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CARTER SUBGROUPS, CHARACTERS AND COMPOSITION SERIES

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ABSTRACT. Let G be a finite solvable group. We construct a set $\mathcal{H}$ of irreducible characters of G such that if C is a Carter subgroup of G, then the members of $\mathcal{H}$ behave well with respect to C-composition series for G, and we show that $\mathcal{H}$ is in bijective correspondence with the set of linear characters of C. Also, if A is a group that acts coprimely on G, then analogously, we characterize in terms of A-composition series for G, the set of A-invariant characters of G that have a linear Glauberman-Isaacs correspondent.

1. INTRODUCTION

Our primary concern in this paper is with finite solvable groups. Every such group contains a unique conjugacy class of nilpotent self-normalizing subgroups. where we say that a subgroup C is **self-normalizing** in G if $\mathbf{N}_G(C) = C$. These self-normalizing nilpotent subgroups of G are referred to as the **Carter subgroups** of G, and it was R. Carter who proved [1] that every finite solvable group G contains a Carter subgroup and that all of the Carter subgroups of G are conjugate.

It is clear that if C is a Carter subgroup of G, then C is also a Carter subgroup of every subgroup of G that contains it. It is perhaps somewhat less obvious that if C is a Carter subgroup of G and $\pi : G \to H$ is a surjective homomorphism, then $\pi(C)$ is a Carter subgroup of H, or equivalently, if $N \triangleleft G$, then NC/N is a Carter subgroup of G/N. It follows easily from this that if G/N is nilpotent, where $N \triangleleft G$, then $G = NC$, and in particular $G = G^\infty C$, where G^∞ is the nilpotent residual of G, which, by definition, is the unique smallest normal subgroup of G for which the corresponding factor group is nilpotent.

Carter subgroups also arise in a more general context: they are the so-called "projectors" for the formation of nilpotent groups. We do not know if results analogous to those in this paper hold more generally for projectors of other saturated formations, and we will not pursue that question here.

One of our principal results is the following.

Theorem A. *Let C be a Carter subgroup of a solvable group G. Then there exists a uniquely defined subset $\mathcal{H}$ of the set* $\mathrm{Irr}(G)$ *of irreducible characters of G such that $\mathcal{H}$ is in bijective correspondence with the set of linear characters of C. The set $\mathcal{H}$ is independent of any choices made in its construction, and also, if χ lies in $\mathcal{H}$, then χ restricts irreducibly to G^∞, and $\chi(1)$ divides $|G : C|$.*

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To prove Theorem A, we construct a family of injective maps from the set $\mathrm{Lin}(C)$ of linear characters of a Carter subgroup C of G into the set $\mathrm{Irr}(G)$ of irreducible characters of G, and although we do not claim to have constructed a uniquely determined map $\mathrm{Lin}(C) \to \mathrm{Irr}(G)$, we show that all of our maps have exactly the same image $\mathcal{H}$, which we refer to as the set of "head characters" of G, and we characterize these head characters unambiguously.

Of course, it follows from Theorem A that the number of characters $\chi \in \mathrm{Irr}(G)$ such that χ restricts irreducibly to G^{∞} and $\chi(1)$ divides $|G : C|$ is at least the number $|C : C'|$ of linear characters of C. Computer experiments show that in fact, the number of characters $\chi \in \mathrm{Irr}(G)$ that satisfy these two conditions can exceed $|C : C'|$, but it is relatively difficult to find examples where this strict inequality actually occurs.

Since $|C : C'| = |\mathrm{Lin}(C)| = |\mathcal{H}| \le |\mathrm{Irr}(G)|$, and $|\mathrm{Irr}(G)|$ is the number of conjugacy classes of G, we have the following immediate consequence of Theorem A, which does not mention characters. It would be interesting to try to find a purely group-theoretic proof of this fact.

Corollary B. *Let C be a Carter subgroup of a solvable group G. Then $|C : C'|$ is at most the number of conjugacy classes of G.*

In general outline, Theorem A is analogous to work of E. C. Dade and D. Gajendragadkar, who carried out a somewhat similar program with system normalizers in place of Carter subgroups. Given a system normalizer S of a solvable group G, Dade constructed a family of injective maps from $\mathrm{Irr}(S)$ into $\mathrm{Irr}(G)$, and then he and Gajendragadkar characterized the common image of these maps. (See [2] or [3].) Note that Dade's maps are defined on the full set of irreducible characters of S, whereas our maps are defined only on the set of linear characters of C. Also, Dade and Gajendragadkar did not limit themselves to the formation of nilpotent groups, as we do.

The head characters of a solvable group G are defined in terms of the C-composition series for G, where C is a Carter subgroup of G. We defer the somewhat technical definition of a head character to Section 5, but it seems appropriate to present a very brief discussion of composition series here.

Suppose that A is a group that acts via automorphisms on some group G. (For example, we could take A to be a subgroup of G, acting by conjugation.) Recall that an A-**composition series** for G is a subnormal series

$$1 = S_0 \lhd S_1 \lhd \cdots \lhd S_r = G$$

pp of A-invariant subgroups S_i of G, where each factor S_{i+1}/S_i is A-simple, which means that it contains no nontrivial proper A-invariant normal subgroup.

If $L \lhd K \subseteq G$, where L and K are A-invariant, we say that K/L is an A-**composition factor** of G if K/L is A-simple and K is subnormal in G. It is easy to see that this happens if and only if L and K are consecutive terms S_i and S_{i+1} in some A-composition series $\{S_i\}$ of G.

If G is solvable, it is easy to see that all A-composition factors of G are abelian, and thus the action of A on an arbitrary A-composition factor K/L of G is either trivial or else it is fixed-point free. In the first case, where $\mathbf{C}_{K/L}(A) = K/L$, we say that K/L has **type TA** (trivial action) with respect to A, and otherwise, we have $\mathbf{C}_{K/L}(A) = 1$, and we say that K/L has **type FPF** (fixed-point free) with respect to A. When we use this notation, we will often neglect to mention the specific group A that we have in mind, but this should be clear from the context.

In addition to our results involving Carter subgroups, we have included in the final section of this paper a theorem analogous to our characterization of head characters. Recall that if A is a group that acts via automorphisms on a group G, where $|A|$ and $|G|$ are relatively prime, the Glauberman-Isaacs correspondence is a uniquely determined bijection $(\ )^*$ from the set of A-invariant irreducible characters of G onto the set $\mathrm{Irr}(C)$, where $C = \mathbf{C}_G(A)$. (See [4, Chapter 13] and [6, Theorem 8.11].) If G is solvable, we use A-composition series to define the A-head characters of G, which turn out to be exactly the set of A-invariant characters $\chi \in \mathrm{Irr}(G)$ such that χ^* is linear.

We close this introduction by recalling some convenient notation. If A is a group that acts via automorphisms on a group G, we write $\mathrm{Irr}_A(G)$ to denote the set of A-invariant members of $\mathrm{Irr}(G)$. Also, if $T \subseteq S$ are subgroups of a group G, and $\sigma \in \mathrm{Irr}(S)$ and $\tau \in \mathrm{Irr}(T)$, we say that σ **lies over** τ or that τ **lies under** σ if τ is a constituent of σ_T, and we write $\mathrm{Irr}(S|\tau)$ to denote the set of irreducible characters of S that lie over τ.

2. FPF SECTIONS

Let C be a Carter subgroup of a solvable group G, and view C as acting on G by conjugation. We generalize slightly the notion of a C-composition factor of G of type FPF. Given C-invariant subgroups $L \lhd K$ of G, we say that K/L is an **FPF section** of G with respect to C if $\mathbf{C}_{K/L}(C) = 1$.

We will often neglect to mention any specific Carter subgroup C of G when we say that K/L is an FPF section. This will generally be clear from the context, but note, however, that if K and L are normal in G, then because all Carter subgroups of G are conjugate, the choice of C is really irrelevant.

Let C be a Carter subgroup of a solvable group G, and suppose that $L \lhd\lhd K$ are C-invariant subgroups of G. Then L and K are subnormal in KC, so there exists a C-composition series $\{S_i\}$ for KC such that L and K appear among the subgroups S_i. It thus makes sense to discuss the C-composition factors "between" L and K, and we note that up to C-isomorphism, these factors are uniquely determined.

Lemma 2.1. *Let C be a Carter subgroup of a solvable group G, and let $L \lhd K$ be C-invariant subgroups of G. The following are then equivalent.*

(a) *K/L is an FPF section.*
(b) *$K \cap LC = L$.*
(c) *For every choice of C-invariant subgroups A and B such that $L \subseteq B \lhd A \subseteq K$, the section A/B is FPF.*
(d) *Every C-composition factor of K/L between L and K is of type FPF.*

Proof. We show first that (a) and (b) are equivalent. If (b) holds, then K/L meets LC/L trivially. Also, since LC/L is a Carter subgroup of KC/L, it is self-normalizing, and it follows that

$$\mathbf{C}_{K/L}(C) = \mathbf{C}_{K/L}(LC/L) \subseteq (LC/L) \cap (K/L) = 1\,,$$

and thus K/L is an FPF section and (a) holds. If (b) is false, on the other hand, then $(K \cap LC)/L$ is a nontrivial normal subgroup of the nilpotent group LC/L, so this subgroup meets $\mathbf{Z}(LC/L)$ nontrivially. Then

$$1 < ((K \cap LC)/L) \cap \mathbf{Z}(LC/L) \subseteq \mathbf{C}_{K/L}(C)\,,$$

and thus (a) is false.

Now assume (b). To prove (c), we must show that A/B is an FPF section, so we want to establish (a) in the situation where A and B replace K and L. We have already seen that (b) implies (a), so it suffices to check that (b) holds with A and B in place of K and L, and thus it suffices to show that $A \cap BC = B$. Now $B \subseteq A$, so by Dedekind's lemma, we have

$$A \cap BC = B(A \cap C) \subseteq B(K \cap LC) = BL = B\,,$$

where the penultimate equality holds because we are assuming (b), which asserts that $K \cap LC = L$. This establishes (c).

Next, we assume (c), and we prove (d). Let A/B be an arbitrary C composition factor between L and K, so $L \subseteq B \lhd A \subseteq K$. Now (c) guarantees that A/B is an FPF section, as wanted.

Finally, assuming (d), we complete the proof by showing that $K \cap LC = L$, so (b) holds, and thus (a) also holds. Let $D = K \cap LC$, so $D \supseteq L$, and we work to obtain a contradiction if $D > L$.

Assuming that $D > L$, let E/L be a C-composition factor with $L < E \subseteq D$, and observe that $E \cap LC \subseteq D \cap LC = D > L$, so (b) fails if E replaces K, Since (a) and (b) are equivalent, (a) also fails in this situation, and thus E/L is not an FPF section. This is a contradiction, however, because we are assuming that all C-composition factors between L and K are of type FPF, and this completes the proof □

Corollary 2.2. *Let C be a Carter subgroup of a solvable group G, and let $L \lhd M \lhd K$ be C-invariant subgroups of G, where $L \lhd K$. Then K/L is an FPF section if and only if both K/M and M/L are FPF sections.*

Proof. By Lemma 2.1, we know that K/L is an FPF section if and only if all C-composition factors between L and K are of FPF type. Up to C-isomorphism (and ignoring multiplicities) the C-composition factors between L and K are exactly the C-composition factors between L and M together with the C-composition factors between M and K. All of these factors are of type FPF, therefore, precisely when all of the factors between L and K are of FPF type. By Lemma 2.1, this happens if and only if both M/L and K/M are FPF sections. □

Corollary 2.3. *Suppose G is solvable but not nilpotent, and let $K = G^\infty$. Then there exists a normal subgroup L of G with $L < K$, and such that K/L is an FPF section of G.*

We mention that in general, there may be many normal subgroups L of G such that $L \subseteq K$ and K/L is an FPF section of G. One of the advantages of the particular construction that we use in the following proof is that it can be implemented fairly easily by a computer program.

Proof of Corollary 2.3. Since G is solvable but not nilpotent, we have $1 < K < G$, so K is solvable and $K > 1$. Now $K' < K = G^\infty$, and thus G/K' is not nilpotent. Then $K'C < G$, where C is a Carter subgroup of G, and writing $L = K \cap K'C$, we see that $K' \subseteq L \subseteq K$, so $L \lhd K$. Also, C normalizes L, so $L \lhd KC = G$, as required.

Now $L \subseteq K'C$, so $LC \subseteq K'C < G = KC$, and hence $L < K$. Finally, since $K' \subseteq L \subseteq K'C$, we see that $LC = K'C$, and thus $K \cap LC = K \cap K'C = L$. Then K/L is an FPF section by Lemma 2.1, and the proof is complete. □

Recall that if K/L is a C-composition factor of G, where C is a Carter subgroup, then K/L is either of type an FPF or of type TA with respect to C, and these possibilities are mutually exclusive.

Lemma 2.4. *Let C be a Carter subgroup of a solvable group G, and let $L \lhd\lhd K$ be C-invariant subgroups of G. Then the following are equivalent.*

(a) *Every C-composition factor of KC between L and K is of type TA.*
(b) $K \subseteq LC$.
(c) $KC = LC$.

Proof. Assuming (a), we prove (b) by induction on $|K : L|$. Suppose first that there exists no C-invariant subgroup M with $L < M \lhd\lhd K$. Then $L \lhd K$ and K/L is a C-composition factor of KC, so in particular, K/L is abelian, and (a) guarantees that K/L must be of type TA. Then K/L is not an FPF factor, and it follows by Lemma 2.1 that $K \cap LC > L$. Now $K \cap LC$ is C-invariant and it is normal in K because K/L is abelian. We deduce that $K \cap LC = K$, and thus (b) holds in this case.

We can assume, therefore, that there exists a C-invariant subgroup M such that $L < M \lhd\lhd K$. All C-composition factors between L and M and between M and K are of type TA, so by the inductive hypothesis, we have $M \subseteq LC$ and $K \subseteq MC$. Then $K \subseteq MC \subseteq (LC)C = LC$, proving (b).

That (b) implies (c) is clear, so it suffices to assume (c) and to prove (a). Let S/T be a C-composition factor of KC between L and K, so we must show that S/T is of type TA. Otherwise, S/T is an FPF section, and we work to derive a contradiction. Now $L \subseteq T \subseteq S \subseteq K$, and by (c), we have $LC = KC$, so $TC = SC$. Also, since S/T is an FPF factor, Lemma 2.1 yields $T = S \cap TC = S \cap SC = S > T$, and this contradiction completes the proof. □

3. Carter-invariant characters

The main result of this section is the following.

Theorem 3.1. *Let C be a Carter subgroup of a solvable group G, and suppose that $L \lhd K$ are C-invariant subgroups of G such that K/L is an FPF section. The following then hold.*

(a) *If $\theta \in \mathrm{Irr}_C(K)$, then θ lies over a unique character $\varphi \in \mathrm{Irr}_C(L)$.*
(b) *If $\varphi \in \mathrm{Irr}_C(L)$, then φ lies under a unique character $\theta \in \mathrm{Irr}_C(K)$.*
(c) *If $\theta \in \mathrm{Irr}_C(K)$ and $\varphi \in \mathrm{Irr}_C(L)$, where θ lies over φ, then θ is extendible to KC if and only if φ is extendible to LC.*

Note that we can paraphrase (b) by saying that there is a map $h : \mathrm{Irr}_C(L) \to \mathrm{Irr}_C(K)$ such that φ lies under $h(\varphi)$. By (a), the map h is both injective and surjective, so (a) and (b) together say that "lying under" defines a bijection from $\mathrm{Irr}_C(L)$ onto $\mathrm{Irr}_C(K)$.

We will need several preliminary results before we can prove Theorem 3.1, but first, we recall that if $L \lhd K$ and $\alpha \in \mathrm{Irr}(K)$ and $\beta \in \mathrm{Irr}(L)$, then we say that α and β are **fully ramified** with respect to each other if α_L is a multiple of β and β^K is a multiple of α. In this situation, α and β uniquely determine each other, and we sometimes say that α and β are fully ramified with respect to the section K/L. Recall also that α and β are fully ramified with respect to each other if and only if $\alpha_L = e\beta$, where $e^2 = |K : L|$.

Lemma 3.2. *Let K/L be an abelian chief factor of a group G, and suppose $\alpha \in \mathrm{Irr}(K)$ lies over $\beta \in \mathrm{Irr}(L)$. Assume either that α is invariant in G or that $G = KT$, where T is the stabilizer of β in G. Then one of the following occurs.*

(a) $\beta^K = \alpha$ *and* $K \cap T = 1$.
(b) $\alpha_L = \beta$.
(c) α *and* β *are fully ramified with respect to each other.*

Proof. See [6, Corollary 7.4]. □

Our next result contains more than we will be need for the proof of Theorem 3.1; the extra information (about zero values of characters) will be needed later, however.

Lemma 3.3. *Let K/L be an abelian chief factor of a group G, and suppose that $\theta \in \mathrm{Irr}(K)$ and $\varphi \in \mathrm{Irr}(L)$ are fully ramified with respect to each other. Suppose also that U/L is a complement for K/L in G/L. Then*

(a) *If $|K : L|$ and $|G : K|$ are relatively prime, then φ is extendible to U if and only if θ is extendible to G. Also, if ξ and χ are extensions of φ and θ to U and G, respectively, then for elements $u \in U$, we have $\xi(u) = 0$ if and only if $\chi(u) = 0$.*
(b) *If U/L is nilpotent, then φ is extendible to U if and only if θ is extendible to G*

Proof. Since φ and θ uniquely determine each other, we see that if θ is not invariant in G, then φ is not invariant in U, and thus neither φ nor θ is extendible. In this case, the lemma is vacuously true, and so we can assume that θ is invariant in G.

Suppose now that $|K : L|$ and $|G : K|$ are relatively prime. By [6, Theorem 8.1], there is a complement X/L for K/L in G/L such that X has certain special properties. By the Schur-Zassenhaus theorem, however, all complements for K/L in G/L are conjugate, and it follows that the given complement U/L enjoys these properties. In particular, there is a character ψ of U such that $\psi(u) \neq 0$ for all elements $u \in U$, and $\psi(1) = e$, where $e^2 = |K : L|$ and $\theta_L = e\varphi$. Also, there exists a bijection $f : \mathrm{Irr}(G|\theta) \to \mathrm{Irr}(U|\varphi)$ such that whenever $f(\chi) = \xi$, we have $\chi_U = \psi\xi$.

If $f(\chi) = \xi$, then $\chi(1) = \psi(1)\xi(1) = e\xi(1)$. Since $\theta(1) = e\varphi(1)$, it follows that $\chi(1)/\theta(1) = \xi(1)/\varphi(1)$, and thus χ is an extension of θ if and only if ξ is an extension of φ. In particular, we see that as required, θ is extendible to G if and only if φ is extendible to U.

Continuing to assume that $f(\chi) = \xi$, we see that for elements $u \in U$, we have $\chi(u) = \psi(u)\xi(u)$, and since $\psi(u) \neq 0$, we conclude that $\chi(u) = 0$ if and only if $\xi(u) = 0$. Now if ξ_0 and χ_0 are arbitrary extensions of φ and θ to U and G, respectively, we must show that $\chi_0(u) = 0$ if and only if $\xi_0(u) = 0$. This follows because the Gallagher correspondence guarantees that χ_0 and ξ_0 can be obtained from χ and φ by multiplying by linear characters, and this completes the proof of (a).

For (b), observe that if $K = G$ then $L = U$, so both φ and θ are extendible to U and G and there is nothing further to prove. We can thus assume that $K < G$, and we proceed by induction on $|G : K|$.

Since K/L is an abelian chief factor of G, we see that K/L must be a p-group for some prime p. If p does not divide $|G : K|$, the result follows by (a), so we can assume that p divides $|G : K| = |U : L|$. By hypothesis, U/L is nilpotent, so we can choose a normal subgroup T/L of order p in U/L, and we let $Z/L = \mathbf{C}_{K/L}(T)$. Now $Z > L$ because K/L is a p-group and T/L has order p. Also, $Z \triangleleft K$ because K/L is abelian, and since U normalizes T, it follows that U normalizes Z, and thus $Z \triangleleft KU = G$.

Now $L < Z \subseteq K$ and K/L is a chief factor of G, and hence $Z = K$. Then T centralizes K/L, and thus K normalizes T, and so $T \triangleleft G$. Let $S = KT$, so $S \triangleleft G$ and S/T is G-isomorphic to K/L, and in particular, S/T is a chief factor of G.

Suppose now that φ extends to U. Then there exists an extension α of φ to T such that α extends to U. Let $\beta \in \mathrm{Irr}(S)$ lie over α, and observe that β lies over φ. It follows that β lies over θ because θ is the unique irreducible character of K that lies over φ. Now S/K is cyclic and θ is invariant in S, and we deduce that β is an extension of θ to S.

If we suppose instead that θ extends to G, there is an extension β of θ to S such that β extends to G, and in this case, we let $\alpha \in \mathrm{Irr}(T)$ lie under β. Now $\beta_L = \theta_L$ and θ_L is a multiple of φ, and since β lies over α, it follows that α lies over φ. Since T/L is cyclic and φ is invariant in T, we deduce that α is an extension of φ.

We are assuming that at least one of φ or θ extends to U or G, respectively, so by the results of the previous two paragraphs, there is an extension α of φ to T and an extension β of θ to S such that β lies over α and either α extends to U or β extends to G.

Now S/T is an abelian chief factor of G and $\beta \in \mathrm{Irr}(S)$ lies over $\alpha \in \mathrm{Irr}(T)$. Also, either β is invariant in G or α is invariant in U, so it follows by Lemma 3.2 that one of the following occurs: either $\alpha^S = \beta$ or $\beta_T = \alpha$ or α and β are fully ramified with respect to each other.

If $\alpha^S = \beta$, then $\beta(1) = |S : T|\alpha(1) = |K : L|\alpha(1) = e^2\alpha(1)$, and if $\beta_T = \alpha$, we have $\beta(1) = \alpha(1)$. We know, however, that $\beta(1) = \theta(1) = e\varphi(1) = e\alpha(1)$, and since $1 < e < e^2$, it follows that only the third alternative can occur, and thus α and β are fully ramified with respect to each other.

Now U/T is a complement for S/T in G/T, and U/T is nilpotent, so we can apply the inductive hypothesis with T and α in place of L and φ, and with S and β in place of K and θ. It follows that α extends to U if and only if β extends to G. We know, however, that at least one of these alternatives is true, so in fact α extends to U and β extends to G. We conclude that φ extends to U and θ extends to G, as required. □

We will also need the following fairly standard result for our proof of Theorem 3.1.

Lemma 3.4. *Suppose that Q acts on K via automorphisms, and let $L \triangleleft K$, where L is Q-invariant, and $|Q|$ and $|K/L|$ are relatively prime. The following then hold.*

(a) *If $\theta \in \mathrm{Irr}_Q(K)$, then θ lies over some character $\varphi \in \mathrm{Irr}_Q(L)$.*
(b) *If $\mathbf{C}_{K/L}(Q) = 1$, then the character φ of (a) is unique.*
(c) *If K/L is abelian and $\varphi \in \mathrm{Irr}_Q(L)$, then φ lies under some character $\theta \in \mathrm{Irr}_Q(K)$.*
(d) *If $\mathbf{C}_{K/L}(Q) = 1$, then the character θ of (c) is unique.*

Proof. Given $\theta \in \mathrm{Irr}_Q(K)$, observe that K/L acts transitively by conjugation on the set $\mathcal{S}$ of irreducible constituents of θ_L. Also, Q acts on K/L, and since θ is

Q-invariant, Q also acts on $\mathcal{S}$. Now $|Q|$ and $|K/L|$ are relatively prime, and it is easy to check that the actions of Q on K/L and on $\mathcal{S}$ are compatible in the sense of Glauberman's lemma with the action of K/L on $\mathcal{S}$. (See [5, Lemma 3.24] or [4, Lemma 13.8 and Corollary 13.9].) It follows by Glauberman's lemma that some character $\varphi \in \mathcal{S}$ is Q-invariant, and this establishes (a).

By the conjugacy part of Glauberman's lemma, the action of $\mathbf{C}_{K/L}(Q)$ on the set of Q-fixed members of $\mathcal{S}$ is transitive, so if $\mathbf{C}_{K/L}(Q) = 1$, there is a unique Q-fixed character in $\mathcal{S}$, and this proves (b).

For (c) and (d), we assume now that K/L is abelian, and we let $\varphi \in \mathrm{Irr}_Q(L)$. Let $\mathcal{T}$ be the set of irreducible constituents of φ^K, and observe that if $\theta \in \mathcal{T}$ and $\lambda \in \mathrm{Irr}(K/L)$, then $\lambda\theta$ is irreducible and lies over φ, so $\lambda\theta \in \mathcal{T}$, and thus the group $\mathrm{Irr}(K/L)$ acts by multiplication on $\mathcal{T}$. This action is transitive because if $\theta \in \mathcal{T}$, then every member of $\mathcal{T}$ is a constituent of $(\theta_L)^K = (1_L\theta_L)^K = (1_L)^K\theta$.

Now Q acts on $\mathcal{T}$ because φ is Q-invariant, and Q also acts on $\mathrm{Irr}(K/L)$. Since K/L is abelian by assumption, we have $|\mathrm{Irr}(K/L)| = |K/L|$, and by hypothesis, this number is relatively prime to $|Q|$. The actions of Q on $\mathcal{T}$ and $\mathrm{Irr}(K/L)$ are easily seen to be compatible (in the sense of Glauberman's lemma) with the multiplication action of $\mathrm{Irr}(K/L)$ on $\mathcal{T}$, and it follows by Glauberman's lemma that Q fixes some character $\theta \in \mathcal{T}$, and (c) follows.

Finally, for (d), we assume that $\mathbf{C}_{K/L}(Q) = 1$, and we show that the character θ is unique. By the conjugacy part of Glauberman's lemma, we know that $\mathbf{C}_{\mathrm{Irr}(K/L)}(Q)$ acts transitively on the set of Q-fixed characters in $\mathcal{T}$, so it suffices to show that only the trivial character of K/L is fixed by Q. Suppose, therefore, that $\lambda \in \mathrm{Irr}(K/L)$ is fixed by Q. Then $[K,Q]L \subseteq \ker\lambda$, and since by hypothesis, $|Q|$ and $|K/L|$ are relatively prime and $\mathbf{C}_{K/L}(Q)$ is trivial, Fitting's lemma guarantees that $[K,Q]L = K$. It follows that $K \subseteq \ker\lambda$, so λ is principal, as required. □

We need one more preliminary result.

Lemma 3.5. *Let $L \lhd G$, where G is solvable, and let C be a Carter subgroup of G. Given $\chi \in \mathrm{Irr}(G)$, there exists at most one C-invariant irreducible character of L that lies under χ.*

Proof. Suppose that α and β are C-invariant irreducible constituents of χ_L, so we can write $\beta = \alpha^g$ for some element $g \in G$. Let T be the stabilizer of α in G, and note that $C \subseteq T$ because α is C-invariant, and thus $C^g \subseteq T^g$. Also, C stabilizes β, and since the stabilizer of $\beta = \alpha^g$ is T^g, we have $C \subseteq T^g$.

Each of C and C^g is a Carter subgroup of T^g, so for some element $x \in T^g$, we have $C^{gx} = C$. Then $gx \in \mathbf{N}_G(C) = C \subseteq T$, and thus $\alpha = \alpha^{gx} = \beta^x = \beta$, where the final equality holds because x lies in the stabilizer T^g of β. □

Proof of Theorem 3.1. If $L = K$, there is nothing to prove, so we can assume that $L < K$, and we proceed by induction on $|K : L|$. Observe that $K \lhd KC$, and since K and C normalize L, we also have $L \lhd KC$.

Suppose first that K/L is a chief factor of KC. Then K/L is an abelian p-group for some prime p, and we let P and Q, respectively, be the Sylow p-subgroup and the Hall p'-subgroup of the nilpotent group C. Write $Z/L = \mathbf{C}_{K/L}(P)$, and observe that $Z > L$ because P is a p-group acting on the nontrivial p-group K/L. Also, $Z \lhd K$ because K/L is abelian, and furthermore, C normalizes Z because C normalizes P. Then $Z \lhd KC$, and since we are assuming that K/L is a chief factor of KC, we deduce that $Z = K$, and thus P acts trivially on K/L. Now $C = PQ$, so

$\mathbf{C}_{K/L}(Q) = \mathbf{C}_{K/L}(C) = 1$, where the second equality holds because by assumption, K/L is an FPF section.

If $\theta \in \mathrm{Irr}_C(K)$, then θ is Q-invariant, and since $|Q|$ and $|K/L|$ are relatively prime, it follows by Lemma 3.4 that there exists a unique Q-invariant character $\varphi \in \mathrm{Irr}(L)$ that lies under θ. Also, since $Q \triangleleft C$ and θ is C-invariant, the uniqueness of φ guarantees that φ is C-invariant, and that in fact, φ is the unique C-invariant member of $\mathrm{Irr}(L)$ lying under θ. This establishes (a) in the case where K/L is a chief factor of KC.

Similarly, given $\varphi \in \mathrm{Irr}_C(L)$, it follows by Lemma 3.4 that there is a unique Q-invariant character $\theta \in \mathrm{Irr}(K)$ that lies over φ. Then θ is the unique C-invariant member of $\mathrm{Irr}(K)$ lying over φ, and this establishes (b) in the case where K/L is chief.

Now suppose that $\theta \in \mathrm{Irr}_C(K)$ and $\varphi \in \mathrm{Irr}_C(L)$, where θ lies over φ. To prove (c), we must show that φ has an extension to LC if and only if θ has an extension to KC.

Since K/L is an abelian chief factor of KC and φ is invariant in LC, we can apply Lemma 3.2 to conclude that one of the following occurs. Either

(a) $\varphi^K = \theta$ or
(b) $\theta_L = \varphi$ or
(c) φ and θ are fully ramified with respect to each other,

and we consider these possibilities in turn.

First, suppose that $\varphi^K = \theta$, so L is the full stabilizer of φ in K, and thus LC is the full stabilizer of φ in KC. By the Clifford correspondence, induction defines a bijection from $\mathrm{Irr}(LC|\varphi)$ onto $\mathrm{Irr}(KC|\varphi) = \mathrm{Irr}(KC|\theta)$. We argue that if this induction map carries $\xi \in \mathrm{Irr}(LC)$ to $\chi \in \mathrm{Irr}(KC)$, then ξ is an extension of φ if and only if χ is an extension of θ. To see this, observe that $\chi(1) = |KC : LC|\xi(1) = |K : L|\xi(1)$, and $\theta(1) = |K : L|\varphi(1)$, so $\chi(1)/\theta(1) = \xi(1)/\varphi(1)$, and thus ξ extends φ if and only if χ extends θ, as claimed. It follows from this that φ has an extension to LC if and only if θ has an extension to KC, as required.

Next, suppose that $\theta_L = \varphi$. By [6, Lemma 2.11(a)], it follows that restriction defines a bijection from $\mathrm{Irr}(KC|\theta)$ onto $\mathrm{Irr}(LC|\varphi)$, and we argue that if this restriction map carries $\chi \in \mathrm{Irr}(KC)$ to $\xi \in \mathrm{Irr}(LC)$, then χ is an extension of θ if and only if ξ is an extension of φ. To see this, observe that $\chi(1) = \xi(1)$ and $\theta(1) = \varphi(1)$, so $\chi(1)/\theta(1) = \xi(1)/\varphi(1)$, and thus χ is an extension of θ if and only if ξ is an extension of φ, as claimed. It follows from this that θ has an extension to KC if and only if φ has an extension to LC, as required.

The remaining possibility is that θ and φ are fully ramified with respect to each other. Since K/L is an FPF section, we see by Lemma 2.1 that $K \cap LC = L$, and thus LC/L is a complement for K/L in KC/L. Also, LC/L is nilpotent, so Lemma 3.3 guarantees that θ has an extension to KC if and only if φ has an extension to LC. This completes the proof of the theorem in the case where K/L is a chief factor of KC.

We can now assume that K/L is not a chief factor, so there exists a C-invariant subgroup M with $L < M < K$, and by hypothesis, K/L is an FPF section, so Corollary 2.2 guarantees that K/M and M/L are FPF sections.

Let $\theta \in \mathrm{Irr}_C(K)$, so by the inductive hypothesis for (a), with M in place of L, there exists a character $\alpha \in \mathrm{Irr}_C(M)$ such that θ lies over α. Similarly, by the inductive hypothesis for (a) with M in place of K and α in place of θ, there exists a character $\varphi \in \mathrm{Irr}_C(L)$ such that α lies over φ. Since θ lies over φ, this establishes existence in (a).

By the inductive hypothesis for (c) with M and α in place of L and φ, we see that θ extends to KC if and only if α extends to MC, and similarly, by the inductive hypothesis for (c) with M and α in place of K and θ, it follows that α extends to MC if and only if φ extends to LC. We conclude that θ extends to KC if and only if φ extends to LC, and this proves (c).

For uniqueness in (a), suppose that $\varphi_0 \in \mathrm{Irr}_C(L)$ also lies under θ, and let $\chi \in \mathrm{Irr}(G)$ lie over θ. Then φ and φ_0 are C-invariant and lie under χ, so by Lemma 3.5, we have $\varphi_0 = \varphi$, and this completes the proof of (a).

By (a), there is a well-defined map $f : \mathrm{Irr}_C(K) \to \mathrm{Irr}_C(L)$, where $f(\theta)$ is the unique member of $\mathrm{Irr}_C(L)$ that lies under θ, and we work next to show that f is surjective.

Let $\varphi \in \mathrm{Irr}_C(L)$ be arbitrary. By the inductive hypothesis for (b), with M in place of K, there exists a character $\beta \in \mathrm{Irr}_C(M)$, such that β lies over φ, and similarly, by the inductive hypothesis for (b) with M in place of L and β in place of φ, there exists a character $\theta \in \mathrm{Irr}_C(K)$ such that θ lies over β. Then θ lies over φ, and since these characters are C-invariant, it follows that $f(\theta) = \varphi$, and thus the map $f : \mathrm{Irr}_C(K) \to \mathrm{Irr}_C(L)$ is surjective, as wanted.

By the inductive hypotheses for both (a) and (b), we have $|\mathrm{Irr}_C(K)|=|\mathrm{Irr}_C(M)|= |\mathrm{Irr}_C(L)|$, and we deduce that our surjective map f is also injective.

For every member $\varphi \in \mathrm{Irr}_C(L)$ we have seen that there exists a character $\theta \in \mathrm{Irr}_C(K)$ such that θ lies over φ. For each such character θ, we have $f(\theta) = \varphi$, and since f is injective, θ is the unique member of $\mathrm{Irr}_C(K)$ that lies over φ. This proves (b). and so the proof of the theorem is now complete. ☐

Corollary 3.6. *Let C be a Carter subgroup of a solvable group G, and suppose that $L \lhd K$ are C-invariant subgroups of G such that K/L is an FPF section. Then $KC/K \cong LC/L$, and there exists a bijection $h : \mathrm{Irr}_C(L) \to \mathrm{Irr}_C(K)$ with the following properties.*

(a) *If $\varphi \in \mathrm{Irr}_C(L)$ and $\theta \in \mathrm{Irr}_C(K)$, then $h(\varphi) = \theta$ if and only if θ lies over φ.*
(b) *If $\varphi \in \mathrm{Irr}_C(L)$ and $\lambda \in \mathrm{Irr}_C(K)$ is linear, then $h(\varphi\lambda_L) = h(\varphi)\lambda$.*
(c) *If $h(\varphi) = \theta$, then the numbers of extensions of φ to LC and θ to KC are equal.*
(d) *If $h(\varphi) = \theta$, then $\theta(1)/\varphi(1)$ is an integer dividing $|K : L| = |KC : LC|$.*

Proof. First, since K/L is an FPF section, we see by Lemma 2.1 that $K \cap LC = L$. Then $KC/K = K(LC)/K \cong LC/(K \cap LC) = LC/L$, and this proves the first assertion.

Now given $\varphi \in \mathrm{Irr}_C(L)$, it follows by Theorem 3.1(b) that φ lies under a unique member θ of $\mathrm{Irr}_C(K)$. Thus $\varphi \mapsto \theta$ defines a map $h : \mathrm{Irr}_C(L) \to \mathrm{Irr}_C(K)$. and by Theorem 3.1(a), this map is a bijection, proving (a).

For (b), observe that if $\lambda \in \mathrm{Irr}_C(K)$ is linear and $\varphi \in \mathrm{Irr}_C(L)$, then $\varphi\lambda_L$ lies in $\mathrm{Irr}_C(L)$, and this character lies under $h(\varphi)\lambda$. Then $h(\varphi\lambda_L) = h(\varphi)\lambda$ by (a), as required.

Now assume that $h(\varphi) = \theta$, where $\varphi \in \mathrm{Irr}_C(L)$ and $\theta \in \mathrm{Irr}_C(K)$. Then θ lies over φ, so it follows by Theorem 3.1(c), that φ is extendible to LC if and only if θ is extendible to KC. If the number of extensions of φ to LC is 0, therefore, then the number of extensions of θ to KC is also 0, and (c) holds in this case.

We can now assume that φ has an extension to LC, and thus θ has an extension to KC. It follows by the Gallagher correspondence ([4, Corollary 6.17]) that the number of extensions of φ to LC is equal to the number of linear characters of LC/L, and similarly, the number of extensions of θ to KC is equal to the number of linear characters of KC/K. Assertion (c) now follows because we have seen that $LC/L \cong KC/K$.

Now for (d), observe that $\theta \in \mathrm{Irr}(K)$ lies over $\varphi \in \mathrm{Irr}(L)$ and $L \triangleleft K$, so it follows that $\theta(1)/\varphi(1)$ is an integer divisor of $|K : L|$. Also, since $LC/L \cong KC/K$, we have $|LC||K| = |KC||L|$, and thus $|K : L| = |KC : LC|$, completing the proof of (d). $\square$

Before we state our next result, we introduce some new notation. If $H \subseteq G$, we write

$$\mathrm{Irr}(G\|H) = \{\chi \in \mathrm{Irr}(G) \mid \ \chi_H \text{is irreducible}\}\,.$$

Thus, for example, $\mathrm{Irr}(G\|G) = \mathrm{Irr}(G)$, and $\mathrm{Irr}(G\|1) = \mathrm{Lin}(G)$. Also, we observe that if $H \subseteq K \subseteq G$, then $\mathrm{Irr}(G\|H) \subseteq \mathrm{Irr}(G\|K)$.

Corollary 3.7. *Let C be a Carter subgroup of a solvable group G, and suppose that $L \triangleleft K$ are C-invariant subgroups of G such that K/L is an FPF section. Then there exists a (not necessarily unique) bijection $k : \mathrm{Irr}(LC\|L) \to \mathrm{Irr}(KC\|K)$, having the following properties.*

(a) *If $k(\xi) = \chi$, then $h(\xi_L) = \chi_K$, where $h : \mathrm{Irr}_C(L) \to \mathrm{Irr}_C(K)$ is the bijection of Corollary* 3.6*, and so in particular, ξ_L lies under χ_K.*
(b) *Let $\mathcal{S} \subseteq \mathrm{Irr}(LC\|L)$ be a subset that is closed under multiplication by linear characters of LC, and write $\mathcal{T} = k(\mathcal{S})$. Then $\mathcal{T}$ is closed under multiplication by linear characters of KC.*
(c) *Let $\mathcal{T} = k(\mathcal{S})$, where as in (b), the set $\mathcal{S}$ is closed under multiplication by linear characters of LC. Then although k is not necessarily unique, the set $\mathcal{T}$ is uniquely determined by $\mathcal{S}$.*
(d) *If $k(\xi) = \chi$, then $\chi(1)/\xi(1)$ is an integer divisor of $|K : L| = |KC : LC|$.*

Proof. Observe that $\mathrm{Irr}(LC\|L)$ is exactly the set of extensions to LC of irreducible characters $\varphi \in \mathrm{Irr}_C(L)$, and similarly, $\mathrm{Irr}(KC\|K)$ is the set of extensions to KC of irreducible characters $\theta \in \mathrm{Irr}_C(K)$.

Now let $h : \mathrm{Irr}_C(L) \to \mathrm{Irr}_C(K)$ be the bijection of Corollary 3.6, and suppose that $h(\varphi) = \theta$. It follows by Corollary 3.6 that the sets

$$\{\xi \in \mathrm{Irr}(LC) \mid \xi_L = \varphi\} \quad \text{and} \quad \{\chi \in \mathrm{Irr}(KC) \mid \chi_X = \theta\}$$

have equal cardinality, so there exists a bijection from the first of these sets onto the second. We can construct a map $k : \mathrm{Irr}(LC\|L) \to \mathrm{Irr}(KC\|K)$ by piecing together these bijections over all pairs φ and θ such that $h(\varphi) = \theta$, and we see that k is a bijection, as required.

Now let $k(\xi) = \chi$, where $\xi \in \mathrm{Irr}(LC\|L)$ and $\chi \in \mathrm{Irr}(KC\|K)$, and write $\varphi = \xi_L$ and $\theta = \chi_K$. Then φ and θ are irreducible, and $h(\varphi) = \theta$ by the construction of the map k, and this proves (a).

Suppose now that $\mathcal{S} \subseteq \mathrm{Irr}(LC\|L)$, where $\mathcal{S}$ is closed under multiplication by linear characters of LC, and let $\mathcal{T} = k(\mathcal{S})$. Let $\chi \in \mathcal{T}$ and let λ be a linear character of KC, so to prove (b), we must show that $\chi\lambda$ lies in $\mathcal{T}$.

Since $\chi \in \mathcal{T} = k(\mathcal{S})$, we have $\chi = k(\xi)$ for some character $\xi \in \mathcal{S}$. Write $\theta = \chi_K$ and $\varphi = \xi_L$, so $\theta = h(\varphi)$ by (a). Also, $(\chi\lambda)_K = \theta\lambda_K$, and this character is irreducible, so $\chi\lambda$ lies in $\mathrm{Irr}(KC\|K)$. Since the map k is surjective, there exists a character $\eta \in \mathrm{Irr}(LC\|L)$ such that $k(\eta) = \chi\lambda$, and so to prove that $\chi\lambda$ lies in $\mathcal{T}$, it suffices to show that η lies in $\mathcal{S}$.

Now $k(\eta) = \chi\lambda$, and so $h(\eta_L) = (\chi\lambda)_K = \theta\lambda_K$. Also, $h(\varphi) = \theta$, so $h(\varphi\lambda_L) = h(\varphi)\lambda_K = \theta\lambda_K$ by Corollary 3.6(b). Then $h(\eta_L) = \theta\lambda_K = h(\varphi\lambda_L)$, and since h is injective, we deduce that $\eta_L = \varphi\lambda_L$.

Also, since $\xi_L = \varphi$, we have $(\xi\lambda_{LC})_L = \varphi\lambda_L$, and we see that η and $\xi\lambda_{LC}$ are irreducible characters of LC that have equal irreducible restrictions to L. By the Gallagher correspondence, therefore, there exists a linear character ν of LC/L such that $\eta = \xi\lambda_{LC}\nu$.

Now recall that $\xi \in \mathcal{S}$, and by hypothesis, $\mathcal{S}$ is closed under multiplication by linear characters of LC. It follows that η lies in $\mathcal{S}$, and thus $k(\eta)$ lies in $\mathcal{T}$, as required.

For (c), we must show that $\mathcal{T}$ is uniquely determined by $\mathcal{S}$, or in other words, that if we replace k by some other bijection k' that also satisfies (a), then this replacement leaves the set $\mathcal{T}$ unchanged. We must show, in other words, that $k(\mathcal{S}) = k'(\mathcal{S})$.

To see this, observe that if $\xi \in \mathcal{S}$, then each of the characters $k(\xi)$ and $k'(\xi)$ is an extension of $h(\xi_L)$ to KC. Then $k'(\xi) = k(\xi)\nu$ for some linear character ν of KC/K, and since we have shown that $\mathcal{T}$ is closed under multiplication by linear character of KC, it follows that $k(\xi)$ lies in $\mathcal{T}$ if and only $k'(\xi)$ lies in $\mathcal{T}$, as required.

Finally for (d), note that if $k(\xi) = \chi$, then writing $\xi_L = \varphi$ and $\chi_K = \theta$, we have $h(\varphi) = \theta$, and thus $\chi(1)/\xi(1) = \theta(1)/\varphi(1)$, and this is an integer divisor of $|KC : LC| = |K : L|$ by Corollary 3.6(d). $\square$

4. Carter chains

Fix a Carter subgroup C of a solvable group G, and suppose that $C < G$, or equivalently, that G is not nilpotent. Write $K = G^\infty$, so by Corollary 2.3 there exists a subgroup $L \lhd G$, where $L < K$, and K/L is an FPF section of G. Write $U = LC$, and observe that $K \cap U = L$ by Lemma 2.1, and since $L < K$, we have $U < G$.

Now C is a Carter subgroup of U, and if $C < U$, we can repeat this construction with U in place of G, and this yields a subgroup V with $G > U > V \supseteq C$. We can continue like this to obtain a strictly decreasing chain of subgroups

$$G = U_0 > U_1 > \cdots > U_m = C\,,$$

and we refer to $\{U_i \mid 0 \le i \le m\}$ as a **Carter chain** for G of length m, and descending to the Carter subgroup C.

Note that at each step in the construction of a Carter chain for G, we must choose a normal subgroup L of G as in Corollary 2.3. There is no requirement, however, that we must use the algorithm of the proof of Corollary 2.3 (or any other specific algorithm) so in general, there may be several different Carter chains for a given group G, and descending to a given Carter subgroup C, and these chains may have different lengths. We can, of course, standardize the construction process, using some specific algorithm (such as the one described in our proof of Corollary 2.3) and if we did this, we would obtain a "canonical" Carter chain. As we shall see, however, this standardization is not necessary.

Given an arbitrary (but fixed) Carter chain $\{U_i\}$ for G, we will define a family of injective maps $f : \mathrm{Lin}(C) \to \mathrm{Irr}(G)$, depending on the chain $\{U_i\}$, and we shall show that the image $f(\mathrm{Lin}(C))$ does not depend on the choice of the map f in the family. Perhaps more surprisingly, we shall see that the image set $f(\mathrm{Lin}(C))$ does not depend on the particular Carter chain $\{U_i\}$, and that is why there is little to be gained by standardizing the construction of a Carter chain.

Consider a consecutive pair of subgroups $U_i > U_{i+1}$ in a Carter chain of length m descending to a Carter subgroup C of a solvable group G. By the construction of the chain, there are normal subgroups $L_i < K_i$ of U_i, where $K_i = (U_i)^\infty$ and K_i/L_i is an FPF section of U_i, and such that $U_{i+1} = L_iC$. Note that since C is a Carter subgroup of U_i and U_i/K_i is nilpotent, we have $U_i = K_iC$. By Corollary 3.6, therefore, we have $U_i/K_i \cong U_{i+1}/L_i$, and it follows that U_{i+1}/L_i is nilpotent, and thus $K_{i+1} \subseteq L_i$. Also, by Corollary 3.7, there is a bijection $k_i : \mathrm{Irr}(U_{i+1}\|L_i) \to \mathrm{Irr}(U_i\|K_i)$.

Holding the Carter chain $\{U_i \mid 0 \le i \le m\}$ fixed, we recursively define subsets $\mathcal{S}_i$ of $\mathrm{Irr}(U_i\|K_i)$, starting with $i = m$ and proceeding to $i = 0$. We set $\mathcal{S}_m = \mathrm{Irr}(U_m\|K_m) = \mathrm{Irr}(C\|1) = \mathrm{Lin}(C)$. Then, assuming that we have already defined the subset $\mathcal{S}_{i+1} \subseteq \mathrm{Irr}(U_{i+1}\|K_{i+1})$, and observing that $\mathrm{Irr}(U_{i+1}\|K_{i+1}) \subseteq \mathrm{Irr}(U_{i+1}\|L_i)$, we can apply the map k_i to $\mathcal{S}_{i+1}$, and we define $\mathcal{S}_i = k_i(\mathcal{S}_{i+1})$. Then $\mathcal{S}_i \subseteq \mathrm{Irr}(U_i\|K_i)$, as wanted.

By composing the maps k_i, we obtain an injective map $f : \mathrm{Lin}(C) \to \mathrm{Irr}(G)$, and we note that this actually yields a family of maps f because the k_i are not uniquely determined in Corollary 3.7.

Next, we study the image $f(\mathrm{Lin}(C))$, where f is one of the maps in the family we have defined. First, we see that f maps $\mathrm{Lin}(C)$ onto the set $\mathcal{S}_0 \subseteq \mathrm{Irr}(U_0\|K_0) = \mathrm{Irr}(G\|K)$, where $K = G^\infty$, and thus for each character $\psi \in f(\mathrm{Lin}(C))$, we see that ψ_K is irreducible.

To compute the degrees of the characters in $f(\mathrm{Lin}(C))$, let $\psi = f(\lambda)$, where $\lambda \in \mathrm{Lin}(C)$ and as before, let $K_i = (U_i)^\infty$. Define the characters $\psi_i \in \mathrm{Irr}(U_i\|K_i)$ by setting $\psi_m = \lambda$, and for $i < m$, set $\psi_i = k_i(\psi_{i+1})$, so $\psi_0 = \psi$. Writing $d_i = \psi_i(1)$, we have

$$\psi(1) = \frac{\psi(1)}{\lambda(1)} = \frac{d_0}{d_m} = \frac{d_0}{d_1}\,\frac{d_1}{d_2}\cdots\frac{d_{m-1}}{d_m}\,.$$

For notational convenience, write $\xi = \psi_{i+1}$ and $\chi = \psi_i$ and $k = k_i$. Then $\chi = k(\xi)$, and $d_i/d_{i+1} = \chi(1)/\xi(1)$. By Corollary 3.7(d), this number divides $|U_i : U_{i+1}|$, and it follows that $\psi(1)$ divides

$$|U_0 : U_1||U_1 : U_2|\cdots|U_{m-1} : U_m| = |U_0 : U_m| = |G : C|\,.$$

We have now proved the following.

Theorem 4.1. *Let C be a Carter subgroup of a solvable group G, and write $K = G^\infty$. Then there exists a (not necessarily unique) injective map f from $\mathrm{Lin}(C)$ into the set of characters $\chi \in \mathrm{Irr}(G)$ such that χ_K is irreducible, and furthermore, the degree of each member of $f(\mathrm{Lin}(C))$ divides $|G : C|$.*

We have not yet shown, however, that the set $f(\mathrm{Lin}(C))$ does not depend on the construction of the map f, so Theorem 4.1 is only a weak form of Theorem A of the introduction. We have proved everything in Theorem A except for the uniqueness of the image set $f(\mathrm{Lin}(C))$, and in particular, we have now proved Corollary B.

In general, the map f of Theorem 4.1 fails to be unique for two distinct reasons: the maps k_i are not necessarily unique, and the construction of the map f depends not only on the k_i but also on a choice of a particular Carter chain for G. It is comparatively easy to see that the ambiguity in the maps k_i does not affect the image set $f(\mathrm{Lin}(C))$, but the fact that $f(\mathrm{Lin}(C))$ does not depend on the choice of the Carter chain seems deeper, and we defer that proof to Section 5.

To prove that the choices of the maps k_i do not affect the image set $f(\mathrm{Lin}(C))$, recall that once we have chosen a Carter chain, our definition of the map f involved constructing subsets $\mathcal{S}_i$ of $\mathrm{Irr}(U_i \| K_i)$, where $\mathcal{S}_m = \mathrm{Lin}(C)$ and $\mathcal{S}_i = k_i(\mathcal{S}_{i+1})$ for $i < m$. Now observe that $\mathcal{S}_m$ is unambiguously defined, and it is invariant under multiplication by linear characters of $C = U_m$. It follows by repeated application of Corollary 3.7(b) and (c) that all of the sets $\mathcal{S}_i$ are invariant under multiplication by linear characters, and thus all of these sets are uniquely determined. In particular, $\mathcal{S}_0 = f(\mathrm{Lin}(C))$ is uniquely determined. (We stress that we have established this uniqueness only when a particular Carter chain has been fixed.)

Because it is possible to make an unambiguous choice of a Carter chain by consistently using a specified algorithm to produce the normal subgroup L of Corollary 2.3, we can define the image set $f(\mathrm{Lin}(C))$ in some unique way. This is still weaker than Theorem A, however, because it requires choosing an algorithm.

5. Composition series and head characters

In proving Theorem 4.1, we constructed injective maps $f : \mathrm{Lin}(C) \to \mathrm{Irr}(G)$, and in this section, we show that all of these maps have the same image, and we present a characterization of this image, expressed in terms of C-composition series for G

Let C be a Carter subgroup of a solvable group G, and let $\{S_i \mid 0 \le i \le r\}$ be a C-composition series for G. Observe that by the Jordan-Hölder theorem, together with the fact that all Carter subgroups of G are conjugate, it follows that the composition length r is an invariant of G.

Now choose C-invariant characters $\theta_i \in \mathrm{Irr}(S_i)$, and suppose that θ_i lies under θ_{i+1} for $0 \le i < r$. In this situation, we say that $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ is a C-**pair series** for G, and we say that this C-pair series is **associated** with the C-composition series $\{S_i\}$. Also, we say that a C-pair series $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ for G is **strong** if each of the characters θ_i is extendible to S_iC

If $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ is a strong C-pair series for G, we say that θ_r is the **head character** of the C-pair series $\{(S_i, \theta_i)\}$. (We have chosen the word “head” to describe the character θ_r because θ_r occurs at the top, or head, of the series.) Also, given a character $\chi \in \mathrm{Irr}(G)$, we say that χ is a **head character** of G if it appears as the head character of some strong C-pair series for G, and we observe that the set of head characters of G is independent of the choice of a particular Carter subgroup C because the Carter subgroups of G are conjugate.

If λ is a linear character of G, then λ is automatically a head character. To see this, let $\{S_i \mid 0 \le i \le r\}$ be an arbitrary C-composition series for G, and let θ_i be the restriction of λ to S_i. Now $\{(S_i, \theta_i)\}$ is clearly a strong C-pair series, and since $\theta_r = \lambda$, we see that λ is a head character of G, as claimed.

If G is nilpotent, then conversely, every head character of G is linear. To prove this, let $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ be a strong C-pair series for G, and observe that G is a Carter subgroup for G, so all of the subgroups S_i are normal in G, and all of the characters θ_i are G-invariant. Also, the index $|S_{i+1} : S_i|$ is prime for all i, and it follows easily by downward induction on i that the restriction of the head character θ_r to S_i is exactly θ_i. Then $\theta_r(1) = \theta_0(1) = 1$, and so the head character θ_r is linear, as claimed.

We will show that the head characters of an arbitrary solvable group G are exactly the members of the image set $f(\mathrm{Lin}(C))$, where f is any of the injective maps constructed as in Theorem 4.1.

Lemma 5.1. *Let C be a Carter subgroup of a solvable group G, and suppose that S/T is a C-composition factor of G. Also, let $\sigma \in \mathrm{Irr}(S)$ lie over $\tau \in \mathrm{Irr}(T)$, and suppose that σ and τ are extendible to SC and TC, respectively. Then τ is the unique C-invariant irreducible character of T lying under σ, and if S/T has type TA, then $\sigma_T = \tau$.*

Proof. Since σ and τ extend to SC and TC, these characters are C-invariant. If S/T is of type FPF, then Theorem 3.1 guarantees that τ is the unique C-invariant member of $\mathrm{Irr}(T)$ lying under σ, as required. Otherwise, S/T is of type TA, so $T \subseteq S \subseteq TC$ by Lemma 2.4. Since τ extends to TC, it follows that τ extends to S, and because S/T is abelian, we conclude by the Gallagher correspondence that every member of $\mathrm{Irr}(S)$ that lies over τ is an extension of τ, and thus $\sigma_T = \tau$, and the proof is complete. □

The following is an immediate consequence of Lemma 5.1.

Corollary 5.2. *Let C be a Carter subgroup of a solvable group G, and let $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ be a strong C-pair series with for G. Then the head character θ_r uniquely determines all of the characters θ_i.*

Next, we state our principal result concerning head characters.

Theorem 5.3. *Let C be a Carter subgroup of a solvable group G, and let $\chi \in \mathrm{Irr}(G)$. Then the following are equivalent.*

(a) *χ lies in the image of the injective map f of Theorem* 4.1.
(b) *χ is a head character of G.*
(c) *Every C-composition series $\{S_i \mid 0 \le i \le r\}$ for G is associated with a unique strong C-pair series $\{(S_i, \theta_i)\}$, such that the head character θ_r is the given character χ.*

Observe first that it is a triviality that assertion (c) of Theorem 5.3 implies assertion (b). To see this, let $\{S_i \mid 0 \le i \le r\}$ be an arbitrary C-composition series for G. By (c) there is a strong C-pair series $\{(S_i, \theta_i)\}$ associated with $\{S_i\}$ such that $\theta_r = \chi$. By definition, therefore, χ is the head character of $\{(S_i, \theta_i)\}$, and so χ is a head character of G. To prove that (b) implies (c), however, is much more difficult, and we need the following technical result.

Lemma 5.4. *Let C be a Carter subgroup of a solvable group G, and suppose that K/L is an FPF section of G with respect to C. Let $\theta \in \mathrm{Irr}(K)$ lie over $\varphi \in \mathrm{Irr}(L)$, where θ and φ are extendible to KC and LC, respectively. Also let X/K and Y/L be C-composition factors of G, where $K \cap Y = L$ and $KY = X$, and assume that Y/L is of type TA. Finally, let $\alpha \in \mathrm{Irr}_C(X)$ lie over $\beta \in \mathrm{Irr}_C(Y)$. Then α lies over θ if and only if β lies over φ.*

Proof. Since Y/L is a C-composition factor, of type TA, it follows by Lemma 2.4 that $L \subseteq Y \subseteq LC$. By assumption, φ is extendible to LC, so φ has a C-invariant extension $\hat{\varphi} \in \mathrm{Irr}(Y)$, and by Gallagher's theorem, every irreducible character of Y that lies over φ has the form $\hat{\varphi}\lambda$ for some character $\lambda \in \mathrm{Irr}(Y/L)$. Also, since C acts trivially on Y/L, we see that λ is C-invariant, and it follows that every member of $\mathrm{Irr}(Y|\varphi)$ is C-invariant.

Now X/K and Y/L are C-isomorphic, and we are assuming that Y/L is a TA factor, so X/K is also a TA factor, and since θ extends to KC by hypothesis, reasoning similar to that in the previous paragraph shows that every member of $\mathrm{Irr}(X|\theta)$ is C-invariant.

Next, observe that X/Y is C-isomorphic to K/L, so X/Y is an FPF section. By Theorem 3.1, therefore, each member of $\mathrm{Irr}_C(X)$ lies over a unique member of $\mathrm{Irr}_C(Y)$, and each member of $\mathrm{Irr}_C(Y)$ lies under a unique member of $\mathrm{Irr}_C(X)$. In particular, β is the unique member of $\mathrm{Irr}_C(Y)$ that lies under α, and furthermore, α is the unique member of $\mathrm{Irr}_C(X)$ that lies over β.

Now suppose that α lies over θ. Then α lies over φ, so some irreducible constituent β_0 of α_Y lies over φ. Then β_0 is C-invariant, and the uniqueness of β guarantees that $\beta = \beta_0$, and thus β lies over φ, as required.

Conversely now, assume that β lies over φ. Then

$$[(\beta^X)_K, \theta] = [(\beta_L)^K, \theta] = [\beta_L, \theta_L] > 0\,,$$

where the final inequality is strict because β and θ each lie over $\varphi \in \mathrm{Irr}(L)$. It follows that some irreducible constituent α_0 of β^X lies over θ, and hence α_0 is C-invariant. Then $\alpha_0 = \alpha$ by the uniqueness of α, and we conclude that α lies over θ, as required. □

Theorem 5.5 shows that (b) implies (c) in Theorem 5.3.

Theorem 5.5. *Let C be a Carter subgroup of a solvable group G, and suppose $\{T_i \mid 0 \le i \le r\}$ is an arbitrary C-composition series for G. Also, let $\chi \in \mathrm{Irr}(G)$ be a head character of G. Then the series $\{T_i\}$ is associated with some unique strong C-pair series $\{(T_i, \varphi_i) | 0 \le i \le r\}$ having head character χ.*

Proof. By assumption, χ is a head character of G, so by definition, there exists a strong C-pair series $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ for G with respect to C, where $\theta_r = \chi$. Our goal is to show that there exist characters $\varphi_i \in \mathrm{Irr}(T_i)$, where $\varphi_r = \chi$, and such that $\{(T_i, \varphi_i)\}$ is a strong C-pair series for G with respect to C. This will establish existence in the statement of the theorem; the uniqueness then follows by Corollary 5.2.

Now, $S_r = G = T_r$, so there is a unique smallest nonnegative integer m such that $S_i = T_i$ for all subscripts i with $m \le i \le r$. If $m = 0$, then $T_i = S_i$ for all i, so we can take $\varphi_i = \theta_i$, and there is nothing further to prove. We can assume, therefore, that $m > 0$, and we proceed by downward induction on m.

Now $S_m = T_m$ and $S_{m-1} \ne T_{m-1}$, and since $S_0 = 1 = T_0$, we have $m - 1 > 0$, and thus $m \ge 2$. For notational simplicity, we define $M = S_m = T_m$ and we write $U = S_{m-1}$ and $V = T_{m-1}$, so U and V are nontrivial and distinct, and we write $D = U \cap V$.

Now M/U and M/V are C-composition factors of G, so U and V are maximal among C-invariant normal subgroups of M, and since $U \ne V$, we have $UV = M$. Then U/D is C-isomorphic to M/V, and thus U/D is a C-composition factor of G, and similarly, V/D is a C-composition factor of G.

The C-composition length of $U = S_{m-1}$ is $m-1$, and it follows that the C-composition length of D is $m-2$, so we can choose a C-composition series

$$1 = D_0 \lhd D_1 \lhd \cdots \lhd D_{m-2} = D$$

for D. Observe that if we append the subgroups S_i for $m-1 \le i \le r$ to the series $\{D_i\}$, we obtain a new C-composition series for G. Writing $\{N_i\}$ to denote this series, we see that $N_i = D_i$ if $0 \le i \le m-2$, and $N_i = S_i$ if $m-1 \le i \le r$. In particular, we have $N_{m-1} = S_{m-1} = U$ and $N_{m-2} = D_{m-2} = D$.

We can now apply the inductive hypothesis with the series $\{N_i\}$ in place of $\{T_i\}$ and with $m-1$ in place of m. We conclude that the series $\{N_i\}$ is associated with some strong C-pair series with head character χ. Also, if $i \ge m$, we have $T_i = S_i = N_i$, so there is no loss if we replace the series $\{S_i\}$ with the series $\{N_i\}$. We can thus assume that $N_i = S_i$ for all i, and in particular, we have $D = D_{m-2} = N_{m-2} = S_{m-2}$. (Note that the original series $\{S_i\}$ is now irrelevant.)

Next, we create yet another C-composition series for G. To do this, we modify the series $\{S_i\}$ by replacing the term $S_{m-1} = U$ with the subgroup V. To see that this actually does yield a C-composition series, it suffices to observe that $S_{m-2} = D \lhd V \lhd M = S_m$, and that V/D is C-isomorphic to the C-composition factor $M/U = S_m/S_{m-1}$ and M/V is the C-composition factor T_m/T_{m-1}. Writing $\{X_i\}$ to denote this new series, we observe that $X_i = S_i$ for $i \ne m-1$ and $X_{m-1} = V$.

Our next goal is to show that the C-composition series $\{X_i\}$ is associated with a strong C-pair series with head character χ. Assuming for the moment that this has been accomplished, we note that $T_i = S_i = X_i$ for $i \ge m$, and $T_{m-1} = V = X_{m-1}$, and thus we can apply the inductive hypothesis with $\{X_i\}$ in place of $\{S_i\}$ and with $m-1$ in place of m. We conclude that the C-composition series $\{T_i\}$ is associated with a strong C-pair series with head character χ, as required.

To complete the proof of the theorem, therefore, we seek characters $\beta_i \in \mathrm{Irr}(X_i)$ with $\beta_r = \chi$, and such that $\{(X_i, \beta_i)\}$ is a strong C-pair series. Recall that if $i \ne m-1$, we have $X_i = S_i$, so we can set $\beta_i = \theta_i$ for these subscripts i. Since $X_{m-1} = V$, it suffices to find a suitable character $\beta \in \mathrm{Irr}(V)$, so that we can set $\beta_{m-1} = \beta$.

To state the conditions that $\beta \in \mathrm{Irr}(V)$ must satisfy, it is convenient to write $\alpha = \theta_m$ and $\gamma = \theta_{m-2}$, so $\alpha \in \mathrm{Irr}(M)$ and $\gamma \in \mathrm{Irr}(D)$.

The conditions on β are thus as follows:

(1) β is extendible to VC,
(2) β lies under α and
(3) β lies over γ.

Also, for further notational convenience, we write $\theta = \theta_{m-1}$, so $\theta \in \mathrm{Irr}(U)$. Observe that each of the pairs (M,α), (U,θ) and (D,γ) is one of the terms of the strong C-pair series $\{(S_i,\theta_i)\}$, so α, θ and γ extend to MC, UC and DC, respectively, and also, α lies over θ and θ lies over γ,

Now M/U and U/D are C-composition factors, so each of them is either of type TA or type FPF. Also V/D is C-isomorphic to M/U and M/V is C-isomorphic to U/D, so there are a total of four distinct cases that must be considered.

Suppose first that M/U is of type TA. It follows by Lemma 5.1 that $\alpha_U = \theta$. Similarly, if U/D is also of type TA, then $\theta_D = \gamma$, and hence $\alpha_D = \gamma$. In particular,

α_D is irreducible, so α_V is irreducible, and we take $\beta = \alpha_V$. Then $\beta_D = \gamma$, and hence conditions (2) and (3) hold.

Now α extends to MC, and since $\alpha_V = \beta$, it follows that β extends to MC. Also, M/V is of type TA because it is C-isomorphic to U/D, so Lemma 2.4 guarantees that $VC = MC$. Then β extends to VC, and condition (1) is satisfied, so the proof is complete in this case.

Continuing to assume that M/U is of type TA, suppose now that U/D is of type FPF, so M/V is also of type FPF. Since α extends to MC, Theorem 3.1 guarantees that α_V has a unique irreducible constituent that extends to VC, and we take β to be this constituent, so conditions (1) and (2) are satisfied. To complete the proof in this case, therefore, we must verify that β lies over γ, and we will use Lemma 5.4 to accomplish this, with U, D, M and V in the roles of K, L, X and Y, respectively.

To check that the hypotheses of Lemma 5.4 are satisfied, recall that $UV = M$, that $U \cap V = D$, and that U/D is an FPF factor, as required. Also, θ and γ extend to UC and DC. respectively, and θ lies over γ. Now M/U is of type TA so V/D is also of type TA. Also, α is C-invariant, and by the choice of β, we know that α lies over β and β extends to VC, so β is C-invariant.

We can now apply Lemma 5.4, to deduce that α lies over θ if and only if β lies over γ. We know, however, that α does lie over θ, so β lies over γ, and thus condition (3) holds. This completes the proof in the case where M/U is of type TA.

We can now assume that M/U is type FPF. Then V/D is also of type FPF, and since $\gamma \in \mathrm{Irr}(D)$ extends to DC, Theorem 3.1 guarantees that there is a unique member of $\mathrm{Irr}(V)$ that lies over γ and extends to VC, and we take β to be this character. Conditions (1) and (3) on β are thus satisfied, so it suffices to verify condition (2), so we argue that β lies under α.

Continuing to assume that M/U is of type FPF, we suppose now that U/D is also of type FPF. By Corollary 2.2, we see that M/D is an FPF section, and since α and γ are C-invariant and α lies over γ, it follows by Theorem 3.1 that α is the unique C-invariant irreducible character of M that lies over γ. Theorem 3.1 also guarantees the existence of a C-invariant irreducible character α_0 of M that lies over β, and since β lies over γ, we see that α_0 lies over γ, and thus $\alpha_0 = \alpha$. Then α lies over β, and thus condition (2) is satisfied, so in this case too, the proof is complete.

Finally, we can assume that U/D is of type TA, and we recall that we have defined $\beta \in \mathrm{Irr}(V)$ so that conditions (1) and (3) are satisfied, so it remains to verify condition (2), which, we recall, asserts that α lies over β. To accomplish this, we apply Lemma 5.4 once again, but this time, with V, D, M and U in the roles of K, L, X and Y, respectively.

To check the hypotheses of Lemma 5.4, observe that V/D is C-isomorphic to M/U, so V/D is an FPF factor. Also, β lies over γ and β and γ extend to VC and DC, respectively. Also U/D is of type TA, so M/V is also of type TA, and we recall that α and θ are C-invariant and α lies over θ, so we can apply Lemma 5.4.

Since θ lies over γ, it follows that α lies over β and the proof is complete. □

To complete the proof of Theorem 5.3, we must show that the set of head characters of a solvable group G is exactly the image of any one of the not-necessarily-unique injective maps f of Theorem 4.1. We recall that to constuct f, we start with an arbitrary Carter chain $\{U_i \mid 0 \le i \le m\}$ for G, where $U_0 = G$ and U_m is a Carter subgroup C. Next, a collection of subsets $\mathcal{S}_i \subseteq \mathrm{Irr}(U_i)$ is defined, where $\mathcal{S}_m = \mathrm{Lin}(C)$ and $\mathcal{S}_i = k_i(\mathcal{S}_{i+1})$ for $i < m$, where k_i is as in Corollary 3.7 and the image set $f(\mathrm{Lin}(C))$ is exactly $\mathcal{S}_0$.

Now C is nilpotent, so as we have seen, the set of head characters of C is $\mathrm{Lin}(C) = \mathcal{S}_m$. To show that the set of head characters of G is $f(\mathrm{Lin}(C)) = \mathcal{S}_0$, therefore, it suffices to observe that for each subscript i with $0 \leq i \leq m$, the set $\mathcal{S}_i$ is exactly the set of head characters of U_i. Since $\mathcal{S}_i = k_i(\mathcal{S}_{i+1})$, this follows by repeated application of the following.

Lemma 5.6. *Let C be a Carter subgroup of a solvable group G, and suppose that K/L is an FPF section of G with respect to C. Also, let $k : \mathrm{Irr}(LC\|L) \to \mathrm{Irr}(KC\|K)$ be the (not necessarily unique) bijection of Corollary* 3.7. *Then k carries the set of head characters of LC onto the set of head characters of KC.*

To prove this, we begin with a preliminary result.

Corollary 5.7. *Let $\chi \in \mathrm{Irr}(G)$ be a head character, where G is solvable, and suppose that $K \vartriangleleft G$ and G/K is nilpotent. Then χ_K is irreducible.*

Proof. Let C be a Carter subgroup of G, and let $\{T_i \mid 0 \leq i \leq r\}$ be a C-composition series for G, where K is one of the subgroups T_i, say $K = T_a$. Since χ is a head character of G, Theorem 5.5 guarantees that there exists a strong C-pair series $\{(T_i, \theta_i)\}$, where $\theta_r = \chi$.

Since G/K is nilpotent, we have $G = KC$, and it follows by Lemma 2.4 that every C-composition factor between K and $KC = G$ is of type TA. In particular, each of the factors T_{i+1}/T_i for $i \geq a$ is of type TA, and hence Lemma 5.1 guarantees that θ_{i+1} restricts irreducibly to T_i for $i \geq a$. Since $K = T_a$ and $\chi = \theta_r$, we conclude that χ_K is irreducible, as wanted. □

Proof of Lemma 5.6. Let $\{T_i \mid 0 \leq i \leq n\}$ be a C-composition series for KC such that L and K appear among its terms, say $L = T_a$ and $K = T_b$, where $0 \leq a \leq b \leq n$, and let $\{S_i \mid 0 \leq i \leq m\}$ be a C-composition series for LC such that $S_i = T_i$ for $0 \leq i \leq a$. Also, observe that for $a \leq i \leq m$, we have $S_iC = LC$, and for $b \leq i \leq n$, we have $T_iC = KC$.

Suppose first that ξ is a head character of LC. Since LC/L is nilpotent, it follows by Corollary 5.7 that ξ_L is irreducible, so $\xi \in \mathrm{Irr}(LC\|L)$, and we write $\varphi = \xi_L$. Then $k(\xi)$ is defined and lies in $\mathrm{Irr}(KC\|K)$, and we write $\chi = k(\xi)$ and $\theta = \chi_K$, so θ is irreducible. Also, $\theta = h(\varphi)$ by Corollary 3.7, and thus θ is the unique C-invariant irreducible character of K that lies over φ.

We must show that χ is a head character of KC, so it suffices to find characters $\theta_i \in \mathrm{Irr}(T_i)$ such that $\{(T_i, \theta_i) \mid 0 \leq i \leq n\}$ is a strong C-pair series for KC with $\theta_n = \chi$. We require, therefore, that θ_i is extendible to T_iC for all i, and we also require that θ_i lies under θ_{i+1} for $i < n$.

By assumption, ξ is a head character of LC, so it follows by Theorem 5.5 that there exist characters $\varphi_i \in \mathrm{Irr}(S_i)$ such that $\{(S_i, \varphi_i) \mid 0 \leq i \leq m\}$ is a strong C-pair series for LC, where $\varphi_m = \xi$. Now $m \geq a$, so $\xi = \varphi_m$ lies over φ_a, and since $T_a = L$ and $\xi_L = \varphi$, we see that $\varphi_a = \varphi$.

To define the characters θ_i, suppose first that $0 \leq i \leq a$. Then $T_i = S_i$, so we can set $\theta_i = \varphi_i$, and in particular, $\theta_a = \varphi_a = \varphi$. Also, since $\{(S_i, \varphi_i)\}$ is a strong C-pair series, it follows for subscripts i such that $0 \leq i \leq a$ that $\theta_i = \varphi_i$ is extendible to $S_iC = T_iC$, as required. Also, if $i < a$, then $\theta_{i+1} = \varphi_{i+1}$ lies over $\varphi_i = \theta_i$, as is also required.

Now $\theta_a = \varphi$, and θ_a is extendible to the character ξ of $LC = T_aC$. Working by induction on i, we proceed to define $\theta_i \in \mathrm{Irr}(T_i)$ for $a < i \leq b$ such that θ_i is extendible to T_iC.

Suppose that we have already defined $\theta_{i-1} \in \mathrm{Irr}(T_{i-1})$, where $a < i \leq b$, and where θ_{i-1} is extendible to $T_{i-1}C$. Observe that T_i/T_{i-1} is a C-composition factor between $L = T_a$ and $K = T_b$, so T_i/T_{i-1} is of type FPF. Since θ_{i-1} is extendible to $T_{i-1}C$, it follows by Theorem 3.1 that there is a unique irreducible character of T_i that lies over T_{i-1} and extends to T_iC, and we let θ_i be this character.

We have now defined θ_i for all subscripts i with $a \leq i \leq b$, and where θ_i is extendible to T_iC, as is required. Furthermore, for $a \leq i < b$, we see that θ_{i+1} lies over θ_i, as is also required.

Now θ_b is a C-invariant character of $T_b = K$, and θ_b lies over $\theta_a = \varphi$. Then $\theta_b = \theta$ because as we have observed, θ is the unique C-invariant irreducible character of K that lies over φ.

Finally, to define θ_i for $b \leq i \leq n$, recall that $K = T_b$ and that $\chi_K = \theta$ is irreducible. Then χ_{T_i} is irreducible for $b \leq i \leq n$, so we can set $\theta_i = \chi_{T_i}$ for i in this range. In particular, this yields $\theta_b = \chi_{T_b} = \chi_K = \theta$, so this is consistent with our earlier definition of θ_b, and furthermore, we have $\theta_n = \chi$. Now for $b \leq i \leq n$, we have $T_iC = KC$, so $\theta_i = \chi_{T_i}$ is extendible to the character χ of T_iC. Also, for $b \leq i < n$, we see that θ_{i+1} lies over θ_i, as required.

We see now that $\{(T_i, \theta_i) \mid 0 \leq i \leq n\}$ is a strong C-pair series for KC with $\theta_n = \chi$, and thus χ is a head character of KC, as wanted.

Conversely now, suppose that χ is a head character of KC. Then χ_K is irreducible by Corollary 5.7, so $\chi \in \mathrm{Irr}(KC\|K)$, and we write $\theta = \chi_K$. Since k maps $\mathrm{Irr}(LC\|L)$ onto $\mathrm{Irr}(KC\|K)$, there exists a character $\xi \in \mathrm{Irr}(LC\|L)$ such that $k(\xi) = \chi$. Also, ξ_L is irreducible because $\chi \in \mathrm{Irr}(LC\|L)$, and we write $\chi_L = \varphi$. Then $h(\varphi) = \theta$ by Corollary 3.7. and thus φ is the unique C-invariant character of L that lies under θ.

To complete the proof, it suffices to show that ξ is a head character of LC, so we seek characters $\varphi_i \in \mathrm{Irr}(S_i)$ with $\varphi_m = \xi$, and such that $\{(S_i, \varphi_i) \mid 0 \leq i \leq m\}$ is a strong C-pair series for LC. In particular, we require φ_i to be extendible to S_iC for all i and we also require that φ_i lies under φ_{i+1} if $i < m$.

Since χ is a head character of KC, it follows by Theorem 5.5 that there exist characters $\theta_i \in \mathrm{Irr}(T_i)$, where $\theta_n = \chi$, and such that $\{(T_i, \theta_i) \mid 0 \leq i \leq n\}$ is a strong C-pair series for KC. In particular, θ_b lies under $\theta_n = \chi$, and since $T_b = K$ and $\chi_K = \theta$, we see that $\theta_b = \theta$.

To define the characters φ_i, suppose first that $0 \leq i \leq a$. Then $S_i = T_i$, so we can set $\varphi_i = \theta_i$, and thus φ_i is extendible to $T_iC = S_iC$ because $\{(T_i, \theta_i)\}$ is a strong C-pair series. Furthermore, if $0 \leq i < a$, then $\varphi_i = \theta_i$ lies under $\theta_{i+1} = \varphi_{i+1}$, as required. Also $\varphi_a = \theta_a$ is a C-invariant irreducible character of $T_a = L$ that lies under $\theta_b = \theta$, and we deduce that $\varphi_a = \varphi$.

To define φ_i for $a \leq i \leq m$, recall that $S_a = T_a = L$, and that $\xi_L = \varphi$, where φ is irreducible. Then ξ_{S_i} is irreducible for $a \leq i \leq m$, and we define $\varphi_i = \xi_{S_i}$. In particular, this yields $\varphi_a = \xi_{S_a} = \xi_L = \varphi$, which agrees with our earlier definition of φ_a. Now φ_i is extendible to the character ξ of $S_iC = LC$ for $a \leq i \leq m$. Also, if $a \leq i < m$, then φ_{i+1} and φ_i are the restrictions of ξ to S_{i+1} and S_i, respectively, and thus φ_i lies under φ_{i+1}, as required. Also, since $S_m = LC$, we have $\varphi_m = \xi$.

We see now that $\{(S_i, \varphi_i) \mid 0 \leq i \leq m$ is a strong C-pair series for LC, and since $\varphi_m = \xi$. It follows that ξ is a head character of LC. $\square$

Now that Lemma 5.6 has been established, we have completed the proof of Theorem 5.3.

6. Carter-nonvanishing characters

We say that a character χ of a group G is **nonvanishing** if for all $x \in G$, we have $\chi(x) \neq 0$. Also, χ is **Carter nonvanishing** if for a Carter subgroup C of G, the restriction χ_C is nonvanishing.

Clearly, every linear character is nonvanishing, and conversely, by a well-known result of W. Burnside (see, for example, Theorem 3.15 of [5]) a nonvanishing irreducible character must be linear. Also, we have seen that if G is nilpotent, then the linear characters of G are exactly the head characters of G. This suggests that perhaps for an arbitrary solvable group G, the Carter-nonvanishing irreducible characters are exactly the head characters.

By Theorem 5.3, the head characters of a solvable group G are exactly the characters χ lying in $f(\mathrm{Lin}(C))$, where f is as in Theorem 4.1, and by Theorem 4.1, the cardinality of $f(\mathrm{Lin}(C))$ is $|\mathrm{Lin}(C)| = |C : C'|$. If it is true, therefore that the head characters of G are exactly the Carter-nonvanishing characters, it would follow that the number of Carter-nonvanishing characters $\chi \in \mathrm{Irr}(G)$ is equal to $|C : C'|$, where C is a Carter subgroup of G.

Abundant computational evidence suggests that this might always be true, as was conjectured by G. Navarro [8]. Further computer experiments indicate that not only does it seem to be true that the number of Carter-nonvanishing irreducible characters of G is equal to $|C : C'|$, but also that these Carter-nonvanishing characters of G actually are the head characters of G.

Unfortunately, we have made only slight progress in proving this, and the following very special case is the best we have been able to obtain so far.

Theorem 6.1. *Suppose that a Carter subgroup C of a solvable group G is a maximal subgroup of G, and let $\chi \in \mathrm{Irr}(G)$. Then χ is Carter nonvanishing if and only if it is a head character.*

We need some preliminary results.

Lemma 6.2. *Let $K \lhd G$, and let $\chi \in \mathrm{Irr}(G)$, where G is solvable and $K \subseteq \ker\chi$. Also, if $\widetilde{\chi}$ is the irreducible character if G/K defined by the formula $\widetilde{\chi}(Kx) = \chi(x)$, then $\widetilde{\chi}$ is Carter nonvanishing if and only if χ is Carter nonvanishing. Also, $\widetilde{\chi}$ is a head character of G/K if and only if χ is a head character of G.*

Proof. Let C be a Carter subgroup of G, so KC/K is a Carter subgroup of G/K. Then $\widetilde{\chi}$ is Carter nonvanishing if and only if $\widetilde{\chi}(Kc) \neq 0$ for every element $c \in C$. Since $\widetilde{\chi}(Kc) = \chi(c)$, this happens if and only if $\chi(c) \neq 0$ for all $c \in C$, and thus the first assertion of the lemma is clear.

Now suppose that $\{S_i \mid 0 \leq i \leq r\}$ is a C-composition series for G, where K is one of its terms, say $K = S_a$. Then $\{S_i/K \mid a \leq i \leq r\}$ is a (CK/K)-composition series for G/K.

If χ is a head character for G, then by Theorem 5.5, there exist characters $\theta_i \in \mathrm{Irr}(S_i)$ with $\theta_r = \chi$, and such that $\{(S_i, \theta_i) \mid 0 \leq i \leq r\}$ is a strong C-pair series for G. Now if $a \leq i \leq r$, then θ_i lies over θ_a, and in particular, χ lies over

θ_a. Since $S_a = K \subseteq \ker\chi$, it follows that θ_a is principal, and thus $K \subseteq \ker\theta_i$. It should now be clear that $\{(S_i/K, \widetilde{\theta}_i) \mid a \le i \le r\}$ is a strong C-pair series for G/K, where $\widetilde{\theta}_r = \widetilde{\chi}$, and thus $\widetilde{\chi}$ is a head character for G/K.

Conversely, if $\widetilde{\chi}$ is a head character for G/K, then by Theorem 5.5, there exist characters $\theta_i \in \mathrm{Irr}(S_i)$ for $a \le i \le r$ with $K \subseteq \ker\theta_i$ and $\theta_r = \chi$, and such that $\{(S_i/K, \widetilde{\theta}_i) \mid a \le i \le r\}$ is a strong G-pair series for G/K. Now for $0 \le i < a$, let θ_i to be the principal character of S_i. Then it is easy to see that $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ is a strong C-pair series for G, and thus χ is a head character for G. □

In order to prove Theorem 6.1, we need some conditions sufficient to guarantee that a character value is or is not zero. The following well-known fact is such a result, and we present the short proof here.

Lemma 6.3. *Let χ be a character of G, and suppose that $\chi(x) = 0$ for some element $x \in G$, where the order of x is a power of a prime q. Then q divides $\chi(1)$.*

Proof. We can assume that $G = \langle x\rangle$, so we can write χ as a sum of linear characters λ_i for $1 \le i \le \chi(1)$. Now $\lambda_i(x)$ is a q-power root of unity, and thus $\lambda_i(x) \equiv 1 \bmod I$, where I is a maximal ideal of the ring R of algebraic integers and $q \in I$, or equivalently, $I \cap \mathbb{Z}$ is the principal ideal (q) of the rational integers $\mathbb{Z}$. Then

$$0 = \chi(x) = \sum_i \lambda_i(x) \equiv \sum_i 1 = \chi(1) \mod I\,,$$

and thus $\chi(1) \in I \cap \mathbb{Z} = (q)$, so q divides $\chi(1)$, as wanted. □

The following is also well known.

Lemma 6.4. *Let $K \lhd G$, where G/K is nilpotent, and let $\chi \in \mathrm{Irr}(G)$. Then χ_K is reducible if and only if there exists a subgroup H with $K \subseteq H < G$, and such that χ vanishes on the elements of $G - H$.*

Proof. Since G/K is nilpotent, it follows by [4, Theorem 6.22] that there exists a subgroup X of G containing K and a character α of X such that α_K is irreducible and $\alpha^G = \chi$. If χ_K is reducible, then χ is not α, and so $X < G$. Then X is contained in some maximal subgroup H of G, and since $K \subseteq H$ and G/K is nilpotent, it follows that $H \lhd G$. Now χ is induced from X, so it is induced from the normal subgroup H, and thus χ vanishes on $G - H$, as wanted.

Conversely, suppose that there is a proper subgroup H of G such that χ vanishes on $G - H$. It follows that $[\chi_H, \chi_H] = |G : H|[\chi, \chi] = |G : H| > 1$, and so χ_H is reducible. If $K \subseteq H$, therefore, we see that χ_K is reducible. □

Proof of Theorem 6.1. Let $M = \ker\chi$, and let $\widetilde{\chi}$ be the character of G/M corresponding to χ as in Lemma 6.2. By that lemma, it suffices to show that $\widetilde{\chi}$ is a Carter-nonvanishing character of G/M if and only if it is a head character of G/M.

Suppose first that G/M is nilpotent. Then $\widetilde{\chi}$ is Carter nonvanishing if and only if it is nonvanishing, and that happens if and only if $\widetilde{\chi}$ is linear. Furthermore, since we are assuming that G/M is nilpotent, $\widetilde{\chi}$ is linear if and only if it is a head character. Then $\widetilde{\chi}$ is Carter nonvanishing if and only if it is a head character, and there is nothing further that we must prove in this case.

We can now assume that G/M is not nilpotent, so $C \subseteq MC < G$. and thus $C = MC$ by the maximality of C. Then $M \subseteq MC = C$, and C/M is a Carter subgroup of G/M. Also, C/M is a maximal subgroup of G/M, so it suffices to prove the theorem with G/M in place of G. We can thus assume that $M = 1$, and hence χ is faithful.

Let $L = \mathrm{core}_G(C)$. Now $L \subseteq C < G$, so we can choose a chief factor K/L of G. Then $K \not\subseteq C$, and thus $KC = G$ by the maximality of C, so G/K is nilpotent. We argue next that K/L is a C-composition factor. Otherwise, there exists a C-invariant subgroup X such that $L < X < K$, and since K/L is abelian, X is also K-invariant. Then $X \triangleleft KC = G$, and this is a contradiction because K/L is a chief factor of G.

Now $L \subseteq K \cap C < K$, and $K \cap C$ is C-invariant. Since C acts irreducibly on K/L, it follows that $L = K \cap C$, so $L \subseteq C$ and C/L is a complement for K/L in G/L. If C acts trivially on K/L, then $C \triangleleft G$, which is not the case. The C-composition factor K/L is thus not of type TA, and hence it is of type FPF. We can thus apply Corollary 3.7 to deduce that the map k of that lemma is a bijection from $\mathrm{Irr}(LC\|L)$ onto $\mathrm{Irr}(KC\|K)$.

We show next that if χ is either a head character or a Carter-nonvanishing character, then χ_K is irreducible. If χ is a head character, this follows by Corollary 5.7 because G/K is nilpotent. Suppose now that χ is Carter nonvanishing. If χ_K is not irreducible, then since G/K is nilpotent, Lemma 6.4 guarantees the existence of a subgroup H with $K \subseteq H < G$, and such that χ vanishes on $G - H$. If $C \subseteq H$, then $G = KC \subseteq H < G$, and this contradiction shows that in fact, $C \not\subseteq H$, and thus there exists an element $x \in C$ such that $x \notin H$. Then $\chi(x) = 0$, and this contradicts the assumption that χ is Carter nonvanishing. We deduce that χ_K is irreducible in this case too.

Our goal is to show that χ is a head character if and only if it is Carter nonvanishing. If χ_K is not irreducible, then by the result of the previous paragraph, χ has neither of these properties, so there is nothing further that we must prove. We can assume, therefore, that χ_K is irreducible, so $\chi \in \mathrm{Irr}(G\|K) = \mathrm{Irr}(KC\|K)$. By Corollary 3.7, therefore, there exists a character $\xi \in \mathrm{Irr}(LC\|L)$ such that $k(\xi) = \chi$

Writing $\theta = \chi_K$ and $\varphi = \xi_L$, we see that θ and φ are irreducible and C-invariant, and that θ lies over φ. Lemma 5.6 guarantees that χ is a head character of G if and only if ξ is a head character of C, and since C is nilpotent, ξ is a head character of C if and only if ξ is linear. We want to show that χ is Carter nonvanishing if and only if ξ is linear, or equivalently, if and only if φ is linear.

Since K/L is an abelian chief factor of G and θ is invariant in G, we can apply Lemma 3.2 to deduce that either $\theta_L = \varphi$, or that θ and φ are fully ramified with respect to each other, or that $\varphi^K = \theta$ and C is the stabilizer of φ in G.

Suppose first that $\theta_L = \varphi$. Then, $\chi_L = \varphi$, so if φ is linear, then χ is linear, and hence χ is Carter nonvanishing. Conversely, if χ is Carter nonvanishing, then since in this case, $\chi_L = \varphi$ and $L \subseteq C$, we see that φ is nonvanishing. Since φ is irreducible, Burnside's theorem guarantees that φ is linear, as wanted.

Next, suppose that θ and φ are fully ramified with respect to each other. Recall that C/L is a complement for K/L in G/L. Also, the irreducible action of C/L on K/L is faithful because $L = \mathrm{core}_G(K)$. Since C/L is nilpotent, it follows that the unique prime divisor of $|K/L|$ cannot divide $|C/L|$, and thus $|G : K|$ and $|K : L|$ are relatively prime. Since χ and ξ are extensions of θ and φ, to G and C respectively, we can apply Lemma 3.3(a) to deduce for elements $x \in C$ that $\chi(x) = 0$ if and only if $\xi(x) = 0$. Then χ is Carter nonvanishing if and only if ξ is nonvanishing, and since $\xi \in \mathrm{Irr}(C)$, this happens if and only if ξ is linear, or equivalently, φ is linear.

Finally, suppose $\varphi^K = \theta$. Assuming first that χ is Carter nonvanishing, we work to show that φ is linear. Now φ is C-invariant, and in fact, C is the full stabilizer of φ in G because φ induces irreducibly to K, and so the stabilizer of φ in G meets K at L.

Since χ lies over φ, it follows by the Clifford correspondence that $\chi = \eta^G$ for some character $\eta \in \mathrm{Irr}(C|\varphi)$. Also

$$\varphi^K = \theta = \chi_K = (\eta^G)_K = (\eta_L)^K \,,$$

and since φ is a constituent of η_L, we deduce that $\eta_L = \varphi$.

Now C is nilpotent and $C > L$, so there exists a subgroup T such that $L \subseteq T \lhd C$ and $|T : L|$ is prime. Then $C \subseteq \mathbf{N}_G(T) < G$, where the final containment is strict because otherwise $T \lhd G$ by the maximality of C, and this is not the case because $T > L = \mathrm{core}_G(C)$.

Let $\tau = \eta_T$, and observe that $\tau_L = \eta_L = \varphi$, and so it suffices to show that τ is linear. If this is false, the restriction of τ to the identity subgroup is reducible, and since T is nilpotent, we can apply Lemma 6.4 to deduce that there exists a proper subgroup H of T such that τ vanishes on $T - H$.

Each of the subgroups L and H is proper in T, so we cannot have $T = L \cup H$, and hence there exists an element $t \in T$ with $t \notin L$ and $t \notin H$. Now $t \in T \subseteq C$, and by assumption, χ is Carter nonvanishing, so $\chi(t) \neq 0$. We proceed now to show that in fact, $\chi(t) = 0$, and from this contradiction, we deduce that τ is linear, as wanted.

Write $S = KT$, so $S \cap C = T$, and $S \lhd G$ because C normalizes T. Now $\tau^S = (\eta_T)^S = (\eta^G)_S = \chi_S$, and thus we have $\chi(t) = \tau^S(t)$. By the definition of character induction, $\tau^S(t)$ is a multiple of $\sum_s \tau(t^s)$, where s runs over elements $s \in S$ such that t^s lies in T.

Suppose that $t^s \in T$, where $s \in S$, and recall that $t \notin L$, and thus also $t^s \notin L$ because $L \lhd G$. Since $|T : L|$ is prime, and both t and t^s lie in T but not in L, we have $\langle L, t\rangle = T = \langle L, t^s\rangle$. Then $T^s = \langle L, t\rangle^s = \langle L, t^s\rangle = T$, and thus $s \in \mathbf{N}_G(T) = C$. Then $s \in S \cap C = T$, and hence $\tau(t^s) = \tau(t) = 0$, where the final equality holds because $t \notin H$. We conclude that $\chi(t) = \tau^S(t) = 0$, and this completes the proof that φ is linear.

Conversely now, continuing to suppose that $\varphi^K = \theta$, we assume that φ is linear, and we work to show that χ is Carter nonvanishing. Recall that χ is faithful and $L \lhd G$, and since χ_L has the linear constituent φ, it follows that L is abelian.

Now $C > L$, so $|C : L| > 1$. Let π be the set of those prime numbers r such that $|C : L|$ is not a power of r. If $q \notin \pi$, therefore, then $|C : L|$ is a nontrivial power of q, and since this can occur for at most one prime, we see that most one prime can fail to lie in π. Also, C/L is a nontrivial p'-group, so $|C : L|$ is not a power of p, and thus $p \in \pi$.

Now let $r \in \pi$, and let R be the unique Sylow r-subgroup of L. Let $B = \mathbf{C}_G(R)$, so $B \lhd G$ and $L \subseteq B$ because L is abelian. Now R is an r-subgroup of the nilpotent group C, so the Hall r'-subgroup of C centralizes R, and thus $|C : B \cap C|$ is a power of r. Since $r \in \pi$, however, $|C : L|$ is not a power if r, and we deduce that $B \cap C > L$, so $B > L$. Since $B \lhd G$ and $L = \mathrm{core}_G(C)$, it follows that $B \not\subseteq C$, and thus $BC = G$ by the maximality of C.

Then G/B is nilpotent, and since also G/K is nilpotent, we see that $G/(K \cap B)$ is nilpotent. Now $L \subseteq (K \cap B)$ and G/L is not nilpotent, and it follows that

$L < K \cap B \subseteq K$. Since K/L is a chief factor of G, we deduce that $K \cap B = K$, so $K \subseteq B$, and thus $R \subseteq \mathbf{Z}(K)$.

Now χ is faithful and χ_K is irreducible, and it follows that $R \subseteq \mathbf{Z}(K) \subseteq \mathbf{Z}(\chi) = \mathbf{Z}(G)$, so $R \subseteq L \cap \mathbf{Z}(G)$. Now writing $Z = L \cap \mathbf{Z}(G)$, we see that no prime $r \in \pi$ can divide $|L : Z|$.

Recall now that we are assuming that $\varphi^K = \theta$. Since θ is irreducible and $L < K$, we deduce that L is not central in G, and thus $|L : Z| > 1$, so there exists a prime q dividing $|L : Z|$. By the result of the previous paragraph, we see that $q \notin \pi$, and thus q is the unique prime not lying in π, and hence $|L : Z|$ is a power of q. Also, we see that $q \neq p$ because $p \in \pi$. Furthermore, since $q \notin \pi$, the definition of π guarantees that $|C : L|$ is a power of q, so $|C : Z|$ is a power of q. We can thus write $C = ZQ$, where Q is a q-group.

Now let $x \in C$, so we have $x = za$, where $z \in Z \subseteq \mathbf{Z}(G)$ and a has q-power order. Then $\chi(z) = \delta\chi(1)$ for some root of unity δ, and it follows that $\chi(x) = \delta\chi(a)$. Now $\chi(1) = \theta(1) = \varphi^K(1) = |K : L|\varphi(1) = |K : L|$, so $\chi(1)$ is a power of p. Since the order of a is a power of q and $q \neq p$, it follows by Lemma 6.3 that $\chi(a) \neq 0$. Then $\chi(x) \neq 0$, and hence χ is Carter nonvanishing, as required. □

7. An analogy

We close this paper by presenting a theorem that is analogous to our results on head characters, but which does not involve Carter subgroups. In the following, we suppose that a group A acts via automorphisms on a group G, where $|A|$ and $|G|$ are relatively prime, and we let $C = \mathbf{C}_G(A)$. In this situation, the Glauberman-Isaacs correspondence is defined, and we recall that it is a canonical bijective map from $\mathrm{Irr}_A(G)$ onto $\mathrm{Irr}(C)$.

If $H \subseteq G$ is an A-invariant subgroup, and $\alpha \in \mathrm{Irr}_A(H)$, we will consistently write α^* to denote the image of the character α under the Glauberman-Isaacs map from $\mathrm{Irr}_A(H)$ onto $\mathrm{Irr}(H \cap C)$. Although we will be using the notation $(\ )^*$ to denote several different maps, which are defined on different sets, we trust that this convention will not result in confusion.

Assuming that G is solvable, the principal result in this section provides a characterization in terms of A-composition series of the characters $\chi \in \mathrm{Irr}_A(G)$ such that the Glauberman-Isaacs correspondent χ^* is linear. To state our result, consider an A-composition factor K/L of G, and note that since G is solvable, K/L is abelian, and thus either A acts trivially on K/L or else $\mathbf{C}_{K/L}(A) = 1$. As before, we say in these cases that K/L has type TA (trivial action) or type FPF (fixed-point-free), respectively.

Now consider an A-composition series

$$1 = S_0 \lhd S_1 \lhd \cdots \lhd S_r = G$$

for G, and let $\theta_i \in \mathrm{Irr}_A(S_i)$, where θ_i lies under θ_{i+1} for $0 \le i < r$. In this situation, we say that $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ is an A-**pair series** for G, and we say that this A-pair series is **strong** if θ_{i+1} restricts irreducibly to S_i whenever the A-composition factor S_{i+1}/S_i has type TA.

We say that θ_r is the **head character** of the strong A-pair series $\{(S_i, \theta_i) \mid 0 \le i \le r\}$, and we say that a character $\chi \in \mathrm{Irr}_A(G)$ is an A-**head character** of G if χ is the head character of some strong A-pair series for G.

The following is the main result of this section.

Theorem 7.1. *Let A act on G via automorphisms, where G is solvable, and $|A|$ and $|G|$ are relatively prime. Then*

(a) *The A-head characters of G are exactly the characters $\chi \in \mathrm{Irr}_A(G)$ such that the Glauberman-Isaacs correspondent χ^* is linear.*

(b) *Suppose χ is an A-head character of G, and let $\{S_i \mid 0 \leq i \leq r\}$ be an arbitrary A-composition series for G. Then there exist uniquely determined A-invariant irreducible characters θ_i of the subgroups S_i such that $\{(S_i, \theta_i) \mid 0 \leq i \leq r\}$ is a strong A-pair series, and $\chi = \theta_r$ is the head character of this series.*

We begin working toward a proof of Theorem 7.1 by stating some known results.

Lemma 7.2. *Suppose A acts on G via automorphisms, where $|A|$ and $|G|$ are relatively prime, and let $C = \mathbf{C}_G(A)$. Let $N \triangleleft G$ be A-invariant, and write $B = N \cap C$. Also, let $\chi \in \mathrm{Irr}_A(G)$ and $\theta \in \mathrm{Irr}_A(N)$, so $\chi^* \in \mathrm{Irr}(C)$ and $\theta^* \in \mathrm{Irr}(B)$. Then*

(a) *If χ lies over θ, then χ^* lies over θ^*.*

(b) *If $\chi_N = \theta$, then $(\chi^*)_B = \theta^*$.*

Proof. Assertion (a) is immediate from [10, Theorem 2.5], and (b) appears as part of [7, Theorem A]. □

Corollary 7.3. *Suppose A acts on G via automorphisms, where $|A|$ and $|G|$ are relatively prime. Let $N \triangleleft G$ be A-invariant, and suppose that $\chi \in \mathrm{Irr}_A(G)$. Then there exists an A-invariant irreducible constituent θ of χ_N, and if χ^* is linear, then θ is unique and θ^* is linear.*

Proof. The existence of an A-invariant irreducible constituent θ of χ_N is guaranteed by Lemma 3.4(a). Now suppose χ^* is linear and that θ_1 and θ_2 are A-invariant irreducible constituents of χ_N. Then χ^* lies over both $(\theta_1)^*$ and $(\theta_2)^*$ by Lemma 7.2(a). By hypothesis, however, χ^* is linear, and hence $(\theta_1)^* = (\theta_2)^*$, and it follows by the injectivity of the Glauberman-Isaacs map that $\theta_1 = \theta_2$. and so θ is unique, as required. Also, since the linear character χ^* lies over θ^*, we see that θ^* must also be linear. □

The following is a converse for Lemma 7.2(b) in the case where the action of A on G/N is trivial. This result is actually a consequence of the more general Theorem 2.12 of Wolf's paper [10], but it is an important step in our proof of Theorem 7.1, so we have decided to state it here, and to give a proof.

Lemma 7.4. *Suppose A acts on G via automorphisms, where $|A|$ and $|G|$ are relatively prime. Let $N \triangleleft G$ be A-invariant, and suppose that the action of A on G/N is trivial. Let $C = \mathbf{C}_G(A)$, and write $B = N \cap C$. Let $\chi \in \mathrm{Irr}_A(G)$, and assume that $(\chi^*)_B$ is irreducible. Then χ_N is irreducible, and $(\chi_N)^* = (\chi^*)_B$.*

Proof. We can assume that $N < G$, and we proceed by induction on $|G : N|$. Since the Glauberman-Isaacs map from $\mathrm{Irr}_A(N)$ to $\mathrm{Irr}(B)$ is surjective and $(\chi^*)_B$ is irreducible, we can write $(\chi^*)_B = \theta^*$ for some character $\theta \in \mathrm{Irr}_A(N)$. We will complete the proof by showing that $\chi_N = \theta$.

We show first that θ is extendible to G. To see this, consider a subgroup H with $N \subseteq H < G$, and observe that H is A-invariant and A acts trivially on H/N because A acts trivially on G/N. Write $D = H \cap C$, so $D \cap N = B$, and thus χ^* restricts irreducibly to D because $B \subseteq D$, and by hypothesis, χ^* restricts irreducibly to B. We can thus write $(\chi^*)_D = \xi^*$ for some character $\xi \in \mathrm{Irr}_A(H)$, and since $(\xi^*)_B = (\chi^*)_B = \theta^*$, it follows by the inductive hypothesis that $\xi_N = \theta$. Thus θ has an A-invariant extension to H.

If G/N is a p-group, then in the notation of the previous paragraph, we can suppose that H has index p in G, so $H \lhd G$. Also, ξ is an A-invariant extension of θ to H. Now $D \lhd C$ and ξ^* is the restriction to D of the character χ^* of C, so ξ^* is C-invariant, and it follows by the injectivity of the Glauberman-Isaacs map that ξ is also C-invariant.

Since A acts trivially on G/N and $|A|$ and $|G|$ are relatively prime, we have $G = NC$, and hence ξ is G-invariant. Now G/H has prime order and $\xi \in \mathrm{Irr}(H)$ is G-invariant, and thus ξ is extendible to G. We conclude that θ is extendible to G, as wanted.

We can now suppose that there is no prime p such that G/N is a p-group. Then if S/N is any Sylow subgroup of G/N, we have $N \subseteq S < G$, and so θ is extendible to S. It follows by [4, Theorem 11.3] that θ is extendible to G, as claimed.

We argue next that every member of the set $\mathcal{S}$ of extensions of θ to G is A-invariant. By the Gallagher correspondence, the group Λ of linear characters of G/N acts transitively on $\mathcal{S}$ by multiplication. Also $|\Lambda|$ and $|A|$ are relatively prime, so we can appeal to Glauberman's lemma to deduce that some member of $\mathcal{S}$ is A-invariant. Each member of Λ is A-invariant, however, and since Λ acts transitively on $\mathcal{S}$, it follows that all members of $\mathcal{S}$ are A-invariant, as claimed.

If $\psi \in \mathcal{S}$, then $\psi_N = \theta$, so Lemma 7.2(b) guarantees that ψ^* is an extension of θ^*. The Glauberman-Isaacs correspondence thus defines an injective map from $\mathcal{S}$ into the set $\mathcal{T}$ of extensions of θ^* to C. Also, $G/N \cong C/B$ because $G = NC$, and thus

$$|\mathcal{S}| = |(G/N) : (G/N)'| = |(C/B) : (C/B)'| = |\mathcal{T}| \,.$$

The injective map $\psi \mapsto \psi^*$ thus carries $\mathcal{S}$ onto $\mathcal{T}$, and since $\chi^* \in \mathcal{T}$, there exists $\psi \in \mathcal{S}$ such that $\psi^* = \chi^*$. Then $\psi = \chi$ so $\chi_N = \theta$, as required, ☐

Proof of Theorem 7.1. First, suppose that χ is an A-head character of G, so by definition, there is a strong A-pair series $\{(S_i, \theta_i) \mid 0 \leq i \leq r\}$, such that $\theta_r = \chi$. All of the characters θ_i are A-invariant, so in particular, χ is A-invariant, and χ^* is defined, and we must show that χ^* is linear.

Now $(\theta_0)^*$ is linear because it is the trivial character of the trivial subgroup of G, If χ^* is not linear, therefore, then since $\chi = \theta_r$, there exists a subscript a with $0 \leq a < r$, such that a is maximal with the property that $(\theta_a)^*$ is linear. For notational convenience, we write $b = a + 1$.

Now θ_b lies over θ_a, so $(\theta_b)^*$ lies over $(\theta_a)^*$ by Lemma 7.2(a). Also $(\theta_b)^*$ is not linear and $(\theta_a)^*$ is linear, so $(\theta_b)^*$ does not restrict irreducibly to $S_a \cap C$, and it follows by Lemma 7.2(b) that θ_b does not restrict irreducibly to S_a. It follows by the definition of a strong A-pair series that S_b/S_a is not of type TA, and we conclude that S_b/S_a has type FPF.

Now A has no nontrivial fixed points on S_b/S_a, so the A-fixed-point subgroups in S_b and S_a are identical, and thus $(\theta_b)^*$ and $(\theta_a)^*$ are characters of the same group. Since $(\theta_b)^*$ lies over $(\theta_a)^*$, it follows that $(\theta_b)^* = (\theta_a)^*$, and this is a contradiction because $(\theta_a)^*$ is linear but $(\theta_b)^*$ is not. This completes the proof that χ^* is linear.

Conversely now, suppose that χ^* is linear, where $\chi \in \mathrm{Irr}_A(G)$, and let $\{S_i \mid 0 \leq i \leq r\}$ be an arbitrary A-composition series for G. We must show that there is a unique choice of A-invariant characters $\theta_i \in \mathrm{Irr}(S_i)$ such that $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ is a strong A-pair series with $\theta_r = \chi$.

We are forced to define $\theta_r = \chi$, so $(\theta_r)^*$ is linear by assumption. Since $S_{r-1} \lhd G$, we can apply Corollary 7.3 (with S_{r-1} in the role of N) to deduce that there is a unique A-invariant irreducible character θ of S_{r-1} that lies under θ_r, so we must define $\theta_{r-1} = \theta$. Also, Corollary 7.3 guarantees that θ^* is linear, so $(\theta_{r-1})^*$ is linear.

We can now apply Corollary 7.3 again, this time with S_{r-1} in the role of G and S_{r-2} in the role of N. We deduce that θ_{r-1} lies over a unique A-invariant irreducible character of S_{r-2}, so we are forced to define θ_{r-2} to be that character, and it follows by Corollary 7.3 that $(\theta_{r-2})^*$ is linear. Continuing like this, with repeated applications of Corollary 7.3, we see that all of the characters θ_i are uniquely determined, and thus there is a unique A-pair series $\{(S_i, \theta_i) \mid 0 \le i \le r\}$ such that $\theta_r = \chi$.

To complete the proof, we must show that this A-pair series is strong. In other words, if $S_a \lhd S_b$ are consecutive terms in the given A-composition series, and S_b/S_a has type TA, we must show that θ_b restricts irreducibly to S_a. Now $(\theta_b)^*$ is linear, so it restricts irreducibly to $S_a \cap C$, and since we are assuming that A acts trivially on S_b/S_a, it follows by Lemma 7.4 that θ_b restricts irreducibly to S_a, as required. $\square$

We close by mentioning a question that was asked a number of years ago by Navarro, and which was communicated to the author privately. Suppose that A acts on G, where $|A|$ and $|G|$ are relatively prime, and let $C = \mathbf{C}_G(A)$. Also, let $\chi \in \mathrm{Irr}_A(G)$, and as usual, let $\chi^* \in \mathrm{Irr}(C)$ be the Glauberman-Isaacs correspondent of χ. Navarro asked if is it true that χ is nonvanishing on C if and only if χ^* is linear.

Of course, it was the analogy between the Glauberman-Isaacs bijection and the material discussed earlier in this paper that motivated the question raised in Section 6: is it true that the head characters of a solvable group G are exactly the Carter nonvanishing irreducible characters of G.

A small piece of evidence that Navarro's question might have an affirmative answer is that if A is a p-group for some prime p, and χ^* is linear, then it is true that χ is nonvanishing on the fixed-point subgroup C. To see this, recall that when A is a p-group, χ^* is the unique irreducible character of C whose multiplicity e as a constituent of χ_C is not divisible by p. Thus if $x \in C$, we have $\chi(x) \equiv e\chi^*(x) \bmod M$, where M is a maximal ideal of the ring R of algebraic integers such that $p \in M$. If χ^* is linear, then $\chi^*(x)$ is some root of unity δ, and thus $\chi(x) \equiv e\delta \bmod M$. If $\chi(x) = 0$, it follows that $e\delta \in M$, and since δ is invertible in R, we have $e \in M$. This is a contradiction, however, because p does not divide e, but $M \cap \mathbb{Z} = p\mathbb{Z}$. We conclude that $\chi(x) \neq 0$, as wanted.

Acknowledgment

The initial version of the final section of this paper considered only the case where the acting group A is solvable. I thank the referee for pointing out that this restriction was not really necessary, and Section 7 was rewritten accordingly. The referee then suggested additional improvements on this section, which was then further revised. What resulted from these changes, is, I think, a cleaner, as well as a more general presentation.

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