{\deg(x_i)\}$. Proposition 3.13 implies that $\deg(xy) = \deg(x) + \deg(y)$ for any $x, y \in K_\zeta(F)$. Hence deg is a degree map. By definition $\deg(\mathscr{S}) = \mathcal{A}$, which by Proposition 3.8 surjects onto $\overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta} = \mathfrak{R}_\zeta$. Since $\mathscr{S} \subset K_\zeta(F)^*$, we also have $\deg_\zeta(K_\zeta(F)^*) = \mathfrak{R}_\zeta$. This completes the proof of Theorem 5.1 for open surface.

5.4. **Proof of Theorem 5.1for closed surfaces.** Suppose $F = F_g$ with $g \geq 2$ is equipped with a coordinate datum $(\mathcal{P}, \mathcal{D})$. In this case $r = 6g - 6$. Let $\kappa : \mathbb{Z}^r \hookrightarrow \mathbb{Z}^{r+1}$ be the group embedding given by

$$\kappa(n_1, \dots, n_r) = (\sum_{i=1}^{3g-3} n_i, n_1, n_2, \dots, n_r). \tag{71}$$

Let the order $\leq$ on $\mathscr{S}$ and $\mathbb{Z}^r$ be the one induced from the lexicographic order of $\mathbb{Z}^{r+1}$ via the embeddings

$$\mathscr{S} \xrightarrow{\nu} \mathcal{A} \hookrightarrow \overline{\mathcal{A}} \hookrightarrow \mathbb{Z}^r \overset{\kappa}{\hookrightarrow} \mathbb{Z}^{r+1}. \tag{72}$$

The first component is used to define the filtrations that appeared in Proposition 3.17. Recall that $S(\mathbf{n})$ denotes a simple diagram with coordinates $\mathbf{n}$. By Proposition 3.17, for $\mathbf{n}, \mathbf{n}' \in \mathcal{A}^\Delta$, there is j such that

$$S(\mathbf{n})S(\mathbf{n}') = \zeta^j S(\mathbf{n} + \mathbf{n}') + \beta, \beta \in G_<(\mathbf{n} + \mathbf{n}') \tag{73}$$

where $G_<(\mathbf{n} + \mathbf{n}')$ is a $\mathbb{C}$-span of $\{\alpha \in \mathscr{S} \mid \alpha < S(\mathbf{n} + \mathbf{n}')\}$. This holds only for triangular $\mathbf{n}, \mathbf{n}'$. There is a better order on $\mathscr{S}$ defined in the following lemma.

Recall that the stable Dehn-Thurston coordinates function $\tilde{\nu} : \mathscr{S} \hookrightarrow \mathbb{Z}^r \times \mathbb{Z}^r$ is defined in Subsection 4.4.

Lemma 5.3. *Let $F = F_g$ be a closed surface with genus with $g \geq 2$ and $r = 6g-6$. Let F be equipped with a coordinate datum $(\mathcal{P}, \mathcal{D})$, and let h_Ω be the composition of Dehn twists along the components of Ω, where Ω is the boundary of a regular neighborhood of the dual graph $\mathcal{D}$.*

There is a total order $\trianglelefteq$ which makes $\mathbb{Z}^r \times \mathbb{Z}^r$ an ordered group, induces an order on the set $\mathscr{S}$ of isotopy classes of simple diagrams of F via $\tilde{\nu} : \mathscr{S} \hookrightarrow \mathbb{Z}^r \times \mathbb{Z}^r$, and has a property that $\alpha \trianglelefteq \beta$ if and only if $h^k_\Omega(\alpha) \leq h^k_\Omega(\beta)$ for sufficiently large k.

Proof. For $\mathbf{n} = (n_1, \dots, n_{6g-6})$ let $\|\mathbf{n}\|_1 = \sum_{i=1}^{3g-3} n_i$. For $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$ let

$$\kappa(k\mathbf{n} + \mathbf{m}) = (k\|\mathbf{n}\|_1 + \|\mathbf{m}\|_1, kn_1 + m_1, kn_2 + m_2, \dots, kn_r + m_r). \tag{74}$$

Define the embedding $\tilde{\kappa} : \mathbb{Z}^r \times \mathbb{Z}^r \hookrightarrow \mathbb{Z}^{2r+2}$ by

$$\tilde{\kappa}(\mathbf{n}, \mathbf{m}) = (\|\mathbf{n}\|_1, \|\mathbf{m}\|_1, n_1, m_1, n_2, m_2, \dots, n_r, m_r). \tag{75}$$

The order $\trianglelefteq$ on $\mathbb{Z}^r \times \mathbb{Z}^r$ induced from the lexicographic order of $\mathbb{Z}^{2r+2}$ via $\tilde{\kappa}$ satisfies the lemma. □

From the definition, $c_\alpha \alpha$ is the ($\trianglelefteq$)-lead term of $x \in K_\zeta(F)^*$ if and only if $c_\alpha h^k_\Omega(\alpha)$ is the ($\leq$)-lead term of $h^k_\Omega(x)$ for sufficiently large k. This yields an ordering that gives us control of lead terms of all diagrams, not just triangular ones. From here we have the following.

Lemma 5.4. *Let F be a closed finite-type surface equipped with a Dehn-Thurston coordinate datum and let ζ be a root of unity. Given skein $x \in K_\zeta(F)^*$, simple diagram $\alpha_k \in \mathscr{S}$ and $0 \neq c_k \in \mathbb{C}$, such that for sufficiently large k we have*

$$h^k_\Omega(x) = c_k \alpha_k + \beta, \beta \in G_<(\alpha_k) \tag{76}$$

where $G_<$ is defined by (67), then $c_k \alpha_k = c_\alpha h^k_\Omega(\alpha)$ for large k, where $c_\alpha \alpha$ is the ($\trianglelefteq$)-lead term of x, and the ordering ($\trianglelefteq$) is given by Lemma 5.3. □

A crucial property of the $\trianglelefteq$ order is that its lead term is a monoid map.

Lemma 5.5. *Let F be a closed finite-type surface equipped with a coordinate datum, and ordering of simple diagrams on F defined by Lemma 5.3. Stable Dehn-Thurston coordinates $\tilde{\nu}$ of simple diagrams satisfy the following monomial product formula.*

Suppose $\alpha, \beta \in \mathscr{S}$ and $\gamma = \max_\trianglelefteq(\mathrm{supp}(\alpha\beta))$, then $\tilde{\nu}(\gamma) = \tilde{\nu}(\alpha) + \tilde{\nu}(\beta)$, i.e. $\mu(\gamma) = \mu(\alpha) + \mu(\beta)$ and $\eta(\gamma) = \eta(\alpha) + \eta(\beta)$.

Proof. By Theorem 4.1 there exists $K \in \mathbb{N}$ such that for all $k > K$ the diagrams $h^k_\Omega(\alpha)$ and $h^k_\Omega(\beta)$ are triangular by. By (73),

$$\begin{aligned} h^k_\Omega(\alpha\beta) &= h^k_\Omega(\alpha) h^k_\Omega(\beta) \\ &= q^{j(k)}(S(\nu(h^k_\Omega(\alpha)) + \nu(h^k_\Omega(\beta))) + \beta, \\ &\text{with } \beta \in G_<(\nu(h^k_\Omega(\alpha)) + \nu(h^k_\Omega(\beta))). \end{aligned}$$

Hence by Lemma 5.4 we have $h^k_\Omega(\gamma) = S(\nu(h^k_\Omega(\alpha)) + \nu(h^k_\Omega(\beta)))$, or

$$\nu(h^k_\Omega(\gamma)) = \nu(h^k_\Omega(\alpha)) + \nu(h^k_\Omega(\beta)). \tag{77}$$

Using (46) we have

$$k\mu(\gamma) + \eta(\gamma) = k\mu(\alpha) + \eta(\alpha) + k\mu(\beta) + \eta(\beta). \tag{78}$$

It follows that $\mu(\gamma) = \mu(\alpha) + \mu(\beta)$ and $\eta(\gamma) = \eta(\alpha) + \eta(\beta)$. $\square$

We use stable Dehn-Thurston coordinates of simple diagrams to define a degree map analogous to (70). Specifically, define the map $\deg : K_\zeta(F)^* \to \overline{\mathcal{A}}$ by

$$\deg(x) = \eta(\alpha) \in \overline{\mathcal{A}}, \quad \text{where } \alpha = \max_{\trianglelefteq}(\operatorname{supp}(x)). \tag{79}$$

We show that this is in fact a degree mapping in the sense of subsection 5.1.

Lemma 5.6. *Let F be a closed finite type surface with fixed Dehn-Thurston coordinate datum.*

(a) $\deg : K_\zeta(F)^* \to \overline{\mathcal{A}}$ *is a monoid homomorphism.*

(b) Suppose $x_1, \dots, x_d \in K_\zeta(F)^*$ *such that* $\deg(x_1), \dots, \deg(x_d)$ *are distinct, then* $\sum_i x_i \neq 0$.

Proof. (a) follows from Lemma 5.5 and the fact that $\overline{\mathcal{A}} \times \overline{\mathcal{A}}$, equipped with $\trianglelefteq$, is an ordered monoid.

(b) Let $c_i \alpha_i$ be the ($\trianglelefteq$)-lead term of x_i. Since $\deg(x_i) = \eta(\alpha_i)$, the α_i are distinct. It follows that $\sum_i x_i \neq 0$. $\square$

Recall that $\mathscr{S}_\zeta$ denotes isotopy classes of central diagrams, as defined by (18) and $\overline{\mathcal{A}_\zeta}$ is the group generated by coordinates of central diagrams.

Lemma 5.7. *Let F be a closed finite type surface with fixed Dehn-Thurston coordinate datum, and let ζ be a root of unity.*

(a) Suppose $\alpha \in \mathscr{S}_\zeta$, then $\deg(\alpha) \in \overline{\mathcal{A}_\zeta}$.

(b) Suppose $z \in Z_\zeta(F)^*$, then $\deg(z) \in \overline{\mathcal{A}_\zeta}$.

Proof.

(a) For sufficiently large k, from (46) we have

$$\eta(\alpha) = \nu(h_\Omega^k(\alpha)) - k\mu(\alpha). \tag{80}$$

Since $\mathscr{S}_\zeta$ is invariant under Dehn twists, $h_\Omega^k(\alpha) \in \mathscr{S}_\zeta$, and $\nu(h_\Omega^k(\alpha)) \in \mathcal{A}_\zeta$. On the other hand, if $k \in 2m\mathbb{Z}$, then $k\mu(\alpha) \in \mathcal{A}_\zeta$ since $2m\mathcal{A} \subset \mathcal{A}_\zeta$. Hence from (80) we see that $\eta(\alpha)$, being the difference of two elements of $\mathcal{A}_\zeta$, is in $\overline{\mathcal{A}_\zeta}$.

(b) Since $\{T(\alpha) \mid \alpha \in \mathscr{S}_\zeta\}$ is a $\mathbb{C}$-basis of $Z_\zeta(F)^*$ (see Theorem 3.5), we have

$$z = \sum_{\alpha \in U \subset \mathscr{S}_\zeta} c_\alpha T(\alpha), \quad c_\alpha \in \mathbb{C}^*. \tag{81}$$

From the definition, $\max_{\trianglelefteq} \operatorname{supp}(T(\alpha)) = \alpha$. Hence $\deg(z) = \eta(\alpha)$, where $\alpha = \max_{\trianglelefteq} U$. Since $\alpha \in \mathscr{S}_\zeta$, the result follows from (a). $\square$

Lemma 5.8. *Let F be a closed finite type surface with fixed Dehn-Thurston coordinate datum $(\mathcal{P}, \mathcal{D})$, and let ζ be a root of unity. The monoid homomorphism* $\deg_\zeta : K_\zeta(F)^* \to \overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta} = \mathfrak{R}_\zeta$ *is surjective.*

Proof. Let $\overline{\eta(\mathscr{S})}$ be the $\mathbb{Z}$-span of $\eta(\mathscr{S})$. One has to show that $\overline{\eta(\mathscr{S})} \supset \overline{\mathcal{A}}$. From the description of $\mathcal{A}$ in Section 3.6 we see that $\overline{\mathcal{A}} = (\overline{\mathcal{A}})_1 \oplus \mathbb{Z}^{3g-3}$, and $(\overline{\mathcal{A}})_1$ is the set of all $\mathbf{n} = (n_1, \dots, n_{3g-3}) \in \mathbb{Z}^{3g-3}$ such that whenever P_i, P_j, P_l bound a pair of pants, $n_i + n_j + n_l$ is even.

The set $\mathcal{A}_1'$ of all $\mathbf{n} = (n_1, \dots, n_{3g-3}) \in \mathbb{N}^{3g-3}$, such that whenever P_i, P_j, P_l bound a pair of pants, $n_i + n_j + n_l$ is even and $n_i \leq n_j + n_l$, spans $(\overline{\mathcal{A}})_1$ over $\mathbb{Z}$. If $\mathbf{n} \in \mathcal{A}_1'$ then there is a simple diagram α lying entirely in a regular neighborhood of the dual graph, $N(\mathcal{D})$, such that $\nu(\alpha) = (\mathbf{n}, \vec{0})$. By Proposition 4.7(c), one has $\eta(\alpha) = \nu(\alpha) = (\mathbf{n}, \vec{0})$. It follows that $\overline{\eta(\mathscr{S})} \supset (\overline{\mathcal{A}})_1 \oplus \{\vec{0}\}$.

Since $\eta(P_i) = -(\vec{0}, \boldsymbol{\delta}_i)$ by Proposition 4.7(e), we have $\overline{\eta(\mathscr{S})} \supset \{\vec{0}\} \oplus \mathbb{Z}^{3g-3}$. Thus, $\overline{\eta(\mathscr{S})} \supset (\overline{\mathcal{A}})_1 \oplus \mathbb{Z}^{3g-3} = \overline{\mathcal{A}}$. □

For closed surfaces Theorem 5.1 follows from Lemmas 5.6, 5.7, and 5.8. □

5.5. **More on** $\deg_\zeta$**.** The degree map is defined on the skein algebra of a surface, and yields a characterization of central skeins. It also allows the exploration of the independence of skeins corresponding to diagrams. Although the definition of the degree map depends on the coordinate datum for the surface, the formulation of the criteria is the same for surfaces with and without punctures.

Recall that $\mathscr{S}^{\mathrm{ev}}$ denotes even simple diagrams.

Proposition 5.9. *Suppose $F = F_{g,p}$ has negative Euler characteristic and a fixed coordinate datum. Let ζ be a root of 1, with $m = \mathrm{ord}(\zeta^4)$.*

(a) If $\alpha \in \mathscr{S}$ then $\deg_\zeta(\alpha) = 0$ if and only if $\alpha \in \mathscr{S}_\zeta$.

(b) Let $C_1, \dots, C_k$ be a sequence of disjoint non-trivial non-peripheral loops such that no two of them are isotopic. For $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ let $C^{\mathbf{n}} = \prod_{i=1}^k (C_i)^{n_i} \in \mathscr{S}$. Suppose $\deg_\zeta(C^{\mathbf{n}}) = 0$.

(i) If $\mathrm{ord}(\zeta) \neq 0 \pmod 4$ then $\mathbf{n} \in m\mathbb{Z}^k$.

(ii) If $\mathrm{ord}(\zeta) = 0 \pmod 4$ then $\mathbf{n} \in m\mathbb{Z}^k$ and $C^{\mathbf{n}/m} \in \mathscr{S}^{\mathrm{ev}}$.

Proof.

(a) By definition, $\deg_\zeta(\alpha) = 0$ if and only if $\deg(\alpha) \in \overline{\mathcal{A}_\zeta}$.

Case 1 ($p > 0$). In this case $\deg(\alpha) = \nu(\alpha)$. If $\alpha \in \mathscr{S}_\zeta$ then $\deg(\alpha) = \nu(\alpha) \in \nu(\mathscr{S}_\zeta) = \mathcal{A}_\zeta \subset \overline{\mathcal{A}_\zeta}$. Conversely, suppose $\nu(\alpha) \in \overline{\mathcal{A}_\zeta}$ then $\nu(\alpha) \in \overline{\mathcal{A}_\zeta} \cap \mathcal{A} = \mathcal{A}_\zeta$. Hence $\alpha \in \mathscr{S}_\zeta$.

Case 2. If $p = 0$ then $\deg(\alpha) = \eta(\alpha)$. If $\alpha \in \mathscr{S}_\zeta$ then $\deg(\alpha) = \eta(\alpha) \in \overline{\mathcal{A}_\zeta}$ by Lemma 5.7.

Suppose $\eta(\alpha) \in \overline{\mathcal{A}_\zeta}$. By Theorem 4.1 for large l we have

$$\nu(h^l_\Omega(\alpha)) = l\mu(\alpha) + \eta(\alpha). \tag{82}$$

When l is a multiple of $2m$, one has $l\mu(\alpha) \in 2m\mathcal{A} \subset \mathcal{A}_\zeta$, and the right hand side of (82) is in $\overline{\mathcal{A}_\zeta}$. It follows that $\nu(h^l_\Omega(\alpha)) \in \overline{\mathcal{A}_\zeta} \cap \mathcal{A} = \mathcal{A}_\zeta$. Hence $h^l_\Omega(\alpha) \in \mathscr{S}_\zeta$. As $\mathscr{S}_\zeta$ is invariant under automorphisms of F, we have $\alpha \in \mathscr{S}_\zeta$.

(b) By part (a), we have $C^{\mathbf{n}} \in \mathscr{S}_\zeta$. From the definition of $\mathscr{S}_\zeta$ (see Section 3.3) one has $C^{\mathbf{n}} = \beta\gamma^m$ where $\beta \in \mathcal{A}^\partial$ and $\gamma \in \mathcal{A}$. Since there are no peripheral elements among the C_i we must have $\beta = \emptyset$ and $C^{\mathbf{n}} = \gamma^m$. This proves $\mathbf{n} \in m\mathbb{N}^k$. Moreover $\gamma = C^{\mathbf{n}/m}$.

If $\mathrm{ord}(\zeta) = 0 \mod 4$, then the definition of $\mathscr{S}_\zeta$ requires $\gamma \in \mathscr{S}^{\mathrm{ev}}$. Hence in this case $C^{\mathbf{n}/m} \in \mathscr{S}^{\mathrm{ev}}$. □

6. Dimension of the skein algebra $K_\zeta(F)$ over its center

The goal of this section is to compute the dimension of the Kauffman bracket skein algebra of a finite type surface over its center.

6.1. **Formulation of result.** Recall that for a finite type surface $F = F_{g,p}$ and a root of unity ζ with $m = \mathrm{ord}(\zeta^4)$,

$$D_\zeta(F) = \begin{cases} m^{6g-6+2p} & \text{if } \mathrm{ord}(\zeta) \not\equiv 0 \pmod 4 \\ 2^{2g}\, m^{6g-6+2p} & \text{if } \mathrm{ord}(\zeta) \equiv 0 \pmod 4. \end{cases} \tag{83}$$

Theorem 6.1. *Suppose F is a finite type surface with negative Euler characteristic and ζ is a root of 1, then* $\dim_{Z_\zeta(F)} K_\zeta(F) = D_\zeta(F)$.

Remark 6.2. There are four cases when the Euler characteristic of $F_{g,p}$ is non-negative: the sphere with zero, one or two punctures, and the torus with zero punctures. The skein algebras of the first three surfaces are commutative so they have dimension 1 over their respective centers. For the torus it was proved in [1] that the dimension is m^2 in the case where $\mathrm{ord}(\zeta)$is odd. The case when $\mathrm{ord}(\zeta)$ has residue 2 on division by 4 is similar and the dimension is also m^2. Finally, when $\mathrm{ord}(\zeta)$ is divisible by 4, the dimension is $4m^2$.

Recall that the localized skein algebra is $\tilde{K}_\zeta(F) = K_\zeta(F) \otimes_{Z_\zeta(F)} \tilde{Z}_\zeta(F)$, where $\tilde{Z}_\zeta(F)$ is the field of fractions of the center $Z_\zeta(F)$. The degree map is defined by (70) and (79), and $\mathfrak{R}_\zeta$ denotes the ζ-residue group, see (31).

Corollary 6.3. *Let $F = F_{g,p}$ be a finite type surface with negative Euler characteristic equipped with a coordinate datum, and let ζ be a root of unity. If X is a $\tilde{Z}_\zeta(F)$-vector subspace of $\tilde{K}_\zeta(F)$ such that* $\deg_\zeta(X \setminus \{0\}) = \mathfrak{R}_\zeta$ *then $X = \tilde{K}_\zeta(F)$.*

Proof. Let $\mathcal{B} \subset X$ be such that $\deg_\zeta$ is a bijection from $\mathcal{B}$ to $\mathfrak{R}_\zeta$. By Corollary 5.2, $\mathcal{B}$ is $\tilde{Z}_\zeta(F)$-linearly independent. Thus $\dim_{\tilde{Z}_\zeta(F)} X \ge |\mathfrak{R}_\zeta| = D_\zeta(F) = \dim_{\tilde{Z}_\zeta(F)} \tilde{K}_\zeta(F)$, and hence $X = \tilde{K}_\zeta(F)$. □

By Corollary 5.2 we have $\dim_{Z_\zeta(F)} K_\zeta(F) \ge D_\zeta(F)$. To prove Theorem 6.1 we need to prove the converse inequality

$$\dim_{Z_\zeta(F)} K_\zeta(F) \le D_\zeta(F). \tag{84}$$

6.2. **Proof of Theorem 6.1, open surface case.** Given a finite type surface $F = F_{g,p}$ assume that $p > 0$. Fix a coordinate datum for F, that is a triangulation $\{e_1, \dots, e_r\}$.

Let $F_k(K_\zeta(F))$ be the $\mathbb{C}$-vector subspace of $K_\zeta(F)$ spanned by $\{\alpha \in \mathscr{S} \mid \sum_{i=1}^r \nu_i(\alpha) \le k\}$. By Proposition 3.7, $(F_k(K_\zeta(F)))_{k=0}^\infty$ is a filtration of $K_\zeta(F)$ compatible with the product. Let $Q \subset \mathbb{R}^r$ be the simplex

$$Q = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid x_i \ge 0, \sum x_i \le 1\}. \tag{85}$$

From Theorem 3.5 it follows that

$$\{T(S(\mathbf{n})) \mid \mathbf{n} \in \mathcal{A} \cap kQ\} \quad \text{is a } \mathbb{C}\text{-basis of } F_k(K_\zeta(F))$$
$$\{T(S(\mathbf{n})) \mid \mathbf{n} \in \mathcal{A}_\zeta \cap kQ\} \quad \text{is a } \mathbb{C}\text{-basis of } F_k(Z_\zeta(F)).$$

It follows that

$$\dim_{\mathbb{C}} F_k(K_\zeta(F)) = |\mathcal{A} \cap kQ| = |\overline{\mathcal{A}} \cap kQ| \tag{86}$$

$$\dim_{\mathbb{C}} F_k(Z_\zeta(F)) = |\mathcal{A}_\zeta \cap kQ| = |\overline{\mathcal{A}_\zeta} \cap kQ|. \tag{87}$$

Hence by Lemma 2.3, there is a positive integer u such that

$$\dim_{Z_\zeta(F)} K_\zeta(F) \le \lim_{k\to\infty} \frac{|\overline{\mathcal{A}} \cap kQ|}{|\overline{\mathcal{A}_\zeta} \cap (k-u)Q|} = |\overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta}| = D_\zeta(F) \tag{88}$$

where the first identity follows from (8) and the second one follows from Proposition 3.12. This proves Theorem 6.1 for open surfaces. □

6.3. **Piecewise-rational-linear functions.** The remainder of this section is dedicated to proving Theorem 6.1 for closed surfaces. For the rest of the section $F = F_g$ is a closed surface of genus $g \ge 2$ with a fixed coordinate datum.

A function $f : \mathbb{R}^k \to \mathbb{R}^l$ is *rational-linear* if there is a matrix A with rational entries such that $f(x) = Ax$. A function $f : \mathbb{R}^k \to \mathbb{R}^l$ is *piecewise-rational-linear* if it is continuous and there are rational-linear functions $f_1, \dots, f_k : \mathbb{R}^k \to \mathbb{R}$ such that on each connected component of the complement of all the hyperplanes $\{x \in \mathbb{R}^k \mid f_i(x) = 0\}$, the function f is equal to the restriction of a rational-linear function.

For $X \subset \mathbb{R}^k$ a function $h : X \to \mathbb{R}^l$ is *piecewise-rational-linear* if there is a piecewise-rational-linear function from $\mathbb{R}^k$ to $\mathbb{R}^l$ restricting to h.

It is clear that sums of piecewise-rational-linear functions are piecewise-rational-linear, and that a piecewise-rational-linear function h is positively homogeneous, i.e. $h(tx) = th(x)$ for all real $t \ge 0$.

A *rational convex polyhedron* is the convex hull of a finite number of points in $\mathbb{Q}^n$. The following proposition follows easily from the definition.

Proposition 6.4. *Suppose $h_1, \dots, h_l : \mathbb{R}^n \to \mathbb{R}$ are piecewise-rational-linear, $c_1, \dots, c_l \in \mathbb{Q}$ and*

$$Q = \{x \in \mathbb{R}^n \mid h_i(x) \ge c_i\}. \tag{89}$$

(a) If Q is not bounded, then Q contains a set of the form $\{tx \mid t \in \mathbb{R}_{\ge 0}\}$ for some non-zero $x \in \mathbb{Q}^n$. We call such set a rational ray.

(b) If Q is bounded, then Q is the union of a finite number of rational convex polyhedra. □

Recall that $\mathcal{A} \subset \mathbb{Z}^{6g-6}$ is the set of all possible Dehn-Thurston coordinates of simple diagrams, and $S(\mathbf{n})$, for $\mathbf{n} \in \mathcal{A}$, is the simple diagram with Dehn-Thurston coordinates $\mathbf{n}$.

Lemma 6.5. *Let $F = F_g$ be a closed surface of genus $g \ge 2$ with a coordinate datum $(\mathcal{P}, \mathcal{D})$. Let α be a simple closed curve in F. The function $\mathcal{A} \to \mathbb{R}$, defined by $\mathbf{n} \to I(S(\mathbf{n}), \alpha)$, is piecewise-rational-linear.*

Proof. This was formulated as Theorem 3 in [31] without proof. Here is a short proof based on [25]. First if α is one of $P_i \in \mathcal{P}$ then the statement is obvious as $I(S(\mathbf{n}), \alpha) = n_i$. Suppose now α is an arbitrary simple closed curve. Choose a coordinate datum $(\mathcal{P}', \mathcal{D}')$ such that α is a curve in $\mathcal{P}'$. By [25], the change from Dehn-Thurston coordinates associated with $(\mathcal{P}', \mathcal{D}')$ to the one associated with $(\mathcal{P}, \mathcal{D})$ is piecewise-rational-linear. The result follows. □

6.4. **Infinite sector Q_∞ and its truncation Q.** Recall that the set of all possible Dehn-Thurston coordinates $\mathcal{A}$ consists of all points $(x_1, \dots, x_{6g-6}) \in \mathbb{Z}^{6g-6}$ satisfying

(i) $x_i \ge 0$ for $i = 1, \dots, 3g-3$,

(ii) if $x_i = 0$ for some $i = 1, \dots, 3g-3$, then $x_{i+3g-3} \geq 0$,

(iii) if P_i, P_j, P_l bound a pair of pants, then $x_i + x_j + x_l$ is even.

Let Q_∞ be the set of all $(x_1, \dots, x_{6g-6}) \in \mathbb{R}^{6g-6}$ satisfying conditions (i) and (ii) above. Note that we allow points in Q_∞ to have real coordinates.

For any submonoid X of $\mathbb{Z}^{6g-6}$, let $\overline{X}$ be the subgroup generated by X. The set Q_∞ was introduced so that $\mathcal{A} = \overline{\mathcal{A}} \cap Q_\infty$.

Lemma 6.6. *Given a closed surface with fixed coordinate datum, and sets Q_∞ and $\mathcal{A}$ defined above, the following hold.*

(a) *For any subset $Q' \subset Q_\infty$ one has*

$$\mathcal{A} \cap Q' = \overline{\mathcal{A}} \cap Q'. \tag{90}$$

(b) *If $x \in Q_\infty \cap \mathbb{Z}^{6g-6}$ then $2x \in \mathcal{A}$.*

Proof.

(a) $\overline{\mathcal{A}} \cap Q' = \overline{\mathcal{A}} \cap (Q_\infty \cap Q') = (\overline{\mathcal{A}} \cap Q_\infty) \cap Q' = \mathcal{A} \cap Q'$.

(b) follows from (iii) above. □

Recall that $S(\mathbf{n})$ denotes a simple diagram with coordinates $\mathbf{n}$. By Lemma 4.5, we can choose additional simple closed curves $P_{3g-2}, \dots, P_{9g-9}$ so that for any simple diagram α the geometric intersection numbers $(I(\alpha, P_i))_{i=1}^{9g-9}$ totally determine the isotopy class of α. From Lemma 6.5 it follows that there is a piecewise-rational-linear function $h : \mathbb{R}^{6g-6} \to \mathbb{R}$ such that

$$h(\mathbf{n}) = \sum_{i=1}^{9g-9} I(S(\mathbf{n}), P_i) \quad \text{for all } \mathbf{n} \in \mathcal{A}. \tag{91}$$

Let $Q := \{x \in Q_\infty \mid h(x) \leq 1\}$. Since $(I(P_i, \alpha))_{i=1}^{9g-9}$ totally determines $\alpha \in \mathscr{S}$, the set $\mathcal{A} \cap kQ$ is finite for any $k \geq 0$.

Lemma 6.7. *The set Q defined above is the union of a finite number of convex polyhedra. Moreover, Q has positive volume in $\mathbb{R}^{6g-6}$.*

Proof. Let us prove that Q is bounded. Suppose to the contrary that Q is not bounded. By Proposition 6.4(a), Q contains a rational ray, which in turns contains infinitely many points whose coordinates are even integers. Since each such point is in $\mathcal{A}$, the set $\mathcal{A} \cap Q$ is infinite, which is a contradiction. Thus Q is bounded, and by Proposition 6.4(b), Q is the union of a finite number of convex polyhedra.

Choose $\alpha \in \mathscr{S}$ with $\nu(\alpha) = (n_1, \dots, n_{6g-6})$ satisfying $n_i > 0$ and whenever P_i, P_j, P_l bound a pair of pants then $n_i < n_j + n_l$. Let $h(\nu(\alpha)) = k$ then the point $\nu(\alpha)/(k+1)$ is an interior point of Q. Hence Q has positive volume. □

6.5. **Proof of Theorem 6.1, closed surface case.** Let $F_k(K_\zeta(F))$ be the $\mathbb{C}$-subspace spanned by $\{\alpha \in \mathscr{S} \mid h(\nu(\alpha)) \leq k\}$. By Proposition 3.7, $(F_k(K_\zeta(F)))_{k=0}^\infty$ is a filtration of $K_\zeta(F)$ compatible with the product. Then $\{S(\mathbf{n}) \mid \mathbf{n} \in \mathcal{A} \cap kQ\}$ is a $\mathbb{C}$-basis of $F_k(K_\zeta(F))$, hence

$$\dim_{\mathbb{C}} F_k(K_\zeta(F)) = |\mathcal{A} \cap kQ| = |\overline{\mathcal{A}} \cap kQ|. \tag{92}$$

If $\mathbf{n} \in \mathcal{A}_\zeta \cap kQ$, then by Theorem 3.5 one has $T(S(\mathbf{n})) \in Z_\zeta(F) \cap F_k(K_\zeta(F)) = F_k(Z_\zeta(F))$. Since the collection $\{T(S(\mathbf{n})), \mathbf{n} \in \mathcal{A}_\zeta \cap kQ\}$ is $\mathbb{C}$-linearly independent, we have

$$\dim_{\mathbb{C}} F_k(Z_\zeta(F)) \geq |\mathcal{A}_\zeta \cap kQ| = |\overline{\mathcal{A}_\zeta} \cap kQ|. \tag{93}$$

Using Lemma 2.3 then (92) and (93), we get, for some integer $u > 0$,

$$\dim_{Z_\zeta(F)} K_\zeta(F) \le \lim_{k\to\infty} \frac{\dim_{\mathbb{C}} F_k(K_\zeta(F))}{\dim_{\mathbb{C}} F_{k-u}(Z_\zeta(F))} \le \lim_{k\to\infty} \frac{|\overline{\mathcal{A}} \cap kQ|}{|\overline{\mathcal{A}_\zeta} \cap (k-u)Q|}. \tag{94}$$

The latter, by (8), is $[\overline{\mathcal{A}} : \overline{\mathcal{A}_\zeta}]$, which is equal to $D_\zeta(F)$ by Proposition 3.16. Thus, $\dim_{Z_\zeta(F)} K_\zeta(F) \le D_\zeta(F)$, completing the proof of Theorem 6.1. □

7. Commutative subalgebras of $\tilde{K}_\zeta(F)$

In this section we study commutative subalgebras of the skein algebra of a surface localized at its center, which are generated by collections of disjoint loops. We compute the dimensions of such subalgebras and describe their bases.

For a finite type surface F and a root of unity ζ recall that $\tilde{Z}_\zeta(F)$ is the field of fractions of the center $Z_\zeta(F)$ of the skein algebra $K_\zeta(F)$, and $\tilde{K}_\zeta(F) = K_\zeta(F) \otimes_{Z_\zeta(F)} \tilde{K}_\zeta(F)$ is a division algebra. Recall that if F is a surface with punctures $\{p_1, \dots, p_k\}$ then $\overline{F} = F \cup \{p_1, \dots, p_k\}$. Isotopy classes of simple diagrams on F are denoted by $\mathscr{S}$. Recall also that $T_k(x)$ denotes the k-th Chebyshev polynomial of the first kind (see section 3.3)

Proposition 7.1. *Suppose $C_1, \dots C_k$ are non-peripheral, non-trivial, disjoint, pairwise non-isotopic loops on a finite type surface $F = F_{g,p}$ of negative Euler characteristic. Let ζ be a root of unity with $m = \operatorname{ord}(\zeta^4)$. Let $\mathcal{C}$ be the $\tilde{Z}_\zeta(F)$-subalgebra of $\tilde{K}_\zeta(F)$ generated by $C_1, \dots, C_k$. For $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$ let $C^{\mathbf{n}} = \prod_{i=1}^k (C_i)^{n_i} \in \mathscr{S}$.*

(a) Suppose $\operatorname{ord}(\zeta) \ne 0 \pmod 4$ then $\dim_{\tilde{Z}_\zeta(F)} \mathcal{C} = m^k$ and the set

$$B = \{C^{\mathbf{n}} \mid 0 \le n_i < m\} \tag{95}$$

is a basis of $\mathcal{C}$ over $\tilde{Z}_\zeta(F)$.

(b) Suppose $\operatorname{ord}(\zeta) = 0 \pmod 4$ then $\dim_{\tilde{Z}_\zeta(F)} \mathcal{C} = 2^t m^k$, where t is the $\mathbb{Z}_2$-rank of the subgroup H of $H_1(\bar{F}, \mathbb{Z}_2)$ generated by $C_1, \dots, C_k$.

Assume that after a re-indexing $\{C_1, \dots, C_t\}$ is a basis for H. The set

$$B = \{C^{\mathbf{n}} \mid n_i < 2m \text{ for } i \le t,\ n_i < m \text{ for } i > t\} \tag{96}$$

is a basis of $\mathcal{C}$ over $\tilde{Z}_\zeta(F)$.

Proof.

(a) In this case $(C_i)^m \in \mathscr{S}_\zeta$ for all each i, where $\mathscr{S}_\zeta$ was defined by (18). By Theorem 3.5, $T_m(C_i) = T((C_i)^m) \in Z_\zeta(F)$, which implies that the degree of C_i over $\tilde{Z}_\zeta(F)$ is $\le m$. Hence B spans $\mathcal{C}$ as a vector space over $\tilde{Z}_\zeta(F)$.

By Corollary 5.2, to prove that B is linearly independent it is enough to show that $\deg_\zeta(x), x \in B$, are distinct. Assume $\deg_\zeta(C^{\mathbf{n}}) = \deg_\zeta(C^{\mathbf{n}'})$. Let $\mathbf{m}$ be the k-tuple all of whose entries are m. Since $\deg_\zeta$ is a monoid homomorphism and $\deg_\zeta(C^{\mathbf{m}}) = 0$, we have $\deg_\zeta(C^{\mathbf{m}-\mathbf{n}+\mathbf{n}'}) = 0$. By Proposition 5.9(b), for each i we have

$$m - n_i + n_i' = 0 \pmod m. \tag{97}$$

Since $0 \le n_i, n_i' \le m-1$, the only way this can happen is if $n_i = n_i'$.

(b) Let $\mathcal{C}_0$ be the $\tilde{Z}_\zeta(F)$-subalgebra of $\tilde{K}_\zeta(F)$ generated by $C_1, \dots, C_t$. Since $T_{2m}(C_i) \in Z_\zeta(F)$ by Theorem 3.5, the set $B_0 = \{C_1^{n_1} \dots C_t^{n_t} \mid n_i < 2m\}$ spans $\mathcal{C}_0$

over $\tilde{Z}_\zeta(F)$. Suppose $i > t$. There are $j_1, \dots j_l \le t$ such that the simple diagram $\beta = C_i \cup C_{j_1} \cup \ldots \cup C_{j_l}$ is even. This implies that $\beta^m \in \mathscr{S}_\zeta$. Hence $T(\beta^m) \in Z_\zeta(F)$ by Theorem 3.5. Using the definition of $T(\beta^m)$,

$$Z_\zeta(F) \ni T(\beta^m) = T_m(C_i)\left[T_m(C_{j_1}) \ldots T_m(C_{j_l})\right].$$

The element in the square bracket is in $\mathcal{C}_0$. It follows that $T_m(C_i) \in \mathcal{C}_0$, which implies that the degree of C_i over $\mathcal{C}_0$ is less than equal to m for each $i \ge t+1$. Hence $B_1 = \{C_{t+1}^{n_{t+1}} \ldots C_k^{n_k} \mid n_i < m\}$ spans $\mathcal{C}$ over $\mathcal{C}_0$. Combining the spanning sets B_0 and B_1, we get that B spans $\mathcal{C}$ over $\tilde{Z}_\zeta(F)$.

Let us show that $\deg_\zeta(x), x \in B$, are distinct. Suppose $\deg_\zeta(C^{\mathbf{n}}) = \deg_\zeta(C^{\mathbf{n}'})$. Let $\mathbf{m} = (m_1, \dots, m_k)$ where $m_i = 2m$ for $i \le t$ and $m_i = m$ for $i > t$. Then $\deg_\zeta(C^{\mathbf{m}}) = 0$. It follows that $\deg_\zeta(C^{\mathbf{m}-\mathbf{n}+\mathbf{n}'}) = 0$. By Proposition 5.9(b), we have $\mathbf{m} - \mathbf{n} + \mathbf{n}' \in m\mathbb{Z}^k$. This forces $n_i = n_i'$ for $i > t$ as in this case $m_i - n_i + n_i'$ is sandwiched between 1 and $2m-1$. Further $C^{(\mathbf{m}-\mathbf{n}+\mathbf{n})/m}$ is even by Proposition 5.9(b). Since $C_1, \dots, C_t$ are linearly independent over $\mathbb{Z}_2$ in $H_1(\bar{F}, \mathbb{Z}_2)$, for each $i \le t$, $(m_i - n_i + n_i')/m$ is even. As $m_i = 2m$ and $0 \le n_i, n_i' < 2m$, this forces $n_i = n_i'$. Thus $\deg_\zeta(x), x \in B$, are distinct, and by Corollary 5.2 the set B' is linearly independent over $\tilde{Z}_\zeta(F)$. □

Corollary 7.2. *Assume the conditions of Proposition* 7.1. *Let $\mathcal{C}'$ be the $\tilde{Z}_\zeta(F)$-subalgebra of $\tilde{K}_\zeta(F)$ generated by $C_1, \dots, C_{k-1}$. The minimal polynomial of C_k over $\mathcal{C}'$ is of the form $T_{m'}(x) - u$, where $u \in \mathcal{C}'$ and*

$$m' = \begin{cases} m & \text{if } \operatorname{ord}(\zeta) \ne 0 \pmod 4, \text{ or } \operatorname{ord}(\zeta) = 0 \pmod 4 \text{ and } k > t, \\ 2m & \text{if } \operatorname{ord}(\zeta) = 0 \pmod 4 \text{ and } k = t. \end{cases}$$

Moreover, u is transcendental over $\mathbb{Q}$.

Proof. Since $\mathcal{C} = \mathcal{C}'(C_k)$, the degree of C_k over $\mathcal{C}'$ is

$$[\mathcal{C} : \mathcal{C}'] = \frac{\dim_{\tilde{Z}_\zeta(F)} \mathcal{C}}{\dim_{\tilde{Z}_\zeta(F)} \mathcal{C}'},$$

which is equal to m' using the formula for $\dim_{\tilde{Z}_\zeta(F)} C$ and $\dim_{\tilde{Z}_\zeta(F)} C'$ given by Proposition 7.1. In the proof of Proposition 7.1 we see that $T_{m'}(C_k) = u \in \mathcal{C}'$. Hence $T_{m'}(x) - u$ is the minimal polynomial of C_k over $\mathcal{C}'$.

Suppose $u = T_{m'}(C_k)$ is algebraic over $\mathbb{Q}$. Since $m' > 0$ this implies C_k is algebraic over $\mathbb{Q}$. But $\{C_k^i, i \ge 0\}$ is a subset of $\mathscr{S}$, which is a $\mathbb{C}$-basis of $K_\zeta(F)$ and hence the non-trivial $\mathbb{Q}$-linear combination of these elements is never 0. This shows u is transcendental over $\mathbb{Q}$. □

8. Calculation of the reduced trace

Let F be a finite type surface and ζ a root of unity. Since the skein algebra $K_\zeta(F)$ is finitely generated as a module over its center $Z_\zeta(F)$, it has a reduced trace (see Section 2.3). The goal of this section is to find a formula for computing it.

By Theorem 3.5 the set $\{T(\alpha) \mid \alpha \in \mathscr{S}\}$ is a $\mathbb{C}$-basis of $K_\zeta(F)$, where $T(\alpha)$ was defined by (17). Therefore it is enough to calculate $\operatorname{tr}(T(\alpha))$ for each simple diagram $\alpha \in \mathscr{S}$. Recall that $\mathscr{S}_\zeta$ was defined by (18).

Theorem 8.1. *Let F be a finite type surface, $\mathscr{S}$ the set of isotopy classes of simple diagrams on F, ζ a root of 1, and let*

$$\mathrm{tr}: \tilde{K}_\zeta(F) \to \tilde{Z}_\zeta(F) \tag{98}$$

be the reduced trace. For $\alpha \in \mathscr{S}$ one has

$$\mathrm{tr}(T(\alpha)) = \begin{cases} T(\alpha) & \text{if } T(\alpha) \text{ is central, i.e., } \alpha \in \mathscr{S}_\zeta \\ 0 & \text{otherwise.} \end{cases} \tag{99}$$

First consider the case when $F_{g,p}$ has non-negative Euler characteristic. The skein algebras of $F_{g,p}$ for $g = 0$ and $p = 0, 1, 2$ are commutative, so the result is trivial, i.e., the normalized trace is the identity map. For $F_{1,0}$ and n not divisible by 4 this is proved in [1]. The remaining case of $K_\zeta(F_{1,0})$ and n divisible by 4 can be proved using similar methods. Hence we will assume that F has negative Euler characteristic.

8.1. **Lemma on traces.** In order to prove Theorem 8.1 we need the following properties of trace algebras.

Recall that TR denotes the non-reduced trace (see Section 2.3) and $T_l(x)$ denotes the l-th Chebyshev polynomial of the first kind, as defined in Section 3.3.

Lemma 8.2. *Suppose $k_1 \subset k_2$ are finite field extensions of a field k_0 and $x_1 \in k_1, x_2 \in k_2$.*

(a) *If $\mathrm{TR}_{k_2/k_1}(x_2) = 0$ then $\mathrm{TR}_{k_2/k_0}(x_1 x_2) = 0$.*
(b) *Assume the minimal polynomial of x_2 over k_1 is $T_l(x) - u$, where $u \in k_1$ is transcendental over $\mathbb{Q}$, and $l \geq 2$. For $0 < s < l$ we have*

$$\mathrm{TR}_{k_2/k_0}(x_1 T_s(x_2)) = 0. \tag{100}$$

Proof.

(a) A property of the trace is that for any $x \in k_2$ we have

$$\mathrm{TR}_{k_2/k_0}(x) = \mathrm{TR}_{k_1/k_0}\left(\mathrm{TR}_{k_2/k_1}(x)\right), \tag{101}$$

see eg [23]. With $x = x_1 x_2$, we have

$$\mathrm{TR}_{k_2/k_0}(x_1 x_2) = \mathrm{TR}_{k_1/k_0}\left(\mathrm{TR}_{k_2/k_1}(x_1 x_2)\right) = \mathrm{TR}_{k_1/k_0}\left(x_1 \mathrm{TR}_{k_2/k_1}(x_2)\right) = 0. \tag{102}$$

(b) From (a) it is enough to show that $\mathrm{TR}_{k_2/k_1}(T_s(x_2)) = 0$.

Let t be the smallest positive integer such that $l|ts$. Note that $t \geq 2$. Denote $m = ts/l$. Define $u_0 = 1$ and $u_i = T_i(u)$ for $i \geq 1$.

Claim. The minimal polynomial of $y = T_s(x_2)$ over k_1 is

$$P = T_t(x) - u_m. \tag{103}$$

Assume the claim for now. Since $t \geq 2$ and T_t is either even or odd polynomial, the second-highest coefficient of $T_t - v$ is 0. By Proposition 2.2(a), we have $\mathrm{TR}_{k_2/k_1}(T_s(x_2)) = 0$. Thus (b) follows from the claim.

Proof of the Claim. First note the $P(y) = 0$. In fact, we have

$$T_t(y) = T_t(T_s(x_2)) = T_{ts}(x_2) = T_{ml}(x_2) = T_m(T_l(x_2)) = T_m(u) = u_m,$$

which shows $P(y) = 0$. Let us show that no polynomial $Q(x)$ of degree $< t$ can annihilate y. Since $\{T_i(x)\}$ forms a basis, we can write $Q(x) = \sum_{i=0}^{d} c_i T_i(x)$ with $d < t$, $c_i \in k_1$, and $c_d = 1$. We have

$$0 = Q(y) = \sum_{i=0}^{d} c_i T_{is}(x_2). \tag{104}$$

Since $T_l(x) - u$ is the minimal polynomial of x_2, we have

$$\sum_{i=0}^{d} c_i T_{is}(x) \equiv 0 \mod (T_l(x) - u). \tag{105}$$

Let $R_i(x)$ be the remainder obtained upon dividing T_{is} by $T_l(x) - u$, then we must have

$$\sum_{i=0}^{d} c_i R_i(x) = 0. \tag{106}$$

To finish the proof we need another lemma:

Lemma 8.3. *Let $T_l(x)$ denote the l-th Chebyshev polynomial of the first kind. Suppose $0 \leq r < l$. When $T_{ql+r}(x)$ is divided by $(T_l(x) - u)$, the remainder is $S_q(u)T_r(x) - S_{q-1}(u)T_{l-r}(x)$, where $S_i(x)$ is the Chebyshev polynomial of the second kind defined recursively by*

$$S_0 = 1, S_1(x) = x, S_n(x) = xS_{n-1}(x) - S_{n-2}(x).$$

Proof of Lemma 8.3. One can easily check that

$$S_i(u) = \sum_{0 \leq j \leq (i/2)} u_{i-2j} = u_i + u_{i-2} + \dots. \tag{107}$$

Using $T_m T_n = T_{m+n} + T_{|m-n|}$, we get

$$\begin{aligned} T_{ql+r}(x) &= T_r(x)T_{ql}(x) - T_{ql-r}(x) = T_r(x)T_q(T_l(x)) - T_{(q-1)l+l-r}(x) \\ &\equiv T_r(x)u_q - T_{(q-1)l+l-r}(x) \pmod{(T_l(x) - u)} \\ &\equiv T_r(x)u_q - T_{l-r}(x)u_{q-1} + T_{(q-2)l+r}(x) \pmod{(T_l(x) - u)} \\ &\equiv T_r(x)u_q - T_{l-r}(x)u_{q-1} + T_r(x)u_{q-2} - T_{(q-3)l+l-r}(x) \pmod{(T_l(x) - u)} \\ &\equiv T_r(x)(u_q + u_{q-2} + u_{q-4} + \dots) \\ &\quad - T_{l-r}(x)(u_{q-1} + u_{q-3} + \dots) \pmod{(T_l(x) - u)} \\ &\equiv S_q(u)T_r(x) - S_{q-1}(u)T_{l-r}(x) \pmod{(T_l(x) - u)}. \end{aligned}$$

□

Suppose $ds = ql + r$, with $0 \leq r < l$. By Lemma 8.3,

$$R_d = S_q(u)T_r(x) - S_{q-1}(u)T_{l-r}(x). \tag{108}$$

Note that

(*) there is no index $j \in [0, d-1]$ such that js has remainder r when divided by l.

Consider two cases: (i) $r = l - r$ and (ii) $r \neq l - r$.

(i) $r = l - r$. Then $R_d = (S_q(u) - S_{q-1}(u))T_r(x)$. From (*) we see that no index $j \neq d$ contributes to the term $T_r(x)$ in (106). Hence $S_q(u) - S_{q-1}(u) = 0$, contradicting the fact that u is transcendental over $\mathbb{Q}$.

(ii) $r \neq l - r$. There is exactly one index $j \in [0, d-1]$ such that js has remainder $l - r$ when divided by l, which is $j = (t - d)$. Suppose $(t - d)s = q'l + (l - r)$, then

$$R_{t-d} = S_{q'}(u)T_{l-r}(x) - S_{q'-1}(u)T_r(x).$$

By looking at the coefficients of $T_r(x)$ and $T_{l-r}(x)$ in (106), we get (with $c = c_{t-d}$)

$$S_q(u) - cS_{q'-1}(u) = 0 \tag{109}$$

$$-S_{q-1}(u) + cS_{q'}(u) = 0. \tag{110}$$

Multiplying the first equation by $S_{q'}(u)$, the second by $S_{q'-1}(u)$, and summing up the two, we get

$$S_q(u)S_{q'}(u)u - S_{q'-1}(u)S_{q-1}(u) = 0 \tag{111}$$

contradicting the fact that u is transcendental over $\mathbb{Q}$. □

8.2. **Proof of Theorem 8.1.**

Proof. Let $n = \text{ord}(\zeta)$ and $m = \text{ord}(\zeta^4)$.

Note that if $z \in Z_\zeta(F)$ then $\text{tr}(z) = z$, and more generally

$$\text{tr}(zx) = z\text{tr}(x) \quad \text{if } z \in Z_\zeta(F). \tag{112}$$

Hence we assume that $T(\alpha) \notin Z_\zeta(F)$ and we will show $\text{tr}(T(\alpha)) = 0$.

Assume $\alpha = \prod_{i=1}^k (C_i)^{m_i}$, where $C_1, \dots, C_k$ are non-trivial loops, no two of which are isotopic, then $T(\alpha) = \prod_{i=1}^k T_{m_i}(C_i)$. If a component C_i is peripheral then $C_i \in Z_\zeta(F)$, and by (112) we can reduce to the case when none of C_i is peripheral.

From the product to sum formula (16), we have

$$T_n(x) = T_{2m}(x)T_{n-2m} - T_{|(n-2m)-2m|}(x) \quad \text{if } n \geq 2m \tag{113}$$

$$T_n(x) = T_m(x)T_{n-m} - T_{|(n-m)-m|}(x) \quad \text{if } n \geq m. \tag{114}$$

Let $\mathcal{C}$ be the $\tilde{Z}_\zeta(F)$-subalgebra of $\tilde{K}_\zeta(F)$ generated by loops $C_1, \dots, C_k$ and $\mathcal{C}'$ be the subalgebra generated by $C_1, \dots, C_{k-1}$. Consider several cases.

(a) Suppose $n \neq 0 \pmod 4$. In this case $T_m(C) \in Z_\zeta(F)$ for any loop C on F. Using (114) and (112) we reduce to the case $m_i < m$ for all i. By Corollary 7.2 the minimal polynomial of C_k over $\mathcal{C}'$ is $T_m - u$ for some $u \in \mathcal{C}'$ and u is transcendental over $\mathbb{Q}$. Lemma 8.2 with $k_0 = \tilde{Z}_\zeta(F), k_1 = \mathcal{C}', k_2 = \mathcal{C}, x_1 = \prod_{i=1}^{k-1} T_{m_i}(C_i)$, and $x_2 = C_k$ shows that $\text{tr}(T(\alpha)) = 0$.

(b) Suppose $n = 0 \pmod 4$. Since $T_{2m}(C_i) \in Z_\zeta(F)$, using (113) and (112) we reduce to the case $m_i < 2m$ for all i.

(i) Suppose $C_1, \dots, C_k$ are $\mathbb{Z}_2$-linearly independent in $H_1(\bar{F}, \mathbb{Z}_2)$. By Corollary 7.2 the minimal polynomial of C_k over $\mathcal{C}'$ is $T_{2m} - u$ for some $u \in \mathcal{C}'$ transcendental over $\mathbb{Q}$. Lemma 8.2 with $k_0 = \tilde{Z}_\zeta(F), k_1 = \mathcal{C}', k_2 = \mathcal{C}, x_1 = \prod_{i=1}^{k-1} T_{m_i}(C_i)$, and $x_2 = C_k$ shows that $\text{tr}(T(\alpha)) = 0$.

(ii) Suppose $C_1, \dots, C_k$ are not $\mathbb{Z}_2$-linearly independent in $H_1(\bar{F}, \mathbb{Z}_2)$. There are indices $j_1, \dots, j_l$ such that $\sum_i C_{j_i} = 0$ in $H_2(\bar{F}, \mathbb{Z}_2)$. This implies that $\prod_i T_m(C_{j_i}) \in Z_\zeta(F)$. If all $m_{j_i} \geq m$, then $2m > m_{j_i} \geq m$. Taking the product of l Equations

(114) with $n = m_{ji}$ then using (112), we reduce to the case when there is i such that $m_{j_i} < m$.

Re-indexing, we assume that $j_i = k$. Thus $m_k < m$, and C_k, as an element of $H_1(\bar{F}, \mathbb{Z}_2)$, is in the $\mathbb{Z}_2$-span of $C_1, \ldots, C_{k-1}$. By Corollary 7.2 the minimal polynomial of C_k over $\mathcal{C}'$ is $T_m - u$ for some $u \in \mathcal{C}'$ transcendental over $\mathbb{Q}$. Again Lemma 8.2 shows that $\mathrm{tr}(T(\alpha)) = 0$. □

8.3. **Trace and** $\deg_\zeta$. In this section we discuss the relationship between the degree map $\deg_\zeta : \tilde{K}_\zeta(F)^* \to \mathfrak{R}_\zeta$ (see (65)) and the reduced trace. The following is a consequence of Theorem 8.1.

Proposition 8.4. *Let F be a finite type surface, ζ a root of unity, and let* $\mathrm{tr} : \tilde{K}_\zeta(F) \to \tilde{Z}_\zeta(F)$ *be the reduced trace. If $x \in K_\zeta(F)^*$ and* $\deg_\zeta(x) = 0$*, then* $\mathrm{tr}(x) \neq 0$.

Proof. This is proved for open surfaces and ζ not divisible by 4 in [12, Lemma 3.8]. Other cases are similar, with a stable lead term replacing the lead term in the argument. □

The following theorem extends the exhaustion criterion from [12].

Theorem 8.5. *Let F be a finite type surface and ζ a root of unity. Suppose $\mathcal{B}$ is a collection of nonzero elements in a $\tilde{Z}_\zeta(F)$-subalgebra A of $\tilde{K}_\zeta(F)$.*

(i) *If* $\deg_\zeta(\mathcal{B}) = \deg_\zeta(A^*)$ *then $\mathcal{B}$ spans $\tilde{K}_\zeta(F)$ over $\tilde{Z}_\zeta(F)$.*

(ii) *The dimension of A over $\tilde{Z}_\zeta(F)$ is equal to* $|\deg_\zeta(A)|$.

Proof.

(i) Since $\deg_\zeta(A^*)$ is a group, for every $x \in A^*$ there exists $b \in \mathcal{B}$ such that $\deg_\zeta(xb) = \deg_\zeta(x) + \deg_\zeta(b) = 0$. By Proposition 8.4, $\mathrm{tr}(xb) \neq 0$. Therefore the set $\mathcal{B}$ exhausts the bilinear form given by the trace on $A \otimes_{\tilde{Z}_\zeta(F)} A$. Consequently $\mathcal{B}$ spans A.

(ii) Choose a subset $\mathcal{B}$ of A^* such that $\deg_\zeta|_{\mathcal{B}} \colon \mathcal{B} \to \deg_\zeta(A^*)$ is bijective. By (i), $\mathcal{B}$ spans A. By Proposition 5.2(a), $\mathcal{B}$ is linearly independent over $\tilde{Z}_\zeta(F)$. Hence $\mathcal{B}$ is a basis. □

9. Pants subalgebra decomposition

In this section we give a splitting of the localized skein algebra $\tilde{Z}_\zeta(F)$ as a module over its center. Throughout, $F = F_{g,p}$ is a finite type surface with negative Euler characteristic, with or without punctures, ζ is a root of unity, and $m = \mathrm{ord}(\zeta^4)$.

9.1. **The splitting.** Recall that a pants decomposition of a surface F is a collection of curves $\mathcal{P} = \{C_1, \ldots, C_{3g-3+p}\}$ such that each component of the complement of $\mathcal{P}$ in F is a planar surfaces of Euler characteristic -1. Also $\tilde{K}_\zeta(F)$ is the localized skein algebra with its center $\tilde{Z}_\zeta(F)$, which is a commutative field.

Given a pants decomposition $\mathcal{P}$ of F, let $\mathcal{C}_\zeta(\mathcal{P})$ be the $\tilde{Z}_\zeta(F)$-subalgebra of $\tilde{K}_\zeta(F)$ generated by the curves in $\mathcal{P}$. By Proposition 7.1 and Theorem 6.1

$$\dim_{\tilde{Z}_\zeta(F)} \mathcal{C}_\zeta(\mathcal{P}) = \sqrt{\dim_{\tilde{Z}_\zeta(F)} \tilde{K}_\zeta(F)}. \tag{115}$$

Hence $\mathcal{C}_\zeta(\mathcal{P})$ is a maximal commutative subalgebra of the division algebra $\tilde{K}_\zeta(F)$. In [12] the first two authors constructed a splitting of $\tilde{K}_\zeta(F)$, when F has at least one puncture and $\mathrm{ord}(\zeta) \neq 0 \pmod 4$. Here we prove that this decomposition works for all surfaces and all roots of unity.

Theorem 9.1. *Let F be a finite type surface of negative Euler characteristic. There exist two pants decompositions $\mathcal{P}$ and $\mathcal{Q}$ of F such that for any root of unity ζ the $\tilde{Z}_\zeta(F)$-linear map*

$$\psi : \mathcal{C}_\zeta(\mathcal{P}) \otimes_{\tilde{Z}_\zeta(F)} \mathcal{C}_\zeta(\mathcal{Q}) \to \tilde{K}_\zeta(F), \tag{116}$$

defined by $\psi(x \otimes y) = xy$, is a $\tilde{Z}_\zeta(F)$-linear isomorphism of vector spaces. Here $\mathcal{C}_\zeta(\mathcal{P})$ and $\mathcal{C}_\zeta(\mathcal{Q})$ denote the $\tilde{Z}_\zeta(F)$-subalgebras of $\tilde{K}_\zeta(F)$ generated by the curves in $\mathcal{P}$ and $\mathcal{Q}$ respectively.

9.2. **Pants decompositions.** Recall that a peripheral loop is a simple closed curve in F bounding a once-puntured disk. Let $\mathbb{C}[\partial]$ be the $\mathbb{C}$-subalgebra of $K_\zeta(F)$ generated by peripheral loops, with the convention that if $p = 0$ then $\mathbb{C}[\partial]$ is the 0 vector space.

For a pants decomposition $\mathcal{P}$ let $\mathbb{C}[\mathcal{P}]$ be the $\mathbb{C}$-subalgebra of $K_\zeta(F)$ generated by loops in $\mathcal{P}$. For a set $U \subset \mathbb{Z}^r$ let $\overline{U}$ be the $\mathbb{Z}$-span of U. When F has a fixed coordinate datum, we use $\mathcal{A} \subset \mathbb{Z}^r$ to denote the submonoid of all possible coordinates of simple diagrams in F. Also, for an algebra X we denote by X^* the set of non-zero elements in X. The map deg is given in Theorem 5.1.

Lemma 9.2. *Suppose $F_{g,p}$ is a finite type surface with negative Euler characteristic. There exist a coordinate datum for $F_{g,p}$ and two pants decompositions $\mathcal{P}$ and $\mathcal{Q}$ such that*

$$\overline{\mathcal{A}} = \overline{\deg(\mathbb{C}[\partial]^*)} + \overline{\deg(\mathbb{C}[\mathcal{P}]^*)} + \overline{\deg(\mathbb{C}[\mathcal{Q}]^*)}. \tag{117}$$

Proof. To prove the lemma we use the following result that follows from the computation of the determinants in the proof of Theorem 4.5 in [12, Section 4]. The proof of the Lemma below was also done independently by Nathan Soedjak. Recall that $\nu : \mathscr{S} \to \mathcal{A}$ is the coordinates map, and $\mathcal{A}^\partial$ is the set of coordinates of peripheral loops.

Lemma 9.3. *Suppose $F_{g,p}$ has negative Euler characteristic and $p > 0$. There exist a coordinate datum $(a_i)_{i=1}^{6g-6+3p}$ and two pants decompositions $\mathcal{P}, \mathcal{Q}$ such that the $\mathbb{Z}$-span of $\nu(\mathcal{P}), \nu(\mathcal{Q})$ and $\mathcal{A}^\partial$ has index $2^{4g-5+2p}$ in $\mathbb{Z}^{6g-6+3p}$. In other words, if $C_1, \dots, C_{6g-6+3p}$ is the set of curves consisting of components of $\mathcal{P}, \mathcal{Q}$ and the p peripheral loops, then*

$$\det \left[I(a_i, C_j)_{i,j=1}^{6g-6+3p} \right] = 2^{4g-5+2p}. \tag{118}$$

$\square$

Pants decompositions for a surface of genus 3 and 1 puncture are shown in Figure 4. Additional curves needed for more than one puncture are shown in Figure 5 for the case of four punctures. For a detailed description of those curves see [12]. Note that the collection of additional cures in the P family needed for more than one puncture differs from the one described in [12]. Following the notation in that paper, the c_g curve has to be replaced by the h_g curve. This fixes the mistake in the computation of the determinant for the case of more than one puncture.

By Lemma 3.9, the index of $\overline{\mathcal{A}}$ in $\mathbb{Z}^{6g-6+3p}$ is also $2^{4g-5+2p}$. Hence $\overline{\mathcal{A}}$ is equal to the $\mathbb{Z}$-span of $\nu(\mathcal{P}), \nu(\mathcal{Q})$ and $\mathcal{A}^\partial$.

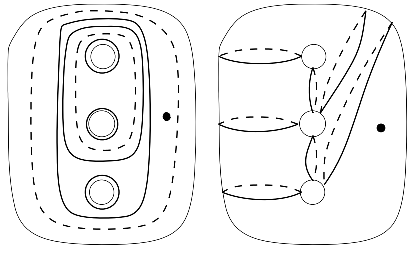

FIGURE 4. P and Q for a surface of genus 3 and 1 puncture

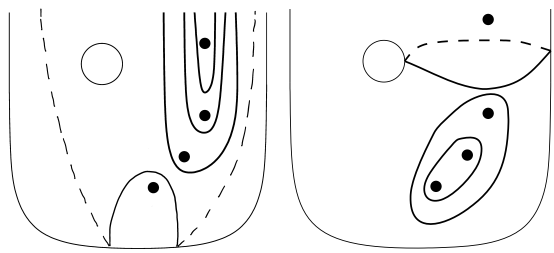

FIGURE 5. Additional curves in families P and Q for 4 punctures

To prove Lemma 9.2 consider cases when $p > 0$ and $p = 0$.

(a) Suppose $p > 0$. Use the ideal triangulation and the two pants decompositions of Lemma 9.3. When $p > 0$, by the definition in Equation (70), one has $\deg(\alpha) = \nu(\alpha)$ for all $\alpha \in \mathscr{S}$. Hence

$$\overline{\deg(\mathbb{C}[\partial]^*)} = \overline{\mathcal{A}}, \ \overline{\deg(\mathbb{C}[\mathcal{P}]^*)} = \overline{\nu(\mathcal{P})}, \ \overline{\deg(\mathbb{C}[\mathcal{Q}]^*)} = \overline{\nu(\mathcal{Q})}.$$

Thus the right hand side of (117) is the $\mathbb{Z}$-span of $\mathcal{A}^\partial, \nu(\mathcal{P}), \nu(\mathcal{Q})$, which by Lemma 9.3 is equal to $\overline{\mathcal{A}}$.

(b) Suppose $p = 0$. Let Σ be a compact planar surface with $g+1$ boundary components, then $\mathring{\Sigma} = \Sigma \setminus \partial\Sigma$ is a finite type surface of type $F_{0,g+1}$. Let $(a_i)_{i=1}^{3g-3}$, $\mathcal{Q}_1$, $\mathcal{Q}_2$ be respectively the ideal triangulation, and the pants decompositions $\mathcal{P}$ and $\mathcal{Q}$ of Lemma 9.3 for the surface $\mathring{\Sigma}$. Let $\mathcal{D} \subset \mathring{\Sigma}$ be the trivalent graph dual to the system $(a_i)_{i=1}^{3g-3}$. We can assume that the topological closure $\bar{a}_i$ of a_i in Σ is a proper embedding of $[0,1]$ into Σ and that the $6g-6$ endpoints of all $3g-3$ arcs $\bar{a}_i$ are distinct. Take another copy Σ' of Σ and assume that $\varphi : \Sigma \to \Sigma'$ is a diffeomorphism. Let F be the result of gluing Σ with Σ' along the boundary by the identification $x \equiv \varphi(x)$ for every $x \in \partial\Sigma$. Let $\mathcal{P} = (P_1, \dots, P_{3g-3})$ where $P_i = \bar{a}_i \cup \varphi(\bar{a}_i)$ and $\mathcal{Q} = (Q_1, \dots, Q_{3g-3})$ be the collection of components of $\mathcal{Q}_1$, $\partial\Sigma$, and $\varphi(\mathcal{Q}_2)$, in some order. We claim that the coordinate datum $(\mathcal{P}, \mathcal{D})$ and the two pants decompositions $\mathcal{P}, \mathcal{Q}$ satisfy (117). Note that we can take $\Sigma = N(\mathcal{D})$, and $\Omega = \partial\Sigma$.

The surface $\mathring{\Sigma}$ has genus $g' = 0$ and puncture number $p' = g+1$. It follows from (118) that

$$\det\left[I(P_i, Q_j)_{i,j=1}^{3g-3}\right] = 2^{4g'-5+2p'} = 2^{2g-3}. \tag{119}$$

Since F is closed, $\deg(\alpha) = \eta(\alpha)$ for $\alpha \in \mathscr{S}$ by Definition (79). Hence

$$\overline{\deg(\mathbb{C}[\mathcal{P}]^*)} = \overline{\eta(\mathcal{P})}, \quad \overline{\deg(\mathbb{C}[\mathcal{Q}]^*)} = \overline{\eta(\mathcal{Q})}. \tag{120}$$

Thus, to prove the lemma we need to show that

$$\overline{\eta(\mathcal{P})} + \overline{\eta(\mathcal{Q})} = \overline{\mathcal{A}}. \tag{121}$$

Since the left hand side is a subgroup of the right hand side, we only need to show that they have the same index in $\mathbb{Z}^{6g-6}$.

Let $\vec{0} \in \mathbb{Z}^{3g-3}$ be the 0 vector and let $\boldsymbol{\delta}_i \in \mathbb{Z}^{3g-3}$ be the vector whose entries are all 0 except for the i-th one which is 1. By Proposition 4.7(e), we have $\eta(P_i) = (\vec{0}, -\boldsymbol{\delta}_i)$. Hence $\overline{\eta(\mathcal{P})} = \{\vec{0}\} \oplus \mathbb{Z}^{3g-3}$. It follows that index of $\overline{\eta(\mathcal{Q})} + \overline{\eta(\mathcal{P})}$ in $\mathbb{Z}^{6g-6}$ is equal to the index of $\left[\overline{\eta(\mathcal{Q})}\right]_1$ in $\mathbb{Z}^{3g-3}$, where $\left[\overline{\eta(\mathcal{Q})}\right]_1$ is the $\mathbb{Z}$-span of $(3g-3)$-tuples which are the first $3g-3$ coordinates of $\eta(Q_i)$, $i = 1, \dots, 3g-3$. Since each Q_i has 0 intersection with Ω, by Proposition 4.7 one has $\eta(Q_i) = \nu(Q_i)$. The first $3g-3$ coordinates of $\nu(Q_i)$ are given by $I(Q_i, P_j)_{j=1}^{3g-3}$. Hence the index of $\left[\overline{\eta(\mathcal{Q})}\right]_1$ in $\mathbb{Z}^{3g-3}$ is $\det\left[I(P_i, Q_j)_{i,j=1}^{3g-3}\right]$, which is equal to 2^{2g-3} by (119).

Thus the index of $\overline{\eta(\mathcal{P})} + \overline{\eta(\mathcal{Q})}$ in $\mathbb{Z}^{6g-6}$ is 2^{2g-3}, equal to the index of $\overline{\mathcal{A}}$ in $\mathbb{Z}^{6g-6}$, by Lemma 3.15. Hence we have (121), which proves the lemma. □

9.3. **Proof of Theorem 9.1.** Let X be the image of ψ defined in (116) and $X^* = X \setminus \{0\}$. Choose the coordinate datum and the pants decompositions $\mathcal{P}$, $\mathcal{Q}$ as in Lemma 9.2. Let

$$\mathcal{B} := \deg(\mathbb{C}[\partial]^*) + \deg(\mathbb{C}[\mathcal{P}]^*) + \deg(\mathbb{C}[\mathcal{Q}]^*).$$

Note that $\mathcal{B}$ is a submonoid of $\overline{\mathcal{A}}$, and Lemma 9.2 implies that $\overline{\mathcal{B}} = \overline{\mathcal{A}}$. By Lemma 3.8 the natural map $\phi : \mathcal{B} \to \overline{\mathcal{B}}/\overline{\mathcal{A}_\zeta} = \overline{\mathcal{A}}/\overline{\mathcal{A}_\zeta} = \mathfrak{R}_\zeta$ is surjective.

Let $Y = \{y_1 y_2 y_3 \mid y_1 \in \mathbb{C}[\partial]^*, y_2 \in \mathbb{C}[\mathcal{P}]^*, y_3 \in \mathbb{C}[\mathcal{Q}]^*\}$, then $\deg(Y) = \mathcal{B}$. Since $Y \subset X$, we have the following commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\deg} & \mathcal{B} \\ \big\downarrow & & \big\downarrow \phi \\ X & \xrightarrow{\deg_\zeta} & \mathfrak{R}_\zeta \end{array}$$

Since deg and ϕ are surjective, $\deg_\zeta$ is also surjective. This means $\deg_\zeta(X) = \mathfrak{R}_\zeta$. By Corollary 6.3 we have $X = \tilde{K}_\zeta(F)$. Thus ψ is surjective. Since the dimension over $\tilde{Z}_\zeta(F)$ of the domain and the codomain of ψ are the same, ψ is a $\tilde{Z}_\zeta(F)$-linear isomorphism. □

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