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HASSE INVARIANT FOR THE TAME BRAUER GROUP OF A HIGHER LOCAL FIELD

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Abstract. We generalize the Hasse invariant of local class field theory to the tame Brauer group of a higher dimensional local field, and use it to study the arithmetic of central simple algebras, which are given *a priori* as tensor products of standard cyclic algebras. We also compute the tame Brauer dimension (or *period-index bound*) and the cyclic length of a general henselian-valued field of finite rank and finite residue field.

1. Introduction

One of the seminal works of 20th century number theory was the 1932 paper [4] by Brauer, Hasse, and Noether. The main theorems, that all $\mathbb{Q}$-division algebras are cyclic, and have equal period and index, formed a basis for the development of class field theory, and were in turn based on Hasse's local class field theory, in Hasse's 1931 article [10], see [21, 6.1]. Key to the latter was a natural isomorphism, the *Hasse invariant map*

$$\mathrm{inv}_F : \mathrm{H}^2(F, \mu_n) \xrightarrow{\sim} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

over a local field F. This map summarizes the Brauer group over a local field, providing not only a functorial group isomorphism, but also the index of a class, and a structure theorem for F-division algebras, showing them all to be cyclic.

The Hasse invariant was generalized to higher dimensional local fields by Kato in his higher local class field theory [17, Theorem 3]. Kato established a natural isomorphism

$$\mathrm{inv}_F : \mathrm{H}^{d+1}(F, \mu_n^{\otimes(d-1)}) \xrightarrow{\sim} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

over a local field F of dimension d, by way of extending the classical reciprocity map. The proof draws on deep results proved with Bloch using algebraic K-theory and Kato's deRham cohomology in characteristic p. We will have nothing more to say about this spectacular result.

For though it was to become a cornerstone of class field theory, Hasse's original motivation in writing [10] was to understand the arithmetic of central simple algebras over local fields [21, Remark 6.3.1]. To extend this aspect of his work, we need to study a higher local field's Brauer group $\mathrm{Br}(F)$.

In this paper we generalize Hasse's invariant map to the prime-to-p part of $2\,\mathrm{Br}(F)$ for a d-dimensional henselian-valued field F with finite residue field k. Our generalization contains a precise index formula; a structure theorem for division algebras, showing them to be tensor products of cyclic division algebras; and an

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algorithm for expressing the underlying division algebra of a given central simple algebra as a tensor product of cyclic division algebras.

We also compute the Brauer dimension, cyclic length, and period-index ratio for the tame Brauer group. Though these results are expected, and easily proved for higher local fields, they are new for general henselian-valued fields with finite residue field.

Our generalization of the Hasse invariant is based on a natural isomorphism

$$_n 2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F,\mu_n) \xrightarrow{\sim} {}_n 2\,\mathrm{Br}(F)$$

induced by the cup product. The designation of a basis then associates to each Brauer class a skew-symmetric matrix, so we have a (non-canonical) injective homomorphism

$$_n 2\,\mathrm{Br}(F) \longrightarrow \mathrm{Alt}_{d+1}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z})\,.$$

This map is functorial with respect to basis change, so that 2-block diagonalization on the right "diagonalizes" the Brauer class. We prove the index of the Brauer class is the order of the pfaffian subgroup of the skew-symmetric matrix, and, like the determinant, has an explicit formula in terms of the matrix coefficients. At the same time we show the underlying division algebra is a tensor product of cyclic division algebras of degrees equal to the abelian group theory invariants of the row space of the skew-symmetric matrix. To prove our index formula we use the valuation theoretic framework of [26]. Our methods thus generalize [26, Example 1.2.8], which was the method used by Hasse in [10] to compute the Hasse invariant over an ordinary local field.

For example, if $F = \mathbb{F}_5((t_1))((t_2))((t_3))$ is the 3-dimensional local field, and we put $t_0 = [2]_5$, then the sum of quaternions

$$\alpha = (t_0,t_1) + (t_0,t_2) + (t_0,t_3) + (t_1,t_2) + (t_1,t_3) + (t_2,t_3)$$

is mapped to the skew-symmetrix matrix

$$\mathrm{alt}_{\underline{t}}(\alpha) = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

The pfaffian formula computes $a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 1/4$ in $\mathbb{Q}/\mathbb{Z}$, so the index is 4. This calculation is unexpectedly easy. In this case the index is computable in principle from the Witt index formula for a Laurent series field, but that approach becomes hopelessly complicated even for low values of d, and does not extend to more general henselian fields.

Elements of $\mathrm{Br}(F)$ that do not fit into our theory include the p-primary part $\mathrm{Br}(F)_p$, where $p = \mathrm{char}(k)$, and elements of $\mathrm{Br}(F)_2 - 2\,\mathrm{Br}(F)_2$. The p-primary part has a much different character, and is not even a candidate for a similar treatment. And though when $2 \neq p$, $\mathrm{Br}(F)_2$ has the right number of summands to define a map into skew-symmetric matrices, the *ad hoc* map is not functorial with respect to basis change, and is therefore useless. The problem with 2-torsion could be anticipated, since totally ramified classes $(t,t)_n$ violate "skew-symmetry" when n is the maximal power of 2 dividing $|\mu(F)|$.

In [5] and [6] the author used the machinery of symplectic modules to derive an index formula and minimal expression for an arbitrary class in the (tame) Brauer group of a strictly henselian field, that is, a henselian field whose residue field is algebraically closed. The central observation was that the cocycles underlying a Brauer class define alternating forms on the Galois group. A more valuation-theoretic treatment using symplectic modules is formalized by Tignol and Wadsworth in [26, Chapter 7]. These ideas predated the present author's work, and are rooted in (more general) work by Amitsur-Rowen-Tignol ([1]), Tignol ([23]), Tignol-Amitsur ([24]), and Tignol-Wadsorth ([25]). Their methods are especially powerful when the residue fields are algebraically closed, and much is reduced to valuation theory. With non algebraically closed residue fields, the augmentation with Galois theoretic methods seems necessary to obtain a comparable understanding. There remains the problem of how to construct a group of Galois symplectic modules isomorphic to the tame Brauer group, as outlined in [26, Chapter 7].

The paper is organized as follows. We first prove structure theorems for the character group and Brauer group of a henselian-valued field of rank d, with finite residue field. These results are well-known; we prove them for convenience, and use them immediately to compute Brauer dimension and cyclic length. Then we prove Theorem 8.6, which shows ${}_n2\,\mathrm{Br}(F)$ is naturally isomorphic to a wedge product, and we define basis and basis change for this wedge product, before proving the main results, Theorem 10.1 and Theorem 10.4, relating the arithmetic of the Brauer group to the arithmetic of skew-symmetric matrices with coefficients in $\mathbb{Q}/\mathbb{Z}$. We end by showing why the missing set ${}_n\mathrm{Br}(F) - {}_n2\,\mathrm{Br}(F)$ resists the same treatment, even when F contains μ_n.

2. Preliminaries

Let F be a field, let n be a number invertible in F, and let $m = |\mu_n(F)|$. Let $\mathrm{Br}(F) = \mathrm{H}^2(F, F_{\mathrm{sep}}^\times)$ and $\mathrm{X}(F) = \mathrm{H}^1(F, \mathbb{Q}/\mathbb{Z})$ denote the Brauer group and character group of F, with n-torsion subgroups ${}_n\mathrm{Br}(F) = \mathrm{H}^2(F, \mu_n)$ and ${}_n\mathrm{X}(F) = \mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. If $\theta \in {}_n\mathrm{X}(F)$ has order n, let $F(\theta)/F$ denote the corresponding cyclic extension of degree n.

The coboundary map $\partial : F_{\mathrm{sep}}^\times \to \mathrm{C}^1(\mathrm{G}_F, F_{\mathrm{sep}}^\times)$, which takes an element $x \in F_{\mathrm{sep}}^\times$ to the function $\sigma \mapsto x^{\sigma-1}$, has kernel $F^\times$ by Galois theory, and image $\mathrm{B}^1(\mathrm{G}_F, F_{\mathrm{sep}}^\times)$, which equals $\mathrm{Z}^1(\mathrm{G}_F, F_{\mathrm{sep}}^\times)$ by Hilbert 90. Since ${}_n(F_{\mathrm{sep}}^\times/F^\times) = F^{\times 1/n}/F^\times$ and ${}_n\mathrm{Z}^1(\mathrm{G}_F, F_{\mathrm{sep}}^\times) = \mathrm{Z}^1(F, \mu_n)$, we have a natural isomorphism

$$\partial : F^{\times 1/n}/F^\times \xrightarrow{\sim} \mathrm{Z}^1(F, \mu_n)$$

We suppress the notation ∂, and write $z^{1/n}$ for the cocycle $\partial(z^{1/n})$. The coboundaries $\mathrm{B}^1(F, \mu_n)$ are the image $\partial(\mu_n F^\times/F^\times) \simeq \mu_n/\mu_m$, and we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mu_n/\mu_m & \longrightarrow & F^{\times 1/n}/F^\times & \xrightarrow{n} & F^\times/F^{\times n} & \longrightarrow & 1 \\
 & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \\
0 & \longrightarrow & \mathrm{B}^1(F, \mu_n) & \longrightarrow & \mathrm{Z}^1(F, \mu_n) & \longrightarrow & \mathrm{H}^1(F, \mu_n) & \longrightarrow & 0
\end{array}$$

Let $(t)_n \in \mathrm{H}^1(F, \mu_n)$ denote the class of $t^{1/n}$.

For a prime p let μ' be the group of all prime-to-p roots of unity, let $(\mathbb{Q}/\mathbb{Z})'$ be the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$, and let

$$\zeta^* : \mu' \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})'$$

be a fixed isomorphism. Put $\zeta_n^* = \zeta^*|_{\mu_n}$ and set $\zeta_n = (\zeta_n^*)^{-1}(1/n)$. Since $\mu_m \le F^\times$, ζ_m^* is a G_F-module isomorphism, and we have an induced isomorphism

$$\begin{aligned} \zeta_m^* : \mathrm{H}^1(F, \mu_m) &\xrightarrow{\sim} \mathrm{H}^1(F, \tfrac{1}{m}\mathbb{Z}/\mathbb{Z}) \\ (t)_m &\longmapsto (t)_m^* := \zeta_m^* \circ t^{1/m} \end{aligned} \tag{2.1}$$

If $\theta = (t)_m^*$ has order m, then $F(\theta) = F(t^{1/m})$ has degree m over F (see [22, XIV.2]).

2.1. **Cyclic classes.** If $|\theta| = n$, the *cyclic Brauer class* $(\theta, t) \in \mathrm{Br}(F)$ is the cup product $\theta \cup (t)_n$. The cyclic class (θ, t) determines the *cyclic algebra of degree n*,

$$\Delta(\theta, t) := \{F(\theta)[y] : y^n = t, xy = y\sigma(x)\ \forall x \in F(\theta)\}$$

where $\sigma \in \mathrm{Gal}(F(\theta)/F)$ satisfies $\theta(\sigma) = 1/n$. Since the cup product is induced by the tensor product, if $m|n$ then $\theta \cup (t)_m = \theta \cup (t^{n/m})_n = (\frac{n}{m}\theta, t)$, and $(s)_m^* \cup (t)_n = (s)_m^* \cup (t)_m = (s, t)_m$. If $\theta = (s)_m^*$, then $(\theta, t) = (s, t)_m$, the *symbol Brauer class*, is represented by the *symbol algebra*

$$\Delta(s, t)_m := \{F[x, y] : x^m = s, y^m = t, [x, y] = \zeta_m\}$$

The *norm criterion* states that $(\theta, t) = 0$ if and only if t is a norm from $F(\theta)$. If $\theta = (t)_m^*$ then $(\theta, t) = (t, t)_m = 0$ if m is odd, and $(\theta, -t) = (t, -t)_m = 0$ if m is even, in which case $(t, t)_m = (t, -1)_m$.

Since 1 is a norm, $(\theta, -1)$ has order dividing 2. If $\theta = 2\theta'$ for some θ', then $(\theta, -1) = 2(\theta', -1) = 0$. Conversely if $(\theta, -1) = 0$ then $\theta = 2\theta'$ for some θ' by [3]. Thus $(\theta, -1) = 0$ if and only if $\theta \in 2\,\mathrm{X}(F)$. We call this result *Albert's criterion.*

2.2. **7-Term sequence.** For an exact sequence $1 \to N \to G \to \overline{G} \to 1$ of profinite groups, with N closed, the Hochschild-Serre spectral sequence $\mathrm{H}^p(\overline{G}, \mathrm{H}^q(N, M)) \Rightarrow \mathrm{H}^{p+q}(G, M)$ yields a 7-term sequence ([18, Appendix B]), which we note for the reader's convenience:

$$\begin{aligned} 0 \longrightarrow &\mathrm{H}^1(\overline{G}, M^N) \longrightarrow \mathrm{H}^1(G, M) \longrightarrow \mathrm{H}^1(N, M)^{\overline{G}} \longrightarrow \mathrm{H}^2(\overline{G}, M^N) \\ &\longrightarrow \ker[\mathrm{H}^2(G, M) \to \mathrm{H}^2(N, M)^{\overline{G}}] \longrightarrow \mathrm{H}^1(\overline{G}, \mathrm{H}^1(N, M)) \longrightarrow \mathrm{H}^3(\overline{G}, M^N) \end{aligned}$$

3. Standard setup

3.1. Let $F = (F, \mathrm{v})$ be a henselian-valued field with finite residue field k of cardinality $|k| = q = p^f$, and totally ordered value group $\Gamma_F \simeq \mathbb{Z}^d$ of rank $d \ge 1$. In the following,

- $\ell \ne p$ is a prime
- n is a power of ℓ
- $m = |\mu_n(k)| = |\mu_n(F)| = \gcd(q - 1, n)$
- $m' = m/2$ if $m = 2^{\mathrm{v}_2(q-1)}$, and $m' = m$ otherwise.

Note $m = 1$ if and only if $\mu_\ell \nleq F^\times$, and $m = n$ if and only if $\mu_n \leq F^\times$.

Let F_{nr} denote the maximal unramified extension of F, which is the strict henselization of F, and let F_{tr} be the maximal tamely ramified extension of F, which is obtained from F_{nr} by adjoining all prime-to-p-th roots of elements of F, by [26, Proposition A.22]. We identify G_k with $\mathrm{Gal}(F_{\text{nr}}/F)$, and put $\mathrm{G}_F^{\text{tr}} = \mathrm{Gal}(F_{\text{tr}}/F)$.

3.2. **Index in the tame Brauer group.** Suppose F is henselian with finite residue field k, and D/F is a division algebra. To compute index we will use the valuation theoretic framework of Tignol-Wadsworth's [25] and Jacob-Wadsworth's [15], which we summarize.

Since k is finite, the residue division algebra $\overline{D}$ is a field extension of k, by Wedderburn's Theorem. By [15, Decomposition Lemma 6.2] (see also [26, Proposition 8.59]), $D \sim S \otimes_F T$ where S and T are nicely semiramified and totally ramified F-division algebras, and $\Gamma_D = \Gamma_S + \Gamma_T$ by [15, Theorem 6.3]. The surjective homomorphism $\theta_D : \Gamma_D/\Gamma_F \to \mathrm{Gal}(Z(\overline{D})/k)$ of [8, p.96] (see [15, Proposition 1.7]) has kernel Γ_T/Γ_F by [15, Theorem 6.3], and it follows that $[\overline{D} : k] = [\Gamma_D : \Gamma_T]$. Therefore by Draxl's Ostrowski Theorem [7, Theorem 2], if the index of D is prime-to-char(k), then

$$\mathrm{ind}(D)^2 = [D : F] = [\Gamma_D : \Gamma_T][\Gamma_D : \Gamma_F]$$

4. Multiplicative group

Assume (3.1). With $\overline{G} = \mathrm{G}_k$, $G = \mathrm{G}_F$, $N = \mathrm{G}_{F_{\text{nr}}}$, and $M = \mu_n$, (2.2) yields

$$1 \longrightarrow k^\times/n \longrightarrow F^\times/n \longrightarrow F_{\text{nr}}^\times/n \longrightarrow 1$$

and composing with the valuation map $\mathtt{v} : F_{\text{nr}}^\times/n \xrightarrow{\sim} \Gamma_{F_{\text{nr}}}/n = \Gamma_F/n$, we obtain the usual valuation sequence on $F^\times/n$. This sequence splits with the choice of a *uniformizer subgroup* T_n of $F^\times/n$, which is the group generated by a basis of *uniformizers* $\{(x_1)_n, \dots, (x_d)_n\}$: elements of $F^\times/n$ whose values $\{\mathtt{v}(x_1), \dots, \mathtt{v}(x_d)\}$ form a basis of Γ_F/n. Since $k^\times$ is a finite cyclic group and $\mu_n(k) = \mu_m(k) \simeq \mu_m(F)$, we have $k^\times/n = k^\times/m \simeq \mathbb{Z}/m$. Thus

$$F^\times/n \simeq k^\times/m \times T_n \simeq \mu_m(F) \times T_n \simeq \mathbb{Z}/m \oplus (\mathbb{Z}/n)^d \tag{4.1}$$

and the natural map $F^\times/n \to F_{\text{nr}}^\times/n$ maps T_n isomorphically onto $F_{\text{nr}}^\times/n$. Similarly, the sequence

$$1 \longrightarrow k^{\times 1/n}/k^\times \longrightarrow F^{\times 1/n}/F^\times \longrightarrow F_{\text{nr}}^{\times 1/n}/F_{\text{nr}}^\times \longrightarrow 1$$

shows

$$F^{\times 1/n}/F^\times \simeq \mu_{mn}/\mu_m \times \langle t_1^{1/n}, \dots, t_d^{1/n} \rangle \simeq (\mathbb{Z}/n)^{d+1}$$

for a uniformizer basis $\{t_1^{1/n}, \dots, t_d^{1/n}\}$ for $F^{\times 1/n}/F^\times$. For since $|k^{\times 1/n}| = |\zeta_{n(q-1)}| = n(q-1)$, $k^{\times 1/n}/k^\times$ has order n. If $t_0^{1/n}$ generates μ_{mn}/μ_m, the set $\{t_0^{1/n}, \dots, t_d^{1/n}\}$ forms a basis for $F^{\times 1/n}/F^\times$. We will say the basis is in *standard form* if its last d elements make a uniformizer basis.

5. Galois group and character group

The structure theory of the Galois group of the maximal tamely ramified extension of a local field goes back at least to Iwasawa ([13]); see [19, VII.5] for additional background. Assume (3.1). We have a split exact sequence

$$1 \longrightarrow \mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}} \longrightarrow \mathrm{G}^{\mathrm{tr}}_{F} \underset{s}{\overset{\longrightarrow}{\leftarrow}} \mathrm{G}_k \longrightarrow 1$$

whereby $\mathrm{G}^{\mathrm{tr}}_F = \mathrm{G}_k \ltimes \mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}}$, $\mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}} \simeq (\widehat{\mathbb{Z}}')^d$, and $\mathrm{G}_k \simeq \widehat{\mathbb{Z}}$. Let $\Phi_0 = s(1)$ be a Frobenius generator, which exponentiates by q. Let $\{x_1, \dots, x_d\}$ be a uniformizer basis for F, generating a subgroup $T \le F^\times$. Let $\{\Phi_1, \dots, \Phi_d\}$ be a (topological) basis for $\mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}}$ dual to $\{x_1, \dots, x_d\}$, so that for each n, $x_j^{1/n}(\Phi_i) = \zeta_n^{\delta_{ij}}$, $i, j \ge 1$. Let $\underline{\Phi} = \{\Phi_0, \dots, \Phi_d\}$ be the resulting basis for $\mathrm{G}^{\mathrm{tr}}_F$. Then we have the presentation

$$\mathrm{G}^{\mathrm{tr}}_F = \left\langle \Phi_0, \dots, \Phi_d : [\Phi_0, \Phi_j] = \Phi_j^{q-1},\ \forall j \ge 1;\ [\Phi_i, \Phi_j] = e,\ \forall i, j \ge 1 \right\rangle \tag{5.1}$$

Theorem 5.1. *Assume* (3.1), *let* $T_{q-1} \le F^\times/(q-1)$ *be a uniformizer group, and let* $T^*_{q-1} = \zeta^*_{q-1}(T_{q-1})$ *as in* (2.1). *Then* $\mathrm{X}(F^{\mathrm{tr}}/F) \simeq \mathrm{X}(k) \times T^*_{q-1} \simeq \mathbb{Q}/\mathbb{Z} \oplus (\frac{1}{q-1}\mathbb{Z}/\mathbb{Z})^{\oplus d}$, *and*

$${}_n\mathrm{X}(F) \simeq \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \oplus (\tfrac{1}{m}\mathbb{Z}/\mathbb{Z})^{\oplus d}$$

Proof. From (5.1) we compute $[\mathrm{G}^{\mathrm{tr}}_F, \mathrm{G}^{\mathrm{tr}}_F] = (\mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}})^{q-1} \simeq (q-1)\widehat{\mathbb{Z}}'^d$, so

$$\mathrm{X}(F^{\mathrm{tr}}/F) = \mathrm{Hom}(\mathrm{G}^{\mathrm{tr}}_F, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(\widehat{\mathbb{Z}} \times \tfrac{\widehat{\mathbb{Z}}'^d}{q-1}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Q}/\mathbb{Z} \oplus (\tfrac{1}{q-1}\mathbb{Z}/\mathbb{Z})^{\oplus d}$$

The first factor is $\mathrm{X}(k)$, and since $\mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}}/(\mathrm{G}^{\mathrm{tr}}_{F_{\mathrm{nr}}})^{q-1} \simeq \mathrm{Gal}(F_{\mathrm{nr}}^{1/(q-1)}/F_{\mathrm{nr}})$, the second factor is T^*_{q-1} by Kummer theory, for any uniformizer subgroup T_{q-1}. The last statement is immediate. □

6. Brauer group

6.1. **Brauer group of F_{nr}.** Let $F = (F, \mathrm{v})$ be henselian-valued of rank d, with residue field k either finite, as in (3.1), or algebraically closed, in which case F is strictly henselian. The tame Brauer group of a strictly henselian field is naturally isomorphic to a group of alternating forms on the absolute Galois group, by [5] and [6], giving a method for finding a minimal expression and an index formula for each class. We prove a slight variant.

Assume $\mu_n \le F^\times$. Let $G = \mathrm{Gal}(L/F)$, where either $L = F^{1/n}$, or $L = \varinjlim F^{1/n}$, with the limit over all prime-to-p numbers n if k is algebraically closed. Then G is isomorphic to either $(\mathbb{Z}/n)^{d+1}$, $(\mathbb{Z}/n)^d$, or $\widehat{\mathbb{Z}}'^d$. Let $\mathrm{Alt}(G, \mu_n)$ denote the set of continuous alternating bilinear functions on G.

Lemma 6.1. *In the setup above, with* $\mu_n \le F^\times$, *there is a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{H}^1(F, \mu_n) & \xrightarrow{\mathrm{alt}} & \mathrm{Alt}(G, \mu_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \cup & & \| & & \\
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(G, \mu_n) & \longrightarrow & \mathrm{H}^2(G, \mu_n) & \xrightarrow{[\mathrm{alt}]} & \mathrm{Alt}(G, \mu_n) & \longrightarrow & 0
\end{array}$$

where $\mathrm{alt}([f])(\sigma,\tau) := f(\sigma,\tau)/f(\tau,\sigma)$ *for* $f \in \mathrm{Z}^2(G,\mu_n)$*, and* $K = \ker(\mathrm{alt})$*. If* G *is torsion-free, then* $[\mathrm{alt}]$ *is an isomorphism, and* $\mathrm{H}^2(G,\mu_n)$ *is generated by cyclic classes. If* $G = (\mathbb{Z}/n)^r$*, then* $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mu_n) \simeq \mu_n^r$*.*

Proof. The result appears in [6, Lemma 2.2, Theorem 2.4] with $\mathbb{Q}/\mathbb{Z}$ in place of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ and μ_n. It also follows from the Universal Coefficient Theorem.

We may assume n is a power of a prime ℓ by primary decomposition. The group G is either isomorphic to $(\mathbb{Z}/n)^r$, where $r = d+1$ or d, or $(\widehat{\mathbb{Z}}')^d$ if k is algebraically closed.

Viewing $\mathrm{H}^1(F,\frac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{H}^1(F,\mu_n)$ as a subgroup of $\mathrm{Z}^2(G,\mu_n)$, extend alt to all of $\mathrm{Z}^2(G,\mu_n)$ by setting $\mathrm{alt}(f) = f/\tilde{f}$ for all $f \in \mathrm{Z}^2(G,\mu_n)$, where $\tilde{f}(\sigma,\tau) := f(\tau,\sigma)$. It is straightforward to verify that $\tilde{f}$ is a 2-cocycle, because G is abelian and acts trivially on μ_n. Moreover, $f/\tilde{f}$ is indeed an alternating form on G. For it is clear from the definition that $(f/\tilde{f})(\sigma,\tau) = (f/\tilde{f})(\tau,\sigma)^{-1}$, and the alternating sum of the cocycle condition on f applied successively to the triples $(\rho,\sigma,\tau),(\rho,\tau,\sigma)$, and (τ,ρ,σ) shows that $f/\tilde{f}$ is linear on the left, hence bilinear (see [2, Section 1]).

If $f \in \mathrm{C}^1(G,\mu_n)$ then since G is abelian and acts trivially on μ_n, $\partial f \in \mathrm{B}^2(G,\mu_n)$ is in the subgroup $\mathrm{Z}^2(G,\mu_n)_{\mathrm{sym}}$ of $\mathrm{Z}^2(G,\mu_n)$ generated by elements $f : f = \tilde{f}$. Therefore $[\mathrm{alt}]$ is well-defined on $\mathrm{H}^2(G,\mu_n)$, and we have a commutative ladder

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & \mathrm{H}^1(F,\frac{1}{n}\mathbb{Z}/\mathbb{Z})\otimes\mathrm{H}^1(F,\mu_n) & \xrightarrow{\mathrm{alt}} & \mathrm{Alt}(G,\mu_n) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\cup} & & \| & & \\
0 & \longrightarrow & \mathrm{H}^2(G,\mu_n)_{\mathrm{sym}} & \longrightarrow & \mathrm{H}^2(G,\mu_n) & \xrightarrow{[\mathrm{alt}]} & \mathrm{Alt}(G,\mu_n) & \longrightarrow & 0
\end{array}$$

If $\{(x_i)_n\}$ is a basis for $\mathrm{H}^1(F,\mu_n)$, the elements $\mathrm{alt}((x_i)_n^* \otimes (x_j)_n)$ for $i<j$ are easily seen to form a basis for $\mathrm{Alt}(G,\mu_n)$, hence alt splits. Therefore $[\mathrm{alt}]$ splits. The group $\mathrm{H}^2(G,\mu_n)_{\mathrm{sym}}$ describes the abelian group-extensions of G by μ_n, hence it is isomorphic to $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mu_n)$. If $G = (\mathbb{Z}/n)^r$, then $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mu_n) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r,\mu_n) \simeq \mu_n^r$ is a standard computation; see e.g. [9, Ch.17]. If G is torsion-free then it is a direct limit of projective $\mathbb{Z}$-modules, hence $\mathrm{Ext}^1_{\mathbb{Z}}(G,\mu_n) = 0$, and $[\mathrm{alt}]$ is an isomorphism. Then the commutative ladder and the Snake Lemma show $\mathrm{H}^2(G,\mu_n)$ is generated by elements $\theta \cup (t)_n$, for $\theta \in \mathrm{H}^1(F,\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ and $(t)_n \in \mathrm{H}^1(F,\mu_n)$. □

Remark 6.2. We study the connection between $\mathrm{H}^2(G,\mu_n)$ and $\mathrm{H}^2(F,\mu_n)$ in Section 12.

6.2. **Brauer group of** F**.**

Theorem 6.3. *Assume* (3.1)*. Then there is an exact sequence*

$$0 \longrightarrow {}_n\mathrm{H}^2(\mathrm{G}_k, F_{\mathrm{nr}}^\times) \longrightarrow \mathrm{H}^2(F,\mu_n) \longrightarrow \mathrm{H}^2(F_{\mathrm{nr}},\mu_m) \longrightarrow 0$$

split by a choice of uniformizer group, so that

$${}_n\mathrm{Br}(F) \simeq (\tfrac{1}{n}\mathbb{Z}/\mathbb{Z})^{\oplus d} \oplus (\tfrac{1}{m}\mathbb{Z}/\mathbb{Z})^{\oplus d(d-1)/2}$$

Suppose $\chi_0 \in {}_n\mathrm{X}(F)$ *is unramified of order* n*, and* $\{(x_1)_n,\dots,(x_d)_n\}$ *is a uniformizer basis. Then each* $\alpha \in {}_n\mathrm{Br}(F)$ *has a unique coordinate expression*

$$\alpha = \sum_{j=1}^{d} n_{0j}(\chi_0,x_j) + \sum_{1\le i<j\le d} m_{ij}(x_i,x_j)_m$$

Proof. This is well-known, see e.g. [26, Theorem 7.84]. Since F_{nr}/F is Galois, $\mathrm{G}_{F_{\mathrm{nr}}}$ is a closed normal subgroup of G_F. Let $\mathrm{G}_k = \mathrm{Gal}(F_{\mathrm{nr}}/F)$. Since $\mathrm{H}^1(\mathrm{G}_{F_{\mathrm{nr}}}, F_{\mathrm{sep}}^\times) = 0$ by Hilbert 90, the n-torsion of the 7-term sequence (2.2) applied to $M = F_{\mathrm{sep}}^\times$ yields the inflation-restriction sequence

$$0 \longrightarrow {}_n\mathrm{H}^2(\mathrm{G}_k, F_{\mathrm{nr}}^\times) \xrightarrow{\mathrm{inf}} \mathrm{H}^2(F, \mu_n) \xrightarrow{\mathrm{res}} \mathrm{H}^2(F_{\mathrm{nr}}, \mu_n)^{\mathrm{G}_k}$$

By Lemma 6.1, $\mathrm{H}^2(F_{\mathrm{nr}}, \mu_n)$ is generated by cyclic classes $(x)_n^* \cup (y)_n$. If $\{(x_1)_n, \dots, (x_d)_n\}$ is a basis for a uniformizer subgroup T_n for $F^\times/n$, then since $T_n \simeq F_{\mathrm{nr}}^\times/n$, every element of $\mathrm{H}^2(F_{\mathrm{nr}}, \mu_n)$ is uniquely expressible in terms of the $(x_i)_n^* \cup (x_j)_n = (x_i, x_j)_n$ with $1 \le i < j \le d$. Thus $\mathrm{H}^2(F_{\mathrm{nr}}, \mu_n) \simeq \mu_n^{d(d-1)/2}$ by counting. The explicit action of G_k shows this is a G_k-module isomorphism. Since $\mu_n^{\mathrm{G}_k} = \mu_m$, it follows that $\mathrm{H}^2(F_{\mathrm{nr}}, \mu_n)^{\mathrm{G}_k} \simeq \mu_m^{d(d-1)/2}$, and we have an exact sequence

$$0 \longrightarrow {}_n\mathrm{H}^2(\mathrm{G}_k, F_{\mathrm{nr}}^\times) \longrightarrow \mathrm{H}^2(F, \mu_n) \longrightarrow \mathrm{H}^2(F_{\mathrm{nr}}, \mu_m) \longrightarrow 0$$

with a splitting defined by the choice of $T_n \le F^\times/n$.

Since $\mathrm{H}^q(\mathrm{G}_k, \mu_n) = 0$ for $q \ge 2$, the 7-term sequence (2.2) with $M = \mu_n$ defines a natural isomorphism

$${}_n\mathrm{H}^2(\mathrm{G}_k, F_{\mathrm{nr}}^\times) \xrightarrow{\sim} \mathrm{H}^1(\mathrm{G}_k, \mathrm{H}^1(\mathrm{G}_{F_{\mathrm{nr}}}, \mu_n))$$

An easy computation shows that the action of G_k on $\mathrm{H}^1(\mathrm{G}_{F_{\mathrm{nr}}}, \mu_n)$ is trivial. Therefore since $\mathrm{H}^1(\mathrm{G}_{F_{\mathrm{nr}}}, \mu_n) \simeq F_{\mathrm{nr}}^\times/n \simeq \Gamma_F/n$, we find ${}_n\mathrm{H}^2(\mathrm{G}_k, F_{\mathrm{nr}}^\times) \simeq \mathrm{H}^1(\mathrm{G}_k, \Gamma_F/n) \simeq \langle\chi_0\rangle^d$, where $\chi_0 \in \mathrm{X}(k)$ is any character of order n. A tracing through the maps shows the splitting $\langle\chi_0\rangle^d \to {}_n\mathrm{H}^2(\mathrm{G}_k, F_{\mathrm{nr}}^\times)$ is induced by the choice of basis $\{(x_1)_n, \dots, (x_d)_n\}$ for T_n, and sends the j-th copy of χ_0 to (χ_0, x_j). Thus a general class is uniquely expressible as a sum $\sum_{j=1}^d n_{0j}(\chi_0, x_j)$. We conclude $\mathrm{H}^2(F, \mu_n) \simeq \mu_n^d \times \mu_m^{d(d-1)/2}$, and ach $\alpha \in \mathrm{H}^2(F, \mu_n)$ is uniquely expressible in the form

$$\begin{array}{cccccccc} n_{01}(\chi_0, x_1) & + & n_{02}(\chi_0, x_2) & + & \cdots & + & n_{0d}(\chi_0, x_d) \\ & + & m_{12}(x_1, x_2)_m & + & \cdots & + & m_{1d}(x_1, x_d)_m \\ & & & & & & \vdots \\ & & & & & + & m_{d-1d}(x_{d-1}, x_d)_m \end{array}$$

□

7. Brauer dimension and cyclic length

Definition 7.1. Let $F = (F, \mathrm{v}, k)$ be a valued field with residue field k.

(a) The *tame cyclic length* of F is the the minimum number of cyclic classes of degree n needed to express all elements of ${}_n\mathrm{Br}(F)$, over all n invertible in k. It is zero if $\mathrm{Br}(F) = 0$, and ∞ if no such number exists.

(b) The *tame Brauer dimension* of F is the supremum of the smallest $d \ge 0$ such that $\mathrm{ind}(\alpha)$ divides $\mathrm{per}(\alpha)^d$ for all tame $\alpha \in \mathrm{Br}(F)$, or ∞ if no such number exists. The relation $\mathrm{ind}(\alpha) \mid \mathrm{per}(\alpha)^d$ for all α is called a *period-index bound.*

(c) The *maximum period-index ratio in* ${}_n\mathrm{Br}(F)$ is the supremum of $\mathrm{ind}(\alpha)/\mathrm{per}(\alpha)$ over all $\alpha \in {}_n\mathrm{Br}(F)$. It is zero if ${}_n\mathrm{Br}(F) = 0$.

Remark 7.2. Since a class of period n and n-cyclic length d has index at most n^d, Brauer dimension is bounded by cyclic length. Cyclic length and Brauer dimension are sensitive to the presence of roots of unity. For example, all tame division algebras over $\mathbb{F}_2((t_1))((t_2))((t_3))$ are cyclic of equal period and index, whereas, as the following example shows, $\mathbb{F}_4((t_1))((t_2))((t_3))$ has elements of cyclic length 2 and unequal period and index.

Example 7.3 gives a lower bound, which turns out to be an upper bound.

Example 7.3. Assume (3.1), with n a prime-power and $m \neq 1$. Let χ_0 be an unramified character of order n, and $\{(x_1)_n, \dots, (x_d)_n\}$ a uniformizer basis. The algebra

$$D = \Delta(\chi_0, x_1) \otimes_F \Delta(x_2, x_3)_m \otimes_F \Delta(x_4, x_5)_m \otimes_F \cdots \otimes_F \Delta(x_{2r}, x_{2r+1})_m$$

where $r = \lfloor \frac{d-1}{2} \rfloor$, is a division algebra of period n and index nm^r, and cyclic length $r+1 = \lfloor \frac{d+1}{2} \rfloor$, by [14, Corollary 2.6] and [25, Extension Lemma 1.6]. This example is well-known.

Theorem 7.4. *Assume* (3.1)*.*

(a) *If $k = \mathbb{F}_2$, the tame Brauer dimension and cyclic length are both* 1*, and the maximum period-index ratio for ${}_n\mathrm{Br}(F)$ is* 1*, for all (prime-to-p) n.*

(b) *If $k \neq \mathbb{F}_2$, the tame Brauer dimension and cyclic length are both $\lfloor \frac{d+1}{2} \rfloor$, and the maximum period-index ratio in ${}_n\mathrm{Br}(F)$ is $m^{\lfloor \frac{d-1}{2} \rfloor}$, where $m = |\mu_n(k)|$, for all n.*

Proof. The case $k = \mathbb{F}_2$ is immediate, since there are no tame totally ramified classes, so ${}_n\mathrm{Br}(F) = \{(\chi_0, t) : t \in F^\times/\mathrm{N}(F(\chi_0)^\times)\}$ for an unramified character χ_0 of order n.

Now suppose $k \neq \mathbb{F}_2$, so that F has a nontrivial tame root of unity and $m \neq 1$ for some prime-to-p number n. We may assume without loss of generality that n is a prime power.

Cyclic length. Suppose $D \simeq D_0 \otimes_F D_1 \otimes_F \cdots \otimes_F D_r$ is a decomposition into cyclic F-division algebras of period dividing n. Since k is finite, $\overline{D}/k$ is a finite cyclic field extension, and since disjoint division algebras cannot have common subfields, at most one of the D_i, say D_0, has a (nontrivial) unramified subfield, which maps onto $\overline{D}$. If D has no unramified subfields, we set $D_0 = F$. Let $T = D_1 \otimes_F \cdots \otimes_F D_r$. Since T is tame and has no unramified subfields, it is totally ramified. The valuation v extends uniquely to each D_i, since F is henselian, and for $i \geq 1$ the Γ_{D_i}/Γ_F are all rank 2 and mutually disjoint, so the rank of Γ_T/Γ_F is $2r$. Since $\Gamma_D \subset \frac{1}{n}\Gamma_F$, the rank of Γ_D/Γ_F is at most d, so $2r \leq d$. Since D_0 is not inertial, Γ_{D_0}/Γ_F has rank at least 1, so D_0 can only contribute to a maximum cyclic length if d is odd, and then we compute the maximum is $1 + (d-1)/2 = (d+1)/2$. If d is even, then D_0 isn't necessary to achieve the maximum, which is $r = d/2$. Thus in any case $\lfloor \frac{d+1}{2} \rfloor$ is an upper bound to the cyclic length, and this is realized by Example 7.3.

Brauer dimension. Let D be an F-division algebra of period n, and as in Section 3.2, let S and T be semiramified and totally ramified division algebras such that $D \sim S \otimes_F T$, $\Gamma_D = \Gamma_S + \Gamma_T$, and $[D : F] = [\Gamma_D : \Gamma_T]^2[\Gamma_T : \Gamma_F]$. Since k is finite and S is semiramified, S is cyclic of degree dividing n, and Γ_D/Γ_T is cyclic of order dividing n. The group Γ_T/Γ_F has even rank at most d, the rank of

$\frac{1}{m}\Gamma_F/\Gamma_F$. If the rank equals d, and the minimum exponent of a summand is m_0, then Γ_D/Γ_T has order dividing n/m_0, hence $[D:F]$ divides n^2m^{d-2}, with d even. If the rank is less than d, then we could have $\Gamma_D/\Gamma_T = n$, and Γ_T/Γ_F has order dividing m^{d-1} if d is odd, and order dividing m^{d-2} if d is even. Therefore $[D:F]$ divides n^2m^{d-1} if d is odd, n^2m^{d-2} if d is even. In any case, we find $\operatorname{ind}(D)$ divides $nm^{\lfloor\frac{d-1}{2}\rfloor}$, which is the lower bound of Example 7.3. We compute from this that the (tame) Brauer dimension is $\lfloor\frac{d+1}{2}\rfloor$, and the maximum period-index ratio is thus $m^{\lfloor\frac{d-1}{2}\rfloor}$, as claimed. Note the latter remains true if $m = 1$. □

8. Wedge product

Since ${}_n\mathrm{Br}(F)$ is generated by cyclic classes, the cup product map

$$ {}_n\mathrm{H}^1(F, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{Z}^1(F, \mu_n) \longrightarrow {}_n\mathrm{Br}(F)$$

is surjective. In this section we explore the extent to which the cokernel is a wedge product.

Example 8.1. Assume (3.1), with n a power of ℓ. Let χ_0 be the Frobenius character of order n, set $x_0^{1/n} = \zeta_{n(q-1)}^q$, and let $\{x_1^{1/n}, \dots, x_d^{1/n}\}$ be a uniformizer basis for $F^{\times 1/n}/F^\times$. If we put $\chi_i = (x_i)_m^*$ for $i \geq 1$ then $\underline{\chi} = \{\chi_0, \dots, \chi_d\}$ is a basis for ${}_n\mathrm{X}(F)$ by Theorem 5.1, and $\underline{x} = \{x_0^{1/n}, \dots, x_d^{1/n}\}$ is a basis for $\mathrm{Z}^1(F, \mu_n)$ by Section (4). In the following we will use two properties of this basis:

(a) $(\chi_0, x_0) = 0$
(b) $\frac{n}{m}\chi_0 = (x_0)_m^*$ and $\chi_i = (x_i)_m^*$

(a) is by Wedderburn's Theorem, since (χ_0, x_0) is defined over the finite field k. The first part of (b) is because $\frac{n}{m}\chi_0$ and $(x_0)_m^*$ are defined over k, and agree on the Frobenius automorphism, the second part is by definition.

Theorem 8.2. *Assume* (3.1)*, with n a power of ℓ. Let $\underline{\chi}\times\underline{x}$ be a basis satisfying* (a),(b) *of Example* 8.1*, put*

$$\theta = \sum_{i=0}^d b_i\chi_i \in {}_n\mathrm{X}(F) \quad \textit{and} \quad t^{1/n} = \prod_{i=0}^d x_i^{c_i/n} \in \mathrm{Z}^1(F, \mu_n)$$

and assume $|t^{1/n}| = n$. Consider the equations

$$(8.1) \qquad b_0c_j = \tfrac{n}{m}c_0b_j (\mathrm{mod}\ n) \quad \textit{and} \quad b_ic_j \equiv c_ib_j (\mathrm{mod}\ m) \quad \forall\ i, j \geq 1$$

(a) *If $\theta \in {}_n2\mathrm{X}(F)$ or $|\theta| \neq m$, then* (8.1) *is equivalent to $(\theta, t) = 0$.*
(b) *If* (8.1) *holds, then $(\theta, t) = 0$ is equivalent to $\theta \in {}_n2\mathrm{X}(F)$ or $|\theta| \neq m$.*
(c) *Let $g = \gcd(|\theta|, m)$. Then* (8.1) *implies $\langle\frac{|\theta|}{g}\theta\rangle = \langle(t)_g^*\rangle$.*

Proof. Compute using properties (a),(b) of Example 8.1,

$$\begin{aligned}(\theta, t) &= (\sum_{i=0} b_i\chi_i, \prod_{i=0} x_i^{c_i}) = (b_0\chi_0, \prod_{i=0} x_i^{c_i}) + (\sum_{i=1} b_i\chi_i, x_0^{c_0}) + (\sum_{i=1} b_i\chi_i, \prod_{i=1} x_i^{c_i}) \\ &= \sum_{i=1}(b_0c_i - \tfrac{n}{m}c_0b_i)(\chi_0, x_i) + \sum_{1\leq i<j}(b_ic_j - b_jc_i)(x_i, x_j)_m + \sum_{i=1} b_ic_i(x_i, x_i)_m\end{aligned}$$

If $\theta \in {}_n2\,\mathrm{X}(F)$ then $b_i c_i (x_i, x_i)_m = 0$ for each $i \geq 1$ by (2.1) and Albert's criterion, so $(\theta, t) = 0$ if and only if (8.1) holds.

Suppose $\theta \notin {}_n2\,\mathrm{X}(F)$, so $\ell = 2$, $m = 2^{\mathrm{v}_2(q-1)}$ divides $|\theta|$, and $2 \nmid b_i$ for some $i \geq 1$. If $|\theta| \neq m$ then $|\theta| = |b_0\chi_0| > m$, so $n/m \nmid b_0$, and (8.1) implies $2|\, c_i$ for each $i \geq 1$, hence $(\theta, t) = 0$. Conversely, if $|\theta| \neq m$ and $(\theta, t) = 0$, then $m(\theta, t) = \sum_{i=1} m b_0 c_i (\chi_0, x_i) = 0$, and since $n/m \nmid b_0$, and (χ_0, x_i) has order n, we must have $2|\, c_i$ for each $i \geq 1$. Therefore $\sum_{i=1} b_i c_i (x_i, x_i)_m = 0$, and (8.1) holds. This proves (a).

Assume (8.1). If $\theta \in {}_n2\,\mathrm{X}(F)$ or $|\theta| \neq m$ then $(\theta, t) = 0$ by (a). Conversely, if $(\theta, t) = 0$, then because of (8.1), $b_i c_i (x_i)^*_m \in {}_n2\,\mathrm{X}(F)$ for each $i \geq 1$ by Albert's criterion, and

$$c_i\theta - b_i(t)^*_m = (b_0 c_i - \tfrac{n}{m} c_0 b_i)\chi_0 + \sum_{j=1} (b_j c_i - b_i c_j)(x_j)^*_m = 0 \qquad (i \geq 1) \tag{8.2}$$

If $(\theta, t) = 0$ and $|\theta| = m$, then since $t^{1/n}$ has order n, $|\theta| = |(t)^*_m| = m$ hence b_i, c_i are units by (8.2). Since $b_i c_i (x_i)^*_m \in {}_n2\,\mathrm{X}(F)$, we must have $m \neq 2^{\mathrm{v}_2(q-1)}$, hence $\theta \in {}_n2\,\mathrm{X}(F)$. This proves (b).

To prove (c), suppose (8.1), then

$$c_0 \tfrac{n}{m}\theta - b_0(t)^*_m = (c_0 b_0 - b_0 c_0)\tfrac{n}{m}\chi_0 + \sum_{j=1} (\tfrac{n}{m} c_0 b_j - b_0 c_j)(x_j)^*_m = 0$$

If $\ell \nmid b_i$ for some $i \geq 0$, then θ is not extendable in ${}_n\mathrm{X}(F)$, and $b_i(t)^*_m$ has order m, hence $\langle \frac{|\theta|}{m}\theta \rangle = \langle (t)^*_m \rangle$ by (8.2). If θ extends to a non-extendable θ' in ${}_n\mathrm{X}(F)$, then $\frac{m}{g}\frac{|\theta'|}{m}\theta' = \frac{|\theta|}{g}\theta$, hence $\langle \frac{|\theta|}{g}\theta \rangle = \langle (t)^*_g \rangle$. Therefore (8.1) implies $\langle \frac{|\theta|}{g}\theta \rangle = \langle (t)^*_g \rangle$ in any case. □

Remark 8.3. The relations (8.1) are that the matrix

$$\begin{bmatrix} \theta \\ t^{1/n} \end{bmatrix} = \begin{bmatrix} b_0 & \frac{n}{m} b_1 & \cdots & \frac{n}{m} b_d \\ c_0 & c_1 & \cdots & c_d \end{bmatrix} (\mathrm{mod}\ n)$$

have rank one, a linear dependence condition. By Theorem 8.2(a), this "linear dependence" is not by itself enough to imply $(\theta, t) = 0$. For example, if $m = 2^{\mathrm{v}_2(q-1)}$, $t^{1/n}$ is a uniformizer, and $\theta = (t)^*_m$, then (8.1) holds, but $(t, t)_m \neq 0$. Therefore to achieve our goal of casting ${}_n\mathrm{Br}(F)$ as a wedge product, we will restrict to the subgroup ${}_n2\,\mathrm{Br}(F) \leq {}_n\mathrm{Br}(F)$ defined by characters in ${}_n2\,\mathrm{X}(F)$. To that end we have Corollary 8.4.

Corollary 8.4. *Assume* (3.1), *and* $\underline{\chi} \times \underline{x}$ *is a basis for* ${}_n2\,\mathrm{X}(F) \times \mathrm{Z}^1(F, \mu_n)$ *satisfying*

(a′) $(\chi_0, x_0) = 0$.
(b′) $\frac{|\chi_i|}{g'_i}\chi_i = (x_i)^*_{g'_i}$, *where* $g'_i = \gcd(|\chi_i|, m')$, *for* $i \geq 0$.

Suppose $\theta = \sum_{i=0}^d b_i \chi_i \in {}_n2\,\mathrm{X}(F)$ *and* $t^{1/n} = \prod_{i=0}^d x_i^{c_i/n} \in \mathrm{Z}^1(F, \mu_n)$, *with* $|t^{1/n}| = n$. *Then* $(\theta, t) = 0$ *if and only if*

$$b_0 c_j = \tfrac{n}{m'} c_0 b_j (\mathrm{mod}\ n) \quad \textit{and} \quad b_i c_j \equiv c_i b_j (\mathrm{mod}\ m') \qquad \forall\, i, j \geq 1 \tag{8.3}$$

If $g' = \gcd(m', |\theta|)$, *then* (8.3) *implies* $\langle \frac{|\theta|}{g'}\theta \rangle = \langle (t)^*_{g'} \rangle$.

Proof. The proof that $(\theta, t) = 0$ is equivalent to (8.3) is exactly as in paragraph one of Theorem 8.2's proof, with m' in place of m, using (a)$'$ and (b)$'$, and noting that $\theta \in {}_n2\,\mathrm{X}(F)$ by hypothesis. Similarly, if (8.3) holds then imitating the proof of Theorem 8.2, we have

$$c_0\tfrac{n}{m'}\theta - b_0(t)^*_{m'} = (c_0b_0 - b_0c_0)\tfrac{n}{m'}\chi_0 + \sum_{j=1}(\tfrac{n}{m'}c_0b_j - b_0c_j)(x_j)^*_{m'} = 0$$

$$c_i\theta - b_i(t)^*_{m'} = (b_0c_i - \tfrac{n}{m'}c_0b_i)\chi_0 + \sum_{j=1}(b_jc_i - b_ic_j)(x_j)^*_{m'} = 0 \qquad (i \geq 1)$$

If $\ell \nmid b_i$ for some $i \geq 0$, then θ is not extendable in ${}_n2\,\mathrm{X}(F)$, and $b_i(t)^*_{m'}$ has order m', hence $\langle\frac{|\theta|}{m'}\theta\rangle = \langle(t)^*_{m'}\rangle$. If θ extends to a non-extendable θ' in ${}_n2\,\mathrm{X}(F)$, then $\frac{m'}{g'}\frac{|\theta'|}{m'}\theta' = \frac{|\theta|}{g'}\theta$, hence $\langle\frac{|\theta|}{g'}\theta\rangle = \langle(t)^*_{g'}\rangle$. Therefore (8.3) implies $\langle\frac{|\theta|}{g'}\theta\rangle = \langle(t)^*_{g'}\rangle$ in any case. □

Definition 8.5. Assume (3.1), with n a power of ℓ.

(a) $\theta \in {}_n2\,\mathrm{X}(F)$ and $t^{1/n} \in \mathrm{Z}^1(F, \mu_n)$ are *matched* if they satisfy (8.3) with respect to a basis satisfying (a)$'$ and (b)$'$ of Corollary 8.4, and we have the normalization $\frac{|\theta|}{g'}\theta = (t)^*_{g'}$, where $g' = \gcd(|\theta|, m')$, which in the above notation is equivalent to

$$b_0 \equiv \tfrac{n}{|\theta|}c_0 (\mathrm{mod}\ \tfrac{ng'}{|\theta|}) \quad \text{and} \quad \tfrac{|\theta|}{g'}b_i \equiv \tfrac{m'}{g'}c_i (\mathrm{mod}\ m') \quad (i \geq 1)$$

(b) Let $C \leq {}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n)$ be the subgroup generated by elements $\theta \otimes t^{1/n}$ such that θ and $t^{1/n}$ are matched, and define

$${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n) := \frac{{}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n)}{C}$$

Write $\theta \wedge t^{1/n}$ in place of $\theta \otimes t^{1/n} + C$.

The definition of matched pairs $(\theta, t^{1/n})$ is independent of the basis used in (8.3), since (8.3) is equivalent to $(\theta, t) = 0$ by Corollary 8.4.

Theorem 8.6. *Assume* (3.1)*, with* n *a power of a prime* ℓ*. The cup product map* ${}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n) \longrightarrow {}_n2\,\mathrm{Br}(F)$ *has kernel* C*, and induces a natural isomorphism*

$${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n) \xrightarrow{\sim} {}_n2\,\mathrm{Br}(F)$$

Proof. Let $\underline{\chi} \times \underline{x}$ be a basis for ${}_n2\,\mathrm{X}(F) \times \mathrm{Z}^1(F, \mu_n)$ of Corollary 8.4, so $|\chi_i| = m'$ for $i \geq 1$. The map ${}_n2\,\mathrm{Br}(F) \longrightarrow {}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n)$ defined by $(\chi_i, x_j) \longmapsto \chi_i \otimes x_j^{1/n}$ for $0 \leq i < j$ obviously splits the cup product map. On the other hand, if $0 \leq i < j$ we have

$$\begin{aligned}\chi_j \otimes x_i^{1/n} &= (\chi_j \otimes x_i^{1/n} + (x_i)^*_{m'} \otimes x_j^{1/n}) - (x_i)^*_{m'} \otimes x_j^{1/n} \\ &= [(\chi_j + (x_i)^*_{m'}) \otimes x_i^{1/n}x_j^{1/n} - (x_i)^*_{m'} \otimes x_i^{1/n} - \chi_j \otimes x_j^{1/n}] - (x_i)^*_{m'} \otimes x_j^{1/n}\end{aligned}$$

Since each character is in ${}_n2\,\mathrm{X}(F)$, the square-bracketed term is in C by Corollary 8.4. Since $\underline{\chi} \otimes \underline{x}$ is a basis, it follows that any element of ${}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n)$ can be expressed as a sum of an element of C and an element in the image of the splitting map. Since $C \subset \ker(\cup)$ by Definition 8.5 and Corollary 8.4, we must have $C = \ker(\cup)$, so we have the natural isomorphism. □

We next generalize the basis of Example 8.1 and Corollary 8.4.

Definition 8.7. A basis $\underline{\theta} \times \underline{t}$ for ${}_n2\,\mathrm{X}(F) \times \mathrm{Z}^1(F, \mu_n)$ is *matched* if it consists of matched pairs $(\theta_i, t_i^{1/n})$, as in Definition 8.5(a) (over only $i = 0$ if $m' = 1$), and if $m' \neq n$, $\underline{t}$ is in *standard form*, meaning $\{t_1^{1/n}, \dots, t_d^{1/n}\}$ is a uniformizer basis for $F^{\times 1/n}/F^\times$.

Remark 8.8. If $\underline{\theta} \times \underline{t}$ is a matched basis for ${}_n2\,\mathrm{X}(F) \times \mathrm{Z}^1(F, \mu_n)$, then by Corollary 8.4 and Definition 8.5(a), the pairs $(\theta_i, t_i^{1/n})$ satisfy the hypotheses (a)$'$ and (b)$'$ of Corollary 8.4, and $\underline{\theta} \times \underline{t}$ may then be used to define matched elements, by Corollary 8.4. In particular, any matched basis may be used to define C in Definition 8.5(b).

Proposition 8.9. *Assume* (3.1). *A matched basis* $\underline{\chi} \times \underline{x}$ *induces a basis* $\underline{\chi} \wedge \underline{x}$ *for* ${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n)$*, and a basis* $\underline{\chi} \cup \underline{x}$ *for* ${}_n2\,\mathrm{Br}(F)$.

Proof. If $m' = 1$ then $C = \langle \chi_0 \otimes x_0^{1/n} \rangle$, and the $\chi_0 \wedge x_j^{1/n}$ and (χ_0, x_j) for $j \geq 1$ clearly form a basis for ${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n)$ and ${}_n2\,\mathrm{Br}(F)$. Assume $m' \neq 1$. Let

$$\boldsymbol{c} = \{\chi_i \otimes x_i^{1/n}, (x_j)^*_{m'} \otimes x_k^{1/n} + \chi_k \otimes x_j^{1/n} \;:\; 0 \leq i \leq d, 0 \leq j < k \leq d\}$$

Since $(x_j)^*_{m'} \otimes x_k^{1/n} + \chi_k \otimes x_j^{1/n} = ((x_j)^*_{m'} + \chi_k) \otimes x_j^{1/n} x_k^{1/n} - (x_j)^*_{m'} \otimes x_j^{1/n} - \chi_k \otimes x_k^{1/n}$, $\langle \boldsymbol{c} \rangle \leq C$. The elements of $\boldsymbol{c}$ are independent: Every element of ${}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n)$ has a unique expression of the form $\prod_{j=0}^d \xi_j \otimes x_j^{1/n}$, for characters ξ_j, and a dependence relation

$$\sum_{i=0}^d a_i \chi_i \otimes x_i^{1/n} + \sum_{0 \leq i < j \leq d} a_{ij}((x_i)^*_{m'} \otimes x_j^{1/n} + \chi_j \otimes x_i^{1/n}) = 0$$

has $x_0^{1/n}$ term $\xi_0 = a_0 \chi_0 + \sum_{j=1}^d a_{0j} \chi_j$ and for $j \geq 1$, $x_j^{1/n}$ term

$$\xi_j = a_j \chi_j + \sum_{0 \leq i < j} a_{ij} (x_i)^*_{m'} + \sum_{0 \leq j < i \leq d} a_{ji} \chi_i$$

Since $\underline{x}$ is a basis, each ξ_j is zero. Therefore since $\underline{\chi}$ is a basis for ${}_n2\,\mathrm{X}(F)$, and each χ_i appears only once in each ξ_j, the dependence relation is trivial. Since $|\langle \boldsymbol{c} \rangle| = n m'^{d(d+3)/2}$, we conclude $\langle \boldsymbol{c} \rangle = C$, so $\boldsymbol{c}$ is a basis for C. Let $\boldsymbol{b} = \{\chi_i \otimes x_j^{1/n} : 0 \leq i < j \leq d\}$. Then $\boldsymbol{b} \cup \boldsymbol{c}$ is another basis for ${}_n2\,\mathrm{X}(F) \otimes \mathrm{Z}^1(F, \mu_n)$, which shows $\boldsymbol{b} + C = \underline{\chi} \wedge \underline{x}$ is a basis for the cokernel, hence that $\underline{\chi} \cup \underline{x}$ is a basis for ${}_n2\,\mathrm{Br}(F)$. □

9. Basis change

To use exterior algebra machinery to find minimal expressions for a given class in ${}_n2\,\mathrm{Br}(F)$, we need to characterize basis change matrices on ${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n)$. To this end we have Lemma 9.1, which characterizes integer matrices with well-defined actions on finite abelian groups that have unequal invariant factors, such as ${}_n2\,\mathrm{X}(F)$. It also lets us transition to coefficients in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, which we use to define the Hasse invariant.

Lemma 9.1. *Let* $G = \mathbb{Z}/d_1 \times \cdots \times \mathbb{Z}/d_r$ *and* ${}^*G = \frac{1}{d_1}\mathbb{Z}/\mathbb{Z} \oplus \cdots \oplus \frac{1}{d_r}\mathbb{Z}/\mathbb{Z}$ *be abelian groups with invariant factors* d_i*, and let* $d_{ij} = \gcd(d_i, d_j)$*.*

(a) *Each* $\rho \in \mathrm{Aut}(G)$ *is representable by a* $P = (p_{ij}) \in \mathrm{M}_r(\mathbb{Z})$ *that satisfies* $\frac{d_i}{d_{ij}} \mid p_{ij}$*.*

(b) *Each* ${}^*\rho \in \mathrm{Aut}({}^*G)$ *is representable by a* ${}^*P = (p_{ij}) \in \mathrm{M}_r(\mathbb{Z})$ *that satisfies* $\frac{d_j}{d_{ij}} \mid p_{ij}$*.*

(c) *Any* $P \in \mathrm{Aut}(G)_{\mathbb{Z}}$ *or* ${}^*P \in \mathrm{Aut}({}^*G)_{\mathbb{Z}}$ *determines a* ρ *or* ${}^*\rho$ *as in* (a) *or* (b)*, and* $P = (p_{ij})$ *and* $P' = (p'_{ij})$ *determine the same* ρ *if and only if* $p_{ij} \equiv p'_{ij} (\mathrm{mod}\ d_i)$*.*

(d) *Let* $\mathrm{Aut}(G)_{\mathbb{Z}}$ *and* $\mathrm{Aut}({}^*G)_{\mathbb{Z}}$ *denote the semigroups of such matrices. There is a natural bijection*

$$\begin{aligned} \mathrm{Aut}(G)_{\mathbb{Z}} &\longleftrightarrow \mathrm{Aut}({}^*G)_{\mathbb{Z}} \\ P = (p_{ij}) &\longmapsto {}^*P = ({}^*p_{ij}) = (\tfrac{d_j}{d_i} p_{ij}) \\ Q^* = (q^*_{ij}) = (\tfrac{d_i}{d_j} q_{ij}) &\longleftarrow\!\shortmid\ Q = (q_{ij}) \end{aligned}$$

making a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{P} & G \\ \downarrow\wr & & \downarrow\wr \\ {}^*G & \xrightarrow[{}^*P]{} & {}^*G \end{array}$$

Proof. See also [11]. It is clear that any $\rho \in \mathrm{Aut}(G)$ is representable with respect to the standard basis by an integer matrix (p_{ij}), whose ij-th entry maps $\mathbb{Z}/d_j$ to $\mathbb{Z}/d_i$, which is well-defined if and only if p_{ij} is divisible by d_i/d_{ij}. Similarly ${}^*\rho$ is representable by a $({}^*p_{ij})$, and ${}^*p_{ij} : \frac{1}{d_j}\mathbb{Z}/\mathbb{Z} \to \frac{1}{d_i}\mathbb{Z}/\mathbb{Z}$ is well-defined if and only if ${}^*p_{ij}$ is divisible by d_j/d_{ij}. Two matrices (p_{ij}) and (p'_{ij}) determine the same ρ if and only if $[p_{ij}]_{d_i} = [p'_{ij}]_{d_i}$ for each i, j, if and only if $p_{ij} \equiv p'_{ij} (\mathrm{mod}\ d_i)$. This proves (a), (b), and (c), and (d) follows from the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/d_j & \xrightarrow{\frac{d_i}{d_{ij}}} & \mathbb{Z}/d_i \\ \downarrow\wr & & \downarrow\wr \\ \frac{1}{d_j}\mathbb{Z}/\mathbb{Z} & \xrightarrow[\frac{d_j}{d_{ij}}]{} & \frac{1}{d_i}\mathbb{Z}/\mathbb{Z} \end{array}$$

□

9.1. **Character bases.** Assume (3.1), with n a power of ℓ. By Theorem 5.1, ${}_n2\,\mathrm{X}(F) \simeq \mathbb{Z}/n$ if $m' = 1$, and ${}_n2\,\mathrm{X}(F) \simeq \mathbb{Z}/n \times (\mathbb{Z}/m')^d$ if $m' \neq 1$. When $m' \neq n$, a given basis for ${}_n2\,\mathrm{X}(F)$ is in standard form as in Definition 8.7, i.e., ordered with the order-n element first. If $\underline{\theta}$ and $\underline{\chi}$ are bases, there is a basis change $[\mathrm{id}]^{\underline{\theta}}_{\underline{\chi}}$, and by Lemma 9.1 it is given by a matrix $R = (r_{ij}) \in \mathrm{Aut}({}_n2\,\mathrm{X}(F))_{\mathbb{Z}}$. If $m' = 1$, $R = [r_{00}]$ with r_{00} prime-to-ℓ. If $m' \neq 1$ then $\frac{n}{m'} \mid r_{0j}$ for $j \geq 1$. If $m' \neq n$, r_{00} and R_{00} are invertible (mod ℓ), where R_{00} is the submatrix obtained by deleting row and column 0.

9.2. **Matched basis change.** If $\underline{\chi} \times \underline{x}$ is a matched basis, then by Remark 8.8, $\underline{\theta} \times \underline{t}$ is a matched basis if and only if $\underline{\theta} \times \underline{t} = \underline{\chi} \times \underline{x}(R \oplus S)$ for some $R = (r_{ij})$ and $S = (s_{ij})$ satisfying

$$(9.1) \quad r_{0j}s_{ij} \equiv \tfrac{n}{m'}s_{0j}r_{ij}(\text{mod } n) \quad \text{and} \quad r_{ij}s_{kj} \equiv r_{kj}s_{ij}(\text{mod } m') \quad (i \geq 1, j \geq 0)$$

and the normalization equations of Definition 8.5(a),

$$\begin{aligned} r_{00} &\equiv s_{00}(\text{mod } m'), \quad r_{0j} \equiv \tfrac{n}{m'}s_{0j}(\text{mod } n), \quad \text{for } j \geq 1 \\ \tfrac{n}{m'}r_{i0} &\equiv s_{i0}(\text{mod } m'), \quad r_{ij} \equiv s_{ij}(\text{mod } m'), \quad \text{for } i, j \geq 1 \end{aligned}$$

If $m' = 1$, $R = [r_{00}]$ is invertible, hence (9.1) implies $s_{i0} \equiv 0(\text{mod } n)$. We call such $R \oplus S \in \text{Aut}({}_n2\,\mathrm{X}(F))_{\mathbb{Z}} \oplus \text{Aut}(\mathrm{Z}^1(F, \mu_n))_{\mathbb{Z}}$ *matched basis changes*, and write $R \oplus S = [\text{id}]^{\underline{\chi} \times \underline{x}}_{\underline{\theta} \times \underline{t}}$.

If $P \in \text{Aut}(\mathrm{Z}^1(F, \mu_n))_{\mathbb{Z}}$ preserves standard form, and $\frac{n}{m'} \mid p_{i0}$, then $P^* \oplus P$ is a matched basis change matrix, since $p^*_{i0} = \frac{m'}{n}p_{i0}$ and $p^*_{0j} = \frac{n}{m'}p_{0j}$ for $i, j \geq 1$, and $p^*_{ij} = p_{ij}$ otherwise. This type will suffice below, though it is not the most general kind.

Proposition 9.2. *Assume* (3.1)*, with n a power of ℓ. A matched basis change $P^* \oplus P = [\text{id}]^{\underline{\chi} \times \underline{x}}_{\underline{\theta} \times \underline{t}}$ induces on ${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n)$ the basis change $P^{*\wedge 2} = [\text{id}]^{\underline{\chi} \wedge \underline{x}}_{\underline{\theta} \wedge \underline{t}}$.*

Proof. We ignore the "zero" basis elements of ${}_n2\,\mathrm{X}(F)$ when $m' = 1$, but keep the degree-$(d+1)$ matrix P^* in order to define $P^{*\wedge 2}$. The matched bases induce bases $\underline{\theta} \wedge \underline{t}$ and $\underline{\chi} \wedge \underline{x}$ on ${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n)$ by Proposition 8.9. Let $P = (p_{ij})$. Since $p^*_{0j} = \frac{n}{m'}p_{0j}$ and $p^*_{jl} = p_{jl}$ for $j, l \geq 1$ by Lemma 9.1, we compute for $l \geq 1$,

$$\begin{aligned} \theta_0 \wedge t_l^{1/n} &= \sum_{i=0} p^*_{i0}\chi_i \wedge \prod_{j=0} x_j^{p_{jl}/n} = \sum_{i \neq j} p^*_{i0}p_{jl}\chi_i \wedge x_j^{1/n} \\ &= \sum_{1 \leq j} (p^*_{00}p_{jl} - p^*_{j0}p_{0l}\tfrac{n}{m'})\chi_0 \wedge x_j^{1/n} + \sum_{1 \leq i < j} (p^*_{i0}p_{jl} - p^*_{j0}p_{il})\chi_i \wedge x_j^{1/n} \\ &= \sum_{1 \leq j} (p^*_{00}p^*_{jl} - p^*_{j0}p^*_{0l})\chi_0 \wedge x_j^{1/n} + \sum_{1 \leq i < j} (p^*_{i0}p^*_{jl} - p^*_{j0}p^*_{il})\chi_i \wedge x_j^{1/n} \end{aligned}$$

When $m' = 1$ the only nonzero entries are $p^*_{00}p^*_{jl}\,\chi_0 \wedge x_j^{1/n}$. For $1 \leq k < l$, when $m' \neq 1$,

$$\begin{aligned} \theta_k \wedge t_l^{1/n} &= \sum_{1 \leq j} (p^*_{0k}p_{jl} - p^*_{jk}p_{0l}\tfrac{n}{m'})\chi_0 \wedge x_j^{1/n} + \sum_{1 \leq i < j} (p^*_{ik}p_{jl} - p^*_{jk}p_{il})\chi_i \wedge x_j^{1/n} \\ &= \sum_{1 \leq j} (p^*_{0k}p^*_{jl} - p^*_{jk}p^*_{0l})\chi_0 \wedge x_j^{1/n} + \sum_{1 \leq i < j} (p^*_{ik}p^*_{jl} - p^*_{jk}p^*_{il})\chi_i \wedge x_j^{1/n} \end{aligned}$$

Thus the basis change is $P^{*\wedge 2}$. If Q and P give the same matched basis change, then $Q \equiv P(\text{mod } n)$, hence $Q^{*\wedge 2}$ and $P^{*\wedge 2}$ produce the same basis change on ${}_n2\,\mathrm{X}(F) \times \mathrm{Z}^1(F, \mu_n)$. Therefore the transformation is well-defined, and this completes the proof. ∎

10. Computing in $2\,\mathrm{Br}(F)$ with skew-symmetric matrices

By Theorem 8.6, for any n not divisible by p we have a natural isomorphism

$$ {}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F,\mu_n) \xrightarrow{\sim} {}_n2\,\mathrm{Br}(F) $$

The bases of Proposition 8.9 determine for each element a skew-symmetric matrix, which we will use to compute the index and decomposition into cyclic division algebras of each Brauer class.

Theorem 10.1. *Assume* (3.1), *with n a power of ℓ. A matched basis $\underline{\chi} \times \underline{x}$ determines an injective homomorphism*

$$ \mathrm{alt}_{\underline{\chi}\times\underline{x}} : {}_n2\,\mathrm{Br}(F) \longrightarrow \mathrm{Alt}_{d+1}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) $$

defined by $[(\chi_i, x_j)]_{\underline{\chi}\times\underline{x}} = \frac{1}{|\chi_i|}(e_{ij} - e_{ji})$ for $0 \le i < j \le d$. A matched basis change $P^* \oplus P = [\mathrm{id}]^{\underline{\theta}\times\underline{t}}_{\underline{\chi}\times\underline{x}}$ induces a commutative diagram

$$ \begin{array}{ccc} & & \mathrm{Alt}_{d+1}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \\ & \nearrow^{\mathrm{alt}_{\underline{\chi}\times\underline{x}}} & \\ {}_n2\,\mathrm{Br}(F) & & \Big\downarrow{\scriptstyle P - P^{\mathrm{t}}} \\ & \searrow_{\mathrm{alt}_{\underline{\theta}\times\underline{t}}} & \\ & & \mathrm{Alt}_{d+1}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \end{array} $$

Conversely, congruence transformation by any $P \in \mathrm{M}_{d+1}(\mathbb{Z})$ satisfying $\frac{n}{m'} \mid p_{i0}$ for $i \ge 1$, and such that P is invertible (mod ℓ)*, and both p_{00} and P_{00} are invertible* (mod ℓ) *when $m' \ne n$, is induced by the matched basis change matrix $P^* \oplus P = [\mathrm{id}]^{\underline{\theta}\times\underline{t}}_{\underline{\chi}\times\underline{x}} \in \mathrm{Aut}({}_n2\,\mathrm{X}(F))_{\mathbb{Z}} \oplus \mathrm{Aut}(\mathrm{Z}^1(F,\mu_n))_{\mathbb{Z}}$.*

Proof. A basis $\underline{\chi} \wedge \underline{x}$ for ${}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F,\mu_n)$ determines a map

$$ {}_n2\,\mathrm{X}(F) \wedge \mathrm{Z}^1(F,\mu_n) \longrightarrow \mathrm{Alt}_{d+1}(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) $$

by $\chi_i \wedge x_j^{1/n} \longmapsto \frac{1}{|\chi_i|}(e_{ij} - e_{ji})$. The basis change $P^* \oplus P = [\mathrm{id}]^{\underline{\theta}\times\underline{t}}_{\underline{\chi}\times\underline{x}}$ induces a basis change $P^{*\wedge 2} = [\mathrm{id}]^{\underline{t}^{\wedge 2}}_{\underline{x}^{\wedge 2}}$ by Proposition 9.2, which on $\mathrm{Alt}_{d+1}(\mathbb{Z}/n)$ is congruence transformation by P^*, but on $\mathrm{Alt}_{d+1}(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ is congruence transformation by $P = {}^*P^*$, as per Lemma 9.1. Composing with the inverse of the natural isomorphism of Theorem 8.6 yields the result.

Conversely, congruence transformation by any P satisfying the stated conditions is induced by the matched basis change $P^* \oplus P$ by (9.2), Proposition 9.2, and what was just proved. □

Remark 10.2. When F is a (1-dimensional) local field, and n is prime-to-p, identifying $\mathrm{Alt}_2(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ with $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ yields the (canonical) Hasse invariant map

$$ \mathrm{inv}_F : {}_n\mathrm{Br}(F) \xrightarrow{\sim} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} $$

It is canonical because the map determined by the Frobenius character χ_0 and any uniformizer give the same map into $\mathrm{Alt}_2(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$: Any basis change matrix P^* for

${}_n 2\,\mathrm{X}(F)$ that preserves χ_0 and preserves the value $1 (\mathrm{mod}\ n)$ of the uniformizer has the form

$$P^* = \begin{bmatrix} 1 & p_{01}^* \\ 0 & 1 \end{bmatrix} (\mathrm{mod}\ n)$$

hence $P^{*\wedge 2} = [1]$. This shows the map ${}_n 2\,\mathrm{Br}(F) \to \mathrm{Alt}_2(\frac{1}{n}\mathbb{Z}/\mathbb{Z}) \simeq \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is independent of the basis used to define it.

We next prove the main result, that the skew-symmetric matrix assigned to a Brauer class computes its index, via the square root of the determinant, and the decomposition of its associated division algebra into cyclic division algebras, via a symplectic basis.

10.1. **Pfaffian.** The determinant of a skew-symmetric matrix over a commutative ring is zero if the matrix has odd degree, and it is a square if the matrix has even degree. The positive square root is called the *pfaffian*. Computations over $\mathbb{Q}/\mathbb{Z}$ can be well-defined as follows.

A degree-r *skew-symmetric submatrix* S of a skew-symmetrix matrix A is the submatrix obtained by intersecting some set of r rows $i_1, \dots, i_r$ with the columns $i_1, \dots, i_r$.

Definition 10.3. Suppose $A \in \mathrm{Alt}_{d+1}(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$.

(a) The *row subgroup* $\mathrm{row}(A)$ is the subgroup of $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^{d+1}$ generated by A's rows.
(b) The *pfaffian subgroup* $\mathrm{pfaff}(A)$ is the subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by the pfaffians of all even-degree skew-symmetric submatrices of A, computed naïvely.

This is [5, Definition 2.7]. "Computed naïvely" means computed from an arbitrary representative of A in $\mathrm{M}_{d+1}(\mathbb{Q})$, then interpreted $(\mathrm{mod}\ \mathbb{Z})$. For a given skew-symmetric A, $\mathrm{pfaff}(A)$ is well-defined by cofactor expansion, and is invariant under congruence transformation by [5, Proposition 2.8, Lemma 2.9]. The isomorphism class of $\mathrm{row}(A)$ is also preserved, since congruence transformations induce automorphisms of $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^{d+1}$.

Since $\mathrm{row}(A)$ and $\mathrm{pfaff}(A)$ are invariant under congruence transformation, they can be computed from a 2-block matrix congruent to A. Thus if

$$A \sim \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} a_{2i\,2i+1}(e_{2i\,2i+1} - e_{2i+1\,2i})$$

for $a_{2i\,2i+1} \in \mathbb{Q}/\mathbb{Z}$ and $d_i = |a_{2i\,2i+1}|$ is the order of the i-th block, then

$$\mathrm{row}(A) \simeq \prod_{i=0}^{r} \tfrac{1}{d_i}\mathbb{Z}/\mathbb{Z} \times \tfrac{1}{d_i}\mathbb{Z}/\mathbb{Z}$$

$$\mathrm{pfaff}(A) = \left\langle \tfrac{1}{d_1 d_2 \cdots d_r} \right\rangle$$

where $r = \lfloor \frac{d-1}{2} \rfloor$. The numbers $d_0, d_0, d_1, d_1, \dots, d_r, d_r$, where $d_i \geq 1$, are the invariant factors of the finite abelian group $\mathrm{row}(A)$.

Theorem 10.4. *Assume* (3.1), *with n a power of ℓ. Let $\underline{\chi}\times\underline{x}$ be a matched basis, with respect to which $\alpha \in {}_n2\,\mathrm{Br}(F)$ has skew-symmetric matrix $\mathrm{alt}_{\underline{\chi}\times\underline{x}}(\alpha) = A \in \mathrm{Alt}_{d+1}(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$. Then*

$$\mathrm{ind}(\alpha) = |\mathrm{pfaff}(A)|$$

If $d_0, d_0, \dots, d_r, d_r$ are the invariant factors of $\mathrm{row}(A)$, where $r = \lfloor \frac{d-1}{2} \rfloor$, then there exists a matched basis $\underline{\theta}\times\underline{t}$ for ${}_n2\,\mathrm{X}(F)\times \mathrm{Z}^1(F,\mu_n)$ with respect to which

$$\alpha = \tfrac{n}{d_0}(\theta_0, t_1) + (t_2, t_3)_{d_1} + \cdots + (t_{2r}, t_{2r+1})_{d_r}$$

and α's division algebra decomposes into cyclic division algebras

$$\Delta(\alpha) = \Delta(\tfrac{n}{d_0}(\theta_0, t_1)) \otimes_F \Delta((t_2, t_3)_{d_1}) \otimes_F \cdots \otimes_F \Delta((t_{2r}, t_{2r+1})_{d_r})$$

Proof. We may assume without loss of generality that $\alpha \in {}_n2\,\mathrm{Br}(F)$ has order n.

Suppose $m' = 1$, so m divides 2. Since $\chi_0 \in {}_n2\mathrm{X}(F)$, χ_0 is unramified by Theorem 5.1. By Theorem 6.3, $\alpha = \sum_{j=1}^d (\chi_0, x_j^{a_j})$, for some integers a_j, and $\mathrm{alt}_{\underline{\chi}\times\underline{x}}(\alpha) = \sum_{j=1}^d a_{0j}(e_{0j} - e_{j0})$, where $a_{0j} = a_j/n$. Since $|\alpha| = n$, at least one of the a_j is a unit (mod n). Therefore $t_1^{1/n} = x_1^{a_1/n} \cdots x_d^{a_d/n}$ is part of a uniformizer basis via a basis change on $\mathrm{Z}^1(F,\mu_n)$ of the form $P = [1] \oplus P_{00}$, hence part of a matched basis $\{\theta_0\}\times\underline{t}$ obtained by $P^* \oplus P$. Then $\alpha = (\theta_0, t_1)$, and the corresponding central simple algebra $\Delta(\theta_0, t_1)$ is a division algebra, since t_1^e is a norm from $F(\theta_0)$ if and only if $\frac{e}{n}\mathrm{v}(t_1) \in \Gamma_{F(\theta_0)} = \Gamma_F$, if and only if $n|e$ (see Reiner [20, Theorem 14.1, p.143]). The resulting congruence transformation on $\mathrm{Alt}_{d+1}(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ is $P - P^{\mathrm{t}}$ by Theorem 10.1, so that $\mathrm{alt}_{\underline{\theta}\times\underline{t}}(\alpha) = PAP^{\mathrm{t}} = \frac{1}{n}(e_{01} - e_{10})$. Since there is only one 2-block in $\mathrm{alt}_{\underline{\theta}\times\underline{t}}(\alpha)$, the row subgroup $\mathrm{row}(A)$ is isomorphic to $\frac{1}{n}\mathbb{Z}/\mathbb{Z}\times\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, and the pfaffian subgroup $\mathrm{pfaff}(A)$ is $\left\langle \frac{1}{n} \right\rangle \le \mathbb{Q}/\mathbb{Z}$, so $|\mathrm{pfaff}(A)| = n$. This completes the proof when $m' = 1$.

Assume $m' \ne 1$. Any matched basis transforms into one with unramified character χ_0 using a matrix of form $P^* \oplus P$, defining the first column of P^* to give an unramified χ_0, using the identity for the other elements, and setting $P = {}^*P^*$. Since the resulting congruence transformation on $\mathrm{Alt}_{d+1}(\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ in Theorem 10.1 does not affect $\mathrm{row}(A)$ or $\mathrm{pfaff}(A)$, we may assume χ_0 is unramified. We now construct a new basis $\underline{\theta} \wedge \underline{t}$, with respect to which α has a "linked 2-block form", and such that $\theta_0 = \chi_0$. Let

(a) $C_{ij}(c) = I + ce_{ij}$ for $0 \le i \ne j \le d$
(b) $D_j(u) = I + (u-1)e_{jj}$ for a unit u, and $0 \le j \le d$
(c) $E_{ij} = I - e_{ii} - e_{jj} + e_{ij} + e_{ji}$ for $0 \le i \ne j \le d$

When $P = C_{ij}(c) = I + ce_{ij}$ for $i \ne j$, PAP^{t} replaces the i-th row and column with $\mathrm{row}(i) + c\cdot\mathrm{row}(j)$ and $\mathrm{col}(i) + c\cdot\mathrm{col}(j)$. When $P = D_j(u)$, PAP^{t} multiplies the j-th row and column by u. When $P = E_{ij}$, PAP^{t} permutes the row/column i and row/column j.

By "apply P" we mean "apply the congruence transformation $P \cdot P^{\mathrm{t}}$".

I. Write $A = (a_{ij})$, where as usual $0 \le i, j \le d$. If any of the a_{0j} are nonzero for $j \ge 1$, apply $C_{1j}(1)$ as necessary so that $|a_{01}|$ has maximal order among the a_{0j}. Then apply $D_1(u)$, if necessary, so that $a_{01} = 1/|a_{01}|$. For each $j : a_{0j} \ne 0$ and $1 < j$, apply $C_{j1}(b)$ as necessary, so that $a_{0j} = 0$ for $1 < j$. The first row and column of the transformed matrix A now have at most one nonzero entry, at locations $(0,1)$ and $(1,0)$. The basis change matrix has the form $P = [1] \oplus B$ for some n-invertible $B \in \mathrm{M}_d(\mathbb{Z})$.

II. Let $c = 1$. Repeat for as long as $c \le d-1$ and $a_{ij} \ne 0$ for some $i, j : c \le i < j \le d$:

(a) Suppose $a_{i_0 j_0}$ has maximal order among all a_{ij} with $c \le i < j \le d$. If $c < i_0$, apply $C_{ci_0}(1)$ as necessary so that $a_{c j_0}$ has this maximal order. Note this operation does not affect $a_{k\,k+1}$ for $k < c$. If $c+1 < j_0$, apply $C_{c+1\,j_0}(1)$ as necessary to that $a_{c\,c+1}$ has maximal order among the a_{ij} with $i, j : c \le i < j \le d$. Then apply $D_{c+1}(u)$ for a unit $u (\mathrm{mod}\ n)$ so that $a_{c\,c+1} = 1/|a_{c\,c+1}|$. The composite matrix P for these operations is again of the form $[1] \oplus B$.

(b) For each $j : c+1 < j \le d$ such that $a_{cj} \ne 0$, apply $C_{j\,c+1}(b)$ as necessary so that $a_{cj} = 0$, for all $j : c+1 < j \le d$. Row c now contains two nonzero entries, $a_{c\,c+1}$ and $a_{c\,c-1}$. The composite P is again of the form $[1] \oplus B$.

(c) Increase c by 1.

If P is the composite of all of the above transformations, P has form $[1] \oplus B$, and $A' = PAP^{\mathrm{t}}$ has the "linked 2-block form"

$$A' = \sum_{i=0}^{d-1} a'_{i\,i+1}(e_{i\,i+1} - e_{i+1\,i})$$

with $|a'_{i\,i+1}| > |a'_{i+1\,i+1}|$ for $i \ge 1$, and $a'_{i\,i+1} = 1/|a'_{i\,i+1}|$. By Theorem 10.1, P is induced by the matched basis change $P^* \oplus P = [\mathrm{id}]^{\underline{\chi}' \times \underline{x}'}_{\underline{\chi} \times \underline{x}}$, so that $A' = \mathrm{alt}_{\underline{x}'}(\alpha)$, and $\alpha = \frac{n}{m_1}(\chi'_0, x'_1) + (x'_1, x'_2)_{m_2} + \cdots + (x'_{d-1}, x'_d)_{m_d}$ where $m_{i+1} | m_i$ for $i \ge 2$. Since P has the form $[1] \oplus B$, the first column of P^* is e_1, so $\chi'_0 = \chi_0$ is unramified.

To conserve notation, assume A is in the linked 2-block form, with respect to the matched basis $\underline{\chi} \times \underline{x}$, and

$$\alpha = \tfrac{n}{m_1}(\chi_0, x_1) + (x_1, x_2)_{m_2} + \cdots + (x_{d-1}, x_d)_{m_d} \tag{10.1}$$

where $m_{i+1} | m_i$ for $i \ge 2$, χ_0 is unramified, and $m_i \ge 1$. Let D be α's F-division algebra, and let S and T be the division algebras defined by

$$[S] = \tfrac{n}{m_1}(\chi_0, x_1), \quad [T] = (x_1, x_2)_{m_2} + (x_2, x_3)_{m_3} + \cdots + (x_{d-1}, x_d)_{m_d}$$

Then S is nicely semiramified and T is totally ramified, since $\underline{x}$ is in standard form, and $D \sim S \otimes_F T$. Since χ_0 is unramified and $x_1^{1/n}$ is a uniformizer, S is isomorphic to $\Delta(\frac{n}{m_1}\chi_0, x_1)$. Thus $\Gamma_S/\Gamma_F = \langle \frac{1}{m_1}\mathrm{v}(x_1) \rangle + \Gamma_F$. As in Section 3.2, we will compute $\mathrm{ind}(\alpha)$ by the formula

$$[D : F] = [\Gamma_D : \Gamma_F][\Gamma_T : \Gamma_F]$$

The linked 2-block form of a skew-symmetric matrix for $[T]$ is put in disjoint 2-block form by applying $C_{31}(m_2/m_3), C_{53}(m_4/m_5), \cdots, C_{2q_0+1\,2q_0-1}(m_{2q_0}/m_{2q_0+1})$ in succession, where $q_0 = \lceil \frac{d}{2} \rceil - 1$, and we index the rows and column starting at $i = 1$ instead of $i = 0$. The composite basis change matrix $P_{00} \in \mathrm{M}_d(\mathbb{Z})$ determines a uniformizer basis $\{t_1^{1/n}, \ldots, t_d^{1/n}\}$, by $t_{2i-1} = x_{2i-1} x_{2i+1}^{-m_{2i}/m_{2i+1}}$, and $t_{2i} = x_{2i}$, with respect to which

$$[T] = (t_1, t_2)_{m_2} + (t_3, t_4)_{m_4} + \cdots + (t_{2r_0-1}, t_{2r_0})_{m_{2r_0}}$$

where $r_0 = \lfloor \frac{d}{2} \rfloor$. The central simple algebra corresponding to this expression is well-known to be a division algebra, since $\{t_1^{1/n}, \dots, t_d^{1/n}\}$ is a uniformizer basis, cf. Example 7.3. Therefore it is isomorphic to T. We compute

$$\Gamma_T/\Gamma_F = \left\langle \tfrac{1}{m_{2i}} \mathrm{v}(t_{2i-1}), \tfrac{1}{m_{2i}} \mathrm{v}(t_{2i}) : 1 \le i \le \lfloor \tfrac{d}{2} \rfloor \right\rangle + \Gamma_F$$

hence $[\Gamma_T : \Gamma_F] = \prod_{i=1}^{\lfloor \frac{d}{2} \rfloor} m_{2i}^2$, and since $\Gamma_D = \Gamma_S + \Gamma_T$,

$$\Gamma_D/\Gamma_F = \left\langle \tfrac{1}{m_1} \mathrm{v}(x_1), \tfrac{1}{m_{2i}} \mathrm{v}(t_{2i-1}), \tfrac{1}{m_{2i}} \mathrm{v}(t_{2i}) : 1 \le i \le \lfloor \tfrac{d}{2} \rfloor \right\rangle + \Gamma_F$$

To compute $\mathrm{ind}(\alpha)$ it remains to compute $[\Gamma_D : \Gamma_F]$. The expression of $x_1^{1/n}$ in terms of the $t_i^{1/n}$ is given by the first column of the composite basis change matrix $P_{00} \in \mathrm{M}_d(\mathbb{Z})$ above. It is easy to show by induction that the first column is $e_1 + \frac{m_2}{m_3} e_3 + \frac{m_2 m_4}{m_3 m_5} e_5 + \cdots + \frac{m_2 m_4 \cdots m_{2q_0}}{m_3 m_5 \cdots m_{2q_0+1}} e_{2q_0+1}$, where $q_0 = \lceil \frac{d}{2} \rceil - 1 = \lfloor \frac{d-1}{2} \rfloor$. Therefore

$$\Gamma_D/\Gamma_F = \left\langle \sum_{i=0}^{\lceil \frac{d}{2} \rceil - 1} \frac{m_2 m_4 \cdots m_{2i}}{m_1 m_3 \cdots m_{2i+1}} \mathrm{v}(t_{2i+1}), \tfrac{1}{m_{2i}} \mathrm{v}(t_{2i-1}), \tfrac{1}{m_{2i}} \mathrm{v}(t_{2i}) : 1 \le i \le \lfloor \tfrac{d}{2} \rfloor \right\rangle + \Gamma_F$$

We replace the term $(m_2 \cdots m_{2i}/m_1 \cdots m_{2i+1}) \mathrm{v}(t_{2i+1})$ by 0 if it is in Γ_F.

Let $R \in \mathrm{M}_{d\,d+1}(\mathbb{Z})$ be the relations matrix, defined by the exact sequence

$$\mathbb{Z}^{d+1} \xrightarrow{R} \mathbb{Z}^d \longrightarrow \Gamma_D/\Gamma_F \longrightarrow 0$$

For example, if $d = 5$ and $m_1 m_3 m_5 > m_2 m_4$, then

$$R = \begin{bmatrix} m_1 & m_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 \\ \frac{m_1 m_3}{m_2} & 0 & 0 & m_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_4 & 0 \\ \frac{m_1 m_3 m_5}{m_2 m_4} & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and if $d = 6$ and $m_1 m_3 m_5 > m_2 m_4$,

$$R = \begin{bmatrix} m_1 & m_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_2 & 0 & 0 & 0 & 0 \\ \frac{m_1 m_3}{m_2} & 0 & 0 & m_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_4 & 0 & 0 \\ \frac{m_1 m_3 m_5}{m_2 m_4} & 0 & 0 & 0 & 0 & m_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & m_6 \end{bmatrix}$$

By e.g. [12, Ch. VII, Section 2, Appendix], the order of Γ_D/Γ_F is the largest minor of R. We have $m_{i+1} | m_i$ for $i \ge 2$ by construction. Thus $m_3 m_5 \cdots m_j < m_2 m_4 \cdots m_{j-1}$ for all j, and the entries in the 0-th column of R are nonincreasing as the row position increases, and if $m_1 m_3 \cdots m_{j_0} / m_2 m_4 \cdots m_{j_0 - 1}$ is greater than one for some j_0, then for $j < j_0$, $m_1 m_3 \cdots m_j / m_2 m_4 \cdots m_{j-1}$ is also greater than one. Thus the 0-th column of R has alternating nonzero entries until some row, past which it is all zeros.

Since at most one column has more than one nonzero entry, cofactor expansion shows every minor consists of a single term. Furthermore, by the Smith Normal Form Theorem, since R has rank d, a minor of degree $c < d$ cannot be larger than all degree-d minors. Thus $[\Gamma_D : \Gamma_F]$ is the supremum of the degree-d minors, which are each defined by a single deleted column. Deleting column 0 yields

$$M_0 := m_2^2 m_4^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2$$

Let j_0 be the largest odd number such that $m_1 m_3 \cdots m_{j_0} > m_2 m_4 \cdots m_{j_0-1}$. Then the entry in row $j_0 - 1$ is the bottom nonzero entry of column 0. Deleting column j_0 deletes m_{j_0+1}, or 1 if d is odd and $j_0 = d$. This yields the minor

$$\begin{aligned} M_{j_0} &:= \tfrac{m_1 m_3 \cdots m_{j_0}}{m_2 m_4 \cdots m_{j_0-1}} m_2^2 m_4^2 \cdots m_{j_0-1}^2 m_{j_0+1} m_{j_0+3}^2 m_{j_0+5}^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2 \\ &= m_1 m_2 m_3 \cdots m_{j_0+1} m_{j_0+3}^2 m_{j_0+5}^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2 \end{aligned}$$

If $j < j_0$ is also odd then $m_1 m_3 \cdots m_j > m_2 m_4 \cdots m_{j-1}$, and deleting column $j+1$ yields the minor $M_j = m_1 m_2 m_3 \cdots m_{j+1} m_{j+3}^2 m_{j+5}^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2$. We compute, using $m_{i+1} | m_i$ for $i \geq 2$,

$$\frac{M_{j_0}}{M_j} = \frac{m_{j+2} m_{j+3} \cdots m_{j_0} m_{j_0+1}}{m_{j+3}^2 m_{j+5}^2 \cdots m_{j_0+1}^2} = \frac{m_{j+2} m_{j+4} \cdots m_{j_0}}{m_{j+3} m_{j+5} \cdots m_{j_0+1}} \geq 1$$

Therefore M_{j_0} is largest of the minors obtained by deleting column j for odd $j \leq j_0$. If j is odd and greater than j_0, then deleting column j makes row $j-1$ equal to zero, and the resulting minor is zero. If j is even and greater than zero, deleting column j removes the only nonzero entry in row $j-1$, and the resulting minor is zero. We conclude

$$|\Gamma_D / \Gamma_F| = \max\{M_0, M_{j_0}\}$$

Since $|\Gamma_T / \Gamma_F| = M_0$, and $[D : F] = [\Gamma_D : \Gamma_T][\Gamma_D : \Gamma_F]$,

$$\begin{aligned} [D : F] &= \max\{M_0, M_{j_0}^2 / M_0\} \\ &= \max\{m_2^2 m_4^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2, m_1^2 m_3^2 \cdots m_{j_0}^2 m_{j_0+3}^2 m_{j_0+5}^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2\} \end{aligned}$$

Suppose that $j_0 < 2\lfloor \frac{d}{2} \rfloor$. Then by definition of j_0 we have

$$m_1 m_3 \cdots m_{j_0} m_{j_0+2} < m_2 m_4 \cdots m_{j_0-1} m_{j_0+1}$$

hence

$$m_1^2 m_3^2 \cdots m_{j_0}^2 m_{j_0+2}^2 m_{j_0+3}^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2 < m_2^2 m_4^2 \cdots m_{2\lfloor \frac{d}{2} \rfloor}^2$$

Thus $[D : F] = M_0$ in this case. Also in this case, if d is even then since $m_{i+1} | m_i$ for $i \geq 2$, we have

$$m_1^2 m_3^2 \cdots m_{j_0+2}^2 m_{j_0+4}^2 \cdots m_{d-1}^2 < m_2^2 m_4^2 \cdots m_{j_0+1}^2 m_{j_0+3}^2 \cdots m_{d-2}^2 m_d^2$$

and if d is odd then similarly

$$m_1^2 m_3^2 \cdots m_{j_0+2}^2 m_{j_0+4}^2 \cdots m_d^2 < m_2^2 m_4^2 \cdots m_{j_0+1}^2 m_{j_0+3}^2 \cdots m_{d-1}^2$$

On the other hand, if d is even and $j_0 = d - 1$,

$$[D : F] = \max\{m_2^2 m_4^2 \cdots m_d^2, m_1^2 m_3^2 \cdots m_{d-1}^2\},$$

and if d is odd and $j_0 = d$, $[D : F] = \max\{m_2^2 m_4^2 \cdots m_{d-1}^2, m_1^2 m_3^2 \cdots m_d^2\}$. We conclude that in any case

$$[D : F] = \begin{cases} \max\{m_2^2 m_4^2 \cdots m_d^2, m_1^2 m_3^2 \cdots m_{d-1}^2\} & \text{if } d \text{ is even} \\ \max\{m_2^2 m_4^2 \cdots m_{d-1}^2, m_1^2 m_3^2 \cdots m_d^2\} & \text{if } d \text{ is odd} \end{cases}$$

The pfaffian subgroup of $\mathrm{alt}_{\underline{\chi}\times\underline{x}}(\alpha)$ is easily computed from the linked 2-block form of α in (10.1). We obtain

$$\mathrm{pfaff}(\mathrm{alt}_{\underline{\chi}\times\underline{x}}(\alpha)) = \begin{cases} \left\langle \frac{1}{m_1 m_3 \cdots m_{d-1}}, \frac{1}{m_2 m_4 \cdots m_d} \right\rangle & \text{if } d \text{ is even} \\ \left\langle \frac{1}{m_1 m_3 \cdots m_{d-2}}, \frac{1}{m_2 m_4 \cdots m_{d-1}} \right\rangle & \text{if } d \text{ is odd} \end{cases}$$

Thus $\mathrm{ind}(\alpha) = |\mathrm{pfaff}(\mathrm{alt}_{\underline{\chi}\times\underline{x}}(\alpha))|$, as desired.

We next show there exists a matched basis $\underline{\theta} \times \underline{t}$ such that $\mathrm{alt}_{\underline{\theta}\times\underline{t}}(\alpha)$ is in disjoint 2-block form. If $m' = n$ then ${}_n\mathrm{X}(F)$ and $\mathrm{Z}^1(F, \mu_n)$ are isomorphic by Kummer theory, and any basis change $P \oplus P$ on ${}_n\mathrm{X}(F) \times \mathrm{Z}^1(F, \mu_n)$ preserves the subgroup C of Definition 8.5, hence induces a basis change $P^{\wedge 2}$ on ${}_n\mathrm{X}(F) \wedge \mathrm{Z}^1(F, \mu_n)$. The algorithm for putting an arbitrary skew-symmetric matrix into disjoint 2-block form with such a matrix is [5, Lemma 1.10], adapted from [16, Section 6.2], and the result follows.

If $m' \neq n$, then since α has order n, the term $\frac{n}{m_1}(\chi_0, x_1)$ in the linked 2-block form (10.1) has order n, so m_1 is divisible by m_2, hence $m_{i+1} \mid m_i$ for all i. Applying

$$P = C_{2q_0\, 2q_0-2}\left(\frac{m_{2q_0-1}}{m_{2q_0}}\right) \cdots C_{42}\left(\frac{m_3}{m_4}\right) C_{20}\left(\frac{m_1}{m_2}\right)$$

where q_0 is at most $\lfloor \frac{d}{2} \rfloor$, then yields the desired 2-block form, leaving at most $2\lfloor \frac{d-1}{2} \rfloor + 1$ nonzero factors. To check that P comes from a matched basis change as in Theorem 10.1, it suffices to check that each factor satisfies the conditions in that theorem. This is immediate for all but possibly $C_{20}(m_1/m_2)$. But since m_2 divides m', and $m_1 = n$, n/m' divides the entries of the 0-th column of $C_{20}(m_1/m_2)$ below the top, so this too satisfies the definition, and the claim is proved.

We have shown that there exists a matched basis $\underline{\theta} \times \underline{t}$ with respect to which $\mathrm{alt}_{\underline{\theta}\times\underline{t}}(\alpha)$ is in disjoint 2-block form. The corresponding expression for α is

$$\alpha = \frac{n}{m_1}(\theta_0, t_1) + (t_2, t_3)_{m_3} + \cdots + (t_{2r}, t_{2r+1})_{m_{2r+1}}$$

where $r = \lfloor \frac{d-1}{2} \rfloor$. The corresponding tensor product of central simple algebras has degree $\mathrm{pfaff}(A)$, which we have shown equals $\mathrm{ind}(\alpha)$. Therefore

$$\Delta\left(\frac{n}{m_1}\theta_0, t_1\right) \otimes_F \Delta(t_2, t_3)_{m_3} \otimes_F \cdots \otimes_F \Delta(t_{2r}, t_{2r+1})_{m_{2r+1}}$$

is a division algebra. This completes the proof. □

The next result generalizes Example 7.3 to any matched basis.

Corollary 10.5. *Assume* (3.1)*, with n a power of a prime ℓ, and either ℓ is odd or $\ell = 2$ and $q \equiv 1 (\mathrm{mod}\ 2n)$. Let $\underline{\theta} \times \underline{t}$ be any matched basis. Then*

$$D = \Delta(\theta_0, t_1) \otimes_F \Delta(t_2, t_3)_{m'} \otimes_F \cdots \otimes_F \Delta(t_{2r}, t_{2r+1})_{m'}$$

where $r = \lfloor \frac{d-1}{2} \rfloor$, is a division algebra of period n and index nm'^r, with $r+1 = \lfloor \frac{d+1}{2} \rfloor$ cyclic tensor factors.

Proof. We have $\text{per}(D) = n$ since θ_0 has order n, and $\text{ind}(D) = nm^r$ by the pfaffian formula. Since $nm^r = \deg(D)$, D is a division algebra, which is a tensor product of $r+1$ cyclic factors. □

Example 10.6. The disjoint 2-block form may not retain the unramified character. Suppose $m = \ell^3$ for ℓ an odd prime, x_0 is defined over k, and

$$\alpha = (x_0, x_1)_{\ell^3} + (x_1, x_2)_{\ell^2} + (x_2, x_3)_\ell$$

Then

$$\text{alt}_{\underline{x}}(\alpha) = \begin{bmatrix} 0 & 1/\ell^3 & 0 & 0 \\ -1/\ell^3 & 0 & 1/\ell^2 & 0 \\ 0 & -1/\ell^2 & 0 & 1/\ell \\ 0 & 0 & -1/\ell & 0 \end{bmatrix}$$

Applying $P = C_{20}(\ell)$ gives a disjoint 2-block form: Defining $\underline{t}$ by $\underline{x} = \underline{t}P$ yields $t_0 = x_0 x_2^{-\ell}$, $x_1 = t_1$, $x_2 = t_2$, $x_3 = t_3$, and

$$\alpha = (t_0, t_1)_{\ell^3} + (t_2, t_3)_\ell = (x_0 x_2^{-\ell}, x_1)_{\ell^3} + (x_2, x_3)_\ell$$

The new leading character $(t_0)^*_{\ell^3}$ is ramified, making the index computation (ℓ^4) nontrivial.

10.2. **Pfaffian formulas.** Here are the first few pfaffian formulas. Let $A = (a_{ij}) \in \text{Alt}_{d+1}(\mathbb{Q}/\mathbb{Z})$ and let $\text{pfaff}_{d+1}(A)$ denote the pfaffian. Then

$$\text{pfaff}_2(A) = a_{01}$$
$$\text{pfaff}_4(A) = a_{01}a_{23} - a_{02}a_{13} + a_{03}a_{12}$$
$$\begin{aligned}\text{pfaff}_6(A) = {} & a_{01}a_{23}a_{45} - a_{01}a_{24}a_{35} + a_{01}a_{25}a_{34} - a_{02}a_{13}a_{45} + a_{02}a_{14}a_{35} - a_{02}a_{15}a_{34} \\ & + a_{03}a_{12}a_{45} - a_{03}a_{14}a_{25} + a_{03}a_{15}a_{24} - a_{04}a_{12}a_{35} + a_{04}a_{13}a_{25} \\ & - a_{04}a_{15}a_{23} + a_{05}a_{12}a_{34} - a_{05}a_{13}a_{24} + a_{05}a_{14}a_{23}.\end{aligned}$$

Thus if $\text{alt}(\alpha) = A = (a_{ij}) \in \text{Alt}_{d+1}(\mathbb{Q}/\mathbb{Z})$, then

(a) If $d = 1$, $\text{per}(\alpha) = \text{ind}(\alpha) = |a_{01}|$.
(b) If $d = 2$, $\text{per}(\alpha) = \text{ind}(\alpha) = \text{lcm}[|a_{01}|, |a_{02}|, |a_{12}|]$.
(c) If $d = 3$, $\text{per}(\alpha) = \text{lcm}[|a_{ij}|]$ and $\text{ind}(\alpha) = \text{lcm}[\text{per}(\alpha), |a_{01}a_{23} - a_{02}a_{13} + a_{03}a_{12}|]$.
(d) If $d = 4$, $\text{per}(\alpha) = \text{lcm}\{|a_{ij}|\}$ and

$$\begin{aligned}\text{ind}(\alpha) = {} & \text{lcm}[\text{per}(\alpha), |a_{01}a_{23} - a_{02}a_{13} + a_{03}a_{12}|, |a_{01}a_{24} - a_{02}a_{14} + a_{04}a_{12}|, \\ & |a_{01}a_{34} - a_{03}a_{14} + a_{04}a_{13}|, |a_{02}a_{34} - a_{03}a_{24} + a_{04}a_{23}|, \\ & |a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}|]\end{aligned}$$

The $d = 3$ case is the first with unequal period and index.

Example 10.7. Suppose $F = \mathbb{F}_5((x_1))((x_2))((x_3))((x_4))((x_5))$, the 5-dimensional local field, $x_0 = [2]_5$, and we have a sum of quaternions

$$\begin{aligned}\alpha = {} & (2, x_1) + (2, x_2) + (2, x_3) + (2, x_4) + (2, x_5) + (x_1, x_2) + (x_1, x_3) + (x_1, x_4) \\ & + (x_1, x_5) + (x_2, x_3) + (x_2, x_4) + (x_2, x_5) + (x_3, x_4) + (x_3, x_5) + (x_4, x_5)\end{aligned}$$

The matrix is

$$A = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

By the formulas above, $\mathrm{pfaff}_2(A) = 1/2$, $\mathrm{pfaff}_4(A) = 1/4$, and, since pfaff_6 is an alternating sum of an odd number of fractions $1/8$, $\mathrm{pfaff}_6(A) = 1/8$. Therefore $\mathrm{per}(\alpha) = 2$ and $\mathrm{ind}(\alpha) = \mathrm{lcm}\{2, 4, 8\} = 8$.

Similarly, since the pfaffian always has an odd number of terms, if $F = \mathbb{F}_p((x_1))\cdots((x_d))$, x_0 generates $\mathbb{F}_p^\times$, and

$$\alpha = \sum_{0 \le i < j \le d} (x_i, x_j)_{m'}$$

then $\mathrm{per}(\alpha) = m'$, and $\mathrm{ind}(\alpha) = m'^r$, where $r = \lfloor \frac{d+1}{2} \rfloor$.

11. Failure in the case $\ell = 2$

Example 11.1. We illustrate the failure of a Brauer class of 2-power order to behave like an alternating form, even when F contains μ_n. Let $F = \mathbb{F}_p((x_1))((x_2))((x_3))$, and let $m = n = 2^{\mathrm{v}_2(p-1)}$. Let $\underline{x}$ be a basis for $\mathrm{Z}^1(F, \mu_n)$, in standard form. Let

$$\alpha = (x_1, x_2)_n + (x_1, x_3)_n + (x_2, x_3)_n$$

then

$$\mathrm{alt}_{\underline{x}}(\alpha) = A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{n} & \frac{1}{n} \\ 0 & -\frac{1}{n} & 0 & \frac{1}{n} \\ 0 & -\frac{1}{n} & -\frac{1}{n} & 0 \end{bmatrix}$$

The pfaffian subgroup is generated by the subpfaffians of degree 1, so $\mathrm{pfaff}(A) = \langle 1/n \rangle$, predicting an index of n. Let

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

be a basis change, and define $\underline{t}$ by $\underline{x} = \underline{t}P$. By substitution,

$$\begin{aligned} \alpha &= (t_1, t_1^{-1} t_2)_n + (t_1, t_1 t_3)_n + (t_1^{-1} t_2, t_1 t_3)_n \\ &= (t_1, t_1^{-1})_n + (t_1, t_2)_n + (t_1, t_1)_n + (t_1, t_3)_n + (t_1^{-1}, t_1)_n \\ &\quad + (t_1^{-1}, t_3)_n + (t_2, t_1)_n + (t_2, t_3)_n \\ &= (t_1^{-1}, t_1)_n + (t_2, t_3)_n \\ &= (-1, t_1)_n + (t_2, t_3)_n \end{aligned}$$

Since $n = 2^{v_2(p-1)}$, $(-1,t_1)_n$ has order 2 by Theorem 8.2, and since the totally ramified class $(t_2,t_3)_n$ has value group disjoint from the semiramified class $(-1,t_1)_n$, we compute $\operatorname{ind}(\alpha) = 2n$ using the methods of Theorem 10.4. If ${}_n\mathrm{Br}(F)$ behaved like a group of skew-symmetric matrices, functorial with respect to basis change, the new matrix would be

$$\operatorname{alt}_{\underline{t}}(\alpha) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n} \\ 0 & 0 & -\frac{1}{n} & 0 \end{bmatrix}$$

But we compute instead

$$PAP^{\mathrm{t}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n} \\ 0 & 0 & -\frac{1}{n} & 0 \end{bmatrix}$$

which is consistent, of course, with the (wrong) prediction of index n from the pfaffian.

Summarizing, though the expression for ${}_n\mathrm{Br}(F)$ determines an *ad hoc* map into the group of skew-symmetric matrices, this map appears not to be functorial with respect to basis change on classes in ${}_n\mathrm{Br}(F) - {}_{n}2\,\mathrm{Br}(F)$. This would defeat the reason for defining the map in the first place, since it means we can't use matrix diagonalization to find a minimal representation for the class, and cannot produce an index formula.

12. Obstruction when ${}_n\mathrm{Br}(F) \neq {}_n2\,\mathrm{Br}(F)$

Assume (3.1), with n a power of a prime ℓ, $\mu_n \le F^\times$, and $G = \mathrm{Gal}(F^{1/n}/F)$. Since $\mu_n \le F^\times$, we substitute $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ for μ_n using ζ_n^*. Consider the diagram

$$\begin{array}{ccccc} & & \mathrm{H}^2(F,\frac{1}{n}\mathbb{Z}/\mathbb{Z}) & & \\ & \overset{\text{inf}}{\nearrow} & & \overset{\text{alt?}}{\searrow} & \\ \mathrm{H}^2(G,\frac{1}{n}\mathbb{Z}/\mathbb{Z}) & & \xrightarrow[\text{[alt]}]{\qquad\qquad} & & \mathrm{Alt}(G,\frac{1}{n}\mathbb{Z}/\mathbb{Z}) \end{array}$$

The bottom arrow is a split surjection by Lemma 6.1. We now show [alt] factors through $\mathrm{H}^2(F,\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ if and only if $q \equiv 1 (\mathrm{mod}\ n^2)$, i.e., $\mu_{n^2} \le F^\times$. Then we show that a natural map $\operatorname{alt} : {}_n\mathrm{Br}(F) \longrightarrow \mathrm{Alt}(G,\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ exists if and only if $n \neq 2^{v_2(q-1)}$, as suggested by Example 11.1. In particular, we always have a map on ${}_n2\,\mathrm{Br}(F)$.

Proposition 12.1. *Assume* (3.1)*, with $m = n$ a power of ℓ. Let $L = F^{1/n}$, and put $G = \mathrm{Gal}(L/F)$. Then we have an exact sequence*

$$0 \longrightarrow \mathrm{H}^1(L,\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\text{tg}} \mathrm{H}^2(G,\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\text{inf}} \mathrm{H}^2(F,\tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \longrightarrow 0$$

Let $[\mathrm{alt}] : \mathrm{H}^2(G,\frac{1}{n}\mathbb{Z}/\mathbb{Z}) \longrightarrow \mathrm{Alt}(G,\frac{1}{n}\mathbb{Z}/\mathbb{Z})$ *be the map of Lemma* 6.1*, let* $\underline{\phi} = \{\phi_0,\phi_1,\dots,\phi_d\}$ *be the image in G of* $\underline{\Phi} = \{\Phi_0,\Phi_1,\dots,\Phi_d\}$ *of* (5.1) *under the* *projection, and let* ξ *be an element of* $\mathrm{H}^1(L, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. *Then for* $i \le j$,

$$[\mathrm{alt}](\mathrm{tg}(\xi))(\phi_i, \phi_j) = \begin{cases} -\frac{q-1}{n}\xi(\Phi_j^n) & \textit{if } i = 0 < j \\ 0 & \textit{otherwise} \end{cases}$$

In particular, $[\mathrm{alt}]$ *factors through* ${}_n\mathrm{Br}(F)$ *if and only if* $q \equiv 1 (\mathrm{mod}\ n^2)$.

Proof. Since $\mu_n \le F^\times$, L contains all cyclic extensions of degree n by Kummer theory, and $G \simeq (\mathbb{Z}/n)^{d+1}$ by Theorem 5.1. All elements of $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ and $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ are split by L, by Theorem 5.1 and Theorem 6.3, respectively. Therefore $\mathrm{H}^1(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) = \mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, and $\ker[\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \mathrm{H}^2(L, \frac{1}{n}\mathbb{Z}/\mathbb{Z})^G] = \mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. The 7-term sequence (2.2) applied to $1 \to \mathrm{G}_L^{\mathrm{tr}} \to \mathrm{G}_F^{\mathrm{tr}} \to G \to 1$ then yields

$$0 \longrightarrow \mathrm{H}^1(L, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z})^G \xrightarrow{\mathrm{tg}} \mathrm{H}^2(G, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\mathrm{inf}} \mathrm{H}^2(F, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z})$$

The cup product $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ is onto by Theorem 6.3, and since $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) = \mathrm{H}^1(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, this map factors through $\mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. Therefore the map on the right is surjective, as desired.

We claim the action of G on $\mathrm{H}^1(L, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ is trivial. Suppose $\xi \in \mathrm{H}^1(L, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, then the action is given by $\xi^{\phi_i} = \xi \circ \Phi_i^{-1}$, where Φ_i^{-1} acts on $\mathrm{G}_L^{\mathrm{tr}}$ by conjugation. Since G is isomorphic to $(\mathbb{Z}/n)^{d+1}$, and G's basis $\underline{\phi}$ is the image of $\underline{\Phi}$, a general element $\prod_{i=0}^d \Phi_i^{a_i} \in \mathrm{G}_F^{\mathrm{tr}}$ is in $\mathrm{G}_L^{\mathrm{tr}}$ if and only if $n | a_i$ for all i, and by (5.1), $\mathrm{G}_L^{\mathrm{tr}}$ has presentation

$$\mathrm{G}_L^{\mathrm{tr}} = \langle \underline{\Phi}^n : [\Phi_0^n, \Phi_j^n] = \Phi_j^{n(q^n-1)},\ [\Phi_i^n, \Phi_j^n] = e,\ \forall i, j \ge 1 \rangle$$

Since $\underline{\Phi}^n$ is a basis for $\mathrm{G}_L^{\mathrm{tr}}$, to prove the claim it suffices to show $\xi(\Phi_i^{-1}\Phi_j^n\Phi_i) = \xi(\Phi_j^n)$ for $i, j : 0 \le i, j \le d$, for any $\xi \in \mathrm{H}^1(L, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$. By (5.1),

$$\Phi_i^{-1}\Phi_j^n\Phi_i = \begin{cases} \Phi_0^n & \text{if } i = j = 0 \\ \Phi_j^{nq^{-1}} & \text{if } i = 0 \text{ and } j > 0 \\ \Phi_i^{q^n-1}\Phi_0^n & \text{if } i > 0 \text{ and } j = 0 \\ \Phi_j^n & \text{if } i, j > 0 \end{cases}$$

Since $n\xi = 0$ and $q \equiv 1 (\mathrm{mod}\ n)$, we have $\xi(\Phi_j^{nq^{-1}}) = q^{-1}\xi(\Phi_j^n) = \xi(\Phi_j^n)$, and

$$\xi(\Phi_i^{q^n-1}\Phi_0^n) = (1 + q + \cdots + q^{n-1})\xi(\Phi_i^{q-1}) + \xi(\Phi_0^n) = n\xi(\Phi_i^{q-1}) + \xi(\Phi_0^n) = \xi(\Phi_0^n)$$

This proves the claim, finishing the construction of the exact sequence.

Let $s : \prod_{i=0}^d \phi_i^{a_i} \mapsto \prod_{i=0}^d \Phi_i^{a_i}$, $0 \le a_i \le n-1$, be a section of the map $\mathrm{G}_F^{\mathrm{tr}} \to G$. By [19, Prop. 1.6.5] and [19, Thm. 2.1.7], $\mathrm{tg}(\xi) \in \mathrm{Z}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ is defined by $\mathrm{tg}(\xi)(\sigma, \tau) = -\xi(s(\sigma\tau)^{-1}s(\sigma)s(\tau))$. Since $[\mathrm{alt}](\mathrm{tg}(\xi))(\phi_i, \phi_j) = \mathrm{tg}(\xi)(\phi_i, \phi_j) - \mathrm{tg}(\xi)(\phi_j, \phi_i)$, it remains to compute $\mathrm{tg}(\xi)(\phi_i, \phi_j)$ for each i, j. Using the relations (5.1), and the fact that G is abelian, we get

$$s(\phi_i\phi_j)^{-1}s(\phi_i)s(\phi_j) = \begin{cases} \Phi_j^{-1}\Phi_i^{-1}\Phi_i\Phi_j = e & \text{if } i \le j \text{ or } j > 0 \\ \Phi_i^{-1}\Phi_0^{-1}\Phi_i\Phi_0 = \Phi_i^{q^{-1}-1} & \text{if } i > j = 0 \end{cases}$$

Thus $\operatorname{tg}(\xi)(\phi_i, \phi_j) = 0$ if $i \le j$ or $j > 0$. Since $q^{-1} - 1 = -q^{-1}(q-1)$, $|\xi|$ divides n, and $q \equiv q^{-1} \equiv 1 (\operatorname{mod} n)$, we have for $i > j = 0$,

$$\operatorname{tg}(\xi)(\phi_i, \phi_0) = -\xi(\Phi_i^{-q^{-1}(q-1)}) = \xi(\Phi_i^{q-1}) = \tfrac{q-1}{n}\xi(\Phi_i^n) \qquad (i > 0)$$

Therefore $[\operatorname{alt}](\operatorname{tg}(\xi))(\phi_0, \phi_i) = \operatorname{tg}(\xi)(\phi_0, \phi_i) - \operatorname{tg}(\xi)(\phi_i, \phi_0) = -\frac{q-1}{n}\xi(\Phi_i^n)$, as claimed.

The map $[\operatorname{alt}] : \mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \to \operatorname{Alt}(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ factors through $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ if and only if $\operatorname{tg}(\mathrm{H}^1(L, \frac{1}{n}\mathbb{Z}/\mathbb{Z}))$ is contained in $\ker([\operatorname{alt}])$, if and only if n divides $\frac{q-1}{n}$, i.e., $q \equiv 1 (\operatorname{mod} n^2)$. □

Theorem 12.2. *Assume* (3.1)*, with $m = n$ a power of a prime ℓ. Let $L = F^{1/n}$, and set $G = \operatorname{Gal}(L/F) \simeq (\mathbb{Z}/n)^{d+1}$. There exists a natural injective homomorphism*

$$\operatorname{alt} : {}_n2\operatorname{Br}(F) \longrightarrow \operatorname{Alt}(G, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z})$$

which does not extend to ${}_n\operatorname{Br}(F)$ when $n = 2^{\mathrm{v}_2(q-1)}$, i.e., when ${}_n\operatorname{Br}(F) \neq {}_n2\operatorname{Br}(F)$.

Proof. We identify $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{Z}^1(F, \mu_n)$ with $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ using ζ_n^*, and the basis elements $\chi_i \otimes x_j^{1/n}$ then have form $\chi_i \otimes \chi_j$. The subgroup $C \le {}_n2\,\mathrm{X}(F) \otimes {}_n\mathrm{X}(F)$ of Theorem 8.6 has basis

$$\{\chi_i \otimes \chi_i, \tfrac{|\chi_j|}{m'}(\chi_j \otimes \chi_k + \chi_k \otimes \chi_j),\ 0 \le i \le d, 0 \le j < k \le d\}$$

by the proof of Proposition 8.9. Application of alt shows C is contained in the kernel K of Lemma 6.1, hence the natural map $\operatorname{alt} : {}_n2\operatorname{Br}(F) \to \operatorname{Alt}(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ exists.

We complete the proof with a lemma that isolates a basis of elements of order 2 in $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ that map to zero in $\operatorname{Alt}(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ but not in $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, foiling the definition of alt on $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ when $n = 2^{\mathrm{v}_2(q-1)}$.

Lemma 12.3. *Assume* (3.1)*, with n a power of ℓ, $\mu_n \le F^\times$, and $G = \operatorname{Gal}(F^{1/n}/F)$. Let $\underline{\phi} = \{\phi_0, \dots, \phi_d\}$ be a basis for G, with dual basis $\underline{\phi}^* = \{\phi_0^*, \dots, \phi_d^*\}$ in $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, and consider the cup product map*

$$\cup\ : \mathrm{H}^1(F, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \otimes \mathrm{H}^1(F, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \longrightarrow \mathrm{H}^2(G, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z})$$

(a) *If ℓ is odd, then $\phi_i^* \cup \phi_i^* = 0$ in $\mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, $\forall i : 0 \le i \le d$.*
(b) *If $\ell = 2$ then $\phi_i^* \cup \phi_i^*$ has order 2 in $\mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$.*
(c) *If $\ell = 2$ then $\phi_i^* \cup \phi_i^* = 0$ in $\mathrm{H}^2(G, \frac{1}{2n}\mathbb{Z}/\mathbb{Z})$.*
(d) *If $\ell = 2$ then $\inf(\phi_i^* \cup \phi_i^*) = 0$ in $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ if and only if $q \equiv 1 (\operatorname{mod} 2e)$, where $e = [\Gamma_{F(\phi_i^*)} : \Gamma_F]$.*

Proof. Since $\mathrm{H}^1(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z}) = \mathrm{H}^1(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, the cup product map makes sense. For each $\sigma \in G$, define $a_\sigma : 0 \le a_\sigma < n$ by $\phi_i(\sigma) = a_\sigma/n$. Then for any $\sigma, \tau \in G$,

$$a_{\sigma\tau} = \begin{cases} a_\sigma + a_\tau & \text{if } a_\sigma + a_\tau < n \\ a_\sigma + a_\tau - n & \text{if } a_\sigma + a_\tau \ge n \end{cases}$$

Let $\mathbb{Z}[\mathbb{Z}]$ denote the group ring of the additive group of integers. Let $\mathbb{Z}[G]$ be the (additive) $\mathbb{Z}[\mathbb{Z}]$-module with the action $a \cdot b := \phi_i^a b$ for all $a \in \mathbb{Z}$ and $b \in \mathbb{Z}[G]$. Let

$\gamma_{\phi_i} \in \mathrm{Z}^1(\mathbb{Z}, \mathbb{Z}[G])$ be the 1-cocycle defined by $\gamma_{\phi_i}(1) = 1$. Then $\gamma_{\phi_i}(0) = 0$, and by the 1-cocycle condition,

$$\gamma_{\phi_i}(a) = 1 + \phi_i + \cdots + \phi_i^{a-1} \qquad (a \in \mathbb{N})$$

We first prove (a). Suppose ℓ is odd, and define $\beta \in \mathrm{C}^1(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ by

$$\beta(\sigma) = \phi_i^*(\gamma_{\phi_i}(a_\sigma)) - (\phi_i^* \otimes \phi_i^*)(\sigma, \sigma)$$

where $\phi_i^* \otimes \phi_i^*$ is the bilinear form with values in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$, and we extend ϕ_i^* to $\mathbb{Z}[G]$ using the rule $\phi_i^*(\tau - \sigma) = \phi_i^*(\tau) - \phi_i^*(\sigma)$. Since the action of G on $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ is trivial, $\partial\beta(\sigma, \tau) := \beta(\sigma) + \beta(\tau) - \beta(\sigma\tau)$, hence

$$\begin{aligned}
\partial\beta(\sigma, \tau) &= \phi_i^*(\gamma_{\phi_i}(a_\sigma) + \gamma_{\phi_i}(a_\tau) - \gamma_{\phi_i}(a_{\sigma\tau})) \\
&\qquad - (\phi_i^* \otimes \phi_i^*)(\sigma, \sigma) - (\phi_i^* \otimes \phi_i^*)(\tau, \tau) + (\phi_i^* \otimes \phi_i^*)(\sigma\tau, \sigma\tau) \\
&= \phi_i^*(\gamma_{\phi_i}(a_\sigma) + \gamma_{\phi_i}(a_\tau) - \gamma_{\phi_i}(a_{\sigma\tau})) + 2\phi_i^* \otimes \phi_i^*(\sigma, \tau) \\
&= \begin{cases} \phi_i^*(\gamma_{\phi_i}(a_\sigma) - \phi_i^{a_\tau}\gamma_{\phi_i}(a_\sigma)) + 2\frac{a_\sigma a_\tau}{n} & \text{if } a_\sigma + a_\tau < n \\ \phi_i^*(\gamma_{\phi_i}(a_\sigma) - \phi_i^{a_\tau}\gamma_{\phi_i}(a_\sigma) - \gamma_{\phi_i}(n)) + 2\frac{a_\sigma a_\tau}{n} & \text{if } a_\sigma + a_\tau \geq n \end{cases} \\
&= \begin{cases} \frac{1}{n} + \cdots + \frac{a_\sigma - 1}{n} - (\frac{a_\tau}{n} + \frac{a_\tau + 1}{n} + \cdots + \frac{a_\tau + a_\sigma - 1}{n}) + 2\frac{a_\sigma a_\tau}{n} \\ \qquad\qquad \text{if } a_\sigma + a_\tau < n \\ \frac{1}{n} + \cdots + \frac{a_\sigma - 1}{n} - (\frac{a_\tau}{n} + \frac{a_\tau + 1}{n} + \cdots + \frac{a_\tau + a_\sigma - 1}{n}) - \phi_i^*(\gamma_{\phi_i}(n)) + 2\frac{a_\sigma a_\tau}{n} \\ \qquad\qquad \text{if } a_\sigma + a_\tau \geq n \end{cases} \\
&= \begin{cases} -\frac{a_\sigma a_\tau}{n} + 2\frac{a_\sigma a_\tau}{n} & \text{if } a_\sigma + a_\tau < n \\ -\frac{a_\sigma a_\tau}{n} + 2\frac{a_\sigma a_\tau}{n} - \phi_i^*(\gamma_{\phi_i}(n)) & \text{if } a_\sigma + a_\tau \geq n \end{cases} \\
&= \begin{cases} \frac{a_\sigma a_\tau}{n} & \text{if } a_\sigma + a_\tau < n \\ \frac{a_\sigma a_\tau}{n} - \frac{n(n-1)}{2n} & \text{if } a_\sigma + a_\tau \geq n \end{cases}
\end{aligned}$$

If ℓ is odd, then $2|n-1$, so $n(n-1)/2n = 0$, and $\partial\beta = \phi_i^* \otimes \phi_i^*$. Therefore $\phi_i^* \cup \phi_i^* = 0$ in $\mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ when ℓ is odd, proving (a).

If $\ell = 2$, then $2|n$, so $n(n-1)/2n = -1/2$. The short exact sequence of trivial G-modules

$$0 \longrightarrow \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \longrightarrow \tfrac{1}{2n}\mathbb{Z}/\mathbb{Z} \longrightarrow \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \longrightarrow 0$$

combined with the fact that G has exponent n yields the long exact sequence fragment

$$0 \longrightarrow \mathrm{H}^1(G, \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}) \xrightarrow{\ \delta\ } \mathrm{H}^2(G, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \longrightarrow \mathrm{H}^2(G, \tfrac{1}{2n}\mathbb{Z}/\mathbb{Z})$$

Since $G \simeq (\mathbb{Z}/n)^{d+1}$ has basis $\{\phi_i : 0 \leq i \leq d\}$, $\mathrm{H}^1(G, \frac{1}{2}\mathbb{Z}/\mathbb{Z})$ is generated by the $\frac{n}{2}\phi_i^*$, and by construction

$$\delta(\tfrac{n}{2}\phi_i^*)(\sigma, \tau) = \frac{a_\sigma + a_\tau - a_{\sigma\tau}}{2n} = \begin{cases} 0 & \text{if } a_\sigma + a_\tau < n \\ -1/2 & \text{if } a_\sigma + a_\tau \geq n \end{cases}$$

Thus we find $\partial\beta = \phi_i^* \otimes \phi_i^* + \delta(\frac{n}{2}\phi_i^*)$. Therefore $\phi_i^* \cup \phi_i^* = \delta(\frac{n}{2}\phi_i^*)$ has order 2 in $\mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, and is trivial in $\mathrm{H}^2(G, \frac{1}{2n}\mathbb{Z}/\mathbb{Z})$. In fact, $\delta(\frac{n}{2}\phi_i^*) = \partial\varepsilon$, where $\varepsilon \in \mathrm{C}^1(G, \frac{1}{2n}\mathbb{Z}/\mathbb{Z})$ is defined by $\varepsilon(\sigma) = a_\sigma/2n \in \frac{1}{2n}\mathbb{Z}/\mathbb{Z}$. This proves (b) and (c).

Let $\phi_i^* = (x_i)_n^*$, where $(x_i)_n \in F^\times/n$. When $\ell = 2$ we have $(\phi_i^*, -x_i) = (\phi_i^*, -1) + (\phi_i^*, x_i) = 0$ in $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, as mentioned in (2.1), and $(\phi_i^*, -1) = 0$ if and only if ϕ_i^* extends to a character of order $2n$, by Albert's criterion. Suppose $[\Gamma_{F(\phi_i^*)} : \Gamma_F] = e$. By Theorem 5.1 we may write $\phi_i^* = \chi + \lambda$, where χ is unramified and λ is totally ramified of order e. The character χ always extends, since it is defined over a finite field, hence ϕ_i^* extends if and only if λ extends, and by Kummer theory, λ extends to a character of order $2e$ if and only if $q \equiv 1 (\mathrm{mod}\ 2e)$. Therefore (ϕ_i^*, x_i) equals zero if and only if $q \equiv 1 (\mathrm{mod}\ 2e)$, and since $(\phi_i^*, x_i) = \inf(\phi_i^* \cup \phi_i^*)$, this proves (d). □

Finish proof of Theorem 12.2. By Lemma 12.3(a) and (d), the natural map

$$\mathrm{alt} : {}_n\mathrm{X}(F) \otimes {}_n\mathrm{X}(F) \to \mathrm{Alt}(G, \tfrac{1}{n}\mathbb{Z}/\mathbb{Z})$$

which factors through $\mathrm{H}^2(G, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$, giving $[\mathrm{alt}]$, factors through $\mathrm{H}^2(F, \frac{1}{n}\mathbb{Z}/\mathbb{Z})$ on precisely the subgroup ${}_n2\,\mathrm{Br}(F)$, the elements not of the form $(t,t)_n$ when $(t)_n^*$ is totally ramified of order $n = 2^{\mathrm{v}_2(q-1)}$. Since ${}_n\mathrm{Br}(F) = {}_n2\,\mathrm{Br}(F) \iff n \neq 2^{\mathrm{v}_2(q-1)}$, this completes the proof. □

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References

[1] S. A. Amitsur, L. H. Rowen, and J.-P. Tignol, *Division algebras of degree 4 and 8 with involution*, Israel J. Math. **33** (1979), no. 2, 133–148, DOI 10.1007/BF02760554. MR571249

[2] J.-P. Tignol and S. A. Amitsur, *Kummer subfields of Mal′cev-Neumann division algebras*, Israel J. Math. **50** (1985), no. 1-2, 114–144, DOI 10.1007/BF02761120. MR788071

[3] Jón Kr. Arason, Burton Fein, Murray Schacher, and Jack Sonn, *Cyclic extensions of* $K(\sqrt{-1})/K$, Trans. Amer. Math. Soc. **313** (1989), no. 2, 843–851, DOI 10.2307/2001431. MR929665

[4] R. Brauer, E. Noether, and H. Hasse, *Beweis eines Hauptsatzes in der Theorie der Algebren* (German), J. Reine Angew. Math. **167** (1932), 399–404, DOI 10.1515/crll.1932.167.399. MR1581351

[5] E. S. Brussel, *The division algebras and Brauer group of a strictly Henselian field*, J. Algebra **239** (2001), no. 1, 391–411, DOI 10.1006/jabr.2000.8639. MR1827890

[6] Eric S. Brussel, *Alternating forms and the Brauer group of a geometric field*, Trans. Amer. Math. Soc. **359** (2007), no. 7, 3025–3069, DOI 10.1090/S0002-9947-07-03988-8. MR2299445

[7] Peter Draxl, *Ostrowski's theorem for Henselian valued skew fields*, J. Reine Angew. Math. **354** (1984), 213–218, DOI 10.1515/crll.1984.354.213. MR767581

[8] Peter Draxl and Martin Kneser (eds.), SK_1 *von Schiefkörpern* (German), Lecture Notes in Mathematics, vol. 778, Springer, Berlin, 1980. Seminar held at Bielefeld and Göttingen, 1976. MR591206

[9] David S. Dummit and Richard M. Foote, *Abstract algebra*, 3rd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004. MR2286236

[10] Helmut Hasse, *Über ℘-adische Schiefkörper und ihre Bedeutung für die Arithmetik hyperkomplexer Zahlsysteme* (German), Math. Ann. **104** (1931), no. 1, 495–534, DOI 10.1007/BF01457954. MR1512683

[11] Christopher J. Hillar and Darren L. Rhea, *Automorphisms of finite abelian groups*, Amer. Math. Monthly **114** (2007), no. 10, 917–923, DOI 10.1080/00029890.2007.11920485. MR2363058

[12] Thomas W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York-Berlin, 1980. Reprint of the 1974 original. MR600654

[13] Kenkichi Iwasawa, *On Galois groups of local fields*, Trans. Amer. Math. Soc. **80** (1955), 448–469, DOI 10.2307/1992998. MR75239

[14] Bill Jacob and Adrian R. Wadsworth, *A new construction of noncrossed product algebras*, Trans. Amer. Math. Soc. **293** (1986), no. 2, 693–721, DOI 10.2307/2000032. MR816320

[15] Bill Jacob and Adrian Wadsworth, *Division algebras over Henselian fields*, J. Algebra **128** (1990), no. 1, 126–179, DOI 10.1016/0021-8693(90)90047-R. MR1031915

[16] Nathan Jacobson, *Basic algebra. I*, 2nd ed., W. H. Freeman and Company, New York, 1985. MR780184

[17] Kazuya Kato, *A generalization of local class field theory by using K-groups. II*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **27** (1980), no. 3, 603–683. MR603953

[18] James S. Milne, *Étale cohomology*, Princeton Mathematical Series, No. 33, Princeton University Press, Princeton, N.J., 1980. MR559531

[19] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2000. MR1737196

[20] I. Reiner, *Maximal orders*, London Mathematical Society Monographs, No. 5, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1975. MR0393100

[21] Peter Roquette, *The Brauer-Hasse-Noether theorem in historical perspective*, Schriften der Mathematisch-Naturwissenschaftlichen Klasse der Heidelberger Akademie der Wissenschaften [Publications of the Mathematics and Natural Sciences Section of Heidelberg Academy of Sciences], vol. 15, Springer-Verlag, Berlin, 2005. MR2222818

[22] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg. MR554237

[23] J.-P. Tignol, *Sur les décompositions des algèbres à division en produit tensoriel d'algèbres cycliques* (French), Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin-New York, 1982, pp. 126–145. MR657427

[24] J.-P. Tignol and S. A. Amitsur, *Symplectic modules*, Israel J. Math. **54** (1986), no. 3, 266–290, DOI 10.1007/BF02764956. MR853452

[25] J.-P. Tignol and A. R. Wadsworth, *Totally ramified valuations on finite-dimensional division algebras*, Trans. Amer. Math. Soc. **302** (1987), no. 1, 223–250, DOI 10.2307/2000907. MR887507

[26] Jean-Pierre Tignol and Adrian R. Wadsworth, *Value functions on simple algebras, and associated graded rings*, Springer Monographs in Mathematics, Springer, Cham, 2015. MR3328410

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