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THE LEGENDRE-HARDY INEQUALITY ON BOUNDED DOMAINS

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Abstract. There have been numerous studies on Hardy's inequality on a bounded domain, which holds for functions vanishing on the boundary. On the other hand, the classical Legendre differential equation defined in an interval can be regarded as a Neumann version of the Hardy inequality with subcritical weight functions. In this paper we study a Neumann version of the Hardy inequality on a bounded C^2-domain in $\mathbb{R}^n$ of the following form

$$\int_\Omega d^\beta(x)|\nabla u(x)|^2 dx \geq C(\alpha,\beta)\int_\Omega \frac{|u(x)|^2}{d^\alpha(x)}dx \quad \text{with} \quad \int_\Omega \frac{u(x)}{d^\alpha(x)}dx = 0,$$

where $d(x)$ is the distance from $x \in \Omega$ to the boundary $\partial\Omega$ and $\alpha, \beta \in \mathbb{R}$. We classify all $(\alpha,\beta) \in \mathbb{R}^2$ for which $C(\alpha,\beta) > 0$. Then, we study whether an optimal constant $C(\alpha,\beta)$ is attained or not. Our study on $C(\alpha,\beta)$ for general $(\alpha,\beta) \in \mathbb{R}^2$ shows that the (classical) Hardy inequality can be regarded as a special case of the Neumann version.

1. Introduction and statement of main results

In this paper, we are interested in the following inequality

$$\int_\Omega \frac{|u(x)|^2}{d^\alpha(x)}dx \leq C\int_\Omega d^\beta(x)|\nabla u(x)|^2 dx \quad \text{with} \quad \int_\Omega \frac{u(x)}{d^\alpha(x)}dx = 0,$$

where $\alpha, \beta \in \mathbb{R}$, $d(x) \equiv \text{dist}(x, \partial\Omega)$ and Ω is a bounded C^2-domain in $\mathbb{R}^n$. This study was motivated by the Legendre differential equation. The Legendre differential equation

$$-((1-t^2)\phi')' = l(l+1)\phi, \quad l = 0, 1, \cdots,$$

has the first kind of solution P_l, which is a polynomial of order l, and the second kind of solution Q_l, which is singular at $t = \pm 1$. Then we see that for any $\phi \in C^1((-1,1)) \cap L^2((-1,1))$ satisfying $\int_{-1}^1 \phi dt = 0$,

$$\int_{-1}^1 (1-t^2)|\phi'(t)|^2 dt \geq 2\int_{-1}^1 (\phi(t))^2 dt. \tag{1}$$

Furthermore, the equality holds for $P_1(x) = x$; on the other hand, for $l = 0, 1, \cdots$,

$$\int_{-1}^1 (Q_l(t))^2 dt < \int_{-1}^1 (1-t^2)|Q_l'(t)|^2 dt = \infty.$$

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This can be compared with Wirtinger's inequality, which says that for any $\phi \in C^1((-1,1)) \cap L^2((-1,1))$ satisfying $\int_{-1}^1 \phi dt = 0$,

$$\int_{-1}^{1} |\phi'(t)|^2 dt \geq \frac{\pi^2}{4} \int_{-1}^{1} (\phi(t))^2 dt, \tag{2}$$

where the equality holds for $\phi(t) = \sin \frac{\pi t}{2}$. On the other hand, a version of Hardy's inequality says that for any $\phi \in W_0^{1,2}((-1,1))$,

$$\int_{-1}^{1} |\phi'(t)|^2 dt \geq \frac{1}{4} \int_{-1}^{1} \frac{(\phi(t))^2}{(1-t^2)^2} dt, \tag{3}$$

where the equality does not hold for any $\phi \in W_0^{1,2}((-1,1)) \setminus \{0\}$.

These inequalities (1), (2), (3) are quite fundamental and there have been numerous studies on higher dimensional versions of (2) and (3). On the other hand, it seems that there have been no studies for the higher dimensional version of (1). In this paper, we study the high dimensional version of (1) and find a general form of inequalities that includes (1), (2), (3). For their generalization on a bounded domain Ω in $\mathbb{R}^N$, we define $d(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$ and

$$\mathcal{X}_{\alpha,\beta}(\Omega) \equiv \Big\{ \phi \in C^1(\Omega) \ \Big|\ \int_\Omega d^\beta(x) |\nabla \phi(x)|^2 + \frac{(\phi(x))^2}{d^\alpha(x)} dx < \infty \Big\}, \ \alpha, \beta \in \mathbb{R}.$$

For $u \in \mathcal{X}_{\alpha,\beta}(\Omega)$, we define a norm $\|u\|_{\alpha,\beta} \equiv \Big(\int_\Omega d^\beta(x)|\nabla u|^2 + d^{-\alpha}(x) u^2 dx \Big)^{1/2}$ and $W^{1,2}_{\alpha,\beta}(\Omega)$ the completion of $\mathcal{X}_{\alpha,\beta}(\Omega)$ with respect to the norm $\|\cdot\|_{\alpha,\beta}$. Then we consider a minimization

$$L_{\alpha,\beta}(\Omega) \equiv \inf \Big\{ \frac{\int_\Omega d^\beta(x) |\nabla u|^2 dx}{\int_\Omega |u|^2 d^{-\alpha}(x) dx} \ \Big|\ u \in W^{1,2}_{\alpha,\beta}(\Omega) \setminus \{0\}, \int_\Omega u d^{-\alpha}(x) dx = 0 \Big\}.$$

In this paper, we characterize $(\alpha, \beta) \in \mathbb{R}^2$ for which $L_{\alpha,\beta}$ is positive, and find conditions of (α,β) and domain Ω under which $L_{\alpha,\beta} > 0$ is attained. In fact, we will see that if (α,β) is in a subcritical region, the attainability of $L_{\alpha,\beta}$ is determined by the comparison of two energy levels $L_{\alpha,\beta}$ and $H_{\alpha,\beta}$, where $H_{\alpha,\beta}$ is defined in (4). For (α,β) in the critical region $\alpha+\beta = 2$, there exists a critical level of $L_{\alpha,\beta}$, where we lose a compactness of a minimizing sequence. Then, for some typical domains, we find conditions of (α,β) in the critical region $\alpha+\beta=2$ under which $L_{\alpha,\beta}$ either is strictly less than the critical level of $L_{\alpha,\beta}$ or is equal to a critical level of $L_{\alpha,\beta}$. In the same direction, there have been many studies (refer to [3],[4],[5],[6],[7],[8], [9], [10],[12],[13],[14] and references therein) on a generalized Hardy's inequality, where they study the following optimal constant

$$H_{\alpha,\beta}(\Omega) \equiv \inf \Big\{ \frac{\int_\Omega d^\beta(x) |\nabla u|^2 dx}{\int_\Omega |u|^2 d^{-\alpha}(x) dx} \ \Big|\ u \in W^{1,2}_{0,\alpha,\beta}(\Omega) \setminus \{0\} \Big\}. \tag{4}$$

Here, $W^{1,2}_{0,\alpha,\beta}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{\alpha,\beta}$. For a generalized Hardy's inequality on general domains, refer to [2], [15], [16] and references therein. Some general results on the positivity and attainability of $H_{\alpha,\beta}$ will be given in Proposition 2.4, Proposition A.2 and Proposition A.4. We will show in Proposition 2.2 that for $\alpha \geq 1$ and $\beta < 1$, $W^{1,2}_{\alpha,\beta}(\Omega) = W^{1,2}_{0,\alpha,\beta}(\Omega)$. Furthermore, we will prove in Appendix, Proposition A.1 that

$$W^{1,2}_{0,\alpha,\beta}(\Omega)\begin{cases}\supsetneq W^{1,2}_0(\Omega) & \text{if } (\alpha,\beta)\in\{(a,b)\mid a\le 2, b>0\},\\ = W^{1,2}_0(\Omega) & \text{if } (\alpha,\beta)\in\{(a,b)\mid a\le 2, b=0\},\\ \subsetneq W^{1,2}_0(\Omega) & \text{if } (\alpha,\beta)\in\{(a,b)\mid b<0 \text{ or } b=0, a>2\},\end{cases}$$

and that for $(\alpha,\beta)\in\{(a,b)\mid a>2, b>0\}$, $W^{1,2}_{0,\alpha,\beta}(\Omega)\setminus W^{1,2}_0(\Omega)$ and $W^{1,2}_0(\Omega)\setminus W^{1,2}_{0,\alpha,\beta}(\Omega)$ are non-empty sets. The property $W^{1,2}_{\alpha,\beta}(\Omega)=W^{1,2}_{0,\alpha,\beta}(\Omega)$ for $\alpha\ge 1$ and $\beta<1$ implies that there is a natural relation between $L_{\alpha,\beta}(\Omega)$ and $H_{\alpha,\beta}(\Omega)$ and that our study on $L_{\alpha,\beta}(\Omega)$ could be a natural extension of previous studies on the Hardy constant $H_{\alpha,\beta}(\Omega)$.

Now we state our main results in this paper.

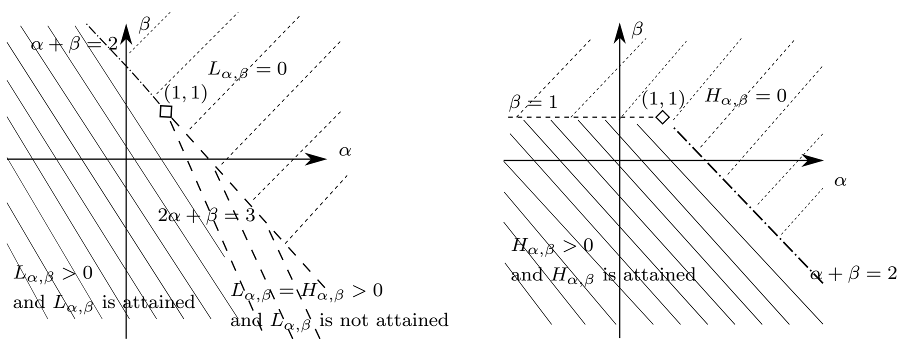
Theorem 1.1. *For any bounded C^2-domain $\Omega\subset\mathbb{R}^N$, $L_{\alpha,\beta}(\Omega)>0$ if and only if $\alpha+\beta\le 2$ and $(\alpha,\beta)\ne(1,1)$.*

Theorem 1.2. *Let Ω be a bounded C^2-domain in $\mathbb{R}^N$. Then it holds that*

(1) *if $(\alpha,\beta)\in\{(a,b)\mid a+b<2, 2a+b<3\}$, $L_{\alpha,\beta}(\Omega)$ is achieved by an element $u_{\alpha,\beta}\in W^{1,2}_{\alpha,\beta}(\Omega)$, which satisfies*

$$-div(d^\beta(x)\nabla u_{\alpha,\beta}) = L_{\alpha,\beta}(\Omega)d^{-\alpha}u_{\alpha,\beta} \text{ in } \Omega \text{ if } \alpha<1,$$
$$-div(d^\beta(x)\nabla u_{\alpha,\beta}) = L_{\alpha,\beta}(\Omega)d^{-\alpha}u_{\alpha,\beta}+\mu d^{-\alpha} \text{ in } \Omega \text{ for some } \mu\in\mathbb{R} \text{ if } \alpha\ge 1;$$

(2) *if $(\alpha,\beta)\in\{(a,b)\mid a+b\le 2, 2a+b\ge 3, (a,b)\ne(1,1)\}$, $L_{\alpha,\beta}(\Omega)=H_{\alpha,\beta}(\Omega)$ and $L_{\alpha,\beta}(\Omega)$ is not achieved.*



The optimal constant $L_{\alpha,\beta}(\Omega)$ The optimal constant $H_{\alpha,\beta}(\Omega)$

FIGURE 1

The key features and their relation for $H_{\alpha,\beta}(\Omega)$ and $L_{\alpha,\beta}(\Omega)$ described in Theorem 1.1 and Theorem 1.2 are illustrated in Figure 1. A result in Theorem 1.2 implies that $L_{\alpha,2-\alpha}(\Omega)$ is not attained for $\alpha>1$. When $\alpha<1$, as in the Hardy inequality, we lose a compactness at a critical level of $L_{\alpha,2-\alpha}(\Omega)$. In fact, we have the following result.

Theorem 1.3. *Let $\alpha<1$ and Ω be a bounded C^2-domain in $\mathbb{R}^N$. Then $L_{\alpha,2-\alpha}(\Omega)\le\frac{(1-\alpha)^2}{4}$. Furthermore, $L_{\alpha,2-\alpha}(\Omega)$ is achieved if $L_{\alpha,2-\alpha}(\Omega)<\frac{(1-\alpha)^2}{4}$.*

As for the optimal constant $H_{2,0}(\Omega)$ on $W^{1,2}_{0,2,0}(\Omega) = W^{1,2}_0(\Omega)$, it was proved [12] that $H_{2,0}(\Omega)$ is achieved if and only if $H_{2,0}(\Omega) < \frac{1}{4}$ when Ω is a bounded C^2-domain in $\mathbb{R}^N$. Recently, the result was extended in [9] to $C^{1,\gamma}-$domains for $\gamma \in (0,1]$. In view of the result in [9], we believe that the result of Theorem 1.3 would be true for bounded $C^{1,\gamma}$ domains. On the other hand, it is known that $H_{2,0}(\Omega)$ is not attained by any function $u \in W^{1,2}_0(\Omega)$ if Ω is convex (see [3]), and if Ω is weakly mean convex (see [10]), that is, the mean curvature of $\partial\Omega$ is nonnegative. On the other hand, as for the optimal constant $L_{\alpha,2-\alpha}(\Omega)$, we have the following results, highly contrasting with the results for $H_{2,0}(\Omega)$.

Theorem 1.4. *For a weakly mean convex C^2-domain $\Omega \subset \mathbb{R}^N$, there exists $\tilde{\alpha} \equiv \tilde{\alpha}(N,\Omega) < 1$ such that $L_{\alpha,2-\alpha}(\Omega)$ is achieved if $-\infty < \alpha < \tilde{\alpha}$.*

We will see that the attainability of the optimal constant $L_{\alpha,2-\alpha}(\Omega)$ strongly depends on a geometry of Ω and the space dimension N. To state the result, we define

$$R^N_+ \equiv \{(x_1,\cdots,x_N) \in \mathbb{R}^N \mid x_N > 0\},$$

$$B^N(0,1) \equiv \Big\{(x_1,\cdots,x_N) \in \mathbb{R}^N \;\Big|\; \sum_{i=1}^N (x_i)^2 < 1\Big\}$$

$$E(a_1,\cdots,a_N) \equiv \Big\{(x_1,\cdots,x_N) \in \mathbb{R}^N \;\Big|\; \sum_{j=1}^N \Big(\frac{x_j}{a_j}\Big)^2 < 1\Big\},$$

where $a_j > 0$. For the most simple domain $\Omega = B^N(0,1)$, we obtain that $L_{\alpha,2-\alpha}(B^N(0,1))$ is attained only when $N > \frac{3-\alpha}{1-\alpha}$.

Theorem 1.5. *Let $\alpha < 1$ and $N \geq 1$. Then $L_{\alpha,2-\alpha}(B^N(0,1)) < \frac{(1-\alpha)^2}{4}$ and $L_{\alpha,2-\alpha}(B^N(0,1))$ is achieved if and only if the space dimension $N > \frac{3-\alpha}{1-\alpha}$.*

When we deform the unit ball $B^N(0,1)$ to an ellipse, we get the following result, which says that even for $N = \frac{3-\alpha}{1-\alpha}$, $L_{\alpha,2-\alpha}(E(a_1,\cdots,a_N))$ is attained if the ellipse $E(a_1,\cdots,a_N)$ is not a ball.

Theorem 1.6. *For $\alpha < 1$, we assume the space dimension $N \geq \frac{3-\alpha}{1-\alpha}$. Then for $(a_1,\cdots,a_{N-1}) \in (0,1]^{N-1}$ with $a_1+\cdots+a_{N-1} < N-1$, we have*

$$L_{\alpha,2-\alpha}(E(a_1,a_2,\cdots,a_{N-1},1)) < \frac{(1-\alpha)^2}{4};$$

thus, the optimal constant $L_{\alpha,2-\alpha}(E(a_1,a_2,\cdots,a_{N-1},1))$ is attained.

In Theorems 1.7–1.9, we state our results for annular domains.

Theorem 1.7. *For $a > 1$ and $N \geq 2$, let $\Omega = B^N(0,a) \setminus \overline{B^N(0,1)}$. Then, for*

$$\alpha < \frac{1 - \frac{2}{N+1}\Big(2-(a+1)^{N+1}2^{-N+1}(1+a^{N+1})^{-1}\Big)}{1 - \frac{1}{N+1}\Big(2-(a+1)^{N+1}2^{-N+1}(1+a^{N+1})^{-1}\Big)},$$

$L_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$ and $L_{\alpha,2-\alpha}(\Omega)$ is achieved.

Theorem 1.8. *For $a > 1$ and $N \geq 2$, let $\Omega = B^N(0,a) \setminus \overline{B^N(0,1)}$. Then there exists $\alpha_0 \equiv \alpha_0(N,\Omega) \in (0,1)$ such that $L_{\alpha,2-\alpha}(\Omega)$ is not achieved for $\alpha \in (\alpha_0,1)$.*

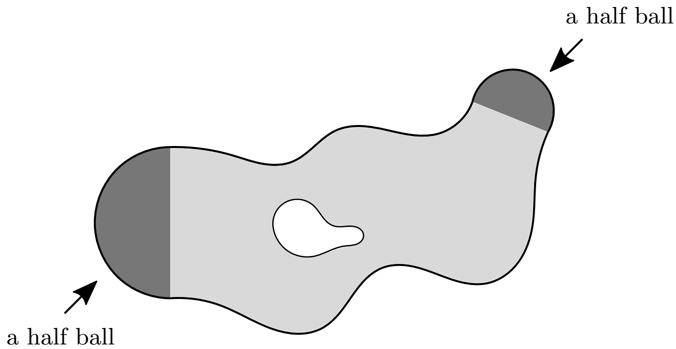


FIGURE 2. Ω containing two half balls

We state the attainable property of $L_{\alpha,2-\alpha}([-1,1]^N)$ without a proof. In fact, using the same type of a test function used in the proof of Theorem 1.5, we can show that $L_{\alpha,2-\alpha}([-1,1]^N) < \frac{(1-\alpha)^2}{4}$ for $N > \frac{2}{3(1-\alpha)} + 1$.

Above results suggest that for a general bounded domain Ω, if N or $-\alpha$ is large, there is a high possibility that $L_{\alpha,2-\alpha}(\Omega)$ is attained; on the other hand, if N or $1-\alpha > 0$ is small, a low possibility. We conjecture that for any given bounded C^2-domain in $\mathbb{R}^N$, $L_{\alpha,2-\alpha}(\Omega)$ is attained if $-\alpha$ is sufficiently large, and not attained if $1-\alpha > 0$ is sufficiently small.

In the following result, we see that the attainable property of $L_{\alpha,2-\alpha}(\Omega)$ depends on a local geometry of Ω rather than a global geometry of Ω. For $\alpha < 1$, we define a class of domains

$$\mathcal{A}_\alpha \equiv \left\{ \text{bounded } \ C^2\text{-domains } D \subset \mathbb{R}^N \ \middle| \ L_{\alpha,2-\alpha}(D) < \frac{(1-\alpha)^2}{4} \right\},$$

and for $D \in \mathcal{A}_\alpha$, a class of functions

$$\mathcal{A}_\alpha(D) \equiv \left\{ u \in C^1(D) \cap W^{1,2}_{\alpha,2-\alpha}(D) \setminus \{0\} \ \middle| \ Q_D(u) < \frac{(1-\alpha)^2}{4}, \int_D d^{-\alpha}(x) u(x) dx = 0 \right\},$$

where

$$Q_D(u) \equiv \frac{\int_D d^{2-\alpha}(x) |\nabla u|^2 dx}{\int_D |u|^2 d^{-\alpha}(x) dx}.$$

For each $u \in \mathcal{A}_\alpha(D)$, we define

$$D^u_+ \equiv \{x \in D \mid u(x) > 0\}, \ u^D_+(x) \equiv \max\{u(x), 0\},$$
$$D^u_- \equiv \{x \in D \mid u(x) < 0\} \ \text{ and } \ u^D_- \equiv \min\{u(x), 0\}.$$

Also, we define $\partial D^u_+ \equiv \partial D^u_+ \cap \partial D, \partial D^u_- \equiv \partial D^u_- \cap \partial D$. For each $u \in \mathcal{A}_\alpha(D)$, it holds that $Q_D(u^D_+) < \frac{(1-\alpha)^2}{4}$ or $Q_D(u^D_-) < \frac{(1-\alpha)^2}{4}$. Defining $\tilde{u} = -u$, we see that $\tilde{u}^D_+ = u^D_-, \tilde{u}^D_- = u^D_+$. Thus, for $u \in \mathcal{A}_\alpha(D)$, we may assume that $Q_D(u^D_+) < \frac{(1-\alpha)^2}{4}$.

Theorem 1.9. *Let $\alpha < 1$ and Ω be a bounded C^2-domain in $\mathbb{R}^N$. We assume that for each $i = 1, 2$, there exist a domain $D_i \in \mathcal{A}_\alpha$, a function $u_i \in \mathcal{A}_\alpha(D_i)$, a rotation $O_i \in O(N)$, a translation $t_i \in \mathbb{R}^N$ and a scale $s_i > 0$ such that for each $i = 1, 2$, $t_i + s_i O_i D^{u_i}_+ \subset \Omega$, $t_i + s_i O_i \partial D^{u_i}_+ \subset \partial\Omega$ with $(t_1 + s_1 O_1 D^{u_1}_+) \cap (t_2 + s_2 O_2 D^{u_2}_+) = \emptyset$ and $dist(t_i + s_i O_i x, \mathbb{R}^N \setminus \Omega) = s_i\, dist(x, \mathbb{R}^N \setminus D_i)$ for any $x \in D^{u_i}_+$. Then, $L_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$ and $L_{\alpha,2-\alpha}(\Omega)$ is achieved.*

In the proofs of Theorems 1.5–1.8 for the estimate $L_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$, we use a test function of form $d^s(x)x_N$. Thus we see from Theorem 1.9 that if Ω has two disjoint parts coming from translations, scalings and rotations of a half ball or a half ellipse or a half annulus as in Figure 2, $L_{\alpha,2-\alpha}(\Omega)$ is attained for α satisfying conditions in the corresponding theorem for ball, ellipse, or annulus.

This paper is organized as follows. Section 2 is devoted to give some preliminary results for proofs of main results. In Section 3, we prove Theorem 1.1 and Theorem 1.2 by examining the positivity and attainability of $L_{\alpha,\beta}(\Omega)$ in a subcritical region. In Section 4, we study the positivity and attainability of $L_{\alpha,\beta}(\Omega)$ in a critical region and prove Theorems 1.3 − 1.9. In the last appendix section, we study the inclusion relation between $W_0^{1,2}(\Omega)$ and $W_{0,\alpha,\beta}^{1,2}(\Omega)$ and prove some general results on the positivity and attainability of $H_{\alpha,\beta}(\Omega)$.

2. Some preliminary Lemmas

Assume that $\Omega \subset \mathbb{R}^N$ is a bounded C^2-domain. Now we follow a scheme in section 1 of [3]. For $\delta > 0$, we define $\Omega_\delta \equiv \{x \in \Omega \mid d(x) < \delta\}$ and $\Sigma_\delta = \{x \in \Omega \mid d(x) = \delta\}$. If $\delta > 0$ is small, for every $x \in \Omega_\delta$, there exists a unique point $\sigma(x) \in \partial\Omega$ such that $d(x) = |x - \sigma(x)|$. For small $\delta > 0$, we define a map $H_\delta : \partial\Omega \to \Sigma_\delta$ defined by $H_\delta(\sigma) = \sigma + \delta n(\sigma)$, where $n(\sigma)$ is the inward unit normal to $\partial\Omega$ at σ. We define a mapping $\Pi : \Omega_\delta \to (0, \delta) \times \partial\Omega$ by $\Pi(x) = (d(x), \sigma(x))$. Then, it is a C^1-diffeomorphism and its inverse is given by $\Pi^{-1}(t, \sigma) = \sigma + tn(\sigma)$ for $t \in (0, \delta)$, $\sigma \in \partial\Omega$. It is easy to see that for small $\delta > 0$, there exists $c > 0$ such that

$$\Big||\nabla H_t(\sigma)| - 1\Big| \le ct, \ \forall (t, \sigma) \in (0, \delta_0) \times \partial\Omega,$$

where $|\nabla H_t|$ is the Jacobin determinant of the Jacobian ∇H_t on $\partial\Omega$.

Then, since for any nonnegative integrable function f in Ω_δ,

$$\int_{\Omega_\delta} f(x)dx = \int_0^\delta \int_{\Sigma_t} f d\sigma_t dt = \int_0^\delta \int_{\partial\Omega} f \circ \Pi^{-1}(t, \sigma)|\nabla H_t(\sigma)|d\sigma dt,$$

it follows that for any nonnegative integrable function f in Ω_δ,

(5)

$$\int_{\partial\Omega}\int_0^\delta f \circ \Pi^{-1}(t, \sigma)(1 - ct)d\sigma dt \le \int_{\Omega_\delta} f(x)dx \le \int_{\partial\Omega}\int_0^\delta f \circ \Pi^{-1}(t, \sigma)(1 + ct)d\sigma dt.$$

Lemma 2.1. *Let $\beta < 1$ and $u \in C^1((0, 1))$ with $u(0) = 0$ and $\int_0^1 t^\beta (u')^2 dt < \infty$. Then we see that for $t \in (0, 1)$,*

(i) $u(t) \le \frac{1}{(1-\beta)^{\frac{1}{2}}} \Big(\int_0^t s^\beta \Big(u'(s)\Big)^2 ds\Big)^{\frac{1}{2}} t^{\frac{1-\beta}{2}}$;

(ii) $\int_0^t x^\beta u^2(x)dx \le \frac{t^2}{2(1-\beta)} \int_0^t s^\beta \Big(u'(s)\Big)^2 ds.$

Proof. Since $u(0) = 0$ and $\int_0^1 t^\beta (u')^2 dt < \infty$, for $x \in (0, 1)$,

$$u(x) = \int_0^x u'(s)ds \le \Big(\int_0^x s^\beta \Big(u'(s)\Big)^2 ds\Big)^{\frac{1}{2}} \Big(\int_0^x s^{-\beta} ds\Big)^{\frac{1}{2}}$$
$$= \Big(\int_0^x s^\beta \Big(u'(s)\Big)^2 ds\Big)^{\frac{1}{2}} \Big(\frac{1}{1-\beta} x^{1-\beta}\Big)^{\frac{1}{2}}.$$

This proves (i). From the inequality (i), we see that for $t \in (0,1)$,

$$\int_0^t x^\beta u^2(x)dx \le \frac{1}{1-\beta}\int_0^t s^\beta \Big(u'(s)\Big)^2 ds \int_0^t x dx = \frac{t^2}{2(1-\beta)}\int_0^t s^\beta \Big(u'(s)\Big)^2 ds;$$

this proves (ii). □

Proposition 2.2. *For $\alpha \ge 1$ and $\beta < 1$, we have*

$$W^{1,2}_{\alpha,\beta}(\Omega) = W^{1,2}_{0,\alpha,\beta}(\Omega).$$

Proof. It is obvious that $W^{1,2}_{\alpha,\beta}(\Omega) \supset W^{1,2}_{0,\alpha,\beta}(\Omega)$. To prove that $W^{1,2}_{\alpha,\beta}(\Omega) \subset W^{1,2}_{0,\alpha,\beta}(\Omega)$, we note that if $\alpha \ge \beta$, for $w \in C^\infty((0,1))$ satisfying $\int_0^1 t^\beta (w')^2 dt < \infty$, $\int_0^1 t^{-\alpha} w^2 dt < \infty$, we have

$$\begin{aligned} w^2(y) - w^2(x) = \int_x^y \Big(w^2(t)\Big)' dt &\le 2\Big(\int_x^y \frac{w^2}{t^\alpha} dt\Big)^{\frac12}\Big(\int_x^y t^\alpha (w')^2 dt\Big)^{\frac12} \\ &\le 2\Big(\int_x^y \frac{w^2}{t^\alpha} dt\Big)^{\frac12}\Big(\int_x^y t^\beta (w')^2 dt\Big)^{\frac12}, \end{aligned}$$

where $0 < x < y < 1$. Then we see that $\lim_{x\to 0} w(x)$ exists. From this and the assumptions that $\alpha \ge 1$ and $\int_0^1 \frac{w^2}{t^\alpha} dt < \infty$, we deduce that $w(0) = 0$. This and (5) imply that if $\alpha \ge \beta$ and $\alpha \ge 1$, then

$$u = 0 \quad \text{on} \quad \partial\Omega \quad \text{for} \quad u \in \mathcal{X}_{\alpha,\beta}(\Omega). \tag{6}$$

We find a function $\varphi \in C^1(\mathbb{R})$ such that $\varphi(t) = 0$ for $t \le 1$ and $\varphi(t) = 1$ for $t \ge 2$. For $u \in \mathcal{X}_{\alpha,\beta}(\Omega)$, we define $u_n(x) = u(x)\varphi(nd(x))$. Then we claim that $\|u_n - u\|_{\alpha,\beta} \to 0$ as $n \to \infty$. To prove the claim, it suffices to show that for a neighborhood $N(x_0)$ of $x_0 \in \partial\Omega$,

$$\lim_{n\to\infty} \int_{\Omega\cap N(x_0)} d^\beta |\nabla(u-u_n)|^2 + d^{-\alpha}(u-u_n)^2 dx = 0.$$

By the dominated convergence theorem, we see that

$$\lim_{n\to\infty} \int_{\Omega\cap N(x_0)} d^{-\alpha}(u-u_n)^2 dx = \lim_{n\to\infty} \int_{\Omega\cap N(x_0)} d^{-\alpha} u^2 \Big(1 - \varphi^2(nd(x))\Big) dx = 0.$$

To prove $\lim_{n\to\infty} \int_{\Omega\cap N(x_0)} d^\beta |\nabla(u-u_n)|^2 dx = 0$, we take a small neighborhood $N(x_0)$ of $x_0 \in \partial\Omega$ and a C^1 flattening map $\Phi : (-\delta,\delta)^N \cap \mathbb{R}^N_+ \to \Omega \cap N(x_0)$ for a small $\delta > 0$ such that $d \circ \Phi(x_1, \cdots, x_N) = x_N$ and

$$\begin{aligned} \frac12 \int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+} (x_N)^\beta |\nabla(v\circ\Phi)|^2 dx &\le \int_{\Omega\cap N(x_0)} d^\beta |\nabla v|^2 dx \\ &\le 2\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+} (x_N)^\beta |\nabla(v\circ\Phi)|^2 dx, \end{aligned}$$

where $v \in W^{1,2}_{\alpha,\beta}(\Omega)$. Thus it suffices to show that

$$\lim_{n\to\infty} \int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+} (x_N)^\beta |\nabla(u\circ\Phi) - \nabla(u_n\circ\Phi)|^2 dx = 0.$$

Since $u_n(\Phi(x)) = u(\Phi(x))\varphi(nd(\Phi(x))) = u(\Phi(x))\varphi(nx_N)$, we see from the dominated convergence theorem that

$$\lim_{n\to\infty}\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}(x_N)^\beta\Big|\frac{\partial}{\partial x_i}u(\Phi(x))-\frac{\partial}{\partial x_i}u_n(\Phi(x))\Big|^2dx=0,\ i=1,\cdots,N-1.$$

For $i=N$, we see that

$$\begin{aligned}&\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}(x_N)^\beta\Big|\frac{\partial}{\partial x_N}u(\Phi(x))-\frac{\partial}{\partial x_N}u_n(\Phi(x))\Big|^2dx\\&=\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}(x_N)^\beta\Big|\frac{\partial}{\partial x_N}u(\Phi(x))(1-\varphi(nx_N))\Big|^2dx\\&\le 2\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}(x_N)^\beta\Big|\frac{\partial}{\partial x_N}u(\Phi(x))\Big|^2(1-\varphi(nx_N))^2dx\\&\quad+2\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}n^2(x_N)^\beta(u(\Phi(x)))^2(\varphi'(nx_N))^2dx.\end{aligned}$$

We see from the dominated convergence theorem that

$$\lim_{n\to\infty}\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}(x_N)^\beta\Big|\frac{\partial}{\partial x_N}u(\Phi(x))\Big|^2(1-\varphi(nx_N))^2dx=0.$$

From (6), $\beta<1$, Lemma 2.1(ii) and the dominated convergence theorem, we see that for some constant $C_1>0$,

$$\begin{aligned}&\int_{(-\delta,\delta)^N\cap\mathbb{R}^N_+}n^2(x_N)^\beta(u(\Phi(x)))^2(\varphi'(nx_N))^2dx\\&=n^2\int_{[-\delta,\delta]^{N-1}}\int_0^{2n^{-1}}(x_N)^\beta(u(\Phi(x)))^2(\varphi'(nx_N))^2dx_Ndx'\\&\le\frac{C_1}{1-\beta}\int_{[-\delta,\delta]^{N-1}}\int_0^{2n^{-1}}(x_N)^\beta\Big|\frac{\partial}{\partial x_N}u(\Phi(x))\Big|^2dx_Ndx'\to0\quad\text{as } n\to\infty.\end{aligned}$$

Thus, $\|u_n-u\|_{\alpha,\beta}\to0$ as $n\to\infty$. This proves $W_{\alpha,\beta}(\Omega)\subset W_{0,\alpha,\beta}(\Omega)$ for $\alpha\ge1$ and $\beta<1$. □

Lemma 2.3. *Let $\delta\in(0,1)$ and $(\alpha,\beta)\in\{(a,b)\mid a<b,\ a+b\le2\}\cup\{(a,b)\mid a\ge1,\ a+b\le2,\ (a,b)\ne(1,1)\}$. For $v\in C^1((0,\delta])$ satisfying $\int_0^\delta x^{-\alpha}v^2dx<\infty$ and $\int_0^\delta x^\beta|v'|^2dx<\infty$, it holds that*

$$\frac{(\beta-\alpha)^2}{16}\int_0^\delta x^{-\alpha}v^2dx\le\int_0^\delta x^\beta(v')^2dx+\frac{\beta-\alpha}{4}\,\delta^{\frac{\beta-\alpha}{2}}v^2(\delta).\tag{7}$$

Proof. Let $v\in C^1((0,\delta])$ satisfying $\int_0^\delta x^{-\alpha}v^2dx<\infty$ and $\int_0^\delta x^\beta|v'|^2dx<\infty$. Note that

(8)
$$\int_0^\delta\left(x^{\frac{\beta}{2}}v'+\frac{\beta-\alpha}{4}x^{\frac{-\alpha}{2}}v\right)^2dx=\int_0^\delta x^\beta(v')^2+\frac{(\beta-\alpha)^2}{16}x^{-\alpha}v^2+\frac{\beta-\alpha}{2}x^{\frac{\beta-\alpha}{2}}vv'dx$$

and for any small $\varepsilon > 0$,

$$\begin{aligned}\int_\varepsilon^\delta x^{\frac{\beta-\alpha}{2}} vv' dx &= \frac{1}{2}\int_\varepsilon^\delta x^{\frac{\beta-\alpha}{2}}(v^2)' dx \\ &= -\frac{\beta-\alpha}{4}\int_\varepsilon^\delta x^{\frac{\beta-\alpha-2}{2}} v^2 dx + \frac{1}{2}\Big(\delta^{\frac{\beta-\alpha}{2}} v^2(\delta) - \varepsilon^{\frac{\beta-\alpha}{2}} v^2(\varepsilon)\Big).\end{aligned} \tag{9}$$

First, we assume that $(\alpha,\beta) \in \{(a,b) \mid a < b,\ a+b \le 2\}$. Then, since $\delta \in (0,1)$, if $(\alpha,\beta) \in \{(a,b) \mid a < b,\ a+b \le 2\}$, it follows that

$$\begin{aligned}\frac{\beta-\alpha}{2}\int_0^\delta x^{\frac{\beta-\alpha}{2}} vv' dx &\le -\frac{(\beta-\alpha)^2}{8}\int_0^\delta x^{\frac{\beta-\alpha-2}{2}} v^2 dx + \frac{\beta-\alpha}{4}\ \delta^{\frac{\beta-\alpha}{2}} v^2(\delta) \\ &\le -\frac{(\beta-\alpha)^2}{8}\int_0^\delta x^{-\alpha} v^2 dx + \frac{\beta-\alpha}{4}\ \delta^{\frac{\beta-\alpha}{2}} v^2(\delta).\end{aligned} \tag{10}$$

Then, combining (8) and (10), we get (7).

Next, we assume that $(\alpha,\beta) \in \{(a,b) \mid a \ge 1,\ a+b \le 2,\ (a,b) \ne (1,1)\}$. Since $\alpha \ge \beta$, we see that for $0 < x < y < 1$,

$$\begin{aligned}v^2(y) - v^2(x) &= \int_x^y \Big(v^2(t)\Big)' dt \le 2\Big(\int_x^y \frac{v^2}{t^\alpha} dt\Big)^{\frac{1}{2}}\Big(\int_x^y t^\alpha (v')^2 dt\Big)^{\frac{1}{2}} \\ &\le 2\Big(\int_x^y \frac{v^2}{t^\alpha} dt\Big)^{\frac{1}{2}}\Big(\int_x^y t^\beta (v')^2 dt\Big)^{\frac{1}{2}}.\end{aligned}$$

Then we see that $\lim_{x\to 0} v(x)$ exists. Since $\int_0^\delta x^{-\alpha} v^2 dx < \infty$, we see that $\lim_{x\to 0} v(x) = 0$. Then, from this and (9), we see that for $\delta \in (0,1)$, $\alpha \ge 1$ and $\alpha+\beta \le 2$,

$$\begin{aligned}\frac{\beta-\alpha}{2}\int_0^\delta x^{\frac{\beta-\alpha}{2}} vv' dx &= -\frac{(\beta-\alpha)^2}{8}\int_0^\delta x^{\frac{\beta-\alpha-2}{2}} v^2 dx + \frac{\beta-\alpha}{4}\ \delta^{\frac{\beta-\alpha}{2}} v^2(\delta) \\ &\le -\frac{(\beta-\alpha)^2}{8}\int_0^\delta x^{-\alpha} v^2 dx + \frac{\beta-\alpha}{4}\ \delta^{\frac{\beta-\alpha}{2}} v^2(\delta).\end{aligned} \tag{11}$$

Combining (8) and (11), we get (7). □

Proposition 2.4. *Let Ω be a bounded C^2-domain in $\mathbb{R}^N$. Then we have*

$$H_{\alpha,\beta}(\Omega)\begin{cases} > 0 & \text{if } (\alpha,\beta) \in \{(a,b) \mid a+b \le 2, b < 1\}, \\ = 0 & \text{if } (\alpha,\beta) \in \mathbb{R}^2 \setminus \{(a,b) \mid a+b \le 2, b < 1\}.\end{cases}$$

Proof. We first prove that $H_{\alpha,\beta}(\Omega) > 0$ for $(\alpha,\beta) \in \{(a,b) \mid a+b \le 2, b < 1\}$. We claim that $H_{\alpha,2-\alpha}(\Omega) > 0$ for $\alpha > 1$. In fact, by (8) and (11), we see that for $\alpha+\beta = 2, \beta < 1$ and $v \in C^1((0,1))$ vanishing in a neighborhood of 0,

$$\frac{(1-\alpha)^2}{4}\int_0^\delta x^{-\alpha} v^2 dx \le \int_0^\delta x^{2-\alpha}(v')^2 dx,$$

where $\delta \in (0,1)$. This and (5) imply that for small $\delta > 0$, there exists a constant $C_1 > 0$ such that for any $u \in C_0^\infty(\Omega)$,

$$\int_{\Omega_\delta} d^{2-\alpha}(x)|\nabla u|^2 dx \ge C_1\int_{\Omega_\delta} |u(x)|^2 d^{-\alpha}(x) dx. \tag{12}$$

We take $\psi \in C_0^\infty(\Omega)$ such that $\psi \in [0,1]$, $\psi(x) = 1$ for $x \in \Omega \setminus \Omega_\delta$ and $\psi(x) = 0$ for $x \in \Omega_{\delta/2}$. By the Poincaré inequality and the fact that

$$0 < \inf_{x\in\Omega\setminus\Omega_{\delta/2}} d(x) \le \sup_{x\in\Omega\setminus\Omega_{\delta/2}} d(x) < \infty,$$

there exists a constant $C_2 > 0$ such that for any $w \in C_0^\infty(\Omega \setminus \overline{\Omega_{\delta/2}})$,

$$\int_\Omega d^{2-\alpha}(x)|\nabla w|^2 dx \ge C_2 \int_\Omega |w|^2 d^{-\alpha}(x)dx. \tag{13}$$

Thus, we see from (12) and (13) that for any $u \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega |u(x)|^2 d^{-\alpha}(x)dx \\
&\quad\le 2\int_\Omega |\psi(x)u(x)|^2 d^{-\alpha}(x)dx + 2\int_{\Omega_\delta} \Big|\big(1-\psi(x)\big)u(x)\Big|^2 d^{-\alpha}(x)dx \\
&\quad\le \frac{2}{C_1}\int_{\Omega_\delta} d^{2-\alpha}(x)\Big|\nabla\big((1-\psi)u\big)\Big|^2 dx + \frac{2}{C_2}\int_\Omega d^{2-\alpha}(x)|\nabla(\psi u)|^2 dx \\
&\quad\le \frac{4}{C_1}\int_{\Omega_\delta} d^{2-\alpha}(x)(|\nabla u|^2 + |\nabla\psi|^2u^2)dx + \frac{4}{C_2}\int_\Omega d^{2-\alpha}(x)(|\nabla u|^2 + |\nabla\psi|^2u^2)dx \\
&\quad\le C\int_\Omega d^{2-\alpha}(x)|\nabla u|^2 dx + C\int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx,
\end{aligned}$$

where C is a positive constant depending only on $C_1, C_2, \max_{x\in\Omega} d(x), \|\nabla\psi\|_{L^\infty}$. Combining the above inequality and (12), we see that for some constant $C > 0$,

$$\int_\Omega |u(x)|^2 d^{-\alpha}(x)dx \le C\int_\Omega d^{2-\alpha}(x)|\nabla u|^2 dx, \qquad u \in C_0^\infty(\Omega).$$

This shows that $H_{\alpha,\beta}(\Omega) > 0$ for $\alpha+\beta = 2$ and $\beta < 1$. On the other hand, for any $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta < 2$ and $\beta < 1$, we can find $\alpha' > \alpha$ such that $\alpha' + \beta = 2$. Since

$$\int_\Omega |u(x)|^2 d^{-\alpha}(x)dx \le \max_{x\in\Omega} d^{\alpha'-\alpha}(x)\int_\Omega |u(x)|^2 d^{-\alpha'}(x)dx$$

and Ω is bounded, we see that for some $C > 0$, $H_{\alpha,\beta}(\Omega) \ge CH_{\alpha',\beta}(\Omega) > 0$. Thus, we prove the claim that $H_{\alpha,\beta}(\Omega) > 0$ for $\alpha + \beta \le 2$ and $\beta < 1$.

Next, we claim that $H_{\alpha,\beta}(\Omega) = 0$ for $(\alpha,\beta) \in \mathbb{R}^2 \setminus \{(a,b) \mid a+b \le 2, b < 1\}$. Then we consider two cases: (i) $\beta \ge 1, \alpha \le 1$ and (ii) $\alpha + \beta > 2$. In the case $\beta \ge 1, \alpha \le 1$, for $\sigma > 0$, we define

$$w(x) = \begin{cases} \left(\frac{d(x)}{\delta}\right)^\sigma & \text{if } d(x) < \delta, \\ 1 & \text{if } d(x) \ge \delta. \end{cases}$$

Then we see from (5) that

$$\int_\Omega d^\beta(x)|\nabla w|^2 dx = \sigma^2\delta^{-2\sigma}\int_{\Omega_\delta} d^{\beta+2\sigma-2}(x)dx \le \begin{cases} C_1\sigma^2\delta^{\beta-1} & \text{if } \beta > 1, \\ C_1\sigma & \text{if } \beta = 1, \end{cases}$$

$$\int_\Omega d^{-\alpha}(x)w^2 dx \ge \delta^{-2\sigma}\int_{\Omega_\delta} d^{-\alpha+2\sigma}(x)dx \ge \begin{cases} C_2\delta^{1-\alpha} & \text{if } \alpha < 1, \\ C_2\sigma^{-1} & \text{if } \alpha = 1, \end{cases}$$

where $C_1, C_2 > 0$ are constants independent of $\sigma > 0$. Thus, there exists $C > 0$ such that for small $\sigma > 0$,

$$H_{\alpha,\beta}(\Omega) \leq \frac{\int_\Omega d^\beta(x)|\nabla w|^2 dx}{\int_\Omega d^{-\alpha}(x) w^2 dx} \leq C\sigma,$$

which implies that $H_{\alpha,\beta}(\Omega) = 0$ if $\alpha \leq 1$ and $\beta \geq 1$.

In the case $\alpha + \beta > 2$, we define $W_n(x) = \phi(nd(x))$ for a nonnegative $\phi \in C_0^\infty((0,\delta)) \setminus \{0\}$. Using (5), we see that for some constants $C_3, C_4 > 0$, independent of n,

$$\begin{aligned}\int_\Omega d^\beta(x)|\nabla W_n|^2 dx &\leq n^2 \int_\Omega d^\beta(x)\Big(\phi'(nd(x))\Big)^2 dx \\ &\leq C_3 n^2 \int_0^\delta t^\beta \Big(\phi'(nt)\Big)^2 dt = C_3 n^{1-\beta} \int_0^\delta t^\beta \Big(\phi'(t)\Big)^2 dt\end{aligned}$$

and

$$\int_\Omega d^{-\alpha}(x) W_n^2 dx \geq C_4 \int_0^\delta t^{-\alpha}\Big(\phi(nt)\Big)^2 dt = C_4 n^{\alpha-1} \int_0^\delta t^{-\alpha}\Big(\phi(t)\Big)^2 dt.$$

Then, we see that as $n \to \infty$,

$$\frac{\int_\Omega d^\beta(x)|\nabla W_n|^2 dx}{\int_\Omega d^{-\alpha}(x) W_n^2 dx} \leq C_3 C_4^{-1} n^{2-\alpha-\beta} \frac{\int_0^\delta t^\beta (\phi')^2 dt}{\int_0^\delta t^{-\alpha}(\phi(t))^2 dt} \to 0.$$

This proves that $H_{\alpha,\beta}(\Omega) = 0$ when $\alpha+\beta > 2$, and thus, we complete the proof. □

Proposition 2.5. *For $\alpha + \beta \leq 2$, $2\alpha + \beta \geq 3$, $(\alpha, \beta) \neq (1,1)$, we have $H_{\alpha,\beta}(\Omega) = L_{\alpha,\beta}(\Omega)$.*

Proof. Assume that $\alpha + \beta \leq 2$, $2\alpha + \beta \geq 3$, $(\alpha,\beta) \neq (1,1)$. In this case, we have $\alpha > 1$ and $\beta < 1$. Clearly, by Proposition 2.2, $H_{\alpha,\beta}(\Omega) \leq L_{\alpha,\beta}(\Omega)$. To prove the reverse inequality $H_{\alpha,\beta}(\Omega) \geq L_{\alpha,\beta}(\Omega)$, we choose a minimizing sequence $\{u_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ of $H_{\alpha,\beta}(\Omega)$ such that $\int_\Omega d^{-\alpha}(x) u_m^2 dx = 1$ and $\int_\Omega d^\beta(x)|\nabla u_m|^2 dx \to H_{\alpha,\beta}(\Omega)$ as $m \to \infty$. We find a function $\varphi \in C^1(\mathbb{R})$ such that $\varphi(t) = 0$ for $t \leq 1$ and $\varphi(t) = 1$ for $t \geq 2$. Define $u_{n,m}(x) \equiv u_m(x)\varphi\Big(nd(x)\Big)$ and

$$w_{n,m} \equiv u_{n,m} + c_{n,m} d^{\alpha-1+n^{-\alpha+\frac{1-\beta}{2}}}(x) \text{ with } c_{n,m} = -\frac{\int_\Omega d^{-\alpha}(x) u_{n,m} dx}{\int_\Omega d^{-1+n^{-\alpha+\frac{1-\beta}{2}}}(x) dx}.$$

Note that $\int_\Omega d^{-\alpha}(x) w_{n,m} dx = 0$, and it is easy to check $w_{n,m} \in W^{1,2}_{\alpha,\beta}(\Omega)$ since $-1 < \alpha - 2 + 2n^{-\alpha+\frac{1-\beta}{2}}$ and $-1 < \beta + 2\alpha - 4 + 2n^{-\alpha+\frac{1-\beta}{2}}$. Then, by the same argument with the proof of Proposition 2.2, we see that $\|u_{n,m} - u_m\|_{\alpha,\beta} \to 0$ as $n \to \infty$. Moreover, it follows from the boundedness of $\{\int_\Omega d^\beta(x)|\nabla u_m|^2 dx\}_m$ that $\int_\Omega d^\beta(x)|\nabla u_{n,m}|^2 dx \leq C$ with $C > 0$ is a constant independent of m, n. We see from Lemma 2.1(i) and (5) that for some C_1, C_2, independent of m, n,

$$\begin{aligned}\int_\Omega d^{-\alpha}(x)u_{n,m}dx &= \int_{\Omega_\delta\setminus\Omega_{1/n}} d^{-\alpha}(x)u_{n,m}dx + \int_{\Omega\setminus\Omega_\delta} d^{-\alpha}(x)u_{n,m}dx \\ &\le C_1\Big(1 + \int_{n^{-1}}^{\delta} t^{-\alpha+\frac{1-\beta}{2}}dt\Big) \\ &\le \begin{cases} C_2(1 + n^{\alpha+\frac{\beta-3}{2}}) & \text{if } 2\alpha+\beta-3>0, \\ C_2(1+\log n) & \text{if } 2\alpha+\beta-3=0.\end{cases}\end{aligned}$$

Then, since for some C_3, C_4, independent of m, n,

$$\int_\Omega d^{-1+n^{-\alpha+\frac{1-\beta}{2}}}(x)dx \ge C_3\int_0^\delta t^{-1+n^{-\alpha+\frac{1-\beta}{2}}}dt \ge C_4 n^{\alpha+\frac{\beta-1}{2}},$$

it follows that for large $n > 0$,

$$|c_{n,m}| \le C_2 C_4^{-1} n^{-1}\log n.$$

Now, we see that

$$\begin{aligned}&(c_{n,m})^2\int_\Omega d^\beta\Big|\nabla\Big(d^{\alpha-1+n^{-\alpha+\frac{1-\beta}{2}}}\Big)\Big|^2dx \\ &= c_{n,m}^2\Big(\alpha-1+n^{-\alpha+\frac{1-\beta}{2}}\Big)^2\int_\Omega d^{2\alpha+\beta-4+2n^{-\alpha+\frac{1-\beta}{2}}}dx \\ &\le \begin{cases} C_5(c_{n,m})^2 = O\Big(n^{-2}(\log n)^2\Big) & \text{if } 2\alpha+\beta-3>0, \\ C_5(c_{n,m})^2 n = O\Big(n^{-1}(\log n)^2\Big) & \text{if } 2\alpha+\beta-3=0.\end{cases}\end{aligned}$$

and

$$(c_{n,m})^2\int_\Omega d^{\alpha-2+2n^{-\alpha+\frac{1-\beta}{2}}} \le C_6(c_{n,m})^2 = O\Big(n^{-2}(\log n)^2\Big),$$

where C_5, C_6 are positive constants independent of m, n. Therefore, from these, we see that

$$L_{\alpha,\beta}(\Omega) \le \frac{\int_\Omega d^\beta(x)|\nabla w_n|^2dx}{\int_\Omega d^{-\alpha}(x)w_n^2dx} = \frac{\int_\Omega d^\beta(x)|\nabla u_{n,m}|^2dx + o(1)}{\int_\Omega d^{-\alpha}(x)u_{n,m}^2dx + o(1)} = \frac{\int_\Omega d^\beta(x)|\nabla u_m|^2dx}{\int_\Omega d^{-\alpha}(x)u_m^2dx}$$

as $n \to \infty$. Lastly, letting $m \to \infty$, we get $L_{\alpha,\beta}(\Omega) \le H_{\alpha,\beta}(\Omega)$. □

Lemma 2.6. *There exists a sequence* $\{u_n\}_n \subset C_0^1(\Omega)$ *such that*

$$\int_\Omega d^{-1}(x)u_ndx = 0, \quad \lim_{n\to\infty}\frac{\int_\Omega d(x)|\nabla u_n|^2dx}{\int_\Omega d^{-1}(x)u_n^2dx} = 0.$$

Proof. By a scaling, we may assume $d(x) < 1$. We find a function $\phi \in C^1(\mathbb{R})$ such that $\phi(t) = \begin{cases} 1 & \text{if } t \ge 2, \\ 0 & \text{if } t \le 1,\end{cases}$ and define

$$u_n(x) = \phi(nd(x))\Big(|\ln(d(x))|^{-1/2} + c_n\Big),$$

$$\text{where } c_n = -\frac{\int_\Omega d^{-1}(x)|\ln(d(x))|^{-1/2}\phi(nd(x))dx}{\int_\Omega d^{-1}(x)\phi(nd(x))dx}.$$

Then we have $\int_\Omega d^{-1}(x)u_n dx = 0$. Using (5), we see that

$$\begin{aligned}
&\int_\Omega d^{-1}(x)|\ln(d(x))|^{-1/2}\phi(nd(x))dx \\
&\quad = \int_{\Omega_\delta - \Omega_{1/n}} d^{-1}(x)|\ln(d(x))|^{-1/2}\phi(nd(x))dx + O(1) \\
&\quad \le C_1 \int_{n^{-1}}^{\delta} t^{-1}|\ln t|^{-1/2}dt + O(1) = 2C_1(\ln n)^{1/2} + O(1)
\end{aligned}$$

and

$$\int_\Omega d^{-1}(x)\phi(nd(x))dx \ge \int_{\Omega_\delta \setminus \Omega_{2/n}} d^{-1}(x)dx \ge C_2 \int_{2n^{-1}}^{\delta_0} t^{-1}dt \ge C_2 \ln n + O(1).$$

Then, it follows that

$$|c_n| = \frac{\int_\Omega d^{-1}(x)|\ln(d(x))|^{-1/2}\phi(nd(x))dx}{\int_\Omega d^{-1}(x)\phi(nd(x))dx} \le \frac{2C_1(\ln n)^{1/2} + O(1)}{C_2 \ln n + O(1)} \le C_3(\ln n)^{-1/2},$$

where C_1, C_2 and C_3 are positive constants, independent of large $n > 0$. Now, using (5) again, we see that for some positive constants C_4, C_5 and C_6, independent of large $n > 0$,

$$\begin{aligned}
&\int_\Omega d(x)|\nabla u_n|^2 dx \\
&\le 2\int_\Omega n^2 d(x)\Big(\phi'(nd(x))\Big)^2\Big(|\ln(d(x))|^{-1/2} + c_n\Big)^2 \\
&\quad + \frac{1}{4}d^{-1}(x)\phi^2(nd(x))|\ln(d(x))|^{-3}dx \\
&\le C_4\Big[\int_{\Omega_{2n^{-1}}} n^2 d(x)\Big(|\ln(d(x))|^{-1} + c_n^2\Big)dx + \int_{\Omega\setminus\Omega_{n^{-1}}} |\ln(d(x))|^{-3}d^{-1}(x)dx\Big] \\
&\le C_5\Big[\int_0^{2n^{-1}} n^2 t\Big(|\ln t|^{-1} + c_n^2\Big)dt + \int_{n^{-1}}^{\delta} |\ln t|^{-3}t^{-1}dt + O(1)\Big] = O(1)
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega d^{-1}(x)(u_n)^2 dx &= \int_\Omega \phi^2(nd(x))d^{-1}(x)\Big(|\ln(d(x))|^{-1} + c_n^2 + 2c_n|\ln(d(x))|^{-1/2}\Big) \\
&\ge C_6 \ln((\ln n)) + O(1).
\end{aligned}$$

This implies that $\lim_{n\to\infty} \frac{\int_\Omega d(x)|\nabla u_n|^2 dx}{\int_\Omega d^{-1}(x)u_n^2 dx} = 0$ and completes the proof. □

Lemma 2.7. *Let $\alpha + \beta > 2$. Then there exists a sequence $\{u_n\}_n \subset C_0^1(\Omega)$ such that*

$$\int_\Omega d^{-\alpha}(x)u_n dx = 0, \quad \lim_{n\to\infty} \frac{\int_\Omega d^{\beta}(x)|\nabla u_n|^2 dx}{\int_\Omega d^{-\alpha}(x)u_n^2 dx} = 0.$$

Proof. We find nonnegative functions $\phi_1, \phi_2 \in C_0^\infty(\mathbb{R})$ such that $\phi_1(t) = \begin{cases} 1 & \text{if } t \in [2,3], \\ 0 & \text{if } t \in \mathbb{R} \setminus [1,4] \end{cases}$ and $\phi_2(t) = \begin{cases} 1 & \text{if } t \in [5,6], \\ 0 & \text{if } t \in \mathbb{R} \setminus [4,7]. \end{cases}$ Then we define

$$u_n(x) \equiv \phi_1(nd(x)) + c_n\phi_2(nd(x)), \quad \text{where} \quad c_n = -\frac{\int_\Omega d^{-\alpha}(x)\phi_1\Big(nd(x)\Big)dx}{\int_\Omega d^{-\alpha}(x)\phi_2\Big(nd(x)\Big)dx}.$$

Then we have $\int_\Omega d^{-\alpha}(x)u_n dx = 0$. We see from (5) that

$$\int_\Omega d^{-\alpha}(x)\phi_1\Big(nd(x)\Big)dx \le C_1\int_0^{\delta_0} t^{-\alpha}\phi_1(nt)dt = C_1 n^{\alpha-1}\int_0^\infty t^{-\alpha}\phi_1(t)dt$$

and

$$\int_\Omega d^{-\alpha}(x)\phi_2\Big(nd(x)\Big)dx \ge C_2\int_0^{\delta_0} t^{-\alpha}\phi_2(nt)dt = C_2 n^{\alpha-1}\int_0^\infty t^{-\alpha}\phi_2(t)dt.$$

These estimations imply that

$$|c_n| = \frac{\int_\Omega d^{-\alpha}(x)\phi_1\Big(nd(x)\Big)dx}{\int_\Omega d^{-\alpha}(x)\phi_2\Big(nd(x)\Big)dx} \le \frac{C_1\int_0^\infty t^{-\alpha}\phi_1(t)dt}{C_2\int_0^\infty t^{-\alpha}\phi_2(t)dt},$$

where C_1 and C_2 are positive constants, independent of n. For large $n > 0$, using (5), we see that

$$\begin{aligned}\int_\Omega d^\beta(x)|\nabla u_n|^2 dx &= n^2\int_\Omega d^\beta(x)\bigg(\Big(\phi_1'(nd(x))\Big)^2 + c_n^2\Big(\phi_2'(nd(x))\Big)^2\bigg)dx \\ &\le C_3 n^2\int_0^\delta t^\beta\bigg(\Big(\phi_1'(nt)\Big)^2 + \Big(\phi_2'(nt)\Big)^2\bigg)dt \\ &= C_3 n^{1-\beta}\int_0^\infty t^\beta\bigg(\Big(\phi_1'(t)\Big)^2 + \Big(\phi_2'(t)\Big)^2\bigg)dt\end{aligned}$$

and

$$\begin{aligned}\int_\Omega d^{-\alpha}(x)u_n^2 dx = \int_\Omega d^{-\alpha}(x)\Big(\phi_1(nd(x))\Big)^2 dx &\ge C_4\int_0^\delta t^{-\alpha}\Big(\phi_1(nt)\Big)^2 dt \\ = C_4 n^{\alpha-1}\int_0^\infty t^{-\alpha}\Big(\phi_1(t)\Big)^2 dt,&\end{aligned}$$

where C_3 and C_4 are positive constants. Thus, from these estimates and the assumption $\alpha + \beta > 2$, we conclude that $\lim_{n\to\infty}\frac{\int_\Omega d^\beta(x)|\nabla u_n|^2dx}{\int_\Omega d^{-\alpha}(x)u_n^2dx} = 0$. □

Lemma 2.8. *Let $(\alpha,\beta) \in \mathbb{R}^2$ and Ω be a bounded C^2-domain. Let $\{r_m\}_{m=0}^\infty$ be a sequence such that $r_m \downarrow 0$ as $m \to \infty$ and $\{x \in \Omega \mid d(x) > r_0\} \ne \emptyset$. Then for $a, b \in \mathbb{R}$ and a sequence u_m satisfying $u_m \rightharpoonup 0$ in $W^{1,2}_{\alpha,\beta}(\Omega)$ as $m \to \infty$, we can find a sequence $\{m_i\}_{i=1}^\infty$ of positive integers such that $m_1 < m_2 < \cdots$ and*

$$\int_{\Omega^{(i)}} d^a(x)|u_{m_i}|dx, \int_{\Omega^{(i)}} d^b(x)u_{m_i}^2 dx < \min\{1/2, (r_{i-1} - r_i)^2\}, \tag{14}$$

where $\Omega^{(i)} \equiv \{x \in \Omega \mid d(x) > r_i\}$.

Proof. Let $a, b \in \mathbb{R}$ and u_m be a sequence satisfying $u_m \rightharpoonup 0$ in $W^{1,2}_{\alpha,\beta}(\Omega)$ as $m \to \infty$. Note that

$$\int_{\Omega'} u_m^2 + |\nabla u_m|^2 dx \leq C_1 \int_{\Omega'} d^{-\alpha} u_m^2 + d^\beta |\nabla u_m|^2 dx \leq C_2,$$

where $\Omega' \subset\subset \Omega$ and $C_1, C_2 > 0$ are constants independent of m. Then, by the Rellich-Kondrachov theorem, we deduce that $u_m \to 0$ in $L^2_{loc}(\Omega)$ as $m \to \infty$, up to a subsequence. Since $u_m \to 0$ in $L^2(\Omega^{(0)})$, there is u_{m_0} such that

$$\int_{\Omega^{(0)}} d^a(x)|u_{m_0}|dx, \int_{\Omega^{(0)}} d^b(x) u_{m_0}^2 dx < 1/2.$$

Next, choose u_{m_1} such that $\int_{\Omega^{(1)}} d^a(x)|u_{m_1}|dx, \int_{\Omega^{(1)}} d^b(x) u_{m_1}^2 dx < \min\{1/2, (r_0 - r_1)^2\}$ and $m_1 > m_0$. We repeat this process to define a subsequence $\{u_{m_i}\} \subset \{u_m\}$ for $i = 1, 2, \cdots$, which satisfies (14). ◻

Lemma 2.9. *Let $(\alpha, \beta) \in \mathbb{R}^2$ and Ω be a bounded C^2-domain. Let u_m be a minimizing sequence of $L_{\alpha,\beta}(\Omega)$ satisfying $\int_\Omega d^{-\alpha}(x) u_m^2 dx = 1$ and $u_m \rightharpoonup u$ in $W^{1,2}_{\alpha,\beta}(\Omega)$ as $m \to \infty$, where $u \not\equiv 0$ and $\int_\Omega d^{-\alpha}(x) u dx = 0$. Then u attains $L_{\alpha,\beta}(\Omega)$.*

Proof. Let $v_m = u_m - u$. Then v_m converges weakly to 0 in $W^{1,2}_{\alpha,\beta}(\Omega)$ as $m \to \infty$. Now we see that

(15)
$$\begin{aligned} L_{\alpha,\beta}(\Omega) + o(1) &= \int_\Omega d^\beta(x)|\nabla u_m|^2 dx = \int_\Omega d^\beta(x)(|\nabla u|^2 + 2\nabla u \cdot \nabla v_m + |\nabla v_m|^2)dx \\ &= \int_\Omega d^\beta(x)|\nabla u|^2 dx + \int_\Omega d^\beta(x)|\nabla v_m|^2 dx + o(1) \end{aligned}$$

and

$$1 = \int_\Omega d^{-\alpha}(x) u_m^2 dx = \int_\Omega d^{-\alpha}(x) u^2 dx + \int_\Omega d^{-\alpha}(x) v_m^2 dx + o(1). \tag{16}$$

Since $u \not\equiv 0$, it follows that

$$\limsup_{m\to\infty} \int_\Omega d^{-\alpha}(x) v_m^2 dx < 1. \tag{17}$$

Thus, by (15)-(17) and the facts that $\int_\Omega d^{-\alpha}(x) u dx = \int_\Omega d^{-\alpha}(x) v_m dx = 0$, we obtain

$$\begin{aligned} L_{\alpha,\beta}(\Omega) &\leq \frac{\int_\Omega d^\beta(x)|\nabla u|^2 dx}{\int_\Omega d^{-\alpha}(x) u^2 dx} = \frac{L_{\alpha,\beta}(\Omega) - \int_\Omega d^\beta(x)|\nabla v_m|^2 dx + o(1)}{1 - \int_\Omega d^{-\alpha}(x) v_m^2 dx + o(1)} \\ &\leq \frac{L_{\alpha,\beta}(\Omega) - L_{\alpha,\beta}(\Omega)\int_\Omega d^{-\alpha}(x) v_m^2 dx + o(1)}{1 - \int_\Omega d^{-\alpha}(x) v_m^2 dx + o(1)} = L_{\alpha,\beta}(\Omega) + o(1). \end{aligned}$$

This implies that u attains $L_{\alpha,\beta}(\Omega)$. ◻

Lemma 2.10. *Let $\alpha < 1$, $b \in [0, \infty)$, $N \geq 2$, and $f \in C^1([0, 1))$ satisfying $\int_0^1 (1-r)^{2-\alpha}(f')^2 dr < \infty$ and $\int_0^1 (1-r)^{-\alpha} f^2 dr < \infty$. For $b > 0$ and $N = 2$, we assume that $f(0) = 0$. Then, for $g(r) = (1-r)^{\frac{1-\alpha}{2}} r^{-b} f(r)$, it holds that*

$$\begin{aligned} \int_0^1 (1-r)^{2-\alpha} r^{N-1} (f')^2 dr = \int_0^1 r^{N-1}(1-r)^{-\alpha} f^2 \Bigg[\frac{(1-\alpha)^2}{4} - b(N+b-2)r^{-2}(1-r)^2 \\ + \left(b - (1-\alpha)\frac{N-1}{2}\right) r^{-1}(1-r)\Bigg] + r^{N+2b-1}(1-r)(g')^2 dr. \end{aligned}$$

Proof. By the arguments in Proposition 2.2 and the assumptions that $\int_0^1 (1-r)^{2-\alpha}(f')^2 dr < \infty$ and $\int_0^1 (1-r)^{-\alpha} f^2 dr < \infty$, we may assume that $g(1) = 0$. We note that $f(r) = r^b(1-r)^{-\frac{1-\alpha}{2}} g(r)$,

$$
\begin{aligned}
(f')^2 &= \Big((r^b(1-r)^{-\frac{1-\alpha}{2}})' g + r^b(1-r)^{-\frac{1-\alpha}{2}} g' \Big)^2 \\
&= \Big[\Big(br^{b-1}(1-r)^{-\frac{1-\alpha}{2}} + \frac{1-\alpha}{2} r^b (1-r)^{-\frac{3-\alpha}{2}} \Big) g + r^b(1-r)^{-\frac{1-\alpha}{2}} g' \Big]^2 \\
&= \Big(br^{b-1}(1-r)^{-\frac{1-\alpha}{2}} + \frac{1-\alpha}{2} r^b (1-r)^{-\frac{3-\alpha}{2}} \Big)^2 g^2 + \Big(r^b(1-r)^{-\frac{1-\alpha}{2}} g' \Big)^2 \\
&\quad + 2\Big(br^{b-1}(1-r)^{-\frac{1-\alpha}{2}} + \frac{1-\alpha}{2} r^b (1-r)^{-\frac{3-\alpha}{2}} \Big) r^b (1-r)^{-\frac{1-\alpha}{2}} g g' \\
&= \Big(b^2 r^{2(b-1)}(1-r)^{-1+\alpha} + \frac{(1-\alpha)^2}{4} r^{2b}(1-r)^{-3+\alpha} + b(1-\alpha) r^{2b-1}(1-r)^{-2+\alpha} \Big) g^2 \\
&\quad + r^{2b}(1-r)^{-1+\alpha}(g')^2 + \Big(br^{2b-1}(1-r)^{-1+\alpha} + \frac{1-\alpha}{2} r^{2b}(1-r)^{-2+\alpha} \Big)(g^2)'.
\end{aligned}
$$

Moreover, by the assumption that $f(0) = 0$ if $b > 0$ and $N = 2$, we see that for $b \in [0, \infty)$ and $N \geq 2$,

$$
\begin{aligned}
&\Big(br^{N+2b-2}(1-r) + \frac{1-\alpha}{2} r^{N+2b-1} \Big) g^2 \Big|_{r=0}^1 \\
&= \Big(br^{N-2}(1-r)^{2-\alpha} + \frac{1-\alpha}{2} r^{N-1}(1-r)^{1-\alpha} \Big) f^2 \Big|_{r=0}^1 \\
&= 0.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
&\int_0^1 (1-r)^{2-\alpha} r^{N-1} (f')^2 dr \\
&= \int_0^1 \Big(b^2 r^{N+2b-3}(1-r) + \frac{(1-\alpha)^2}{4} r^{N+2b-1}(1-r)^{-1} + b(1-\alpha) r^{N+2b-2} \Big) g^2 \\
&\quad + \Big(br^{N+2b-2}(1-r) + \frac{1-\alpha}{2} r^{N+2b-1} \Big)(g^2)' + r^{N+2b-1}(1-r)(g')^2 dr \\
&= \int_0^1 \Big[b^2 r^{N+2b-3}(1-r) + \frac{(1-\alpha)^2}{4} r^{N+2b-1}(1-r)^{-1} + b(1-\alpha) r^{N+2b-2} \\
&\quad - b(N+2b-2) r^{N+2b-3}(1-r) + br^{N+2b-2} - \frac{1-\alpha}{2}(N+2b-1) r^{N+2b-2} \Big] g^2 \\
&\quad + r^{N+2b-1}(1-r)(g')^2 dr \\
&= \int_0^1 r^{N+2b-1}(1-r)^{-1} g^2 \Big[\frac{(1-\alpha)^2}{4} - b(N+b-2) r^{-2}(1-r)^2 \\
&\quad + \big(b - (1-\alpha)\frac{N-1}{2} \big) r^{-1}(1-r) \Big] + r^{N+2b-1}(1-r)(g')^2 dr \\
&= \int_0^1 r^{N-1}(1-r)^{-\alpha} f^2 \Big[\frac{(1-\alpha)^2}{4} - b(N+b-2) r^{-2}(1-r)^2 \\
&\quad + \Big(b - (1-\alpha)\frac{N-1}{2} \Big) r^{-1}(1-r) \Big] + r^{N+2b-1}(1-r)(g')^2 dr. \qquad \square
\end{aligned}
$$

3. Proofs of main results for subcritical cases

3.1. **Proof of Theorem 1.1.** By Lemma 2.6 and Lemma 2.7, if $\alpha, \beta \in \{(a,b) \mid a+b>2\} \cup \{(1,1)\}$, we have $L_{\alpha,\beta}(\Omega)=0$. Thus it suffices to prove that for $\alpha+\beta \leq 2$ and $(\alpha,\beta) \neq (1,1)$,

$$L_{\alpha,\beta}(\Omega) > 0. \tag{18}$$

If $\alpha<1, \alpha+\beta \leq 2$ and $\alpha \geq \beta$, we can take $\beta' \in \mathbb{R}$ satisfying $\alpha+\beta' \leq 2, \beta' > \alpha \geq \beta$. Then, if $L_{\alpha,\beta'}(\Omega)>0$, we see that

$$L_{\alpha,\beta'}(\Omega)\int_\Omega d^{-\alpha}(x)u^2dx \leq \int_\Omega d^{\beta'}(x)|\nabla u|^2dx \leq \bar{C}\int_\Omega d^{\beta}(x)|\nabla u|^2dx,$$

where $\bar{C} \equiv \max\{d^{\beta'-\beta}(x) \mid x \in \Omega\} < \infty$. This implies that for $\alpha<1, \alpha+\beta \leq 2$ and $\alpha \geq \beta$, $L_{\alpha,\beta}>0$ if $L_{\alpha,\beta'}(\Omega)>0$ for $\alpha+\beta' \leq 2, \beta'>\alpha \geq \beta$. Thus it suffices to show that $L_{\alpha,\beta}>0$ when

$$(\alpha,\beta) \in \{(a,b) \mid a+b \leq 2,\ a<b\} \cup \{(a,b) \mid a \geq 1, a+b \leq 2, a=b \neq 1\}.$$

First, take any $(\alpha,\beta) \in \{(a,b) \mid a \geq 1, a+b \leq 2, a=b \neq 1\}$ and $u \in W^{1,2}_{\alpha,\beta}(\Omega)$. By Proposition 2.2, we may assume that $u \in C_0^\infty(\Omega)$. We see from Lemma 2.3 that for $\alpha \geq 1, \alpha+\beta \leq 2, \alpha=\beta \neq 1$ and $v \in C^1((0,1))$ vanishing in a neighborhood of 0,

$$\frac{(\beta-\alpha)^2}{16}\int_0^\delta x^{-\alpha}v^2dx \leq \int_0^\delta x^\beta (v')^2dx.$$

This and (5) imply that for small $\delta>0$, there exists a constant $\tilde{C}_1>0$ such that for any $u \in C_0^\infty(\Omega)$,

$$\int_{\Omega_\delta} d^\beta(x)|\nabla u|^2dx \geq \tilde{C}_1\int_{\Omega_\delta}|u(x)|^2d^{-\alpha}(x)dx. \tag{19}$$

We take $\psi \in C_0^\infty(\Omega)$ such that $\psi \in [0,1]$, $\psi(x)=1$ for $x \in \Omega \setminus \Omega_\delta$ and $\psi(x)=0$ for $x \in \Omega_{\delta/2}$. Since

$$0 < \inf_{x \in \Omega\setminus\Omega_{\delta/2}} d(x) \leq \sup_{x \in \Omega\setminus\Omega_{\delta/2}} d(x) < \infty, \tag{20}$$

there exists a constant $\tilde{C}_2>0$ such that for any $u \in C_0^\infty(\Omega \setminus \overline{\Omega_{\delta/2}})$,

$$\int_\Omega d^\beta(x)|\nabla u|^2dx \geq \tilde{C}_2\int_\Omega |u(x)|^2d^{-\alpha}(x)dx. \tag{21}$$

Now, we see from (19) and (21) that for any $u \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&\int_\Omega |u(x)|^2d^{-\alpha}(x)dx \\
&\quad\leq 2\int_\Omega |\psi(x)u(x)|^2d^{-\alpha}(x)dx + 2\int_{\Omega_\delta}\Big|\big(1-\psi(x)\big)u(x)\Big|^2d^{-\alpha}(x)dx \\
&\quad\leq \frac{2}{\tilde{C}_1}\int_{\Omega_\delta} d^\beta(x)\Big|\nabla\big((1-\psi)u\big)\Big|^2dx + \frac{2}{\tilde{C}_2}\int_\Omega d^\beta(x)|\nabla(\psi u)|^2dx \\
&\quad\leq \frac{4}{\tilde{C}_1}\int_{\Omega_\delta} d^\beta(x)(|\nabla u|^2+|\nabla\psi|^2u^2)dx + \frac{4}{\tilde{C}_2}\int_\Omega d^\beta(x)(|\nabla u|^2+|\nabla\psi|^2u^2)dx \\
&\quad\leq \hat{C}\int_\Omega d^\beta(x)|\nabla u|^2dx + \hat{C}\int_{\Omega_\delta} d^{-\alpha}(x)u^2dx,
\end{aligned}$$

where $\hat{C}$ is a positive constant depending only on $\delta, \tilde{C}_1, \tilde{C}_2$, $\|\nabla\psi\|_{L^\infty}$. Combining the above inequality and (19), we see that for $(\alpha,\beta) \in \{(a,b) \mid a \geq 1, a+b \leq 2, a = b \neq 1\}$.

$$\int_\Omega |u(x)|^2 d^{-\alpha}(x)dx \leq C \int_\Omega d^\beta(x)|\nabla u|^2 dx, \qquad u \in C_0^\infty(\Omega),$$

where $C > 0$ is a constant; thus $L_{\alpha,\beta} > 0$.

Next, take any $(\alpha,\beta) \in \{(a,b) \mid a+b \leq 2,\ a < b\}$. We note that in this case, $\alpha < 1$. Take any function $u \in C^1(\Omega)$ satisfying

$$\int_\Omega d^\beta(x)|\nabla u|^2 dx < \infty, \int_\Omega d^{-\alpha}(x)u^2 dx < \infty \text{ and } \int_\Omega d^{-\alpha}(x)u dx = 0.$$

For any small $\delta \in (0,\delta_0)$, it follows from the Poincaré-Wirtinger inequality and (20) that for a constant C, which depends only on N and $\Omega \setminus \overline{\Omega_\delta}$,

$$\int_{\Omega\setminus\overline{\Omega_\delta}} \left(u - \left(\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(z)dz\right)^{-1} \int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(z)u(z)dz\right)^2 d^{-\alpha}(x)dx$$
$$\leq C \int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla u|^2 dx.$$

Then we see that

$$\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(x)u^2 dx \tag{22}$$
$$\leq C_1 \max\{\delta^{-\beta},1\} \int_{\Omega\setminus\overline{\Omega_\delta}} d^\beta(x)|\nabla u|^2 dx + C_2 \left(\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(x)u dx\right)^2,$$

where $C_1, C_2 > 0$ are independent of u. Since $\int_\Omega d^{-\alpha}(x)u dx = 0$, we see that for some C_3, independent of $\delta > 0$ and u,

$$\left(\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(x)u dx\right)^2 = \left(-\int_{\Omega_\delta} d^{-\alpha}(x)u dx\right)^2$$
$$\leq \int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx \int_{\Omega_\delta} d^{-\alpha}(x)dx \leq C_3\delta^{1-\alpha} \int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx.$$

Then, it follows from this estimate and (22) that

$$\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(x)u^2 dx \tag{23}$$
$$\leq C_1 \max\{\delta^{-\beta},1\} \int_{\Omega\setminus\overline{\Omega_\delta}} d^\beta(x)|\nabla u|^2 dx + C_2C_3\delta^{1-\alpha} \int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx.$$

Thus, we deduce that for small $\delta > 0$,

$$\int_\Omega d^{-\alpha}(x)u^2 dx = \int_{\Omega\setminus\Omega_\delta} d^{-\alpha}(x)u^2 dx + \int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx$$
$$\leq C_1 \max\{\delta^{-\beta},1\} \int_\Omega d^\beta(x)|\nabla u|^2 dx + 2\int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx.$$

Now, for the completion of the proof, it suffices to show that for small $\delta > 0$,

$$\int_{\Omega_\delta} d^{-\alpha}(x)u^2 dx \leq C_4 \int_\Omega d^\beta(x)|\nabla u|^2 dx \tag{24}$$

for some constant $C_4 > 0$, independent of u. By Lemma 2.3 and (5), we see that for some $C_5, C_6 > 0$ independent of u and δ,

$$\begin{aligned} \int_{\Omega_\delta} d^{-\alpha}(x)u^2dx &= \int_0^\delta \int_{\Sigma_t} t^{-\alpha}u^2 d\sigma_t dt \le C_5 \int_0^\delta \int_{\partial\Omega} t^{-\alpha}u^2(t,\sigma)d\sigma dt \\ &\le \Big(\frac{4}{\beta-\alpha}\Big)^2 C_5 \Big[\int_0^\delta \int_{\partial\Omega} t^\beta \Big(\frac{du}{dt}(t,\sigma)\Big)^2 d\sigma dt + \frac{(\beta-\alpha)\delta^{\frac{\beta-\alpha}{2}}}{4}\int_{\partial\Omega} u^2(\delta,\sigma)d\sigma\Big] \\ &\le C_6\Big(\int_\Omega d^\beta(x)|\nabla u|^2dx + \delta^{\frac{\beta-\alpha}{2}}\int_{\partial\Omega} u^2(\delta,\sigma)d\sigma\Big). \end{aligned} \tag{25}$$

Note that for some $C_7 > 0$, independent of u and small $\delta > 0$,

$$\int_{\partial\Omega} u^2(\delta,\sigma)d\sigma \le C_7 \int_{\Sigma_\delta} u^2 d\sigma_\delta, \tag{26}$$

and by the trace inequality, there exists a constant $\tilde{C}$, independent of small $\delta > 0$ such that for small $\delta > 0$,

$$\begin{aligned} \int_{\Sigma_\delta} u^2 d\sigma_\delta &\le \tilde{C}\int_{\Omega\setminus\overline{\Omega_\delta}} |\nabla u|^2 + u^2 dx \\ &\le \tilde{C}\Bigg(\max\{\delta^{-\beta}, (\mathrm{diam}(\Omega))^{-\beta}\}\int_{\Omega\setminus\overline{\Omega_\delta}} d^\beta(x)|\nabla u|^2dx \\ &\quad + \max\{\delta^\alpha, (\mathrm{diam}(\Omega))^\alpha\}\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(x)u^2dx\Bigg). \end{aligned} \tag{27}$$

By (23) and (25)-(27), we see that for small $\delta > 0$,

$$\begin{aligned} &\int_{\Omega_\delta} d^{-\alpha}(x)u^2dx \\ &\quad\le C_8\Big(\big(1+\delta^{-\frac{\alpha+\beta}{2}}\max\{1,\delta^\beta\}\big)\int_\Omega d^\beta(x)|\nabla u|^2dx \\ &\qquad + \delta^{\frac{\beta-\alpha}{2}}\max\{\delta^\alpha, (\mathrm{diam}(\Omega))^\alpha\}\int_{\Omega\setminus\overline{\Omega_\delta}} d^{-\alpha}(x)u^2dx\Big) \\ &\quad\le C_9\int_\Omega d^\beta(x)|\nabla u|^2dx + \frac{1}{2}\int_{\Omega_\delta} d^{-\alpha}(x)u^2dx, \end{aligned}$$

where $C_8, C_9 \equiv C_9(\delta) > 0$ are independent of u, which implies (24). This completes the proof.

3.2. **Proof of Theorem 1.2.** We first prove the second result (2). We assume that $\alpha+\beta \le 2$, $2\alpha+\beta \ge 3$, $(\alpha,\beta) \ne (1,1)$. By Proposition 2.5, $L_{\alpha,\beta}(\Omega) = H_{\alpha,\beta}(\Omega)$ in this case. Suppose that there exists a minimizer $u_{\alpha,\beta}$ of $L_{\alpha,\beta}(\Omega)$ such that $\int_\Omega d^{-\alpha}(x)u_{\alpha,\beta}^2dx = 1$. By Proposition 2.5, we deduce that $|u_{\alpha,\beta}|$ is also a minimizer of $H_{\alpha,\beta}(\Omega)$. Then, we see from the strong maximum principle that $|u_{\alpha,\beta}| > 0$ in Ω. This contradicts that $\int_\Omega d^{-\alpha}(x)u_{\alpha,\beta}dx = 0$. Thus, $L_{\alpha,\beta}(\Omega)$ is not achieved for $(\alpha,\beta) \in \{(a,b) \mid a+b \le 2, 2a+b \ge 3, a = b \ne 1\}$.

Next, we prove the first result (1). We assume $(\alpha,\beta) \in \{(a,b) \mid a+b < 2, 2a+b < 3\}$. Let $\{u_m\}_{m=1}^\infty \subset C^1(\Omega)$ be a minimizing sequence of $L_{\alpha,\beta}(\Omega)$ with

$$\int_\Omega d^{-\alpha}(x)u_m dx = 0, \quad \int_\Omega d^{-\alpha}(x)u_m^2dx = 1 \quad \text{and} \quad \lim_{m\to\infty}\int_\Omega d^\beta(x)|\nabla u_m|^2dx = L_{\alpha,\beta}(\Omega).$$

Taking a subsequence if it is necessary, we may assume that as $m \to \infty$, u_m converges weakly to u in $W^{1,2}_{\alpha,\beta}(\Omega)$.

Suppose that $u \equiv 0$. Let $\{r_m\}_{m=0}^{\infty}$ be a sequence such that r_0 is sufficiently small and $r_m \downarrow 0$ as $m \to \infty$. Then, taking a subsequence of $\{u_i\}$ if it is necessary, we see from Lemma 2.8 that $\{u_i\}$ satisfies

$$\int_{\Omega^{(i)}} d^{-\alpha}(x)|u_i|dx < \min\{1/2, (r_{i-1} - r_i)^2\} \quad \text{and} \tag{28}$$

$$\int_{\Omega^{(i)}} d^{\beta}(x)u_i^2dx, \int_{\Omega^{(i)}} d^{-\alpha}(x)u_i^2dx < \min\{1/2, (r_{i-1} - r_i)^2\}, \tag{29}$$

where $\Omega^{(i)} \equiv \{x \in \Omega \mid d(x) > r_i\}$. For each $m \geq 1$, we define $\phi_m \in C^1(\Omega)$ such that $|\nabla \phi_m| < 2(r_{m-1} - r_m)^{-1}$.

$$\phi_m(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus \Omega^{(m)}, \\ 0 & \text{if } x \in \Omega^{(m-1)}. \end{cases}$$

Then, by (28) and the assumption $\int_\Omega d^{-\alpha}(x)u_m dx = 0$, we have

$$\Big|\int_\Omega d^{-\alpha}(x)u_m\phi_m dx\Big| = \Big|\int_\Omega d^{-\alpha}(x)u_m\Big(1 - \phi_m\Big)dx\Big| \to 0 \tag{30}$$

as $m \to \infty$. For $\delta \in (0, \delta_0)$, define $\psi \in C^1(\mathbb{R})$ such that $\psi(t) = \begin{cases} 1 & \text{if } t \geq \delta, \\ 0 & \text{if } t \leq \delta/2 \end{cases}$ and

$$v_m = u_m\phi_m + c_m\psi(d(x)) \text{ with } c_m \equiv -\frac{\int_\Omega d^{-\alpha}(x)u_m\phi_m dx}{\int_\Omega d^{-\alpha}(x)\psi(d(x))dx}.$$

Note that for large m, $\mathrm{supp}(u_m\phi_m) \cap \mathrm{supp}(\psi(d(x))) = \emptyset$, $\int_\Omega d^{-\alpha}(x)v_m dx = 0$, and from (30), $|c_m| \to 0$ as $m \to \infty$. We take a small $\iota > 0$ so that $\alpha + \beta + \iota < 2$. Then, by (29), there exists a constant $C > 0$, independent of m, such that

$$\begin{aligned}
&\int_\Omega d^{\beta+\iota}(x)|\nabla v_m|^2dx = \int_\Omega d^{\beta+\iota}(x)\left(|\nabla(u_m\phi_m)|^2 + c_m^2\Big(\psi'(d(x))\Big)^2\right)dx \\
&\leq (r_{m-1})^\iota \int_\Omega d^\beta(x)|\nabla(u_m\phi_m)|^2dx + c_m^2 \int_{\Omega_\delta\setminus\Omega_{\delta/2}} d^{\beta+\iota}(x)\Big(\psi'(d(x))\Big)^2dx \\
&\leq 2(r_{m-1})^\iota \int_\Omega d^\beta(x)u_m^2|\nabla\phi_m|^2dx + 2(r_{m-1})^\iota \int_\Omega d^\beta(x)|\nabla u_m|^2\phi_m^2dx + o(1) \\
&\leq C(r_{m-1})^\iota + o(1)
\end{aligned} \tag{31}$$

as $m \to \infty$. Note that, by (29), for each $m \geq 1$,

$$\int_\Omega d^{-\alpha}(x)v_m^2dx \geq \int_\Omega d^{-\alpha}(x)(u_m\phi_m)^2dx \geq \int_{\Omega\setminus\Omega^{(m)}} d^{-\alpha}(x)u_m^2dx \geq 1/2. \tag{32}$$

Thus, by (31), (32) and the property $\int_\Omega d^{-\alpha}(x)v_m dx = 0$, we see that for large $m \geq 1$,

$$L_{\alpha,\beta+\iota}(\Omega) \leq \frac{\int_\Omega d^{\beta+\iota}(x)|\nabla v_m|^2dx}{\int_\Omega d^{-\alpha}(x)v_m^2dx} \leq 2\Big(C(r_{m-1})^\iota + o(1)\Big).$$

This is a contradiction to the fact that $L_{\alpha,\beta+\iota}(\Omega) > 0$. Thus we see that $u \not\equiv 0$.

Now we claim that $\int_\Omega d^{-\alpha}(x)udx = 0$. If $\alpha < 1$, since u_m converges weakly to u in $W^{1,2}_{\alpha,\beta}(\Omega)$ as $m \to \infty$ and any constant functions are in $W^{1,2}_{\alpha,\beta}(\Omega)$, it follows that $\int_\Omega d^{-\alpha}(x)udx = 0$. If $\alpha \geq 1$ and $2\alpha + \beta < 3$, by Proposition 2.2, we may assume that $u_m \in C_0^\infty(\Omega)$. As before, we find a function $\psi \in C^1(\mathbb{R}, [0,1])$ such that $\psi(t) = \begin{cases} 1 & \text{if } t \geq \delta, \\ 0 & \text{if } t \leq \delta/2 \end{cases}$ and write $u_m = u_{m,1} + u_{m,2}$, where $u_{m,1} = u_m\Big(1 - \psi(d(x))\Big)$ and $u_{m,2} = u_m\psi(d(x))$. Since $\psi \circ d \in W^{1,2}_{\alpha,\beta}(\Omega)$ and u_m converges weakly to u in $W^{1,2}_{\alpha,\beta}(\Omega)$ as $m \to \infty$, we see that

$$\lim_{m\to\infty} \int_\Omega d^{-\alpha}(x)u_{m,2}dx = \int_\Omega d^{-\alpha}(x)u\psi(d(x))dx. \tag{33}$$

Since $\beta < 1$, we see from Lemma 2.1(ii) that

$$\int_\Omega d^\beta(x)|\nabla u_{m,1}|^2 dx \leq \int_\Omega d^\beta(x)|\nabla u_m|^2 + d^\beta(x)u_m^2\Big(\psi'(d(x))\Big)^2 dx \leq C_1,$$

where $C_1 > 0$ is a constant independent of m. Now, since

$$\begin{aligned} |u_{m,1}(t,\sigma)| &\leq \int_0^t \left|\frac{\partial u_{m,1}(s,\sigma)}{\partial s}\right| ds \\ &\leq \Big(\int_0^t s^\beta |\nabla u_{m,1}(s,\sigma)|^2 ds\Big)^{1/2} \Big(\int_0^t s^{-\beta} ds\Big)^{1/2}, \quad \sigma \in \partial\Omega, \end{aligned}$$

we see that for $\sigma \in \partial\Omega$,

$$\int_0^\delta t^{-\alpha}|u_{m,1}(t,\sigma)|dt \leq \Big(\int_0^\delta t^\beta |\nabla u_{m,1}(t,\sigma)|^2 dt\Big)^{1/2} \frac{2}{(1-\beta)^{1/2}(3-2\alpha-\beta)} \delta^{\frac{3-\beta-2\alpha}{2}}.$$

Then, we see from (5) and the Cauchy inequality that for some constants $C_2, C_3 > 0$,

$$\begin{aligned} &\Big|\int_0^\delta \int_{\Sigma_t} t^{-\alpha}|u_{m,1}| d\sigma_t dt\Big| \\ &\leq C_2 \int_0^\delta \int_{\partial\Omega} t^{-\alpha}|u_{m,1}(t,\sigma)| d\sigma dt \\ &\leq \frac{2C_2}{(1-\beta)^{1/2}(3-2\alpha-\beta)} \delta^{\frac{3-2\alpha-\beta}{2}} \int_{\partial\Omega} \Big(\int_0^\delta t^\beta |\nabla u_{m,1}(t,\sigma)|^2 dt\Big)^{1/2} d\sigma \\ &\leq \frac{2C_2}{(1-\beta)^{1/2}(3-2\alpha-\beta)} \delta^{\frac{3-2\alpha-\beta}{2}} |\partial\Omega|^{1/2} \Big(\int_{\partial\Omega} \int_0^\delta t^\beta |\nabla u_{m,1}(t,\sigma)|^2 dt d\sigma\Big)^{1/2} \\ &\leq \frac{2C_3}{(1-\beta)^{1/2}(3-2\alpha-\beta)} \delta^{\frac{3-2\alpha-\beta}{2}} |\partial\Omega|^{1/2} \Big(\int_\Omega d^\beta(x)|\nabla u_{m,1}|^2 dx\Big)^{1/2} \\ &\leq \frac{2C_3(C_1)^{1/2}}{(1-\beta)^{1/2}(3-2\alpha-\beta)} \delta^{\frac{3-2\alpha-\beta}{2}} |\partial\Omega|^{1/2}. \end{aligned}$$

This and (33) imply that $\lim_{m\to\infty} \int_\Omega d^{-\alpha}(x)u_m dx = \int_\Omega d^{-\alpha}(x)udx = 0$. Thus, by Lemma 2.9, we see that u attains $L_{\alpha,\beta}(\Omega)$. The above estimate implies that for any $u \in W^{1,2}_{\alpha,\beta}(\Omega)$, $\int_\Omega d^{-\alpha}(x)u(x)dx$ is well-defined and a functional $u \in W^{1,2}_{\alpha,\beta}(\Omega) \mapsto \int_\Omega d^{-\alpha}(x)u(x)dx$ is differentiable.

We denote the minimizer u by $u_{\alpha,\beta}$. We may assume that $\int_\Omega d^{-\alpha}(x)(u_{\alpha,\beta})^2 dx = 1$. Then, there are Lagrange multipliers $\lambda, \mu \in \mathbb{R}$ such that

$$-div(d^\beta(x)\nabla u_{\alpha,\beta}) = \lambda d^{-\alpha}(x)u_{\alpha,\beta} + \mu d^{-\alpha}(x) \text{ in } \Omega.$$

Since

$$\int_\Omega d^\beta(x)|\nabla u_{\alpha,\beta}|^2dx = \lambda\int_\Omega d^{-\alpha}(x)(u_{\alpha,\beta})^2dx + \mu\int_\Omega d^{-\alpha}(x)u_{\alpha,\beta}dx$$
$$= \lambda\int_\Omega d^{-\alpha}(x)(u_{\alpha,\beta})^2dx = \lambda,$$

it follows that $\lambda = L_{\alpha,\beta}(\Omega)$. For $\alpha < 1$, a constant function $1_\Omega(x) = 1$ on Ω is in $W^{1,2}_{\alpha,\beta}(\Omega)$. This implies that

$$0 = \lambda\int_\Omega d^{-\alpha}(x)u_{\alpha,\beta}dx + \mu\int_\Omega d^{-\alpha}(x)dx = \mu\int_\Omega d^{-\alpha}(x)dx.$$

Then, we get that $\mu = 0$ and $\lambda = L_{\alpha,\beta}(\Omega)$ if $\alpha < 1$. □

4. Proof of main results for critical cases

In this section, we study whether or not there exists a minimizer of

$$L_{\alpha,2-\alpha}(\Omega) \equiv \inf\left\{\frac{\int_\Omega d^{2-\alpha}(x)|\nabla u|^2dx}{\int_\Omega |u|^2d^{-\alpha}(x)dx}\ \Big|\ u\in W^{1,2}_{\alpha,2-\alpha}(\Omega)\setminus\{0\}, \int_\Omega ud^{-\alpha}(x)dx = 0\right\},$$

where $\alpha < 1$. We give some criteria for the (non-)attainability of the optimal constant $L_{\alpha,2-\alpha}(\Omega)$ when Ω is the unit ball, an ellipse or an annulus. Moreover, we prove that the attainability of $L_{\alpha,2-\alpha}(\Omega)$ depends on a local geometry of a domain Ω.

4.1. **Proof of Theorem 1.3.** We first prove that for $\alpha < 1$, $L_{\alpha,2-\alpha}(\Omega) \le \frac{(1-\alpha)^2}{4}$. Assume $N \ge 1$. For small $\delta > 0$ and $m > 1$, we define

$$u(x) = \begin{cases} |\log(d(x)/\delta)|^m & \text{if } d(x)\le\delta,\\ 0 & \text{if } d(x) > \delta.\end{cases}$$

Since $|\nabla u(x)| = m|\log(d(x)/\delta)|^{m-1}\frac{1}{|d(x)|}$, using a change of variables $-\log(t/\delta) = s$ and the fact that $\int_0^\infty s^ae^{-bs}ds = b^{-1-a}\Gamma(1+a)$ with $b > 0$ and $a > -1$, we see that for some constant $c > 0$, independent of small $\delta > 0$

$$\int_{\Omega_\delta} d^{2-\alpha}(x)|\nabla u|^2dx \le m^2(1+c\delta)|\partial\Omega|\int_0^\delta t^{-\alpha}|\log(t/\delta)|^{2m-2}dt$$
$$= m^2\delta^{1-\alpha}(1+c\delta)|\partial\Omega|\int_0^\infty s^{2m-2}e^{-(1-\alpha)s}ds$$
$$= m^2\delta^{1-\alpha}(1+c\delta)|\partial\Omega|(1-\alpha)^{1-2m}\Gamma(2m-1),$$

$$\int_{\Omega_\delta} d^{-\alpha}(x)u^2dx \ge (1-c\delta)|\partial\Omega|\int_0^\delta t^{-\alpha}|\log(t/\delta)|^{2m}dt$$
$$= \delta^{1-\alpha}(1-c\delta)|\partial\Omega|\int_0^\infty s^{2m}e^{-(1-\alpha)s}ds$$
$$= \delta^{1-\alpha}(1-c\delta)|\partial\Omega|(1-\alpha)^{-1-2m}\Gamma(2m+1)$$

and

$$\begin{aligned}\frac{1}{|\Omega|}\Big(\int_{\Omega_\delta} d^{-\alpha}(x)udx\Big)^2 &\le \frac{1}{|\Omega_\delta|}\Big((1+c\delta)|\partial\Omega|\int_0^\delta t^{-\alpha}|\log(t/\delta)|^m dt\Big)^2\\ &= \frac{\delta^{2(1-\alpha)}(1+c\delta)^2|\partial\Omega|^2}{|\Omega_\delta|}\Big(\int_0^\infty s^m e^{-(1-\alpha)s}ds\Big)^2\\ &\le \frac{\delta^{2(1-\alpha)}(1+c\delta)^2|\partial\Omega|^2}{\delta(1-c\delta)|\partial\Omega|}\Big((1-\alpha)^{-1-m}\Gamma(m+1)\Big)^2\\ &= \frac{\delta^{1-2\alpha}(1+c\delta)^2|\partial\Omega|}{(1-c\delta)}(1-\alpha)^{-2-2m}\big(\Gamma(m+1)\big)^2.\end{aligned}$$

Hence, it follows that

$$\begin{aligned}L_{\alpha,2-\alpha}(\Omega) &\le \frac{\int_\Omega d(x)^2|\nabla u|^2dx}{\int_\Omega d^{-\alpha}(x)u^2dx - \frac{1}{|\Omega|}(\int_\Omega d^{-\alpha}(x)udx)^2}\\ &\le \frac{m^2\delta^{1-\alpha}(1+c\delta)|\partial\Omega|(1-\alpha)^{1-2m}\Gamma(2m-1)}{\delta^{1-\alpha}(1-c\delta)|\partial\Omega|(1-\alpha)^{-1-2m}\Gamma(2m+1)-\frac{\delta^{1-2\alpha}(1+c\delta)^2|\partial\Omega|}{(1-c\delta)}(1-\alpha)^{-2-2m}\big(\Gamma(m+1)\big)^2}\\ &= \frac{(1-\alpha)^2(1+c\delta)}{(1-c\delta)\frac{2m(2m-1)}{m^2} - \frac{(1-\alpha)^{-1}\delta^{-\alpha}(1+c\delta)^2}{(1-c\delta)}\frac{(\Gamma(m+1))^2}{m^2\Gamma(2m-1)}}\\ &= \frac{(1-\alpha)^2}{\frac{1-c\delta}{1+c\delta}(4-\frac{2}{m}) - (1-\alpha)^{-1}\delta^{-\alpha}\frac{1+c\delta}{1-c\delta}\frac{(\Gamma(m))^2}{\Gamma(2m-1)}}.\end{aligned}$$

Since

$$\frac{(\Gamma(m))^2}{\Gamma(2m-1)} = \frac{(m-1)(m-2)\cdots(2)1}{(2m-2)(2m-3)\cdots(m+1)m} \le 2^{1-m} \to 0 \text{ as } m\to\infty,$$

letting $m\to\infty$ and then $\delta\to 0$ in the above inequality, we get the inequality $L_{\alpha,2-\alpha}(\Omega) \le \frac{(1-\alpha)^2}{4}$.

Next, we prove that if $L_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$, $L_{\alpha,2-\alpha}(\Omega)$ is achieved, where $\alpha<1$. Let $\{u_m\}_{m=1}^\infty \subset C^1(\Omega)$ be a minimizing sequence of $L_{\alpha,2-\alpha}(\Omega)$ with

$$\int_\Omega d^{-\alpha}(x)u_m^2dx = 1, \int_\Omega d^{-\alpha}(x)u_mdx = 0$$
$$\text{and } \lim_{m\to\infty}\int_\Omega d^{2-\alpha}(x)|\nabla u_m|^2dx = L_{\alpha,2-\alpha}(\Omega).$$

Taking a subsequence if it is necessary, we may assume that as $m\to\infty$, u_m converges weakly to some u in $W^{1,2}_{\alpha,2-\alpha}(\Omega)$.

Suppose that $u\equiv 0$. Then, by Lemma 2.8, taking a subsequence if it is necessary, we may assume that u_i satisfies

$$\int_{\Omega^{(i)}} d^{-\alpha}(x)u_i^2dx < \min\{1/2, (r_{i-1}-r_i)^2\},$$

where $\Omega^{(i)} \equiv \{x\in\Omega \mid d(x) > r_i\}$ and $\{r_m\}_{m=0}^\infty$ is a sequence such that $r_m \downarrow 0$ as $m\to\infty$. For each $m\ge 1$, we define $\phi_m \in C^1(\Omega,[0,1])$ such that $|\nabla\phi_m| < 2(r_{m-1}-r_m)^{-1}$ and

$$\phi_m(x) = \begin{cases} 1 & \text{if } x\in\Omega\setminus\Omega^{(m)},\\ 0 & \text{if } x\in\Omega^{(m-1)}.\end{cases}$$

Then we have

$$\int_\Omega d^{2-\alpha}(x)|\nabla u_m|^2dx = \int_\Omega d^{2-\alpha}(x)|\nabla(u_m\phi_m) + (1-\phi_m)\nabla u_m - u_m\nabla\phi_m|^2dx$$

$$= \int_\Omega d^{2-\alpha}(x)\Big[|\nabla(u_m\phi_m)|^2 + (1-\phi_m)^2|\nabla u_m|^2 + u_m^2|\nabla\phi_m|^2$$

$$+ 2(1-\phi_m)\nabla(u_m\phi_m)\cdot\nabla u_m - 2u_m\nabla(u_m\phi_m)\cdot\nabla\phi_m - 2u_m(1-\phi_m)\nabla u_m\cdot\nabla\phi_m\Big]dx$$

$$\geq \int_\Omega d^{2-\alpha}(x)\Big[|\nabla(u_m\phi_m)|^2 - 2u_m\nabla(u_m\phi_m)\cdot\nabla\phi_m\Big]dx.$$

From this, Lemma 2.3 and the fact that

$$\int_\Omega d^{2-\alpha}(x)u_m^2|\nabla\phi_m|^2dx \leq 4r_{m-1}^2(r_{m-1}-r_m)^{-2}\int_{\Omega_m} d^{-\alpha}(x)u_m^2dx \leq 4r_{m-1}^2,$$

we see that for some $c > 0$,

$$\begin{aligned}
\int_\Omega d^{2-\alpha}(x)|\nabla u_m|^2dx &\geq \int_\Omega d^{2-\alpha}(x)|\nabla(u_m\phi_m)|^2dx + o(1)\\
&= \int_{\Omega\setminus\Omega^{(m-1)}} d^{2-\alpha}(x)|\nabla(u_m\phi_m)|^2dx + o(1)\\
&\geq (1-cr_{m-1})\int_{\partial\Omega}\int_0^{r_{m-1}} t^{2-\alpha}\Big(\frac{d}{dt}(u_m\phi_m)\Big)^2dtd\sigma + o(1)\\
&\geq \frac{(1-\alpha)^2}{4}(1-cr_{m-1})\int_{\partial\Omega}\int_0^{r_{m-1}} t^{-\alpha}(u_m\phi_m)^2dtd\sigma + o(1)\\
&\geq \frac{(1-\alpha)^2(1-cr_{m-1})}{4(1+cr_{m-1})}\int_{\Omega\setminus\Omega^{(m-1)}} d^{-\alpha}(x)(u_m\phi_m)^2dx + o(1)\\
&= \frac{(1-\alpha)^2(1-cr_{m-1})}{4(1+cr_{m-1})}\int_\Omega d^{-\alpha}(x)(u_m\phi_m)^2dx + o(1).
\end{aligned}$$

Since $\int_\Omega d^{-\alpha}(x)u_mdx = 0$, $\int_\Omega d^{-\alpha}(x)u_m^2dx = 1$ and $\int_{\Omega^{(m)}} d^{-\alpha}(x)u_m^2dx = o(1)$, we deduce that

$$L_{\alpha,2-\alpha}(\Omega) = \lim_{m\to\infty}\int_\Omega d^{2-\alpha}(x)|\nabla u_m|^2dx \geq \frac{(1-\alpha)^2}{4},$$

which is a contradiction to the assumption $L_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$.

Now, it holds that $u \not\equiv 0$. Thus, by Lemma 2.9, we see that u attains $L_{\alpha,2-\alpha}(\Omega)$. □

To prove Theorem 1.4, we prepare some geometric results of a domain Ω. For a bounded C^2-domain $\Omega \subset \mathbb{R}^N$, let $\kappa(y_0) = (\kappa_1(y_0), \cdots, \kappa_{N-1}(y_0))$ be the principal curvatures with respect to the outward unit normal of $\partial\Omega$ at y_0. Then the mean curvature of $\partial\Omega$ at y_0 is given by $H(y_0) = \frac{1}{N-1}\sum_{i=1}^{N-1} k_i(y_0)$. We adopt the convention that a standard unit sphere $S^{N-1} \subset \mathbb{R}^N$ has mean curvature 1 everywhere. Let $G \subset \Omega$ be the largest open subset of Ω such that for every x in G there is a unique nearest point $x' \in \partial\Omega$ with $d(x) = |x - x'|$. We call G the good set of Ω and $\Omega \setminus G$ a singular set. We denote $S = \Omega \setminus G$. By [10, Lemma 3.8], there exists a C^1-domain $\Omega_\epsilon \subset \Omega$ such that $\Omega \setminus \overline{\Omega_\epsilon} \subset G$, $\cup_{\epsilon>0}(\Omega \setminus \overline{\Omega_\epsilon}) = G$ and $\nu_\epsilon \cdot \nabla d \geq 0$ on $\partial\Omega_\epsilon$, where ν_ϵ is the unit inner normal of the boundary of $\partial\Omega_\epsilon$. [10, Lemma 2.3] implies that the distance function d is in $C^2(G)$ if $\partial\Omega \in C^2$ and [10, Corollary 3.2] (see also [11]) implies that the measure of S is 0. Thus, $d \in W^{1,\infty}(\Omega)$.

4.2. **Proof of Theorem 1.4.** We assume that Ω is a weakly mean convex C^2-domain in $\mathbb{R}^N$. We note that if Ω is a weakly mean convex C^2-domain in $\mathbb{R}^N$, it follows from [10, Theorem 1.7] that $\Delta d \le 0$ in G. For each $\alpha < 1$, we take $p_\alpha \in \mathbb{R}$ such that

$$p_\alpha = -\frac{\int_\Omega d^{-\frac{\alpha}{2}} x_1 dx}{\int_\Omega d^{-\frac{\alpha}{2}} dx}.$$

Then $\{|p_\alpha|\}_\alpha$ is bounded uniformly in $\alpha < 1$. We define $u_\alpha = (x_1 + p_\alpha) d^{\frac{\alpha}{2}}$ and $v_\alpha = d^{\frac{1-\alpha}{2}}(x) u_\alpha = (x_1 + p_\alpha) d^{\frac{1}{2}}$. Then we see that

$$\int_\Omega d^{-\alpha}(x) u_\alpha dx = 0, \quad \int_\Omega d^{-\alpha}(x) u_\alpha^2 dx = \int_\Omega (x_1 + p_\alpha)^2 dx < \infty,$$
$$\int_\Omega d^{2-\alpha}(x) |\nabla u_\alpha|^2 dx = \int_\Omega \frac{\alpha^2}{4}(x_1 + p_\alpha)^2 + \Big(1 - \frac{\alpha}{2}\Big) d^2(x) dx < \infty.$$

Thus, $u_\alpha \in W^{1,2}_{\alpha,\beta}(\Omega)$ and $\int_\Omega d^{-\alpha}(x) u_\alpha dx = 0$. Then, since $v_\alpha = 0$ on $\partial\Omega$, $\nu_\epsilon \cdot \nabla d \ge 0$ on $\partial\Omega_\epsilon$ and

$$\begin{aligned}
&d^{2-\alpha}(x)|\nabla u_\alpha|^2 \\
&\quad = d^{2-\alpha}(x) \Big| \frac{\alpha - 1}{2} d^{\frac{\alpha-1}{2}-1} (\nabla d) v_\alpha + d^{\frac{\alpha-1}{2}}(x) \nabla v_\alpha \Big|^2 \\
&\quad = d^{2-\alpha}(x) \Big(\frac{(1-\alpha)^2}{4} d^{\alpha-3}(x) |\nabla d|^2 v_\alpha^2 + d^{\alpha-1}(x) |\nabla v_\alpha|^2 \\
&\qquad - (1-\alpha) d^{\alpha-2}(x) v_\alpha \nabla d \cdot \nabla v_\alpha \Big) \\
&\quad = \frac{(1-\alpha)^2}{4} d^{-1}(x) |\nabla d|^2 v_\alpha^2 + d |\nabla v_\alpha|^2 - \frac{1-\alpha}{2} \nabla d \cdot \nabla (v_\alpha^2) \\
&\quad = \frac{(1-\alpha)^2}{4} d^{-\alpha}(x) u_\alpha^2 + d |\nabla v_\alpha|^2 - \frac{1-\alpha}{2} \nabla d \cdot \nabla (v_\alpha^2) \ \text{ in } \ G,
\end{aligned}$$

we see that

$$\begin{aligned}
&\int_\Omega d^{2-\alpha}(x) |\nabla u_\alpha|^2 - \frac{(1-\alpha)^2}{4} d^{-\alpha}(x) u_\alpha^2 dx \\
&\quad = \int_\Omega d |\nabla v_\alpha|^2 - \frac{1-\alpha}{2} \nabla d \cdot \nabla (v_\alpha^2) dx \\
&\quad = \int_\Omega d |\nabla v_\alpha|^2 dx + \frac{1-\alpha}{2} \Big(\int_{\Omega \setminus \overline{\Omega_\epsilon}} (\Delta d) v_\alpha^2 dx - \int_{\Omega_\epsilon} \nabla d \cdot \nabla (v_\alpha^2) dx \Big) \\
&\qquad - \frac{1-\alpha}{2} \int_{\partial\Omega_\epsilon} \frac{\partial d}{\partial \nu_\epsilon} v_\alpha^2 d\sigma_\epsilon \\
&\quad \le \int_\Omega d |\nabla v_\alpha|^2 dx + \frac{1-\alpha}{2} \Big(\int_{\Omega \setminus \overline{\Omega_\epsilon}} (\Delta d) v_\alpha^2 dx - \int_{\Omega_\epsilon} \nabla d \cdot \nabla (v_\alpha^2) dx \Big),
\end{aligned} \tag{34}$$

where ν_ϵ is the unit inner normal of the boundary of $\partial\Omega_\epsilon$. Since $\{|p_\alpha|\}_\alpha$ is bounded uniformly in $\alpha < 1$ and

$$\begin{aligned}
\int_\Omega d |\nabla v_\alpha|^2 dx &= \int_\Omega \frac{1}{4}(x_1 + p_\alpha)^2 + d^2 + (x_1 + p_\alpha) \frac{1}{2} \frac{\partial}{\partial x_1}(d^2) dx \\
&= \int_\Omega \frac{1}{4}(x_1 + p_\alpha)^2 + \frac{1}{2} d^2 dx,
\end{aligned}$$

we see that

$$\left\{ \int_\Omega d|\nabla v_\alpha|^2 dx \right\}_{\alpha<1} \text{ is bounded.} \tag{35}$$

An estimate [10, Corollary 2.7] says that for any $x \in G$,

$$-\Delta d(x) \geq \frac{H(y)}{1-d(x)H(y)}, \tag{36}$$

where $y \in \partial\Omega$ satisfying $d(x) = |x-y|$. Since $H \geq 0$ on $\partial\Omega$ and $\partial\Omega$ is compact, H cannot be identically zero. In fact, a classical result [1] of Alexandroff says that any compact hypersurface in $\mathbb{R}^N$ with a constant mean curvature should be a sphere; thus H is positive in an open subset A of $\partial\Omega$. Then, we see from (36) that $\Delta d < 0$ in an open subset Ω' of $G \subset \Omega$. This implies that for small $\varepsilon > 0$, the set $\{\int_{\Omega\setminus\overline{\Omega_\epsilon}}(\Delta d)(x_1+p_\alpha)^2 d(x)dx\}_{\alpha<-1}$ is bounded away from 0. Then, for small $\varepsilon > 0$, there exists $C > 0$, independent of large $-\alpha > 0$, satisfying

$$\begin{aligned}&\int_{\Omega\setminus\overline{\Omega_\epsilon}}(\Delta d)v_\alpha^2 dx - \int_{\Omega_\epsilon}\nabla d\cdot\nabla(v_\alpha^2)dx\\ &= \int_{\Omega\setminus\overline{\Omega_\epsilon}}(\Delta d)(x_1+p_\alpha)^2 d(x)dx - \int_{\Omega_\epsilon}(x_1+p_\alpha)^2 + 2(x_1+p_\alpha)d\frac{\partial d}{\partial x_1}dx < -C.\end{aligned} \tag{37}$$

Combining (34)-(37), we see that for large $-\alpha > 0$,

$$\int_\Omega d^{2-\alpha}(x)|\nabla u_\alpha|^2 - \frac{(1-\alpha)^2}{4}d^{-\alpha}(x)u_\alpha^2 dx < 0,$$

which implies that $L_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$. This completes the proof. $\square$

4.3. **Proof of Theorem 1.5.** We note that Theorem 1.3 says that $L_{\alpha,2-\alpha}(\Omega) \leq \frac{(1-\alpha)^2}{4}$ for $\alpha < 1$ and any C^2-domain Ω. First, we assume that $\alpha < 1$ and $N = 1$. We denote $I = (-1,1)$. Then we claim that $L_{\alpha,2-\alpha}(I) = \frac{(1-\alpha)^2}{4}$ and $L_{\alpha,2-\alpha}(I)$ is not achieved. We see that

$$\begin{aligned}L_{\alpha,2-\alpha}(I) &= \inf_{u\in W^{1,2}_{\alpha,2-\alpha}(I)\setminus\{0\}, \int_I d^{-\alpha}(t)u\,dt=0} \frac{\int_I d^{2-\alpha}(t)|u'|^2dt}{\int_I d^{-\alpha}(t)u^2dt}\\ &= \inf_{u\in W^{1,2}_{\alpha,2-\alpha}(I)\setminus\{0\}} \frac{\int_I d^{2-\alpha}(t)|u'|^2dt}{\int_I d^{-\alpha}(t)\big(u-(\int_I d^{-\alpha}(t)dt)^{-1}\int_I d^{-\alpha}(t)udt\big)^2dt},\end{aligned}$$

where $d(t) = \begin{cases} 1+t & \text{if } t\in(-1,0),\\ 1-t & \text{if } t\in[0,1).\end{cases}$ Since the functional

$$u \mapsto \frac{\int_I d^{2-\alpha}(t)|u'|^2dt}{\int_I d^{-\alpha}(t)\big(u-(\int_I d^{-\alpha}(t)dt)^{-1}\int_I d^{-\alpha}(t)udt\big)^2dt}$$

is invariant under the addition of any constant, it suffices to prove that if $u \in W^{1,2}_{\alpha,2-\alpha}(I)$ and $u(0) = 0$,

$$\int_I d^{2-\alpha}(t)|u'|^2dt \geq \frac{(1-\alpha)^2}{4}\int_I d^{-\alpha}(t)\Big(u-\Big(\int_I d^{-\alpha}(t)dt\Big)^{-1}\int_I d^{-\alpha}(t)udt\Big)^2dt. \tag{38}$$

By the density, it is sufficient to prove (38) for any function $u \in C^1(I)$ satisfying

$$u(0) = 0, \int_I d^{-\alpha}(t)u^2 dt < \infty \text{ and } \int_I d^{2-\alpha}(t)|u'|^2 dt < \infty.$$

Defining $v(t) \equiv d^{\frac{1-\alpha}{2}}(t)u(t)$, we see that

(39)

$$\begin{aligned} & d^{2-\alpha}(t)|u'(t)|^2 \\ & = d^{2-\alpha}(t)\Big(- \frac{1-\alpha}{2} d^{\frac{\alpha-3}{2}}(t)d'(t)v(t) + d^{\frac{\alpha-1}{2}}(t)v'(t)\Big)^2 \\ & = d^{2-\alpha}(t)\Big[\Big(\frac{1-\alpha}{2}\Big)^2 d^{\alpha-3}(t)|v(t)|^2 - (1-\alpha)d^{\alpha-2}(t)d'(t)v(t)v'(t) + d^{\alpha-1}(t)|v'(t)|^2\Big] \\ & = \Big(\frac{1-\alpha}{2}\Big)^2 d^{-\alpha}(t)|u(t)|^2 - (1-\alpha)d'(t)v(t)v'(t) + d(t)|v'(t)|^2. \end{aligned}$$

Since for $-1 < x < y < 1$,

$$\begin{aligned} |v^2(y) - v^2(x)| = \Big| \int_x^y \Big(v^2(t)\Big)' dt \Big| & = \Big| \int_x^y \Big(d^{1-\alpha}(t)u^2(t)\Big)' dt \Big| \\ & \le (1-\alpha) \int_x^y d^{-\alpha}(t)u^2(t)dt \\ & \quad + 2\Big(\int_x^y d^{-\alpha}(t)u^2(t)dt\Big)^{1/2} \Big(\int_x^y d^{2-\alpha}(t)\Big(u'(t)\Big)^2 dt\Big)^{1/2}, \end{aligned}$$

we see that $\lim_{t\to\pm1} v(t)$ exists. If $\lim_{t\to\pm1} v(t)$ is not 0, then $\int_I d^{-\alpha}(t)u^2 dt = \infty$, which is a contradiction. Thus $v(0) = v(1) = v(-1) = 0$. Then

$$2\int_{-1}^1 d'(t)v(t)v'(t)dt = \int_{-1}^1 d'(t)(v^2(t))'dt = -\int_0^1 (v^2(t))'dt + \int_{-1}^0 (v^2(t))'dt = 0.$$

Thus, we see from (39) that

(40)

$$\begin{aligned} & \int_I d^{2-\alpha}(t)|u'|^2 dt - \frac{(1-\alpha)^2}{4} \int_I d^{-\alpha}(t)\Big(u - \Big(\int_I d^{-\alpha}(t)dt\Big)^{-1} \int_I d^{-\alpha}(t)udt\Big)^2 dt \\ & = \int_I d^{2-\alpha}(t)|u'|^2 dt - \frac{(1-\alpha)^2}{4} \int_I d^{-\alpha}(t)u^2 dx \\ & \quad + \frac{(1-\alpha)^2}{4}\Big(\int_I d^{-\alpha}(t)dt\Big)^{-1}\Big(\int_I d^{-\alpha}(t)udt\Big)^2 \\ & = \int_I d(t)|v'(t)|^2 dt + \frac{(1-\alpha)^2}{4}\Big(\int_I d^{-\alpha}(t)dt\Big)^{-1}\Big(\int_I d^{-\alpha}(t)udt\Big)^2 \\ & \ge 0, \end{aligned}$$

which implies that $L_{\alpha,2-\alpha}(I) = \frac{(1-\alpha)^2}{4}$. Moreover, (40) implies that if $L_{\alpha,2-\alpha}(I)$ is attained, then $u \equiv 0$ on $(-1,1)$.

Next, assume $\alpha < 1$ and $N \ge 2$. We first prove that $L_{\alpha,2-\alpha}(B^N(0,1)) < \frac{(1-\alpha)^2}{4}$ and $L_{\alpha,2-\alpha}(B^N(0,1))$ is achieved when the space dimension $N > \frac{3-\alpha}{1-\alpha}$. We take

$u(x) = x_N d^{-s}(x)$ with $s \in (0, \frac{1-\alpha}{2})$. Since

$$
\begin{aligned}
& d^{2-\alpha}|\nabla u|^2 \\
& = d^{2-\alpha}| - sd^{-s-1}(\nabla d)x_N + (0,0,\cdots,0,d^{-s})|^2 \\
& = d^{2-\alpha}\Big(s^2 d^{-2s-2}|\nabla d|^2 x_N^2 + d^{-2s} - 2sd^{-2s-1}x_N\frac{\partial d}{\partial x_N}\Big) \qquad (41)\\
& = s^2 d^{-\alpha-2s}x_N^2|\nabla d|^2 + d^{-\alpha-2s+2} - \frac{2s}{-\alpha-2s+2}x_N\frac{\partial}{\partial x_N}(d^{-\alpha-2s+2}),
\end{aligned}
$$

and

$$\int_0^1 r^a(1-r)^b dr = \frac{\Gamma(1+a)\Gamma(1+b)}{\Gamma(2+a+b)} \quad \text{for } a,b>-1,$$

it follows that

$$
\begin{aligned}
& \frac{\int_{B^N(0,1)} d^{2-\alpha}(x)|\nabla u|^2}{\int_{B^N(0,1)} d^{-\alpha}(x)u^2 dx} \\
& = \frac{\int_{B^N(0,1)} s^2 d^{-\alpha-2s}x_N^2 + d^{-\alpha-2s+2} - \frac{2s}{-\alpha-2s+2}x_N\frac{\partial}{\partial x_N}(d^{-\alpha-2s+2})dx}{\int_{B^N(0,1)} d^{-\alpha-2s}x_N^2 dx} \\
& = s^2 + \Big(1+\frac{2s}{-\alpha-2s+2}\Big)\frac{\int_{B^N(0,1)} d^{-\alpha-2s+2}dx}{\int_{B^N(0,1)} d^{-\alpha-2s}x_N^2 dx} \\
& = s^2 + \frac{N(\alpha-2)}{\alpha+2s-2}\frac{\int_{B^N(0,1)} d^{-\alpha-2s+2}dx}{\int_{B^N(0,1)} d^{-\alpha-2s}|x|^2 dx} \\
& = s^2 + \frac{N(\alpha-2)}{\alpha+2s-2}\frac{\int_0^1 r^{N-1}(1-r)^{-\alpha-2s+2}dr}{\int_0^1 r^{N+1}(1-r)^{-\alpha-2s}dr} \\
& = s^2 + \frac{N(\alpha-2)}{\alpha+2s-2}\frac{\Gamma(N)\Gamma(-\alpha-2s+3)}{\Gamma(N+2)\Gamma(-\alpha-2s+1)} \\
& = s^2 + \frac{(2-\alpha)(1-\alpha-2s)}{N+1} \equiv g(s).
\end{aligned}
$$

Then since for $N > \frac{3-\alpha}{1-\alpha}$,

$$\frac{dg(s)}{ds}\Big|_{s=\frac{1-\alpha}{2}} = 1-\alpha-\frac{2(2-\alpha)}{N+1} = \frac{N-3}{N+1}-\frac{N-1}{N+1}\alpha > 0, \qquad g\Big(\frac{1-\alpha}{2}\Big) = \frac{(1-\alpha)^2}{4},$$

$g(s) < \frac{(1-\alpha)^2}{4}$ for s less than and close to $\frac{1-\alpha}{2}$. This implies that $L_{\alpha,2-\alpha}(B^N(0,1)) < \frac{(1-\alpha)^2}{4}$ for $N > \frac{3-\alpha}{1-\alpha}$.

Lastly, we prove that for $\alpha < 1$ and the space dimension $N \le \frac{3-\alpha}{1-\alpha}$, $L_{\alpha,2-\alpha}(B_N(0,1)) = \frac{(1-\alpha)^2}{4}$ and $L_{\alpha,2-\alpha}(B_N(0,1))$ is not achieved. If $L_{\alpha,2-\alpha}(B_N(0,1))$ is achieved by u, we see that $u \in C^2(B^N(0,1))$. Moreover, since $C^2(B^N(0,1)) \cap W^{1,2}_{\alpha,2-\alpha}(B^N(0,1))$ is dense in $W^{1,2}_{\alpha,2-\alpha}(B^N(0,1))$, it suffices to prove that

$$\frac{\int_{B^N(0,1)} d^{2-\alpha}(x)|\nabla u|^2 dx}{\int_{B^N(0,1)} d^{-\alpha}(x)u^2 dx} \ge \frac{(1-\alpha)^2}{4} \qquad (42)$$

and the equality does not hold for $u \in \Big(C^2(B^N(0,1)) \cap W^{1,2}_{\alpha,2-\alpha}(B^N(0,1))\Big) \setminus \{0\}$ with $\int_{B^N(0,1)} d^{-\alpha}(x)u dx = 0$. As far as it makes no confusion, we abuse $d(x) =$

$d(r)$, where $r = |x|$. We define $\bar{u}(x) \equiv u(x) - \psi_0(r)$, where $r = |x|$ and $\psi_0(r) = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} u(r\theta)d\theta$. We note that $\psi_0, \bar{u} \in C^1(B^N(0,1))$,

$$\int_{S^{N-1}} \bar{u}(r\theta)d\theta = \int_{S^{N-1}} u(r\theta)d\theta - |S^{N-1}|\psi_0(r) = 0 \text{ for } r \in [0,1) \tag{43}$$

and

$$\begin{aligned} 0 &= \int_{B^N((0,1))} d^{-\alpha}(x)udx = \int_0^1 \int_{S^{N-1}} r^{N-1}d^{-\alpha}(r)ud\theta dr \\ &= |S^{N-1}|\int_0^1 d^{-\alpha}r^{N-1}\psi_0(r)dr. \end{aligned}$$

By (43),

$$\int_{S^{N-1}} \frac{\partial \bar{u}}{\partial r}(r\theta)d\theta = \frac{\partial}{\partial r}\int_{S^{N-1}} \bar{u}(r\theta)d\theta = 0 \text{ for } r \in [0,1).$$

Hence, for $r \in [0,1)$,

$$\int_{S^{N-1}} \Big(\frac{\partial u}{\partial r}\Big)^2 d\theta = |S^{N-1}|\Big(\psi_0'(r)\Big)^2 + \int_{S^{N-1}} \Big(\frac{\partial \bar{u}}{\partial r}\Big)^2 d\theta.$$

Then we deduce that

$$\begin{aligned} &\int_0^1 \int_{S^{N-1}} d^{2-\alpha}r^{N-1}\Big(\frac{\partial u}{\partial r}\Big)^2 d\theta dr \\ &\quad = |S^{N-1}|\int_0^1 d^{2-\alpha}r^{N-1}\Big(\psi_0'(r)\Big)^2 dr + \int_0^1 \int_{S^{N-1}} d^{2-\alpha}r^{N-1}\Big(\frac{\partial \bar{u}}{\partial r}\Big)^2 d\theta dr. \end{aligned}$$

From this and the fact that $\nabla_{S^{N-1}}u = \nabla_{S^{N-1}}\bar{u}$, we have

$$\begin{aligned} &\int_{B^N((0,1))} d^{2-\alpha}|\nabla u|^2 dx \\ &= \int_0^1 \int_{S^{N-1}} d^{2-\alpha}r^{N-1}\Big(\frac{\partial u}{\partial r}\Big)^2 + d^{2-\alpha}r^{N-3}|\nabla_{S^{N-1}}u|^2 d\theta dr \\ &= |S^{N-1}|\int_0^1 d^{2-\alpha}r^{N-1}\Big(\psi_0'(r)\Big)^2 dr \\ &\quad + \int_0^1 \int_{S^{N-1}} d^{2-\alpha}r^{N-1}\Big(\frac{\partial \bar{u}}{\partial r}\Big)^2 + d^{2-\alpha}r^{N-3}|\nabla_{S^{N-1}}\bar{u}|^2 d\theta dr \\ &= |S^{N-1}|\int_0^1 d^{2-\alpha}r^{N-1}\Big(\psi_0'(r)\Big)^2 dr + \int_{B^N((0,1))} d^{2-\alpha}|\nabla \bar{u}|^2 dx. \end{aligned} \tag{44}$$

Similarly, by (43),

$$\begin{aligned} \int_{B^N((0,1))} d^{-\alpha}u^2 d &= \int_0^1 \int_{S^{N-1}} d^{-\alpha}r^{N-1}u^2 dx \\ &= |S^{N-1}|\int_0^1 d^{-\alpha}r^{N-1}\psi_0^2 dr + \int_0^1 \int_{S^{N-1}} d^{-\alpha}r^{N-1}\bar{u}^2 d\theta dr \\ &= |S^{N-1}|\int_0^1 d^{-\alpha}r^{N-1}\psi_0^2 dr + \int_{B^N((0,1))} d^{-\alpha}\bar{u}^2 dx. \end{aligned} \tag{45}$$

By (44), (45) and the fact that for $r \in [0,1)$,

$$\int_{S^{N-1}} |\nabla_{S^{N-1}} \bar{u}|^2 d\theta \geq (N-1) \int_{S^{N-1}} \bar{u}^2 d\theta,$$

we get

(46)

$$\begin{aligned}
&\frac{\int_{B^N((0,1))} d^{2-\alpha} |\nabla u|^2 dx}{\int_{B^N((0,1))} d^{-\alpha} u^2 dx} \\
&= \frac{|S^{N-1}| \int_0^1 d^{2-\alpha} r^{N-1} \Big(\psi_0'(r)\Big)^2 dr + \int_{B^N((0,1))} d^{2-\alpha} |\nabla \bar{u}|^2 dx}{|S^{N-1}| \int_0^1 d^{-\alpha} r^{N-1} \psi_0^2 dr + \int_{B^N((0,1))} d^{-\alpha} \bar{u}^2 dx} \\
&\geq \frac{\int_{S^{N-1}} \int_0^1 d^{2-\alpha} r^{N-1} \Big(\psi_0'(r)\Big)^2 + d^{2-\alpha} r^{N-1} \Big(\frac{\partial \bar{u}}{\partial r}\Big)^2 + (N-1) d^{2-\alpha} r^{N-3} \bar{u}^2 dr d\theta}{|S^{N-1}| \int_0^1 d^{-\alpha} r^{N-1} \psi_0^2 dr + \int_{S^{N-1}} \int_0^1 d^{-\alpha} r^{N-1} \bar{u}^2 dr d\theta}
\end{aligned}$$

where $\int_0^1 d^{-\alpha} r^{N-1} \psi_0(r) dr = 0$ and $\int_{S^{N-1}} \bar{u}(r\theta) d\theta = 0$ for $r \in [0,1)$. We claim that for $\theta \in S^{N-1}$ and $\int_0^1 (1-r)^{-\alpha} r^{N-1} \psi_0 dr = 0$,

$$\begin{aligned}
&\int_0^1 (1-r)^{2-\alpha} r^{N-1} \Big(\frac{\partial \bar{u}}{\partial r}(r,\theta)\Big)^2 + (N-1)(1-r)^{2-\alpha} r^{N-3} (\bar{u}(r,\theta))^2 dr \\
&\qquad \geq \frac{(1-\alpha)^2}{4} \int_0^1 (1-r)^{-\alpha} r^{N-1} (\bar{u}(r,\theta))^2 dr, \qquad (47)\\
&\int_0^1 (1-r)^{2-\alpha} r^{N-1} (\psi_0'(r))^2 \geq \frac{(1-\alpha)^2}{4} \int_0^1 (1-r)^{-\alpha} r^{N-1} (\psi_0(r))^2 dr
\end{aligned}$$

which implies (42). We first show that for $f \in C^1([0,1))$,

$$\begin{aligned}
&\int_0^1 (1-r)^{2-\alpha} r^{N-1} (f'(r))^2 dr + (N-1) \int_0^1 (1-r)^{2-\alpha} r^{N-3} f^2(r) dr \\
&\geq \frac{(1-\alpha)^2}{4} \int_0^1 f^2(r) (1-r)^{-\alpha} r^{N-1} dr
\end{aligned}$$

when the integrals involved above are all finite. For $N = 2$, the second integral in (48) is not finite if $f(0) \neq 0$. Thus, we may assume that $f(0) = 0$ for $N = 2$. Define $g(r) = (1-r)^{\frac{1-\alpha}{2}} r^{-1} f(r)$. By Lemma 2.10 and the assumption $\alpha \geq \frac{N-3}{N-1}$, we see that

$$\begin{aligned}
&\int_0^1 (1-r)^{2-\alpha} r^{N-1} (f'(r))^2 dr + (N-1)\int_0^1 (1-r)^{2-\alpha} r^{N-3} f^2(r) dr \\
&= \int_0^1 r^{N+1}(1-r)^{-1} g^2 \Big[\frac{(1-\alpha)^2}{4} - (N-1) r^{-2}(1-r)^2 \\
&\quad + \Big(1-(1-\alpha)\frac{N-1}{2}\Big) r^{-1}(1-r)\Big] \\
&\quad + r^{N+1}(1-r)(g')^2 dr + (N-1)\int_0^1 (1-r) r^{N-1} g^2(r) dr \qquad (48)\\
&= \int_0^1 r^{N+1}(1-r)^{-1} g^2 \Big[\frac{(1-\alpha)^2}{4} + \Big(1-(1-\alpha)\frac{N-1}{2}\Big) r^{-1}(1-r)\Big] \\
&\quad + r^{N+1}(1-r)(g')^2 dr \\
&\geq \frac{(1-\alpha)^2}{4}\int_0^1 f^2(r)(1-r)^{-\alpha} r^{N-1} dr.
\end{aligned}$$

This implies (48). Next, we show that for $N \leq \frac{3-\alpha}{1-\alpha}$ and $f \in C^1([0,1))$ satisfying $\int_0^1 r^{N-1}(1-r)^{-\alpha} f dr = 0$,

$$\int_0^1 (1-r)^{2-\alpha} r^{N-1} (f')^2 dr \geq \frac{(1-\alpha)^2}{4}\int_0^1 (1-r)^{-\alpha} r^{N-1} f^2 dr \qquad (49)$$

when the integrals involved above are all finite. This is equivalent to show without the average condition $\int_0^1 r^{N-1}(1-r)^{-\alpha} f dr = 0$ that

$$\frac{\int_0^1 (1-r)^{2-\alpha} r^{N-1}(f')^2 dr}{\int_0^1 (1-r)^{-\alpha} r^{N-1}\big(f - (\int_0^1 s^{N-1}(1-s)^{-\alpha} ds)^{-1}\int_0^1 s^{N-1}(1-s)^{-\alpha} f ds\big)^2 dr} \geq \frac{(1-\alpha)^2}{4}.$$

Note that for any $\kappa \in \mathbb{R}$,

$$\begin{aligned}
&\frac{\int_0^1 (1-r)^{2-\alpha} r^{N-1}(f')^2 dr}{\int_0^1 (1-r)^{-\alpha} r^{N-1}\big(f - (\int_0^1 s^{N-1}(1-s)^{-\alpha} ds)^{-1}\int_0^1 s^{N-1}(1-s)^{-\alpha} f ds\big)^2 dr} \\
&= \frac{\int_0^1 (1-r)^{2-\alpha} r^{N-1}\big((f+\kappa)'\big)^2 dr}{\int_0^1 (1-r)^{-\alpha} r^{N-1}\big[f+\kappa - (\int_0^1 s^{N-1}(1-s)^{-\alpha} ds)^{-1}\int_0^1 s^{N-1}(1-s)^{-\alpha}(f+\kappa) ds\big]^2 dr}.
\end{aligned}$$

Thus it suffices to prove that for $f \in C^1([0,1)) \setminus \{0\}$ with $f(1-e^{1-N}) = 0$, $\int_0^1 (1-r)^{2-\alpha}(f')^2 dr < \infty$ and $\int_0^1 (1-r)^{-\alpha} f^2 dr < \infty$,

$$\frac{\int_0^1 (1-r)^{2-\alpha} r^{N-1}(f')^2 dr}{\int_0^1 (1-r)^{-\alpha} r^{N-1}\big[f - (\int_0^1 s^{N-1}(1-s)^{-\alpha} ds)^{-1}\int_0^1 s^{N-1}(1-s)^{-\alpha} f ds\big]^2 dr} \geq \frac{(1-\alpha)^2}{4}.$$

Defining $g(r) = (1-r)^{\frac{1-\alpha}{2}} f(r)$, we see from Lemma 2.10 that

$$\int_0^1 (1-r)^{2-\alpha} r^{N-1} (f')^2 dr$$
$$= \int_0^1 \frac{(1-\alpha)^2}{4} r^{N-1}(1-r)^{-1} g^2 - (1-\alpha)\frac{N-1}{2} r^{N-2} g^2 + r^{N-1}(1-r)(g')^2 dr.$$

Then it follows that

$$\begin{aligned}
&\frac{\int_0^1 (1-r)^{2-\alpha} r^{N-1}(f')^2 dr}{\int_0^1 (1-r)^{-\alpha} r^{N-1}\big[f - (\int_0^1 s^{N-1}(1-s)^{-\alpha} ds)^{-1} \int_0^1 s^{N-1}(1-s)^{-\alpha} f ds\big]^2 dr} \\
&= \frac{\int_0^1 (1-r)^{2-\alpha} r^{N-1}(f')^2 dr}{\int_0^1 (1-r)^{-\alpha} r^{N-1} f^2 - (\int_0^1 s^{N-1}(1-s)^{-\alpha} ds)^{-1}\big(\int_0^1 s^{N-1}(1-s)^{-\alpha} f ds\big)^2} \\
&\geq \frac{\int_0^1 \frac{(1-\alpha)^2}{4} r^{N-1}(1-r)^{-1} g^2 - (1-\alpha)\frac{N-1}{2} r^{N-2} g^2 + r^{N-1}(1-r)(g')^2 dr}{\int_0^1 r^{N-1}(1-r)^{-1} g^2(r) dr} \\
&= \frac{(1-\alpha)^2}{4} + \frac{\int_0^1 r^{N-1}(1-r)(g')^2 - (1-\alpha)\frac{N-1}{2} r^{N-2} g^2 dr}{\int_0^1 r^{N-1}(1-r)^{-1} g^2(r) dr}.
\end{aligned} \tag{50}$$

Denoting $a_N \equiv 1 - e^{\frac{3}{2}-N}$, we see that by integration by parts,

$$\int_0^{a_N} r^{N-2}\Big(\int_r^{a_N} \frac{1}{t^{N-1}(1-t)} dt\Big) dr = \frac{1}{N-1}\int_0^{a_N} \frac{1}{1-r} dr = \frac{N-\frac{3}{2}}{N-1} < 1. \tag{51}$$

Since $\frac{1}{s}(e^{(N-1)s} - 1) \leq N-1$ for $s < 0$, it follows that by a change of variables $\ln r = s$,

$$\begin{aligned}
&\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r \frac{1}{t^{N-1}(1-t)} dt\Big) dr \\
&= \frac{1}{N-1}\int_{a_N}^1 \frac{1-r^{N-1}}{r^{N-1}(1-r)} dr = \frac{1}{N-1}\sum_{i=1}^{N-1}\int_{a_N}^1 \frac{1}{r^i} dr \\
&\leq \frac{1}{N-1}\int_1^N \int_{a_N}^1 \frac{1}{r^m} dr dm = \frac{1}{N-1}\int_{a_N}^1 -\frac{1}{r^N \ln r} + \frac{1}{r\ln r} dr \\
&= \frac{1}{N-1}\int_{\ln a_N}^0 \frac{e^{(N-1)s}-1}{s e^{(N-1)s}} ds \leq \int_{\ln a_N}^0 \frac{1}{e^{(N-1)t}} dt \\
&= \frac{1}{N-1}(-1 + a_N^{-(N-1)}) = \frac{1}{N-1}\Big((1-e^{\frac{3}{2}-N})^{-(N-1)} - 1\Big).
\end{aligned} \tag{52}$$

Since

$$e^x > 2\sqrt{e}x \quad \text{for} \quad x \geq 2 \quad \text{and} \quad (1-x)^{-\frac{1}{x}} \leq e^2 \quad \text{for} \quad 0 < x < \frac{1}{2},$$

we deduce that

$$(1-e^{\frac{3}{2}-N})^{1-N} = (1-e^{\frac{3}{2}-N})^{-\frac{1}{e^{\frac{3}{2}-N}}\frac{N-1}{e^{N-\frac{3}{2}}}} \leq e^{2\frac{N-1}{e^{N-\frac{3}{2}}}} < e \text{ for } N \geq 3.$$

From this and (52), for $N \geq 3$,

$$\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r \frac{1}{t^{N-1}(1-t)} dt\Big) dr < 1.$$

When $N=2$, since $e-\sqrt{e}>1$,

$$\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r \frac{1}{t^{N-1}(1-t)}dt\Big)dr=\int_{a_N}^1 \frac{1}{r}dr=-\ln(1-e^{-\frac{1}{2}})<1.$$

Thus, we have for $N\geq 2$,

$$\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r \frac{1}{t^{N-1}(1-t)}dt\Big)dr<1. \tag{53}$$

Then it follows from (51) and (53) that

(54)

$$\begin{aligned}
\int_0^1 r^{N-2}g^2(r)dr &= \int_0^{a_N} r^{N-2}\Big(\int_r^{a_N} g'(t)dt\Big)^2dr+\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r g'(t)dt\Big)^2dr\\
&\leq \int_0^{a_N} r^{N-2}\Big(\int_r^{a_N} t^{N-1}(1-t)(g'(t))^2dt\Big)\Big(\int_r^{a_N}\frac{1}{t^{N-1}(1-t)}dt\Big)dr\\
&\quad+\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r t^{N-1}(1-t)(g'(t))^2dt\Big)\Big(\int_{a_N}^r\frac{1}{t^{N-1}(1-t)}dt\Big)dr\\
&\leq \int_0^{a_N} t^{N-1}(1-t)(g'(t))^2dt\int_0^{a_N} r^{N-2}\Big(\int_r^{a_N}\frac{1}{t^{N-1}(1-t)}dt\Big)dr\\
&\quad+\int_{a_N}^1 t^{N-1}(1-t)(g'(t))^2dt\int_{a_N}^1 r^{N-2}\Big(\int_{a_N}^r\frac{1}{t^{N-1}(1-t)}dt\Big)dr\\
&\leq \int_0^{a_N} t^{N-1}(1-t)(g'(t))^2dt+\int_{a_N}^1 t^{N-1}(1-t)(g'(t))^2dt\\
&= \int_0^1 t^{N-1}(1-t)(g'(t))^2dt.
\end{aligned}$$

Then, from this, (50) and the assumption $\alpha\geq\frac{N-3}{N-1}$, we see that

$$\begin{aligned}
&\frac{\int_0^1(1-r)^{2-\alpha}r^{N-1}(f')^2dr}{\int_0^1(1-r)^{-\alpha}r^{N-1}\big[f-(\int_0^1 s^{N-1}(1-s)^{-\alpha}ds)^{-1}\int_0^1 s^{N-1}(1-s)^{-\alpha}fds\big]^2dr}\\
&\geq \frac{(1-\alpha)^2}{4}+\frac{\int_0^1 r^{N-1}(1-r)(g')^2-(1-\alpha)\frac{N-1}{2}r^{N-2}g^2dr}{\int_0^1 r^{N-1}(1-r)^{-1}g^2(r)dr}\\
&\geq \frac{(1-\alpha)^2}{4}+\frac{\int_0^1\big(1-(1-\alpha)\frac{N-1}{2}\big)r^{N-1}(1-r)(g')^2dr}{\int_0^1 r^{N-1}(1-r)^{-1}g^2(r)dr}\geq\frac{(1-\alpha)^2}{4},
\end{aligned} \tag{55}$$

which implies (49). Thus, combining (46), (48) and (49), we get (42).

We note that if $L_{\alpha,2-\alpha}(B_N(0,1))$ is achieved, the equalities in the estimations (48) and (55) instead of inequalities should hold. In (48), the equality holds only when $g'\equiv 0$. This means that $f(r)=cr(1-r)^{\frac{\alpha-1}{2}}$ for some $c\in\mathbb{R}$; in this case, $c=0$ since $\int_0^1(1-r)^{-\alpha}r^{N-1}(r(1-r)^{\frac{\alpha-1}{2}})^2dr=\infty$. Next, in (55), the equality holds only when $\int_0^1 s^{N-1}(1-s)^{-\alpha}fds=0$ and $g'\equiv 0$ because the estimations (51), (53) and the equality in (54) should hold. Since $g'\equiv 0$, $f=c_1(1-s)^{\frac{\alpha-1}{2}}$ for some $c_1\in\mathbb{R}$; in this case, $c_1=0$ since $\int_0^1 s^{N-1}(1-s)^{-\alpha}fds=0$. These imply that $L_{\alpha,2-\alpha}(B_N(0,1))$ is not achieved by any element in $W^{1,2}_{\alpha,2-\alpha}(B^N(0,1))\setminus\{0\}$. $\square$

4.4. **Proof of Theorem 1.6.** We assume that $\alpha < 1$, $N \geq \frac{3-\alpha}{1-\alpha}$ and $(a_1, \cdots, a_{N-1}) \in (0,1]^{N-1}$ with $a_1+\cdots+a_{N-1} < N-1$. We define diffeomorphisms $\Psi : B^N(0,1) \to E(a_1, \cdots, a_{N-1}, 1)$ by

$$\Psi(y) = (a_1 y_1, a_2 y_2, \cdots, a_{N-1} y_{N-1}, y_N)$$

and $\Phi : E(a_1, \cdots, a_{N-1}, 1) \to B^N(0,1)$ by

$$\Phi(x) = \Big(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \cdots, \frac{x_{N-1}}{a_{N-1}}, x_N\Big).$$

For $r(t) \equiv t\Phi(x) + (1-t)\frac{\Phi(x)}{|\Phi(x)|}$ and $x \in \Omega = E(a_1, \cdots, a_{N-1}, 1)$, we see that

$$d(x) \leq \int_0^1 \Big|\frac{d}{dt}\Psi(r(t))\Big| dt = \frac{|x|}{|\Phi(x)|}(1 - |\Phi(x)|). \tag{56}$$

Define $u(x) = x_N d^{-s}(x)$, where $s \in (0, \frac{1-\alpha}{2})$ and $\alpha + 2s > 0$. From (41) and (56), we see that

(57)

$$\begin{aligned}
\frac{\int_\Omega d^{2-\alpha}(x)|\nabla u|^2}{\int_\Omega d^{-\alpha}(x)u^2 dx} &= \frac{\int_\Omega s^2 d^{-\alpha-2s}x_N^2 + d^{-\alpha-2s+2} - \frac{2s}{-\alpha-2s+2}x_N\frac{\partial}{\partial x_N}(d^{-\alpha-2s+2})dx}{\int_\Omega d^{-\alpha-2s}x_N^2 dx} \\
&= s^2 + \Big(1 + \frac{2s}{-\alpha-2s+2}\Big)\frac{\int_\Omega d^{-\alpha-2s+2}dx}{\int_\Omega d^{-\alpha-2s}x_N^2 dx} \\
&\leq s^2 + \Big(\frac{\alpha-2}{\alpha+2s-2}\Big)\frac{\int_\Omega \Big(\frac{|x|}{|\Phi(x)|}(1-|\Phi(x)|)\Big)^{-\alpha-2s+2}dx}{\int_\Omega \Big(\frac{|x|}{|\Phi(x)|}(1-|\Phi(x)|)\Big)^{-\alpha-2s}x_N^2 dx}.
\end{aligned}$$

Since $\frac{|\Psi(y)|}{|y|} \leq 1$ and

$$\begin{aligned}
\Big(\frac{|\Psi(y)|}{|y|}\Big)^{-\alpha-2s+2} &= \Big(1 + \frac{\big(\sum_{i=1}^{N-1}(a_i^2-1)y_i^2\big)}{|y|^2}\Big)^{\frac{-\alpha-2s+2}{2}} \\
&\leq 1 + \Big(\frac{-\alpha-2s+2}{2}\Big)\frac{\big(\sum_{i=1}^{N-1}(a_i^2-1)y_i^2\big)}{|y|^2},
\end{aligned}$$

we see by a change of variables $x = \Psi(y)$ that

$$\begin{aligned}
&\frac{\int_\Omega \Big(\frac{|x|}{|\Phi(x)|}(1-|\Phi(x)|)\Big)^{-\alpha-2s+2}dx}{\int_\Omega \Big(\frac{|x|}{|\Phi(x)|}(1-|\Phi(x)|)\Big)^{-\alpha-2s}x_N^2 dx} = \frac{\int_{B^N(0,1)} \Big(\frac{|\Psi(y)|}{|y|}(1-|y|)\Big)^{-\alpha-2s+2}dy}{\int_{B^N(0,1)} \Big(\frac{|\Psi(y)|}{|y|}(1-|y|)\Big)^{-\alpha-2s}y_N^2 dy} \\
&\leq \frac{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s+2}\Big[1 + \Big(\frac{-\alpha-2s+2}{2}\Big)\frac{\big(\sum_{i=1}^{N-1}(a_i^2-1)y_i^2\big)}{|y|^2}\Big]dy}{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s}y_N^2 dy} \\
&= \frac{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s+2}\Big[1 + N^{-1}\Big(\frac{-\alpha-2s+2}{2}\Big)\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big)\Big]dy}{N^{-1}\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s}|y|^2 dy} \\
&= \Big[N + \Big(\frac{-\alpha-2s+2}{2}\Big)\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big)\Big]\frac{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s+2}dy}{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s}|y|^2 dy}.
\end{aligned} \tag{58}$$

Then, by (57) and (58) and the fact that

$$\frac{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s+2}dy}{\int_{B^N(0,1)}(1-|y|)^{-\alpha-2s}|y|^2dy} = \frac{\int_0^1(1-r)^{-\alpha-2s+2}r^{N-1}dr}{\int_0^1(1-r)^{-\alpha-2s}r^{N+1}dr}$$
$$= \frac{\Gamma(N)\Gamma(-\alpha-2s+3)}{\Gamma(N+2)\Gamma(-\alpha-2s+1)} = \frac{(-\alpha-2s+2)(-\alpha-2s+1)}{N(N+1)},$$

we see that for $s \in \Big(\max\{0,-\frac{\alpha}{2}\},\frac{1-\alpha}{2}\Big)$,

$$\frac{\int_\Omega d^{2-\alpha}(x)|\nabla u|^2}{\int_\Omega d^{-\alpha}(x)u^2dx}$$
$$\leq s^2+\Big(\frac{\alpha-2}{\alpha+2s-2}\Big)\Big[N+\Big(\frac{-\alpha-2s+2}{2}\Big)\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big)\Big]\frac{(-\alpha-2s+2)(-\alpha-2s+1)}{N(N+1)}$$
$$= s^2+\frac{(\alpha-2)(\alpha+2s-1)}{N+1}\Big[1+\Big(\frac{-\alpha-2s+2}{2N}\Big)\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big)\Big] \equiv h(s).$$

Note that $h\Big(\frac{1-\alpha}{2}\Big) = \Big(\frac{1-\alpha}{2}\Big)^2$ and

$$h'(s) = 2s + \frac{2(\alpha-2)}{N+1}\Big[1+\Big(\frac{-\alpha-2s+2}{2N}\Big)\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big)\Big]$$
$$-\frac{(\alpha-2)(\alpha+2s-1)}{(N+1)N}\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big).$$

Thus, for $N \geq \frac{3-\alpha}{1-\alpha}$,

$$h'\Big(\frac{1-\alpha}{2}\Big) = 1-\alpha+\frac{2(\alpha-2)}{N+1}\Big[1+\frac{1}{2N}\Big(\sum_{i=1}^{N-1}(a_i^2-1)\Big)\Big] > 1-\alpha+\frac{2(\alpha-2)}{N+1} \geq 0.$$

Taking s strictly less than and sufficiently close to $\frac{1-\alpha}{2}$, we see that $L_{\alpha,2-\alpha}(\Omega) < \Big(\frac{1-\alpha}{2}\Big)^2$ when $N \geq \frac{3-\alpha}{1-\alpha}$. This completes the proof. □

4.5. **Proof of Theorem 1.7.** Define $u = x_N d^{-s}(x)$ for s strictly less than and close to $\frac{1-\alpha}{2}$. Note that $d(x) = \begin{cases} a-|x| & \text{if } x \in \Omega_1, \\ |x|-1 & \text{if } x \in \Omega_2, \end{cases}$ where $\Omega_1 \equiv \{x \in \Omega \mid |x| > \frac{a+1}{2}\}$

and $\Omega_2 \equiv \{x \in \Omega \mid |x| \leq \frac{a+1}{2}\}$. By (41), it follows that

(59)

$$\begin{aligned}
&\int_\Omega d^{2-\alpha}|\nabla u|^2 dx \\
&= \int_\Omega s^2 d^{-\alpha-2s} x_N^2 |\nabla d|^2 + d^{-\alpha-2s+2} - 2s d^{-\alpha-2s+1} x_N \frac{\partial d}{\partial x_N} dx \\
&= \int_{\Omega_1} s^2 (a-|x|)^{-\alpha-2s} x_N^2 + (a-|x|)^{-\alpha-2s+2} + 2s(a-|x|)^{-\alpha-2s+1} x_N^2 |x|^{-1} dx \\
&\quad + \int_{\Omega_2} s^2 (|x|-1)^{-\alpha-2s} x_N^2 + (|x|-1)^{-\alpha-2s+2} - 2s(|x|-1)^{-\alpha-2s+1} x_N^2 |x|^{-1} dx \\
&= |S^{N-1}| \Big[\int_{\frac{a+1}{2}}^a \Big(s^2 (a-r)^{-\alpha-2s} \frac{r^2}{N} + (a-r)^{-\alpha-2s+2} + 2s(a-r)^{-\alpha-2s+1} \frac{r}{N} \Big) r^{N-1} dr \\
&\quad + \int_1^{\frac{a+1}{2}} \Big(s^2 (r-1)^{-\alpha-2s} \frac{r^2}{N} + (r-1)^{-\alpha-2s+2} - 2s(r-1)^{-\alpha-2s+1} \frac{r}{N} \Big) r^{N-1} dr \Big] \\
&= |S^{N-1}| \frac{1}{N} \Big[\int_{\frac{a+1}{2}}^a s^2 r^{N+1} (a-r)^{-\alpha-2s} \\
&\quad + N r^{N-1} (a-r)^{-\alpha-2s+2} + 2s r^N (a-r)^{-\alpha-2s+1} dr \\
&\quad + \int_1^{\frac{a+1}{2}} s^2 r^{N+1} (r-1)^{-\alpha-2s} + N r^{N-1} (r-1)^{-\alpha-2s+2} - 2s r^N (r-1)^{-\alpha-2s+1} dr \Big]
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega d^{-\alpha}(x) u^2 dx &= \int_{\Omega_1} (a-|x|)^{-\alpha-2s} x_N^2 dx + \int_{\Omega_2} (|x|-1)^{-\alpha-2s} x_N^2 dx \\
&= |S^{N-1}| \frac{1}{N} \Big[\int_{\frac{a+1}{2}}^a r^{N+1} (a-r)^{-\alpha-2s} dr + \int_1^{\frac{a+1}{2}} r^{N+1} (r-1)^{-\alpha-2s} dr \Big].
\end{aligned} \tag{60}$$

Since

$$\begin{aligned}
&\int_{\frac{a+1}{2}}^a r^N (a-r)^{-\alpha-2s+1} dr \\
&\qquad = \frac{1}{N+1} \Big[- \Big(\frac{a+1}{2}\Big)^{N+1} \Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1} \\
&\qquad\qquad + (-\alpha-2s+1) \int_{\frac{a+1}{2}}^a r^{N+1} (a-r)^{-\alpha-2s} dr \Big]
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{a+1}{2}}^a r^{N-1} (a-r)^{-\alpha-2s+2} dr \\
&= \frac{1}{N} \Big[- \Big(\frac{a+1}{2}\Big)^N \Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2} + (-\alpha-2s+2) \int_{\frac{a+1}{2}}^a r^N (a-r)^{-\alpha-2s+1} dr \Big] \\
&= \frac{1}{N} \Big[- \Big(\frac{a+1}{2}\Big)^N \Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2} \\
&\quad + (-\alpha-2s+2) \frac{1}{N+1} \Big(- \Big(\frac{a+1}{2}\Big)^{N+1} \Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}
\end{aligned}$$

$$+(-\alpha-2s+1)\int_{\frac{a+1}{2}}^{a} r^{N+1}(a-r)^{-\alpha-2s}dr\Big)\Big]$$
$$=\frac{1}{N(N+1)}\Big[-(N+1)\Big(\frac{a+1}{2}\Big)^{N}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2}$$
$$-(-\alpha-2s+2)\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}$$
$$+(-\alpha-2s+2)(-\alpha-2s+1)\int_{\frac{a+1}{2}}^{a} r^{N+1}(a-r)^{-\alpha-2s}dr\Big],$$

we deduce that

(61)
$$\int_{\frac{a+1}{2}}^{a} s^2r^{N+1}(a-r)^{-\alpha-2s}+Nr^{N-1}(a-r)^{-\alpha-2s+2}+2sr^{N}(a-r)^{-\alpha-2s+1}dr$$
$$=\frac{1}{N+1}\Big[-(N+1)\Big(\frac{a+1}{2}\Big)^{N}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2}$$
$$-(-\alpha-2s+2)\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}$$
$$-2s\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}\Big]$$
$$+\Big[s^2+\frac{1}{N+1}(-\alpha-2s+2)(-\alpha-2s+1)$$
$$+\frac{2s}{N+1}(-\alpha-2s+1)\Big]\int_{\frac{a+1}{2}}^{a} r^{N+1}(a-r)^{-\alpha-2s}dr$$
$$=\frac{1}{N+1}\Big[-(N+1)\Big(\frac{a+1}{2}\Big)^{N}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2}$$
$$+(\alpha-2)\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}\Big]$$
$$+\Big[s^2+\frac{1}{N+1}(-\alpha+2)(-\alpha-2s+1)\Big]\int_{\frac{a+1}{2}}^{a} r^{N+1}(a-r)^{-\alpha-2s}dr.$$

Similarly, since

$$\int_{1}^{\frac{a+1}{2}} r^{N}(r-1)^{-\alpha-2s+1}dr$$
$$=\frac{1}{N+1}\Big[\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}$$
$$-(-\alpha-2s+1)\int_{1}^{\frac{a+1}{2}} r^{N+1}(r-1)^{-\alpha-2s}dr\Big]$$

and

$$\int_{1}^{\frac{a+1}{2}} r^{N-1}(r-1)^{-\alpha-2s+2}dr$$
$$=\frac{1}{N}\Big[\Big(\frac{a+1}{2}\Big)^{N}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2}-(-\alpha-2s+2)\int_{1}^{\frac{a+1}{2}} r^{N}(r-1)^{-\alpha-2s+1}dr\Big]$$

$$
\begin{aligned}
&= \frac{1}{N}\Big[\Big(\frac{a+1}{2}\Big)^N\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2} \\
&- (-\alpha-2s+2)\frac{1}{N+1}\Big(\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1} \\
&- (-\alpha-2s+1)\int_1^{\frac{a+1}{2}} r^{N+1}(r-1)^{-\alpha-2s}dr\Big)\Big] \\
&= \frac{1}{N(N+1)}\Big[(N+1)\Big(\frac{a+1}{2}\Big)^N\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2} \\
&\qquad - (-\alpha-2s+2)\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1} \\
&\qquad + (-\alpha-2s+2)(-\alpha-2s+1)\int_1^{\frac{a+1}{2}} r^{N+1}(r-1)^{-\alpha-2s}dr\Big],
\end{aligned}
$$

we see that

(62)

$$
\begin{aligned}
&\int_1^{\frac{a+1}{2}} s^2 r^{N+1}(r-1)^{-\alpha-2s} + Nr^{N-1}(r-1)^{-\alpha-2s+2} - 2sr^N(r-1)^{-\alpha-2s+1}dr \\
&= \frac{1}{N+1}\Big[(N+1)\Big(\frac{a+1}{2}\Big)^N\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2} \\
&\qquad - (-\alpha-2s+2)\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1} \\
&\qquad - 2s\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}\Big] \\
&\quad + \Big[s^2 + \frac{1}{N+1}(-\alpha-2s+2)(-\alpha-2s+1) \\
&\qquad + \frac{1}{N+1}2s(-\alpha-2s+1)\Big]\int_1^{\frac{a+1}{2}} r^{N+1}(r-1)^{-\alpha-2s}dr \\
&= \frac{1}{N+1}\Big[(N+1)\Big(\frac{a+1}{2}\Big)^N\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+2} \\
&\qquad + (\alpha-2)\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}\Big] \\
&\quad + \Big[s^2 + \frac{1}{N+1}(-\alpha+2)(-\alpha-2s+1)\Big]\int_1^{\frac{a+1}{2}} r^{N+1}(r-1)^{-\alpha-2s}dr.
\end{aligned}
$$

Then, by (59)-(62) and the identity

$$
\begin{aligned}
&\int_0^1 r^{-\alpha-2s}\Big[\Big(r+\frac{2}{a-1}\Big)^{N+1} + \Big(\frac{2a}{a-1}-r\Big)^{N+1}\Big]dr \\
&= (-\alpha-2s+1)^{-1}\Big[2\Big(\frac{a+1}{a-1}\Big)^{N+1} \\
&\quad - (N+1)\int_0^1 r^{-\alpha-2s+1}\Big(\Big(r+\frac{2}{a-1}\Big)^N - \Big(\frac{2a}{a-1}-r\Big)^N\Big)dr\Big],
\end{aligned}
$$

we obtain that

$$\begin{aligned}\frac{\int_\Omega d^{2-\alpha}|\nabla u|^2dx}{\int_\Omega d^{-\alpha}(x)u^2dx} &= s^2+\frac{1}{N+1}(-\alpha+2)(-\alpha-2s+1)\\ &+2\frac{\alpha-2}{N+1}\Big(\frac{a+1}{2}\Big)^{N+1}\Big(\frac{a-1}{2}\Big)^{-\alpha-2s+1}\Big[\int_1^{\frac{a+1}{2}} r^{N+1}(r-1)^{-\alpha-2s}dr\\ &+\int_{\frac{a+1}{2}}^a r^{N+1}(a-r)^{-\alpha-2s}dr\Big]^{-1}\end{aligned}$$

$$\begin{aligned}&= s^2+\frac{1}{N+1}(-\alpha+2)(-\alpha-2s+1)\\ &+2\frac{\alpha-2}{N+1}\Big(\frac{a+1}{a-1}\Big)^{N+1}\Big[\int_0^1 r^{-\alpha-2s}\Big[\Big(r+\frac{2}{a-1}\Big)^{N+1}+\Big(\frac{2a}{a-1}-r\Big)^{N+1}\Big]dr\Big]^{-1}\\ &= s^2+\frac{1}{N+1}(-\alpha+2)(-\alpha-2s+1)\\ &+2\frac{\alpha-2}{N+1}\Big(\frac{a+1}{a-1}\Big)^{N+1}(-\alpha-2s+1)\\ &\times\Big[2\Big(\frac{a+1}{a-1}\Big)^{N+1}-(N+1)\int_0^1 r^{-\alpha-2s+1}\Big(\Big(r+\frac{2}{a-1}\Big)^N\\ &-\Big(\frac{2a}{a-1}-r\Big)^N\Big)dr\Big]^{-1}\equiv f(s).\end{aligned}$$

Define $h(s)\equiv\int_0^1 r^{-\alpha-2s+1}\Big[\Big(r+\frac{2}{a-1}\Big)^N-\Big(\frac{2a}{a-1}-r\Big)^N\Big]dr$. Note that

$$\Big|h'\Big(\frac{1-\alpha}{2}\Big)\Big|=2\Big|\int_0^1\ln r\Big[\Big(r+\frac{2}{a-1}\Big)^N-\Big(\frac{2a}{a-1}-r\Big)^N\Big]dr\Big|<C_a$$

and

$$\begin{aligned}&2\Big(\frac{a+1}{a-1}\Big)^{N+1}-(N+1)\int_0^1\Big(r+\frac{2}{a-1}\Big)^N-\Big(\frac{2a}{a-1}-r\Big)^N dr\\ &\quad=\Big(\frac{2}{a-1}\Big)^{N+1}(1+a^{N+1}),\end{aligned}$$

where $C_a>0$ is a constant depending on $a>0$. Then, since $f\Big(\frac{1-\alpha}{2}\Big)=\Big(\frac{1-\alpha}{2}\Big)^2$ and

$$\begin{aligned}f'\Big(\frac{1-\alpha}{2}\Big)&=1-\alpha+\frac{1}{N+1}(-\alpha+2)(-2)\\ &\quad+2\frac{\alpha-2}{N+1}\Big(\frac{a+1}{a-1}\Big)^{N+1}(-2)\Big(\frac{a-1}{2}\Big)^{N+1}(1+a^{N+1})^{-1}\\ &=1-\alpha+\frac{2}{N+1}(\alpha-2)-\frac{\alpha-2}{N+1}(a+1)^{N+1}2^{-N+1}(1+a^{N+1})^{-1}>0\end{aligned}$$

if $\alpha<\dfrac{1-\frac{2}{N+1}\Big(2-(a+1)^{N+1}2^{-N+1}(1+a^{N+1})^{-1}\Big)}{1-\frac{1}{N+1}\Big(2-(a+1)^{N+1}2^{-N+1}(1+a^{N+1})^{-1}\Big)}$. This implies that $f(s)<\frac{(1-\alpha)^2}{4}$ for s strictly less than and sufficiently close to $\frac{1-\alpha}{2}$. This completes the proof. □

4.6. **Proof of Theorem 1.8.**

Lemma 4.1. *Let $\Omega = B^N(0,a) \setminus \overline{B^N(0,1)}$, where $N \geq 2$ and $a > 1$. Then there exists $\alpha_1 \in (0,1)$ such that for $\alpha \in (\alpha_1, 1)$ and $u \in C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)$ satisfying $\int_{S^{N-1}} u(r\theta)d\theta = 0$ with any $r \in (1,a)$,*

$$\int_\Omega d^{2-\alpha}|\nabla u|^2 dx \geq \frac{(1-\alpha)^2}{4}\int_\Omega d^{-\alpha}u^2 dx. \tag{63}$$

Moreover, for $\alpha \in (\alpha_1, 1)$, the best constant $\frac{(1-\alpha)^2}{4}$ in (63) is not achieved.

Proof. Denote $\Omega_1 = B^N(0,a) \setminus \overline{B^N(0,\frac{1+a}{2})}$ and $\Omega_2 = B^N(0,\frac{1+a}{2}) \setminus \overline{B^N(0,1)}$. For $u \in C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)$, we define $v = d^{\frac{1-\alpha}{2}}(x)u$. Then, we see that

$$\int_\Omega d^{-1}(x)v^2 dx = \int_\Omega d^{-\alpha}(x)u^2 dx < \infty \quad \text{and} \quad \int_{S^{N-1}} v(r\theta)d\theta = 0 \text{ for } r \in (1,a).$$

Then, since

$$\begin{aligned}
&d^{2-\alpha}(x)|\nabla u|^2 \\
&\quad= d^{2-\alpha}(x)\Big|\frac{\alpha-1}{2}d^{\frac{\alpha-1}{2}-1}(\nabla d)v + d^{\frac{\alpha-1}{2}}(x)\nabla v\Big|^2 \\
&\quad= d^{2-\alpha}(x)\Big(\frac{(1-\alpha)^2}{4}d^{\alpha-3}(x)|\nabla d|^2v^2 + d^{\alpha-1}(x)|\nabla v|^2 - (1-\alpha)d^{\alpha-2}(x)v\nabla d\cdot\nabla v\Big) \\
&\quad= \frac{(1-\alpha)^2}{4}d^{-1}(x)|\nabla d|^2 v^2 + d|\nabla v|^2 - \frac{1-\alpha}{2}\nabla d\cdot\nabla(v^2),
\end{aligned}$$

it follows that $\int_\Omega d|\nabla v|^2 dx < \infty$ and

$$\begin{aligned}
&\int_\Omega d^{2-\alpha}(x)|\nabla u|^2 - \frac{(1-\alpha)^2}{4}d^{-\alpha}(x)u^2 dx \\
&= \int_\Omega d|\nabla v|^2 - \frac{1-\alpha}{2}\nabla d\cdot\nabla(v^2)dx \\
&= \int_{\Omega_1} d|\nabla v|^2 + \frac{1-\alpha}{2}(\Delta d)v^2 dx + \int_{\Omega_2} d|\nabla v|^2 \\
&\quad + \frac{1-\alpha}{2}(\Delta d)v^2 dx - (1-\alpha)\int_{\partial\Omega_1\cap\partial\Omega_2} v^2 d\sigma \\
&= \int_{\Omega_1} d|\nabla v|^2 + \frac{1-\alpha}{2}\frac{1-N}{|x|}v^2 dx + \int_{\Omega_2} d|\nabla v|^2 \\
&\quad + \frac{1-\alpha}{2}\frac{N-1}{|x|}v^2 dx - (1-\alpha)\int_{\partial\Omega_1\cap\partial\Omega_2} v^2 d\sigma.
\end{aligned} \tag{64}$$

By the trace inequality, we see that for some $C_1 > 0$,

$$\int_{\partial\Omega_1\cap\partial\Omega_2} v^2 d\sigma \leq C_1 \int_{\Omega_3} |\nabla v|^2 + v^2 dx, \tag{65}$$

where $\Omega_3 = B^N(0,\frac{3a+1}{4}) \setminus \overline{B^N(0,\frac{1+a}{2})}$. For small $\delta > 0$, define $\psi \in C^\infty([1,a])$ such that $\psi \in [0,1]$, $\psi(r) = 1$ if $r \in [a-\delta/2, a]$ and $\psi(r) = 0$ if $r \in [1, a-\delta]$. Denote

$v = v_1 + v_2$, where $v_1 = v\psi(d(x))$ and $v_2 = v\Big(1 - \psi(d(x))\Big)$. Since $\int_{S^{N-1}} v(r\theta)d\theta = 0$ for $r \in (1, a)$, it holds that

$$\begin{aligned}\int_{\Omega_1} d|\nabla v|^2 dx &= \int_{\frac{1+a}{2}}^{a}\int_{S^{N-1}} (a-r)r^{N-1}\Big|\frac{\partial v}{\partial r}\Big|^2 + (a-r)r^{N-3}|\nabla_{S^{N-1}} v|^2 d\sigma dr \\ &\geq \int_{\frac{1+a}{2}}^{a}\int_{S^{N-1}} (a-r)r^{N-1}\Big|\frac{\partial v}{\partial r}\Big|^2 + (N-1)(a-r)r^{N-3}v^2 d\sigma dr.\end{aligned} \tag{66}$$

Then, from (66) and the fact that

$$\int_{\Omega_1} |\nabla v_2|^2 dx \geq \int_{\frac{1+a}{2}}^{a}\int_{S^{N-1}} r^{N-1}\Big|\frac{\partial v_2}{\partial r}\Big|^2 + (N-1)r^{N-3}v_2^2 d\sigma dr \geq C_2^{-1}\int_{\Omega_1} v_2^2 dx,$$

we have

$$\begin{aligned}\int_{\Omega_1} v_2^2 dx \leq C_2\int_{\Omega_1}|\nabla v_2|^2 dx &\leq 2C_2\int_{\Omega_1}|\nabla v|^2\Big(1-\psi(d(x))\Big)^2 + v^2\Big(\psi'(d(x))\Big)^2 dx \\ &\leq C_3\int_{\Omega_1} d|\nabla v|^2 dx,\end{aligned} \tag{67}$$

where C_2, C_3 are constants. On the other hand, by (5), (66) and Lemma 2.3,

$$\begin{aligned}\int_{\Omega_1} v_1^2 dx &\leq 2\int_{\partial\Omega}\int_{a-\delta}^{a} v_1^2(t,\sigma)dtd\sigma \leq 32\int_{\partial\Omega}\int_{a-\delta}^{a}(a-t)\Big(\frac{\partial}{\partial t}v_1(t,\sigma)\Big)^2 dtd\sigma \\ &\leq 64\int_{\Omega_1} d|\nabla v_1|^2 dx \leq 128\int_{\Omega_1} d|\nabla v|^2\Big(\psi(d(x))\Big)^2 + d\Big(\psi'(d(x))\Big)^2 v^2 dx \\ &\leq C_4\int_{\Omega_1} d|\nabla v|^2,\end{aligned} \tag{68}$$

where C_4 is a constant. Then, by (67) and (68), we see that

$$\int_{\Omega_1} v^2 dx \leq 2\int_{\Omega_1} v_1^2 dx + 2\int_{\Omega_1} v_2^2 dx \leq 2(C_3 + C_4)\int_{\Omega_1} d|\nabla v|^2 dx.$$

Then, from this and (65), there exists $\alpha_1 = \alpha_1(N, \Omega) \in (0,1)$ such that for $\alpha \in (\alpha_1, 1)$

$$\begin{aligned}&\int_{\Omega} d^{2-\alpha}(x)|\nabla u|^2 - \frac{(1-\alpha)^2}{4}d^{-\alpha}(x)u^2 dx \\ &= \int_{\Omega_1} d|\nabla v|^2 - \frac{1-\alpha}{2|x|}(N-1)v^2 dx \\ &\quad + \int_{\Omega_2} d|\nabla v|^2 + \frac{1-\alpha}{2|x|}(N-1)v^2 dx - (1-\alpha)\int_{\partial\Omega_1\cap\partial\Omega_2} v^2 d\sigma \\ &\geq \int_{\Omega_1} d|\nabla v|^2 - \frac{1-\alpha}{2|x|}(N-1)v^2 dx \\ &\quad + \int_{\Omega_2} d|\nabla v|^2 + \frac{1-\alpha}{2|x|}(N-1)v^2 dx - C_1(1-\alpha)\int_{\Omega_3}|\nabla v|^2 + v^2 dx \\ &\geq \frac{1}{2}\int_{\Omega_1} d|\nabla v|^2 dx + \int_{\Omega_2} d|\nabla v|^2 + \frac{1-\alpha}{2|x|}(N-1)v^2 dx.\end{aligned}$$

This proves the inequality (63). If the equality in (63) holds, we see from the estimation above that $v(x) = 0$ for $x \in \Omega_2$ and v is constant on Ω_1. Since $v = d^{\frac{1-\alpha}{2}} u$, we conclude that $u \equiv 0$ in Ω. This completes the proof. $\square$

Lemma 4.2. *Let $N \geq 2$ and $a > 1$. Then there exists $\alpha_2 = \alpha_2(N, a) \in (0, 1)$ such that for $\alpha \in (\alpha_2, 1)$,*

$$\int_1^a (d(r))^{2-\alpha} r^{N-1} \Big(f'(r)\Big)^2 dr \geq \frac{(1-\alpha)^2}{4} \int_1^a (d(r))^{-\alpha} r^{N-1} f^2 dr \tag{69}$$

for any radially symmetric $f(x) = f(r) \in C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)$ satisfying

$$\int_1^a (d(r))^{-\alpha} r^{N-1} f(r) dr = 0,$$

where $d(r) = \begin{cases} r-1 & \text{if } r \in [1, \frac{1+a}{2}], \\ a-r & \text{if } r \in [\frac{1+a}{2}, a]. \end{cases}$ Moreover, for $\alpha \in (\alpha_2, 1)$, the best constant $\frac{(1-\alpha)^2}{4}$ in (69) is not achieved.

Proof. We note that for all $c \in \mathbb{R}$,

$$\begin{aligned}
&\inf\left\{ \frac{\int_1^a d^{2-\alpha} r^{N-1} \Big(f'(r)\Big)^2 dr}{\int_1^a d^{-\alpha} r^{N-1} f^2 dr} \;\Big|\; f(r) \right.\\
&\qquad \left. \in \Big(C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)\Big) \setminus \{0\}, \int_1^a d^{-\alpha} r^{N-1} f(r) dr = 0 \right\}\\
&= \inf\left\{ \frac{\int_1^a d^{2-\alpha} r^{N-1} \Big(f'(r)\Big)^2 dr}{\int_1^a d^{-\alpha} r^{N-1} \Big(f - (\int_1^a d^{-\alpha} r^{N-1} dr)^{-1} \int_1^a d^{-\alpha} r^{N-1} f dr\Big)^2 dr} \;\Big|\; f(r) \right.\\
&\qquad \left. \in \Big(C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)\Big) \setminus \{0\} \right\}\\
&= \inf\left\{ \frac{\int_1^a d^{2-\alpha} r^{N-1} \Big((f(r)+c)'\Big)^2 dr}{\int_1^a \frac{r^{N-1}}{d^\alpha} \Big(f + c - (\int_1^a \frac{r^{N-1}}{d^\alpha} dr)^{-1} \int_1^a \frac{r^{N-1}}{d^\alpha} (f+c) dr\Big)^2 dr} \;\Big|\; f(r) \right.\\
&\qquad \left. \in \Big(C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)\Big) \setminus \{0\} \right\}.
\end{aligned}$$

Then it suffices to prove that for $f \in C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)$ and $f\Big(\frac{1+a}{2}\Big) = 0$,

$$\frac{\int_1^a d^{2-\alpha} r^{N-1} (f')^2 dr}{\int_1^a d^{-\alpha} r^{N-1} f^2 dr} \geq \frac{(1-\alpha)^2}{4}.$$

Defining $f(r) = d^{\frac{\alpha-1}{2}}(r) h(r)$, we see that

(70)

$$\begin{aligned}
d^{2-\alpha} r^{N-1} (f')^2 &= d^{2-\alpha} r^{N-1} \Big(\frac{\alpha-1}{2} d^{\frac{\alpha-3}{2}} d' h + d^{\frac{\alpha-1}{2}} h'\Big)^2\\
&= d^{2-\alpha} r^{N-1} \left[\Big(\frac{\alpha-1}{2}\Big)^2 d^{\alpha-3} h^2 + d^{\alpha-1} (h')^2 + (\alpha-1) d^{\alpha-2} h d' h'\right]\\
&= \Big(\frac{\alpha-1}{2}\Big)^2 r^{N-1} d^{-1} h^2 + d r^{N-1} (h')^2 + \frac{\alpha-1}{2} r^{N-1} d' (h^2)'.
\end{aligned}$$

By (70), we obtain that

(71)
$$\begin{aligned}&\int_{\frac{1+a}{2}}^{a}(a-r)^{2-\alpha}r^{N-1}(f')^2dr\\&=\int_{\frac{1+a}{2}}^{a}\Big(\frac{\alpha-1}{2}\Big)^2r^{N-1}(a-r)^{-1}h^2+(a-r)r^{N-1}(h')^2-\frac{\alpha-1}{2}r^{N-1}(h^2)'dr\\&\geq\int_{\frac{1+a}{2}}^{a}\Big(\frac{\alpha-1}{2}\Big)^2r^{N-1}(a-r)^{-1}h^2+(a-r)r^{N-1}(h')^2+\frac{\alpha-1}{2}(N-1)r^{N-2}h^2dr.\end{aligned}$$

Since

$$h(r)=\int_{\frac{1+a}{2}}^{r}h'(t)dt\leq\Big(\int_{\frac{1+a}{2}}^{r}t^{N-1}(a-t)(h'(t))^2dt\Big)^{\frac12}\Big(\int_{\frac{1+a}{2}}^{r}\frac{1}{t^{N-1}(a-t)}dt\Big)^{\frac12},$$

we have

$$\begin{aligned}\int_{\frac{1+a}{2}}^{a}r^{N-2}h^2(r)dr&\leq\int_{\frac{1+a}{2}}^{a}r^{N-1}(a-r)(h')^2dr\int_{\frac{1+a}{2}}^{a}\int_{\frac{1+a}{2}}^{r}\frac{r^{N-2}}{t^{N-1}(a-t)}dtdr\\&\leq c_{N,a}\int_{\frac{1+a}{2}}^{a}r^{N-1}(a-r)(h')^2dr,\end{aligned}$$

where $c_{N,a}>0$ is a constant depending only on N and a. Thus, we see that

$$\begin{aligned}&\int_{\frac{1+a}{2}}^{a}(a-r)r^{N-1}(h')^2+\frac{\alpha-1}{2}(N-1)r^{N-2}h^2dr\\&\geq\Big(\frac{1}{c_{N,a}}-\frac{1-\alpha}{2}(N-1)\Big)\int_{\frac{1+a}{2}}^{a}r^{N-2}h^2dr.\end{aligned}$$

From this and (71), there exists $\alpha_2=\alpha_2(N,a)\in(0,1)$ and $C_4>0$ such that for $\alpha\in(\alpha_2,1)$,

$$\begin{aligned}&\int_{\frac{1+a}{2}}^{a}(1-r)^{2-\alpha}r^{N-1}(f')^2dr\\&\geq\Big(\frac{\alpha-1}{2}\Big)^2\int_{\frac{1+a}{2}}^{a}r^{N-1}(1-r)^{-1}h^2dr+C_4\int_{\frac{1+a}{2}}^{a}r^{N-2}h^2dr\\&=\Big(\frac{\alpha-1}{2}\Big)^2\int_{\frac{1+a}{2}}^{a}r^{N-1}(1-r)^{-\alpha}f^2dr+C_4\int_{\frac{1+a}{2}}^{a}r^{N-2}h^2dr.\end{aligned}\tag{72}$$

On the other hand, we see from (70) and the fact $f(\frac{1+a}{2})=0$ that

(73)
$$\begin{aligned}&\int_{1}^{\frac{1+a}{2}}(r-a)^{2-\alpha}r^{N-1}(f')^2dr\\&=\int_{1}^{\frac{1+a}{2}}\Big(\frac{\alpha-1}{2}\Big)^2r^{N-1}(r-a)^{-1}h^2+(r-a)r^{N-1}(h')^2+\frac{\alpha-1}{2}r^{N-1}(h^2)'dr\\&\geq\int_{1}^{\frac{1+a}{2}}\Big(\frac{\alpha-1}{2}\Big)^2r^{N-1}(r-a)^{-1}h^2+(r-a)r^{N-1}(h')^2+\frac{1-\alpha}{2}(N-1)r^{N-2}h^2dr.\end{aligned}$$

Thus, combining (72) and (73), we get (69). If the equality holds in (69), we see from (72) and (73) that $h\equiv0$; thus $f\equiv0$. This completes the proof. □

Now we are ready to complete the proof of Theorem 1.8. As far as it makes no confusion, we abuse $d(x) = d(r)$, where $r = |x|$. Let $u \in C^1(\Omega) \cap W^{1,2}_{\alpha,\beta}(\Omega)$ such that $\int_\Omega d^{-\alpha}(x) u dx = 0$. We define $\bar{u}(x) \equiv u(x) - \psi_0(r)$, where $r = |x|$ and $\psi_0(r) = \frac{1}{|S^{N-1}|}\int_{S^{N-1}} u(r\theta) d\theta$. Then, following the arguments in Theorem 1.5 (see (46) above), we deduce that

$$\frac{\int_\Omega d^{2-\alpha}|\nabla u|^2 dx}{\int_\Omega d^{-\alpha} u^2 dx} = \frac{|S^{N-1}|\int_a^1 d^{2-\alpha} r^{N-1}\Big(\psi_0'(r)\Big)^2 dr + \int_\Omega d^{2-\alpha}|\nabla \bar{u}|^2 dx}{|S^{N-1}|\int_a^1 d^{-\alpha} r^{N-1}\psi_0^2 dr + \int_\Omega d^{-\alpha}\bar{u}^2 dx},$$

where $\int_1^a d^{-\alpha} r^{N-1}\psi_0(r) dr = 0$ and $\int_{S^{N-1}} \bar{u}(r\theta) d\theta = 0$ for $r \in (1, a)$. Lastly, applying Lemma 4.1 and Lemma 4.2, we conclude to the claim. □

4.7. **Proof of Theorem 1.9.** Define

$$v(x) = \begin{cases} \lambda_1 u_1\Big(\frac{1}{s_1}(O_1)^{-1}(x - t_1)\Big) & \text{if } x \in t_1 + s_1 O_1 D_+^{u_1}, \\ -\lambda_2 u_2\Big(\frac{1}{s_2}(O_2)^{-1}(x - t_2)\Big) & \text{if } x \in t_2 + s_2 O_2 D_+^{u_2}, \\ 0 & \text{if } x \in \Omega \setminus (t_1 + s_1 O_1 D_+^{u_1}) \cup (t_2 + s_2 O_2 D_+^{u_2}), \end{cases}$$

where we take $\lambda_1, \lambda_2 > 0$ so that $\int_\Omega d^{-\alpha}(x) v(x) dx = 0$. Then, we see that $v \in W^{1,2}_{\alpha,2-\alpha}(\Omega) \setminus \{0\}$ and $\int_\Omega d^{-\alpha}(x) v(x) dx = 0$. Moreover, we see that

$$\begin{aligned} &\frac{\int_\Omega d_\Omega^{2-\alpha}(x)|\nabla v|^2 dx}{\int_\Omega d_\Omega^{-\alpha}(x) v^2 dx} \\ &= \frac{(\lambda_1)^2 (s_1)^{N-\alpha}\int_{D_1} d_{D_1}^{2-\alpha}(x)|\nabla(u_1)_+|^2 dx + (\lambda_2)^2 (s_2)^{N-\alpha}\int_{D_2} d_{D_2}^{2-\alpha}(x)|\nabla(u_2)_+|^2 dx}{(\lambda_1)^2 (s_1)^{N-\alpha}\int_{D_1} d_{D_1}^{-\alpha}(x)|(u_1)_+|^2 dx + (\lambda_2)^2 (s_2)^{N-\alpha}\int_{D_2} d_{D_2}^{-\alpha}(x)|(u_2)_+|^2 dx} \\ &< \frac{(1-\alpha)^2}{4}, \end{aligned}$$

where $d_D(x) = \mathrm{dist}(x, \mathbb{R}^N \setminus D)$ for any domain $D \subset \mathbb{R}^N$. This proves Theorem 1.9. □

Appendix A

In this last section, we study the relationship between $W_0^{1,2}(\Omega)$ and $W^{1,2}_{0,\alpha,\beta}(\Omega)$ and whether $H_{\alpha,\beta}$ is attained or not.

Proposition A.1. *Let $(\alpha, \beta) \in \mathbb{R}^2$. Then we have*

$$W^{1,2}_{0,\alpha,\beta}(\Omega) \begin{cases} \supsetneq W_0^{1,2}(\Omega) & \text{if } (\alpha,\beta) \in \{(a,b) \mid a \le 2, b > 0\}, \\ = W_0^{1,2}(\Omega) & \text{if } (\alpha,\beta) \in \{(a,b) \mid a \le 2, b = 0\}, \\ \subsetneq W_0^{1,2}(\Omega) & \text{if } (\alpha,\beta) \in \{(a,b) \mid b < 0 \text{ or } b = 0, a > 2\}. \end{cases}$$

If $(\alpha, \beta) \in \{(a, b) \mid a > 2, b > 0\}$, $W^{1,2}_{0,\alpha,\beta}(\Omega) \setminus W_0^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega) \setminus W^{1,2}_{0,\alpha,\beta}(\Omega)$ are non-empty.

Proof. We note by Hardy's inequality, for $u \in W_0^{1,2}(\Omega)$ and $\alpha \le 2$,

$$\int_\Omega u^2 d^{-\alpha} dx = \int_\Omega u^2 d^{-2} d^{2-\alpha} dx \le C_1 \int_\Omega |\nabla u|^2 dx, \tag{74}$$

where $C_1 > 0$ is a constant. Denote

$$A_1 = \{(a,b) \mid a \le 2, b > 0\}, \quad A_2 = \{(a,b) \mid a \le 2, b = 0\}$$

and

$$A_3 = \{(a,b) \mid b<0 \text{ or } b=0, a>2\}.$$

First, assume that $(\alpha,\beta)\in A_1$. Then we see that for a constant $C_2>0$, $\int_\Omega d^\beta|\nabla u|^2dx \le C_2\int_\Omega |\nabla u|^2dx$. From this and (74), it follows that $W^{1,2}_{0,\alpha,\beta}(\Omega)\supset W^{1,2}_0(\Omega)$. On the other hand, if $\alpha=2$ and $\beta>0$, $d\sin(d^{-\frac12})\in W^{1,2}_{0,2,\beta}(\Omega)$, but $d\sin(d^{-\frac12})\notin W^{1,2}_0(\Omega)$. If $\alpha<2$ and $\beta>0$, for $m\in(\max\{\frac{\alpha-1}{2},\frac{1-\beta}{2},\frac14\},\frac12)$, $d^m(x)\in W^{1,2}_{0,\alpha,\beta}(\Omega)$, but $d^m(x)\notin W^{1,2}_0(\Omega)$. Thus, we deduce that $W^{1,2}_{0,\alpha,\beta}(\Omega)\supsetneq W^{1,2}_0(\Omega)$.

Assume that $(\alpha,\beta)\in A_2$. Then, by (74), we see that $W^{1,2}_{0,\alpha,\beta}(\Omega)=W^{1,2}_0(\Omega)$.

Assume that $(\alpha,\beta)\in A_3$. Since $\int_\Omega|\nabla u|^2dx\le C_3\int_\Omega d^\beta|\nabla u|^2dx$ for some constant $C_3>0$, we see that $W^{1,2}_{0,\alpha,\beta}(\Omega)\subset W^{1,2}_0(\Omega)$. On the other hand, for $m\in(\frac12,\max\{\frac{\alpha-1}{2},\frac{1-\beta}{2}\})$, $d^m\in W^{1,2}_0(\Omega)$, but $d^m\notin W^{1,2}_{0,\alpha,\beta}(\Omega)$, which implies that $W^{1,2}_{0,\alpha,\beta}(\Omega)\subsetneq W^{1,2}_0(\Omega)$.

If $(\alpha,\beta)\in\{(a,b)\mid a>2,b>0\}$, for $m\in(\frac12,\frac{\alpha-1}{2})$, $d^m(x)\in W^{1,2}_0(\Omega)$, but $d^m(x)\notin W^{1,2}_{0,\alpha,\beta}(\Omega)$. On the other hand, for $n>\frac{\alpha-1}{2}$, $d^n\sin(d^{-n+\frac12})\in W^{1,2}_{0,\alpha,\beta}(\Omega)$, but $d^n\sin(d^{-n+\frac12})\notin W^{1,2}_0(\Omega)$. In fact, note that, since $\int_1^\infty\frac{\cos^2(t)}{t}dt\ge \frac12\int_\pi^\infty\frac{\sin^2(t)}{t}dt$, we have

$$\int_0^1\frac{\cos^2(t^{-n+\frac12})}{t}dt=\Big(n-\frac12\Big)^{-1}\int_1^\infty\frac{\cos^2(t)}{t}dt=\infty.$$

Then, from this and the facts that $\beta>0$ and $n>\frac{\alpha-1}{2}>\frac12$, we see that

$$\begin{aligned}
&\int_\Omega\Big|\nabla\Big(d^n\sin(d^{\frac12-n})\Big)\Big|^2dx\\
&=\int_\Omega\Big|nd^{n-1}\sin(d^{\frac12-n})\nabla d+\Big(\frac12-n\Big)d^{-\frac12}\cos(d^{-n+\frac12})\nabla d\Big|^2dx\\
&=\int_\Omega n^2d^{2n-2}\sin^2(d^{\frac12-n})+\Big(\frac12-n\Big)^2d^{-1}\cos^2(d^{\frac12-n})\\
&\quad+2n\Big(\frac12-n\Big)d^{n-\frac32}\sin(d^{\frac12-n})\cos(d^{\frac12-n})dx\\
&=+\infty,
\end{aligned}$$

$$\int_\Omega d^\beta\Big|\nabla\Big(d^n\sin(d^{\frac12-n})\Big)\Big|^2dx<\infty \quad\text{and}\quad \int_\Omega d^{2n-\alpha}\sin^2(d^{\frac12-n})dx\le\int_\Omega d^{2n-\alpha}dx<\infty.$$

This completes the proof. □

Proposition A.2. *$H_{\alpha,\beta}(\Omega)$ is achieved by an element $v_{\alpha,\beta}\in W^{1,2}_{0,\alpha,\beta}(\Omega)$ if $(\alpha,\beta)\in\{(a,b)\mid a+b<2,b<1\}$.*

Proof. We assume $(\alpha,\beta)\in\{(a,b)\mid a+b<2,b<1\}$. Note that, by Proposition 2.4, $H_{\alpha,\beta}(\Omega)>0$. Let $\{u_m\}_{m=1}^\infty\subset C_0^\infty(\Omega)$ be a minimizing sequence of $H_{\alpha,\beta}(\Omega)$ with

$$\int_\Omega d^{-\alpha}(x)u_m^2dx=1 \quad\text{and}\quad \lim_{m\to\infty}\int_\Omega d^\beta(x)|\nabla u_m|^2dx=H_{\alpha,\beta}(\Omega).$$

Taking a subsequence if it is necessary, we may assume that as $m\to\infty$, u_m converges weakly to u in $W^{1,2}_{0,\alpha,\beta}(\Omega)$.

Then, by Lemma 2.8, the same argument in the proof of Theorem 1.2 and the fact that $H_{\alpha,\beta}(\Omega) > 0$ for $(\alpha,\beta) \in \{(a,b) \mid a+b<2, b<1\}$, we deduce that $u \not\equiv 0$.

Now, it holds that $u \not\equiv 0$. Let $w_m = u_m - u$. Then w_m converges weakly to 0 in $W^{1,2}_{0,\alpha,\beta}(\Omega)$ as $m \to \infty$. Then, we see that

(75)
$$H_{\alpha,\beta}(\Omega) + o(1) = \int_\Omega d^\beta(x)|\nabla u_m|^2 dx = \int_\Omega d^\beta(x)(|\nabla u|^2 + 2\nabla u \cdot \nabla w_m + |\nabla w_m|^2)dx$$
$$= \int_\Omega d^\beta(x)|\nabla u|^2 dx + \int_\Omega d^\beta(x)|\nabla w_m|^2 dx + o(1)$$

and

$$1 = \int_\Omega d^{-\alpha}(x)u_m^2 dx = \int_\Omega d^{-\alpha}(x)u^2 dx + \int_\Omega d^{-\alpha}(x)w_m^2 dx + o(1). \tag{76}$$

Then, since $u \not\equiv 0$, it follows that

$$\limsup_{m\to\infty} \int_\Omega d^{-\alpha}(x)w_m^2 dx < 1. \tag{77}$$

Thus, by (75)-(77) and the fact that $w_m \in W^{1,2}_{0,\alpha,\beta}$, we obtain

$$H_{\alpha,\beta}(\Omega) \le \frac{\int_\Omega d^\beta(x)|\nabla u|^2 dx}{\int_\Omega d^{-\alpha}(x)u^2 dx} = \frac{H_{\alpha,\beta}(\Omega) - \int_\Omega d^\beta(x)|\nabla w_m|^2 dx + o(1)}{1 - \int_\Omega d^{-\alpha}(x)w_m^2 dx + o(1)}$$
$$\le \frac{H_{\alpha,\beta}(\Omega) - H_{\alpha,\beta}(\Omega)\int_\Omega d^{-\alpha}(x)w_m^2 dx + o(1)}{1 - \int_\Omega d^{-\alpha}(x)w_m^2 dx + o(1)} = H_{\alpha,\beta}(\Omega) + o(1)$$

as $m \to \infty$. This implies that u attains $H_{\alpha,\beta}(\Omega)$. □

We recall

$$H_{\alpha,2-\alpha}(\Omega) = \inf_{u \in W^{1,2}_{0,\alpha,2-\alpha}(\Omega)\setminus\{0\}} \frac{\int_\Omega d^{2-\alpha}(x)|\nabla u|^2 dx}{\int_\Omega |u(x)|^2 d^{-\alpha}(x)dx}. \tag{78}$$

Lemma A.3. *Let $\alpha \in (1,2]$. For small $\delta > 0$, it holds that*

$$\int_{\Omega_\delta} d^{2-\alpha}|\nabla u|^2 dx \ge \Big(\frac{\alpha-1}{2}\Big)^2 \int_{\Omega_\delta} d^{-\alpha}u^2 dx \textit{ for all } u \in W^{1,2}_{0,\alpha,2-\alpha}(\Omega), \tag{79}$$

where $\Omega_\delta = \{x \in \Omega \mid d(x) < \delta\}$ and $d(x) = dist(x, \mathbb{R}^N \setminus \Omega)$.

Proof. By the density of $C_0^\infty(\Omega)$ in $W^{1,2}_{0,\alpha,2-\alpha}(\Omega)$, it suffices to prove (79) for functions $u \in C_0^\infty(\Omega)$. Since

$$\int_{\Omega_\delta} d^{2-\alpha}|\nabla u|^2 dx \ge \int_{\partial\Omega}\int_0^\delta t^{2-\alpha}\Big(\frac{\partial u}{\partial t}\Big)^2(1-Ct)dtd\sigma$$

and

$$\int_{\Omega_\delta} d^{-\alpha}u^2 dx \le \int_{\partial\Omega}\int_0^\delta t^{-\alpha}u^2(t)(1+Ct)dtd\sigma,$$

where $C > 0$ is a constant, we see that

$$\begin{aligned}&\int_{\Omega_\delta} d^{2-\alpha}|\nabla u|^2 - \Big(\frac{\alpha-1}{2}\Big)^2 d^{-\alpha}u^2 dx\\ &\geq \int_{\partial\Omega}\int_0^\delta t^{2-\alpha}\Big(\frac{\partial u}{\partial t}\Big)^2 - \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}u^2(t)\\ &\qquad - Ct\bigg(t^{2-\alpha}\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}u^2(t)\bigg)dtd\sigma.\end{aligned}$$

Then, by a scaling, it suffices to prove that for $v \in C^\infty([0,1])$ such that v vanishes in a neighborhood of the origin,

$$\int_0^1 t^{2-\alpha}(v')^2 - \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}v^2(t) - t\bigg(t^{2-\alpha}(v')^2 + \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}v^2(t)\bigg)dt \geq 0.$$

Defining $v = t^{\frac{\alpha-1}{2}}w$, we see that $v' = \frac{\alpha-1}{2}t^{\frac{\alpha-3}{2}}w + t^{\frac{\alpha-1}{2}}w'$,

$$\begin{aligned}&\int_0^1 t^{2-\alpha}(v')^2 - \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}v^2(t)dt\\ &= \int_0^1 t^{2-\alpha}\bigg(\Big(\frac{\alpha-1}{2}\Big)^2 t^{\alpha-3}w^2 + t^{\alpha-1}(w')^2 + (\alpha-1)t^{\alpha-2}ww'\bigg)\\ &\quad - \Big(\frac{\alpha-1}{2}\Big)^2 t^{-1}w^2 dt\\ &= \int_0^1 t(w')^2 + (\alpha-1)ww'dt = \int_0^1 t(w')^2 dt + \frac{\alpha-1}{2}w^2(1)\end{aligned} \tag{80}$$

and

$$\begin{aligned}&\int_0^1 t\bigg(t^{2-\alpha}(v')^2 + \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}v^2(t)\bigg)dt\\ &= \int_0^1 t^2(w')^2 + \frac{(\alpha-1)^2}{2}w^2 + (\alpha-1)tww'dt\\ &= \int_0^1 t^2(w')^2 + \frac{(\alpha-1)(\alpha-2)}{2}w^2 dt + \frac{\alpha-1}{2}w^2(1).\end{aligned} \tag{81}$$

Thus, by (80), (81) and the assumption $\alpha \in (1,2]$, we see that

$$\begin{aligned}&\int_0^1 t^{2-\alpha}(v')^2 - \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}v(t) - t\bigg(t^{2-\alpha}(v')^2 + \Big(\frac{\alpha-1}{2}\Big)^2 t^{-\alpha}v(t)\bigg)dt\\ &= \int_0^1 t(1-t)(w')^2 - \frac{(\alpha-1)(\alpha-2)}{2}w^2 dt \geq 0.\end{aligned}$$

□

For the inequality in Lemma A.3 on Ahlfors regular domains, refer to [16].

Proposition A.4. *Assume $\alpha > 1$. Then $H_{\alpha,2-\alpha}(\Omega) \leq \frac{(1-\alpha)^2}{4}$, and if $H_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$, then $H_{\alpha,2-\alpha}(\Omega)$ is achieved. Moreover, for $\alpha \in (1,2]$, if $H_{\alpha,2-\alpha}(\Omega)$ is achieved, then $H_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$.*

Proof. We first prove that $H_{\alpha,2-\alpha}(\Omega) \leq \frac{(1-\alpha)^2}{4}$. Define

$$f(t) = \begin{cases}(t/\delta)^m & \text{if } t \in (0,\delta),\\ 2 - t/\delta & \text{if } t \in (\delta, 2\delta)\end{cases}$$

and

$$u(x) = \begin{cases} f(d(x)) & \text{if } x \in \Omega_{2\delta}, \\ 0 & \text{if } x \in \Omega \setminus \Omega_{2\delta}, \end{cases}$$

where $m > \frac{\alpha-1}{2}$. Then it holds that $u \in W^{1,2}_{0,\alpha,2-\alpha}(\Omega)$. Moreover, we see that $\nabla u = f'(d(x))\nabla d$,

$$\begin{aligned}\int_\Omega d^{2-\alpha}(x)|\nabla u|^2 dx &= \int_{\Omega_{2\delta}} d^{2-\alpha}(x)\Big(f'(d(x))\Big)^2 dx \\ &\le (1 + C\delta)|\partial\Omega| \int_0^{2\delta} t^{2-\alpha}\Big(f'(t)\Big)^2 dt\end{aligned}$$

and

$$\begin{aligned}\int_\Omega d^{-\alpha}(x)u^2 dx &= \int_{\Omega_{2\delta}} d^{-\alpha}(x) f^2(d(x))dx \\ &\ge (1 - C\delta)|\partial\Omega| \int_0^{2\delta} t^{-\alpha} f^2(t)dt,\end{aligned}$$

where $C > 0$ is a constant independent of δ. Since

$$\int_0^{2\delta} t^{2-\alpha}\Big(f'(t)\Big)^2 dt = m^2\delta^{-2m}\int_0^\delta t^{2m-\alpha}dt + \delta^{-2}\int_\delta^{2\delta} t^{2-\alpha}dt = \frac{m^2\delta^{-\alpha+1}}{2m-\alpha+1} + C_1(\delta).$$

$$\int_0^{2\delta} t^{-\alpha} f^2(t)dt = \delta^{-2m}\int_0^\delta t^{2m-\alpha}dt + \int_\delta^{2\delta} t^{-\alpha}\Big(2 - \frac{t}{\delta}\Big)^2 dt = \frac{\delta^{-\alpha+1}}{2m-\alpha+1} + C_2(\delta),$$

where $C_1(\delta), C_2(\delta) > 0$ are constants independent of m, we see that

$$\begin{aligned}H_{\alpha,2-\alpha}(\Omega) &\le \frac{\int_\Omega d^{2-\alpha}(x)|\nabla u|^2 dx}{\int_\Omega d^{-\alpha}(x)u^2 dx} \\ &= \Big(\frac{1 + C\delta}{1 - C\delta}\Big)\frac{m^2 + (2m-\alpha+1)\delta^{\alpha-1}C_1(\delta)}{1 + (2m-\alpha+1)\delta^{\alpha-1}C_2(\delta)} \to \frac{1+C\delta}{1-C\delta}\Big(\frac{\alpha-1}{2}\Big)^2\end{aligned}$$

as $m \downarrow \frac{\alpha-1}{2}$, which implies that $H_{\alpha,2-\alpha}(\Omega) \le \Big(\frac{\alpha-1}{2}\Big)^2$.

Next, by the same arguments in the proof of Theorem 1.3, we can prove that $H_{\alpha,2-\alpha}(\Omega)$ is achieved if $H_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$ (See also [4]). Indeed, let $\{u_m\}_{m=1}^\infty \subset C_0^\infty(\Omega)$ be a minimizing sequence of $H_{\alpha,2-\alpha}(\Omega)$ with

$$\int_\Omega d^{-\alpha}(x)u_m^2 dx = 1 \text{ and } \lim_{m\to\infty}\int_\Omega d^{2-\alpha}(x)|\nabla u_m|^2 dx = H_{\alpha,2-\alpha}(\Omega).$$

Taking a subsequence, if it is necessary, we may assume that as $m \to \infty$, u_m converges weakly to some u in $W^{1,2}_{0,\alpha,2-\alpha}(\Omega)$. If $u \equiv 0$, by Lemma 2.8, we deduce that u_m concentrates near $\partial\Omega$. Then by Lemma 2.3, the argument in Theorem 1.3, we see that $H_{\alpha,2-\alpha}(\Omega) \ge \frac{(1-\alpha)^2}{4}$, which is a contradiction to the assumption that $H_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$. Now, it follows from $u \not\equiv 0$ and the argument in (75) below, that u attains $H_{\alpha,2-\alpha}(\Omega)$.

Finally, we prove that for $\alpha \in (1, 2]$, if $H_{\alpha,2-\alpha}(\Omega)$ is achieved, $H_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$. Suppose that $H_{\alpha,2-\alpha}(\Omega)$ is achieved and $H_{\alpha,2-\alpha}(\Omega) = \frac{(1-\alpha)^2}{4}$. Then there exists a nonnegative function $u \in W^{1,2}_{0,\alpha,2-\alpha}(\Omega)$ such that

$$-\mathrm{div}(d^{2-\alpha}\nabla u) = \frac{(1-\alpha)^2}{4}d^{-\alpha}u \text{ in } \Omega.$$

By the maximum principle, $u > 0$ in Ω. Based on the idea of [3], we define

$$Y(t) = t^{\frac{\alpha-1}{2}} X^s(t) \quad \text{and} \quad w_s(x) = Y(d(x)),$$

where $s > \frac{1}{2}$ and $X(t) = \begin{cases} (1-\ln t)^{-1} & \text{if } 0 < t \le 1, \\ 1 & \text{if } 1 < t. \end{cases}$ We take sufficiently small $\delta \in (0,1)$ such that $d \in C^2(\Omega_\delta)$ and the statement of Lemma A.3 holds in Ω_δ. Then we see that $w_s \in W^{1,2}_{0,\alpha,2-\alpha}(\Omega)$ for $s > \frac{1}{2}$. Moreover, since

$$\begin{aligned} Y'(t) &= \frac{\alpha-1}{2} t^{\frac{\alpha-3}{2}} X^s(t) \\ &\quad + s t^{\frac{\alpha-3}{2}} X^{s+1}(t) = t^{\frac{\alpha-3}{2}} \Big(\frac{\alpha-1}{2} X^s(t) + s X^{s+1}(t) \Big) \quad \text{for} \quad t \in (0,1) \end{aligned}$$

and for $x \in \Omega_\delta$,

$$\begin{aligned} &\mathrm{div}(d^{2-\alpha}\nabla w_s) = \mathrm{div}\Big(d^{2-\alpha} Y'(d(x)) \nabla d\Big) \\ &= \mathrm{div}\Big(d^{\frac{-\alpha+1}{2}} (\nabla d) \Big(\frac{\alpha-1}{2} X^s(d) + s X^{s+1}(d)\Big)\Big) \\ &= -\frac{\alpha-1}{2} d^{\frac{-\alpha-1}{2}} \Big(\frac{\alpha-1}{2} X^s(d) + s X^{s+1}(d)\Big) + d^{\frac{-\alpha+1}{2}} (\Delta d) \Big(\frac{\alpha-1}{2} X^s(d) + s X^{s+1}(d)\Big) \\ &\quad + d^{\frac{-\alpha-1}{2}} \Big(s\frac{\alpha-1}{2} X^{s+1}(d) + s(s+1) X^{s+2}(d)\Big) \\ &= -\Big(\frac{\alpha-1}{2}\Big)^2 d^{\frac{-\alpha-1}{2}} X^s(d) + s(s+1) d^{\frac{-\alpha-1}{2}} X^{s+2}(d) \\ &\quad + d^{\frac{-\alpha+1}{2}} (\Delta d) \Big(\frac{\alpha-1}{2} X^s(d) + s X^{s+1}(d)\Big), \end{aligned}$$

it follows that for small d,

$$\begin{aligned} &- \mathrm{div}(d^{2-\alpha}\nabla w_s) - \Big(\frac{\alpha-1}{2}\Big)^2 d^{-\alpha} w_s \\ &\quad = -s(s+1) d^{\frac{-\alpha-1}{2}} X^{s+2}(d) - d^{\frac{-\alpha+1}{2}} (\Delta d) \Big(\frac{\alpha-1}{2} X^s(d) + s X^{s+1}(d)\Big) \\ &\quad = -d^{\frac{-\alpha-1}{2}} X^{s+2}(d) \Big(s(s+1) + (\Delta d) d X^{-2}(d) \Big(\frac{\alpha-1}{2} + s X(d)\Big)\Big) \le 0. \end{aligned}$$

Take $\epsilon > 0$ such that for all $s \in (\frac{1}{2}, 1)$, $\epsilon w_s \le u$ on $\{x \in \Omega \mid d(x) = \delta\}$ and define $v_s \equiv \epsilon w_s - u$. Then $(v_s)_+ \in W^{1,2}_{0,\alpha,2-\alpha}(\Omega)$ and

$$-\mathrm{div}(d^{2-\alpha}\nabla v_s) - \Big(\frac{\alpha-1}{2}\Big)^2 d^{-\alpha} v_s \le 0 \text{ in } \Omega_\delta,$$

where $u_+ = \max\{u, 0\}$. Then, multiplying $(v_s)_+$ to the above inequality and then integrating over Ω_δ,

$$\int_{\Omega_\delta} d^{2-\alpha} |\nabla (v_s)_+|^2 - \Big(\frac{\alpha-1}{2}\Big)^2 d^{-\alpha} ((v_s)_+)^2 dx \le 0.$$

On the other hand, by Lemma A.3,

$$\int_{\Omega_\delta} d^{2-\alpha} |\nabla (v_s)_+|^2 - \Big(\frac{\alpha-1}{2}\Big)^2 d^{-\alpha} ((v_s)_+)^2 dx \ge 0.$$

We deduce that $\epsilon w_s \leq u$ in Ω_δ for every $s \in (\frac{1}{2}, 1)$. Hence, $\epsilon d^{\frac{\alpha-1}{2}} X^{\frac{1}{2}}(d) \leq u$ in Ω_δ, which is a contradiction to the fact the $d^{-\alpha/2}u \in L^2(\Omega)$. Thus we conclude that if $H_{\alpha,2-\alpha}(\Omega)$ is achieved, $H_{\alpha,2-\alpha}(\Omega) < \frac{(1-\alpha)^2}{4}$. □

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