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ON FLOER MINIMAL KNOTS IN SUTURED MANIFOLDS

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Abstract. Suppose (M,γ) is a balanced sutured manifold and K is a rationally null-homologous knot in M. It is known that the rank of the sutured Floer homology of $M\backslash N(K)$ is at least twice the rank of the sutured Floer homology of M. This paper studies the properties of K when the equality is achieved for instanton homology. As an application, we show that if $L\subset S^3$ is a fixed link and K is a knot in the complement of L, then the instanton link Floer homology of $L\cup K$ achieves the minimum rank if and only if K is the unknot in $S^3\backslash L$.

1. Introduction

Suppose Y is a closed oriented 3-manifold and K is a rationally null-homologous knot in Y. The spectral sequence in [OS04, Lemma 3.6] implies the following[1]:

$$\operatorname{rank}\widehat{\mathrm{HFK}}(Y,K;\mathbb{Z}/2)\ge \operatorname{rank}\widehat{\mathrm{HF}}(Y;\mathbb{Z}/2). \tag{1.1}$$

The knot K is called *Floer simple* or *Floer minimal* if (1.1) attains equality (see [Hed11, Ras07]). The properties of Floer minimal knots have been studied in [NW14, RR17, GL21].

More generally, suppose (M,γ) is a balanced sutured 3-manifold and K is a knot in the interior of M such that $[K]=0\in H_1(M,\partial M;\mathbb{Q})$. Let $N(K)\subset M$ be an open tubular neighborhood of K. Let γ_K be the union of γ and a pair of oppositely oriented meridians on $\partial N(K)$. Then the following inequality holds for sutured Heegaard Floer homology:

$$\operatorname{rank}\mathrm{SFH}(M\backslash N(K),\gamma_K;\mathbb{Z}/2)\ge 2\cdot\operatorname{rank}\mathrm{SFH}(M,\gamma;\mathbb{Z}/2). \tag{1.2}$$

Remark 1.1. The proof of (1.2) is the same as (1.1): a spectral sequence similar to the one in [OS04, Lemma 3.6] implies that

$$\operatorname{rank}\mathrm{SFH}(M\backslash N(K),\gamma_K;\mathbb{Z}/2)\ge \operatorname{rank}\mathrm{SFH}(M\backslash B^3,\gamma\cup\delta;\mathbb{Z}/2),$$

where B^3 is a 3–ball inside M and δ is a simple closed curve on ∂B^3. Inequality (1.2) then follows from [Juh06, Proposition 9.14].

To see that (1.1) follows from (1.2), take $M=Y\backslash B^3$ and take γ to be a simple closed curve on ∂B^3.

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[1]In the original setup of [OS04], the knot K is assumed to be null-homologous in $\mathbb{Z}$ instead of in $\mathbb{Q}$, but the same spectral sequence still works for the latter case.

Another interesting special case of (1.2) is when M is a link complement. Let L be a link in S^3, let $N(L)$ be an open tubular neighborhood of L, let $M = S^3\backslash N(L)$, and let $\gamma \subset \partial M$ be the union of a pair of oppositely oriented meridians for every component of L. In this case, by [Juh06, Proposition 9.2], inequality (1.2) is equivalent to:

$$\operatorname{rank} \widehat{\mathrm{HFK}}(L \cup K; \mathbb{Z}/2) \geq 2 \cdot \operatorname{rank} \widehat{\mathrm{HFK}}(L; \mathbb{Z}/2), \tag{1.3}$$

where K is an arbitrary knot in $S^3\backslash L$. It is natural to ask about the properties of the pair (L, K) when the equality of (1.3) is achieved. This question has been studied by Ni [Ni14] and Kim [Kim20]. Ni [Ni14] proved that if L is the unlink and the equality of (1.3) holds, then $L \cup K$ is the unlink. (This property was then used in [Ni14, Proposition 1.4] to prove that the rank of $\widehat{\mathrm{HFK}}$ detects the unlink.) Kim [Kim20, Proposition 4] proved that if the equality of (1.3) holds and $L \cup K$ is non-split, then L is non-split and the linking number of K with every component of L is zero.

It is conjectured by Kronheimer and Mrowka [KM10b, Conjecture 7.25] that sutured *Heegaard* Floer homology is isomorphic to sutured *instanton* Floer homology. This paper studies an analogue of (1.2) for instanton Floer homology. From now on, all instanton Floer homology groups are defined with $\mathbb{C}$–coefficients. An inequality on instanton Floer homology analogous to (1.2) was proved by Li-Ye [LY22, Proposition 3.14]: letting (M, γ), K, and γ_K be as above, we have

$$\dim_{\mathbb{C}} \mathrm{SHI}(M\backslash N(K), \gamma_K) \geq 2 \cdot \dim_{\mathbb{C}} \mathrm{SHI}(M, \gamma). \tag{1.4}$$

Remark 1.2. To see that (1.4) follows from [LY22, Proposition 3.14], remove a small 3–ball B^3 from M centered on K such that $B^3 \cap K$ is an arc. Let δ be a simple closed curve on ∂B^3 separating the two points in $\partial B^3 \cap K$. Consider the balanced sutured manifold $(M\backslash B^3, \gamma \cup \delta)$, and let $T = K\backslash B^3$ be a vertical tangle in $M\backslash B^3$. Then $(M\backslash N(K), \gamma_K)$ can be obtained from $(M\backslash B^3, \gamma \cup \delta)$ by removing a neighborhood of T and adding a meridian of T to the suture. By [LY22, Proposition 3.14] and [Li20, Proposition 4.15],

$$\dim_{\mathbb{C}} \mathrm{SHI}(M\backslash N(K), \gamma_K) \geq \dim_{\mathbb{C}} \mathrm{SHI}(M\backslash B^3, \gamma \cup \delta) = 2 \cdot \dim_{\mathbb{C}} \mathrm{SHI}(M, \gamma).$$

In this paper, we will study the properties of K when the equality of (1.4) is achieved. To simplify notation, we introduce Definition 1.3:

Definition 1.3. Suppose (M, γ) is a sutured manifold. Suppose K is a knot in M such that $[K] = 0 \in H_1(M, \partial M; \mathbb{Q})$. Let $N(K) \subset M$ be an open tubular neighborhood of K. Let γ_K be the union of γ and a pair of oppositely oriented meridians on $\partial N(K)$. We say that K is *instanton Floer minimal*, if

$$\dim_{\mathbb{C}} \mathrm{SHI}(M\backslash N(K), \gamma_K) = 2 \cdot \dim_{\mathbb{C}} \mathrm{SHI}(M, \gamma).$$

Suppose (M, γ) is a connected balanced sutured manifold. Let $\{\gamma_i\}_{1 \leq i \leq s}$ be the components of γ, then the homotopy class of every γ_i defines a conjugacy class $[\gamma_i]$ in $\pi_1(M)$. Define

$$\pi_1'(M) := \pi_1(M)/\langle [\gamma_1], \cdots, [\gamma_s] \rangle, \tag{1.5}$$

where $\langle [\gamma_1], \cdots, [\gamma_s] \rangle$ is the normal subgroup of $\pi_1(M)$ generated by the conjugacy classes $[\gamma_1]$, $\cdots$, $[\gamma_s]$.

The first main result of this paper is Theorem 1.4.

Theorem 1.4. *Let (M,γ) be a connected balanced sutured manifold with*

$$\mathrm{SHI}(M,\gamma)\neq 0.$$

Suppose M is decomposed as

$$M = M_1\#\cdots\#M_k\#Y, \tag{1.6}$$

where each M_i is an irreducible sutured manifold (in particular, ∂M_i is non-empty), and Y is a closed manifold, which may be S^3 or reducible. Let K be a knot in the interior of M that represents the zero element in $\pi_1'(M)$. If K is instanton Floer minimal, then K is contained in Y after isotopy.

Remark 1.5. If the knot K represents the zero element in $\pi_1'(M)$, then $[K]=0\in H_1(M,\partial M;\mathbb{Q})$, and hence Definition 1.3 is applicable to K.

Moreover, when $Y=S^3$ in (1.6), we have the following result.

Theorem 1.6. *Suppose (M,γ) is a balanced taut sutured manifold or a connected sum of balanced taut sutured manifolds. Let K be a knot in the interior of M that represents the zero element in $\pi_1'(M)$. Then K is instanton Floer minimal if and only if K is the unknot.*

Suppose L is a link in S^3 and K is a knot in $S^3\backslash L$. By (1.4),

$$\dim_{\mathbb{C}}\mathrm{KHI}(L\cup K)\geq 2\cdot\dim_{\mathbb{C}}\mathrm{KHI}(L). \tag{1.7}$$

The following result characterizes all the pairs (L,K) such that the equality of (1.7) is achieved.

Theorem 1.7. *Let $L\subset S^3$ be a link and $K\subset S^3\backslash L$ be a knot. Then*

$$\dim_{\mathbb{C}}\mathrm{KHI}(L\cup K)= 2\cdot\dim_{\mathbb{C}}\mathrm{KHI}(L) \tag{1.8}$$

if and only if K is the unknot in $S^3\backslash L$.

Theorem 1.7 has the following immediate corollary:

Corollary 1.8. *Suppose L is an n–component link in S^3. Then L is an unlink if and only if* $\dim_{\mathbb{C}}\mathrm{KHI}(L)=2^{n-1}$.

Proof. Suppose K is a knot in S^3, then by [KM10b, Proposition 7.16], we have

$$\dim_{\mathbb{C}}\mathrm{KHI}(K)\geq 1. \tag{1.9}$$

Moreover, K is the unknot if and only if $\dim_{\mathbb{C}}\mathrm{KHI}(K)=1$. If L is the unlink with n components, then $\dim_{\mathbb{C}}\mathrm{KHI}(L)=2^{n-1}$ by Theorem 1.7 and induction on n. On the other hand, suppose L is a link with n components such that $\dim_{\mathbb{C}}\mathrm{KHI}(L)=2^{n-1}$. Let $K_1,\cdots,K_n$ be the components of L. Then by (1.7) and (1.9), we have

$$\dim_{\mathbb{C}}\mathrm{KHI}(K_1\cup\cdots\cup K_{m+1})=2\cdot\dim_{\mathbb{C}}\mathrm{KHI}(K_1\cup\cdots\cup K_m)$$

for every $m<n$. Therefore by Theorem 1.7, K_{m+1} is an unknot in

$$S^3\backslash(K_1\cup\cdots\cup K_m)$$

for every $m<n$, so L is the unlink. □

Theorems 1.4, 1.6 and 1.7 will be proved in Section 3.3.

2. Sutured manifolds and tangles

2.1. **Balanced sutured manifolds and tangles.** This subsection reviews some terminologies on balanced sutured manifolds and tangles.

Definition 2.1 ([Gab83], [Juh06])**.** Let M be a compact oriented 3-manifold with boundary. Suppose $\gamma \subset \partial M$ is an embedded closed oriented 1-submanifold such that

(1) every component of ∂M contains at least one component of γ,
(2) $[\gamma] = 0 \in H_1(\partial M; \mathbb{Z})$.

Let $A(\gamma) \subset \partial M$ be an open tubular neighborhood of γ, and let $R(\gamma) = \partial M \backslash A(\gamma)$. Let $R_+(\gamma)$ be the subset of $R(\gamma)$ consisting of components whose boundaries have the same orientation as γ, and let $R_-(\gamma) = R(\gamma)\backslash R_+(\gamma)$. We say that (M, γ) is a *balanced sutured manifold* if it satisfies the following properties:

(1) The 3-manifold M has no closed components.
(2) $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$.

Definition 2.2. Suppose (M, γ) is a balanced sutured manifold.

(1) A *tangle* in (M, γ) is a properly embedded 1-manifold $T \subset M$ so that $\partial T \cap \gamma = \emptyset$.
(2) A component T_0 of T is called *vertical* if $\partial T_0 \cap R_+(\gamma) \neq \emptyset$ and $\partial T_0 \cap R_-(\gamma) \neq \emptyset$.
(3) A tangle T is called *vertical* if every component of T is vertical.
(4) A tangle T is called *balanced* if $|T \cap R_+(\gamma)| = |T \cap R_-(\gamma)|$.

Remark 2.3. In the following, we will use the notation $T \subset (M, \gamma)$ to indicate that the tangle T is in (M, γ).

Remark 2.4. Note that a tangle $T \subset (M, \gamma)$ is, a priori, un-oriented. When the tangle is vertical, we may orient each component of T so that it goes from $R_+(\gamma)$ to $R_-(\gamma)$.

Remark 2.5. The tangle T in Definition 2.2 is allowed to be empty.

Definition 2.6 ([Sch89])**.** Suppose (M, γ) is a balanced sutured manifold, $T \subset (M, \gamma)$ is a (possibly empty) tangle, and $S \subset M$ is an oriented properly embedded surface so that S intersects T transversely. If S is connected, define the *T–Thurston norm* of S to be

$$x_T(S) = \max\{|T \cap S| - \chi(S), 0\}.$$

Here $|T \cap S|$ is the number of intersection points between T and S. If S is disconnected and $S_1, \ldots, S_n$ are the connected components of S, then define the *T–Thurston norm* of S to be

$$x_T(S) = x_T(S_1) + \cdots + x_T(S_n).$$

Suppose $N \subset \partial M$ is a subset. For a homology class $\alpha \in H_2(M, N; \mathbb{Z})$, define the *$T$–Thurston norm* of α to be

$$x_T(\alpha) = \min\{x_T(S) \mid S \text{ is properly embedded in } M,\ \partial S \subset N,\ [S] = [\alpha] \in H_2(M, N; \mathbb{Z})\}.$$

A surface S properly embedded in M is said to be *T-norm-minimizing* if

$$x_T(S) = x_T([S]).$$

Here $[S]$ is the homology class in $H_2(M, \partial S; \mathbb{Z})$ represented by S.

When $T = \emptyset$, we simply write x_T as x.

Definition 2.7 ([Sch89])**.** Suppose (M, γ) is a balanced sutured manifold and $T \subset (M, \gamma)$ is a (possibly empty) tangle. We say that (M, γ) is *T-taut*, if (M, γ, T) satisfies the following properties:

(1) $M \backslash T$ is irreducible.
(2) $R_+(\gamma)$ and $R_-(\gamma)$ are both incompressible in $M \backslash T$.
(3) $R_+(\gamma)$ and $R_-(\gamma)$ are both T-norm-minimizing.
(4) Every component of T is either vertical or closed.

When $T = \emptyset$, we simply say that (M, γ) is *taut*.

2.2. **Sutured manifold hierarchy.** Suppose (M, γ) is a balanced sutured manifold and $T \subset (M, \gamma)$ is a tangle. Suppose $S \subset M$ is an oriented properly embedded surface. Under certain mild conditions, Scharlemann [Sch89] generalized the concepts from [Gab83] and introduced a process to decompose the triple (M, γ, T) into a new triple (M', γ', T'), and this process is called a *sutured manifold decomposition.* Extending the work of Gabai in [Gab83], Scharlemann [Sch89] also proved the existence of *sutured manifold hierarchies* for a T-taut balanced sutured manifold (M, γ). In this paper, we will use a modified version of the sutured manifold hierarchy, which is presented in the current subsection.

Definition 2.8 (c.f. [GL19])**.** Suppose (M, γ) is a balanced sutured manifold and $T \subset (M, \gamma)$ is a tangle. An oriented properly embedded surface $S \subset M$ is called *admissible* in (M, γ, T) if

(1) Every component of ∂S intersects γ nontrivially and transversely.
(2) For every component T_0 of T, choose an arbitrary orientation for T_0, then all intersections between T_0 and S are of the same sign.
(3) S has no closed components.
(4) The number $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is even.

Theorem 2.9 is essentially a combination of [Sch89, Theorem 4.19] and [Juh08, Theorem 8.2].

Theorem 2.9. *Suppose (M, γ) is a balanced sutured manifold. Suppose K is a knot in the interior of M and view K as a tangle in (M, γ). If (M, γ) is K-taut, then there is a finite sequence of sutured manifold decompositions*

$$(M, \gamma, K) \overset{S_1}{\rightsquigarrow} (M_1, \gamma_1, K) \rightsquigarrow \cdots \rightsquigarrow (M_{n-1}, \gamma_{n-1}, K) \overset{S_n}{\rightsquigarrow} (M_n, \gamma_n, T)$$

so that the following is true.

(1) *For $i = 1, \ldots, n$, the surface S_i is an admissible surface in (M_{i-1}, γ_{i-1}). ($M_0 = M$ and $\gamma_0 = \gamma$.)*
(2) *For $i = 1, \ldots, n-1$, we have $S_i \cap K = \emptyset$.*
(3) *T is a vertical tangle in (M_n, γ_n).*
(4) *For $i = 1, \ldots, n-1$, (M_i, γ_i) is K-taut.*
(5) *(M_n, γ_n) is T-taut.*

Proof. The proof of the theorem is essentially a combination of the proofs of [Sch89, Theorem 4.19] and [Juh08, Theorem 8.2]. We will only sketch the proof here and point out the adaptions to our current setup.

First, suppose there is a homology class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ so that $\partial_*(\alpha) \neq 0 \in H_1(\partial M; \mathbb{Z})$. The argument in [Sch89, Section 3] then produces a surface S properly embedded in M so that

(1) $[S] = [\alpha] \in H_2(M, \partial M; \mathbb{Z})$.
(2) The sutured manifold decomposition of (M, γ, K) along S yields a taut sutured manifold.

Note that S must have a non-empty boundary. We further perform the following modifications on S:

(a) Disregard any closed components of S. This makes the decomposition along S resulting in a balanced sutured manifold.
(b) Apply the argument in [Gab83, Section 5] to make S satisfy the following extra property: for every component V of $R(\gamma)$, the intersection of ∂S with V is a collection of parallel oriented non-separating simple closed curves or arcs. Note the argument in [Gab83, Section 5] happens in a collar of ∂M inside M, and the tangle K is in the interior of M. So the argument applies to our case.
(c) After step (b), we can make S admissible by further isotoping ∂S on ∂M via positive stabilizations on S, in the sense of [Li19, Definition 3.1]. Note that by definition, a positive stabilization creates a pair of intersection points between ∂S and γ, and thus it is possible to modify S so that every component of S intersects γ and $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is even. By [Li19, Lemma 3.2], we know that after positive stabilizations, the sutured manifold decomposition along S still yields a K-taut (or T-taut, if $S \cap K \neq \emptyset$) balanced sutured manifold.

Suppose S_1 is the resulting surface after the above modifications. If S_1 has a non-trivial intersection with K then we are done. If $S_1 \cap K = \emptyset$, we then know that S_1 is admissible and the sutured manifold decomposition

$$(M, \gamma, K) \stackrel{S_1}{\rightsquigarrow} (M_1, \gamma_1, K)$$

yields a K-taut balanced sutured manifold (M_1, γ_1). We can run the above argument repeatedly. By [Sch89, Section 4], the triple (M_1, γ_1, K) has a smaller complexity than (M, γ, K), so this decomposition procedure must end after finitely many steps. There are two possibilities when the procedure ends:

Case 1. We have a surface S_n that intersects K nontrivially, and $S_1, \ldots, S_{n-1}$ are all disjoint from K. After the decomposition along S_n, the knot K becomes a collection of arcs T inside (M_n, γ_n), and we have a sutured manifold decomposition

$$(M_{n-1}, \gamma_{n-1}, K) \stackrel{S_n}{\rightsquigarrow} (M_n, \gamma_n, T)$$

so that (M_n, γ_n) is T-taut. Note that T has no closed components and by Condition (4) in Definition 2.7, T is a vertical tangle. Hence the desired properties of the theorem are verified.

Case 2. All decomposing surfaces S_n are disjoint from K, and we end up with a balanced sutured manifold (M_n, γ_n) so that

$$\partial_* : H_2(M_n, \partial M_n; \mathbb{Z}) \to H_1(\partial M_n; \mathbb{Z})$$

is the zero map. In this case, the map

$$i_* : H_1(\partial M_n; \mathbb{Z}) \to H_1(M_n; \mathbb{Z})$$

is injective. It then follows from the "half lives, half dies" principle (see, for example, [How11, Theorem A.3]) that ∂M_n must be a union of 2-spheres. Let $\hat{M}_n$ be the component of M_n that contains K, and let $\hat{S}$ be a sphere in $\hat{M}_n$ that is parallel to one of the boundary components. Then $\hat{S}$ is a reducing sphere for $\hat{M}_n\backslash K$, therefore (M_n, γ_n) cannot be K-taut, which yields a contradiction.

□

2.3. **Instanton Floer homology.** Suppose (M, γ) is a balanced sutured manifold. Kronheimer and Mrowka [KM10b] constructed the *sutured instanton Floer homology* for the couple (M, γ). Given a balanced tangle $T \subset (M, \gamma)$ (see Definition 2.2), an instanton Floer homology theory for the triple (M, γ, T) was constructed in [XZ19b]. In this subsection, we review the constructions of these instanton Floer theories.

Suppose (M, γ) is a balanced sutured manifold and $T \subset (M, \gamma)$ is a balanced tangle. We construct a tuple (Y, R, L, u) as follows: Pick a connected surface F so that

$$\partial F \cong -\gamma.$$

Pick $n \geq 2$ points $p_1, \ldots, p_n$ on F. Let u be an embedded simple arc connecting p_1 and p_2. Glue $[-1,1] \times F$ to M along $\overline{A(\gamma)}$:

$$\widetilde{M} := M \bigcup_{\overline{A(\gamma)} \sim [-1,1]\times \partial F} [-1,1] \times F,$$

where $\sim$ is given by a diffeomorphism from $\overline{A(\gamma)}$ to $[-1,1] \times \gamma$. The boundary of $\widetilde{M}$ consists of two components:

$$\partial \widetilde{M} = R_+ \cup R_-,$$

where

$$R_\pm = R_\pm(\gamma) \cup \{\pm 1\} \times F$$

are closed surfaces. Let

$$\widetilde{T} = T \cup [-1,1] \times \{p_1, \ldots, p_n\}.$$

Now pick an orientation-preserving diffeomorphism

$$f : R_+ \to R_-$$

so that $f(\widetilde{T} \cap R_+) = \widetilde{T} \cap R_-$ and $f(\{1\} \times u) = \{-1\} \times u$. Gluing R_+ to R_- via f yields a closed oriented 3-manifold Y. Let $R \subset Y$ be the image of $R_\pm$. Let $L \subset Y$ be the image of T, then L is a link in Y. We will also abuse notation and let u denote the image of $\{\pm 1\} \times u \subset [-1,1] \times F$ in Y.

Definition 2.10. The tuple (Y, R, L, u) is called a *closure* of (M, γ, T).

By the construction of Kronheimer and Mrowka in [KM11], there is an instanton homology group $\mathrm{I}(Y, L, u)$, which is a finite-dimensional $\mathbb{C}$–vector space, associated to the triple (Y, L, u). The homology $\mathrm{I}(Y, L, u)$ is defined using instantons which are singular along L. The surface R induces a complex linear map

$$\mu^{\mathrm{orb}}(R) : \mathrm{I}(Y, L, u) \to \mathrm{I}(Y, L, u).$$

Given a point $p \in Y\backslash L$, there is a complex linear map

$$\mu(p) : \mathrm{I}(Y, L, u) \to \mathrm{I}(Y, L, u).$$

The maps $\mu^{\text{orb}}(R)$ and $\mu(p)$ are commutative to each other. The reader may refer to [XZ19b, Section 2.1] or [Str12, Section 2.3.2] for a review of the definition of $\mu^{\text{orb}}(R)$ and $\mu(p)$. From now on, if μ is a linear map on $\mathrm{I}(Y,L,u)$, we will use $\mathrm{Eig}(\mu,\lambda)$ to denote the *generalized* eigenspace of μ with eigenvalue λ in $\mathrm{I}(Y,L,u)$,

Definition 2.11 ([XZ19b, Definition 7.6]). Suppose (M,γ) is a balanced sutured manifold, and $T \subset (M,\gamma)$ is a balanced tangle. Let (Y,R,L,u) be an arbitrary closure of (M,γ,T) such that $|L\cap R|$ is odd and at least 3 (this can always be achieved by increasing the value of n in the construction of the closure). Let $p \in Y\backslash L$ be a point. Then the *sutured instanton Floer homology* of the triple (M,γ,T), which is denoted by $\mathrm{SHI}(M,\gamma,T)$, is defined as

$$\mathrm{SHI}(M,\gamma,T) = \mathrm{Eig}\left(\mu^{\text{orb}}(R), |L\cap R| - \chi(R)\right) \cap \mathrm{Eig}\left(\mu(p),2\right) \subset \mathrm{I}(Y,L,u).$$

The next statement guarantees that the isomorphism class of $\mathrm{SHI}(M,\gamma,T)$ is well-defined:

Theorem 2.12 ([XZ19b, Proposition 7.7]). *For a triple (M,γ,T), the isomorphism class of the sutured instanton Floer homology $\mathrm{SHI}(M,\gamma,T)$ as a complex vector space is independent of the choice of the closure of (M,γ,T) and the point p.*

We also have the following non-vanishing theorem:

Theorem 2.13 ([XZ19b, Theorem 7.12]). *Suppose (M,γ) is a balanced sutured manifold and $T \subset (M,\gamma)$ is a balanced tangle so that (M,γ) is T-taut. Then we have*

$$\mathrm{SHI}(M,\gamma,T) \neq 0.$$

Now we establish a theorem relating sutured instanton Floer homology to sutured manifold decompositions, which is analogous to [KM10b, Proposition 6.9]:

Theorem 2.14. *Suppose (M,γ) is a balanced sutured manifold and $T \subset (M,\gamma)$ is a balanced tangle. Suppose further that $S \subset (M,\gamma)$ is an admissible surface in the sense of Definition 2.8 and we have a sutured manifold decomposition:*

$$(M,\gamma,T) \stackrel{S}{\rightsquigarrow} (M',\gamma',T').$$

Then there is a closure (Y,R,L,u) of (M,γ,T) and a closure (Y,R',L,u) of (M',γ',T') so that

1. *the two closures share the same 3-manifold Y, the same link L and the same arc u.*
2. *there is a closed oriented surface $\bar{S} \subset Y$ extending $S \subset (M,\gamma)$ so that R' is obtained from R and $\bar{S}$ by a double curve surgery in the sense of* [Sch89, Definition 1.1].

Let (Y,R,L,u) and (Y,R',L,u) be as above and let $p \in Y\backslash L$. We have

$$\mathrm{SHI}(M,\gamma,T) = \mathrm{Eig}\left(\mu^{\text{orb}}(R), |L\cap R| - \chi(R)\right) \cap \mathrm{Eig}\left(\mu(p),2\right) \tag{2.1}$$

and

$$\begin{aligned}\mathrm{SHI}(M',\gamma',T') = &\,\mathrm{Eig}\left(\mu^{\text{orb}}(R), |L\cap R| - \chi(R)\right) \cap \mathrm{Eig}\left(\mu(p),2\right)\\ &\cap \mathrm{Eig}\left(\mu^{\text{orb}}(\bar{S}), |L\cap \bar{S}| - \chi(\bar{S})\right).\end{aligned} \tag{2.2}$$

Proof. The proof of the theorem is almost the same as that of [KM10b, Proposition 7.11], which referred to the argument of [KM10b, Proposition 6.9]: the constructions of the closure (Y, R, L, u) and the extension $\bar{S}$ follow verbatim from the constructions in the proof of [KM10b, Proposition 6.9]. Equation (2.1) is precisely the definition of $\mathrm{SHI}(M, \gamma, T)$ in Definition 2.11. The verification of Equation (2.2) is also similar to the proof of [KM10b, Proposition 6.9], except that the surface in the current setup might intersect L transversely at finitely many points, so we need to replace [KM10b, Corollary 7.2] in the proof of [KM10b, Proposition 6.9] by [XZ19b, Proposition 6.1] for our current setup. □

Remark 2.15. The original proof of [KM10b, Proposition 6.9] only works with a surface $S \subset M$ where $\frac{1}{2}|S \cap \gamma| - \chi(S)$ is even (or else it is impossible to extend S to a closed surface $\bar{S}$ in the closures of (M, γ).) This is the reason why we needed to introduce the definition of admissible surfaces in Definition 2.8. This subtlety was not explicitly mentioned in [KM10b].

2.4. **A module structure on** SHI. We introduce an additional module structure on $\mathrm{SHI}(M, \gamma, T)$ when $T \neq \emptyset$.

As before, suppose (M, γ) is a balanced sutured manifold and $T \subset (M, \gamma)$ is a balanced tangle. Pick an arbitrary closure (Y, R, L, u) of (M, γ) such that $|L \cap R|$ is odd and at least 3 (so that the condition of Definition 2.11 is satisfied). Pick a point $p \in Y \backslash L$ and a point $q \in T$. The point q determines a complex linear map

$$\sigma(q) : \mathrm{I}(Y, L, u) \to \mathrm{I}(Y, L, u)$$

as described in [XZ19b, Section 2.1] (this map was denoted by σ_q in [XZ19b]), and we have $\sigma(q)^2 = 0$ by [XZ19b, Equation (2) and Proposition 2.1]. Moreover, $\sigma(q)$ commutes with $\mu(R)$ and $\mu(p)$. Hence $\sigma(q)$ defines a complex linear map on

$$\mathrm{SHI}(M, \gamma, T) = \mathrm{Eig}\Big(\mu^{\mathrm{orb}}(R), |L \cap R| - \chi(R)\Big) \cap \mathrm{Eig}\Big(\mu(p), 2\Big)$$

that squares to zero. Therefore, we can view $\mathrm{SHI}(M, \gamma, T)$ as a $\mathbb{C}[X]/(X^2)$–module where the action of X is defined by $\sigma(q)$. The proof of [XZ19b, Theorem 7.12] applies verbatim to verify Theorem 2.16:

Theorem 2.16. *For a triple (M, γ, T), the isomorphism class of the sutured instanton Floer homology $\mathrm{SHI}(M, \gamma, T)$ as a $\mathbb{C}[X]/(X^2)$–module is independent of the choice of the closures of (M, γ, T). Moreover, if $q, q' \in T$ are in the same component of T, then the module structure induced by q is isomorphic to the module structure induced by q'.* □

For $q \in T$, we will call the $\mathbb{C}[X]/(X^2)$–module structure on $\mathrm{SHI}(M, \gamma, T)$ defined as above the *module structure induced by q*.

Remark 2.17. If T has only one component, then the isomorphism class of the module structure is independent of q. In this case, it makes sense to refer to the isomorphism class of the module structure on $\mathrm{SHI}(M, \gamma, T)$ without specifying the point q.

Corollary 2.18. *Suppose (M, γ) is a balanced sutured manifold and $T \subset (M, \gamma)$ is a balanced tangle. Suppose further that $S \subset (M, \gamma)$ is an admissible surface in the sense of Definition* 2.8 *and we have a sutured manifold decomposition:*

$$(M, \gamma, T) \stackrel{S}{\rightsquigarrow} (M', \gamma', T').$$

Suppose $q \in T$, and let $q' \in T'$ be the image of q in M'. If the module structure on $\mathrm{SHI}(M,\gamma,T)$ *induced by q is a free $\mathbb{C}[X]/(X^2)$–module, then so is the module structure on* $\mathrm{SHI}(M',\gamma',T')$ *induced by q'.*

Proof. Notice that if Q is a finitely generated $\mathbb{C}[X]/(X^2)$–module, then

$$2 \cdot \mathrm{rank}_{\mathbb{C}}(m_X) \leq \dim_{\mathbb{C}} Q,$$

where $m_X : Q \to Q$ is the multiplication of X. Moreover, Q is a free $\mathbb{C}[X]/(X^2)$–module if and only if

$$2 \cdot \mathrm{rank}_{\mathbb{C}}(m_X) = \dim_{\mathbb{C}} Q.$$

As a consequence, if Q is a finitely generated free $\mathbb{C}[X]/(X^2)$–module, then every $\mathbb{C}[X]/(X^2)$–module that is a direct summand of Q is also free.

The corollary then follows from Equation (2.2) and the fact that the action of X commutes with the action of $\mu(\bar{S})$. □

2.5. SHI **with local coefficients.** We also introduce a version of sutured instanton Floer homology defined with local coefficients. Let (M,γ) be a balanced sutured manifold and let $T \subset (M,\gamma)$ be a tangle. Suppose $L \subset T$ is a link in the interior of M. Let (Y,R,L',u) be a closure of (M,γ,T) such that $|L' \cap R|$ is odd and at least 3.

Let Γ_L be the local system for the triple (Y,L',u) associated with $L \subset L'$ as defined in [XZ19a, Section 3]. By definition, Γ_L is a local system over the ring $\mathcal{R} = \mathbb{C}[t,t^{-1}]$. For $h \in \mathbb{C}$, define $\Gamma_L(h) = \Gamma_L \otimes \mathcal{R}/(t-h)$, then $\Gamma_L(h)$ is a local system over the ring $\mathcal{R}/(t-h) \cong \mathbb{C}$, and hence $\mathrm{I}(Y,L',u;\Gamma_L(h))$ is a $\mathbb{C}$–linear space. Let $p \in Y \backslash L'$ be a point.

Definition 2.19. Define $\mathrm{SHI}(M,\gamma,T;\Gamma_L(h))$ to be the simultaneous generalized eigenspace of $\mu^{\mathrm{orb}}(R)$ and $\mu(p)$ in $\mathrm{I}(Y,L',u;\Gamma_L(h))$ with eigenvalues $|L' \cap R| - \chi(R)$ and 2 respectively.

Since $L \cap R = \emptyset$, the proof of [XZ19b, Theorem 7.12] applies verbatim to verify that:

Theorem 2.20. *For M,γ,T,L as above and $h \in \mathbb{C}$, the isomorphism class of the sutured instanton Floer homology* $\mathrm{SHI}(M,\gamma,T;\Gamma_L(h))$ *as a $\mathbb{C}$–linear space is well-defined. In other words, it is independent of the choice of the closures of (M,γ,T) and the point p.* □

Lemma 2.21.

$$\limsup_{h \to 1} \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,T;\Gamma_L(h)) \leq \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,T).$$

Proof. This is an immediate consequence of [XZ19a, Lemma 4.1 (1)] and the observation that $\Gamma_L(1)$ is the trivial local system over $\mathbb{C}$, thus

$$\mathrm{SHI}(M,\gamma,T;\Gamma_L(1)) = \mathrm{SHI}(M,\gamma,T). \qquad \square$$

Lemma 2.22. *Suppose T_0 is a tangle in (M,γ), and $L, L' \subset M$ are two links in the interior of M that are homotopic to each other in M and are both disjoint from T_0. Let $T = T_0 \cup L$, $T' = T_0 \cup L'$, and suppose $h \notin \{\pm 1\}$. Then*

$$\dim_{\mathbb{C}}(M,\gamma,T;\Gamma_L(h)) = \dim_{\mathbb{C}}(M,\gamma,T';\Gamma_{L'}(h)).$$

Proof. This is an immediate consequence of [XZ19a, Proposition 3.2]. □

Lemma 2.23. *Suppose T is a tangle in (M,γ) and U is an unknot in $M\backslash T$. Then for every $h\in\mathbb{C}$, we have*

$$\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T\cup U,\Gamma_U(h))=2\cdot\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T).$$

Proof. Let B be a 3–ball in $M\backslash(T\cup U)$, let δ be a simple closed curve on ∂B. Let $U'\subset S^3$ be an unknot in S^3, let B' be a 3–ball in $S^3\backslash U'$, let δ' be a simple closed curve on $\partial B'$. By the connected sum formula for SHI (c.f. [Li20, Proposition 4.15]), we have

$$\begin{aligned}&\dim_{\mathbb{C}}\mathrm{SHI}(M\backslash B,\gamma\cup\delta,T\cup U;\Gamma_U(h))\\&\qquad\qquad=2\cdot\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T\cup U;\Gamma_U(h))\cdot\dim_{\mathbb{C}}\mathrm{SHI}(S^3\backslash B',\delta'),\end{aligned}$$

and

$$\begin{aligned}&\dim_{\mathbb{C}}\mathrm{SHI}(M\backslash B,\gamma\cup\delta,T\cup U;\Gamma_U(h))\\&\qquad\qquad=2\cdot\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T)\cdot\dim_{\mathbb{C}}\mathrm{SHI}(S^3\backslash B',\delta',U';\Gamma_{U'}(h)).\end{aligned}$$

Therefore

$$\begin{aligned}&\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T\cup U;\Gamma_U(h))\cdot\dim_{\mathbb{C}}\mathrm{SHI}(S^3\backslash B',\delta')\\=\;&\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T)\cdot\dim_{\mathbb{C}}\mathrm{SHI}(S^3\backslash B',\delta',U';\Gamma_{U'}(h)).\end{aligned}$$

Since $(S^3\backslash B',\delta')$ is a product sutured manifold, we have

$$\dim_{\mathbb{C}}\mathrm{SHI}(S^3\backslash B',\delta')=1.$$

Let

$$c(h)=\dim_{\mathbb{C}}\mathrm{SHI}(S^3\backslash B',\delta',U';\Gamma_{U'}(h)),$$

then $c(h)$ is a function depending on h, and we have

$$\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T\cup U,\Gamma_U(h))=c(h)\cdot\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T),$$

for all M,γ,T. Recall that T is allowed to be empty.

Now consider special the case where M is a solid torus, γ is a pair of oppositely oriented meridians on ∂M, and $T=\emptyset$. In this case, we have

$$\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T)=1.$$

Moreover, the sutured instanton Floer homology $\mathrm{SHI}(M,\gamma,T\cup U,\Gamma_U(h))$ is by definition the same as the annular instanton Floer homology of the unknot with local coefficients given by $\Gamma_U(h)$ (see [Xie21b] and [XZ19a, Sections 2 and 3]). By [Xie21b, Example 4.2], the perturbed Chern-Simons functional for the annular instanton Floer homology of the unknot has two critical points with homological degrees differ by 2. Therefore, we have

$$\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma,T\cup U,\Gamma_U(h))=2$$

for all $h\in\mathbb{C}$. In conclusion, we must have $c(h)=2$ for all h. □

Remark 2.24. The $\mathbb{C}[X]/(X^2)$–module structure introduced in Section 2.4 cannot be straightforwardly extended to SHI with local coefficients, because we may no longer have $\sigma(q)^2=0$ on $\mathrm{SHI}(M,\gamma,T;\Gamma_L(h))$ when $q\in L$.

3. Properties of Floer minimal knots

3.1. **Irreducibility and module structure.** This subsection proves the following theorem. The result is inspired by the work of Wang [Wan21].

Theorem 3.1. *Suppose (M,γ) is a balanced sutured manifold with*

$$\mathrm{SHI}(M,\gamma)\neq 0 \tag{3.1}$$

and K is a knot in the interior of M. View K as a balanced tangle in (M,γ). If $M\backslash K$ is irreducible, then $\mathrm{SHI}(M,\gamma,K)$ is not free as a $\mathbb{C}[X]/(X^2)$–module.

Proof. Since $M\backslash K$ is irreducible, by (3.1) and the adjunction inequality for instanton Floer homology [KM10b, Corollary 7.2], we conclude that (M,γ) is K-taut.

Let

$$(M,\gamma,K)\overset{S_1}{\rightsquigarrow}(M_1,\gamma_1,K)\rightsquigarrow\cdots\rightsquigarrow(M_{n-1},\gamma_{n-1},K)\overset{S_n}{\rightsquigarrow}(M_n,\gamma_n,T)$$

be the sutured hierarchy given by Theorem 2.9. In particular, T is a vertical tangle and (M_n,γ_n) is T-taut.

Assume $\mathrm{SHI}(M,\gamma,K)$ is a free $\mathbb{C}[X]/(X^2)$-module. Applying Corollary 2.18 repeatedly and applying Theorem 2.13, we conclude that for every point $q\in T$, the module structure on $\mathrm{SHI}(M_n,\gamma_n,T)$ induced by q is a non-trivial free $\mathbb{C}[X]/(X^2)$–module.

Pick any closure (Y,R,L,u) of (M_n,γ_n,T) such that $|R\cap L|$ is odd and at least 3. Suppose $R\cap L=\{p_1,\cdots,p_m\}$. We may also suppose $u\cap R=\emptyset$.

Let $W=[-1,1]\times(Y,L,u)$ be the product cobordism from (Y,L,u) to itself. We use R_0 to denote $\{0\}\times R\subset W$. Removing a neighborhood $N(R_0)\cong D^2\times R_0$ from W, we obtain a cobordism

$$W':(Y,L,u)\sqcup(S^1\times R,S^1\times\{p_1,\cdots,p_m\},\emptyset)\to(Y,L,u).$$

Now W' induces a map

$$\mathrm{I}(W'):\mathrm{I}(Y,L,u|R)\otimes\mathrm{I}(S^1\times R,S^1\times\{p_1,\cdots,p_m\},\emptyset|R)\to\mathrm{I}(Y,L,u|R).$$

Since the original cobordism W induces the identify map on the Floer homology, $\mathrm{I}(W')$ is a surjection. Moreover $\mathrm{I}(W')$ intertwines the $\sigma(q)$ maps on the three ends since we can move the point q from one end to any other end.

By Section 2.4, we have $\sigma(q)^2=0$ on $\mathrm{I}(S^1\times R,S^1\times\{p_1,\cdots,p_m\},\emptyset|R)$. On the other hand, by [XZ19b, Proposition 6.5],

$$\mathrm{I}(S^1\times R,S^1\times\{p_1,\cdots,p_m\},\emptyset|R)\cong\mathbb{C}.$$

This implies that $\sigma(q)=0$ on the above space. Since $\mathrm{I}(W')$ is surjective and intertwines the $\sigma(q)$ maps on the three ends, we conclude that $\sigma(q)=0$ on $\mathrm{I}(Y,L,u|R)\cong\mathrm{SHI}(M_n,\gamma_n,T)$, which yields a contradiction. □

3.2. **Floer minimality and module structure.** Recall that for a sutured manifold (M,γ), the group $\pi_1'(M)$ is defined by Equation (1.5) as a quotient group of $\pi_1(M)$. The main result of this subsection is Theorem 3.2.

Theorem 3.2. *Let (M,γ) be a balanced sutured manifold. Suppose K is a knot in the interior of M that represents the zero element in $\pi_1'(M)$. If K is instanton Floer minimal (see Definition 1.3), then $\mathrm{SHI}(M,\gamma,K)$ is a free module over $\mathbb{C}[X]/(X^2)$ (see Section 2.4).*

We first establish the following rank inequality for instanton homology.

Lemma 3.3. *Suppose (M,γ) is a balanced sutured manifold, and K is a knot in the interior of M such that K represents the zero element in $\pi_1'(M)$. Then we have*

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K) \geq 2\cdot \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma). \tag{3.2}$$

Proof. Suppose $\gamma=\gamma_1\cup\cdots\cup\gamma_m$ has m components. For each i, let γ_i',γ_i'' be a pair of oppositely-oriented simple closed curves on ∂M parallel to γ_i. Moreover, let γ_i'' have the same orientation as γ_i, let γ_i' have the opposite orientation as γ_i, and let γ_i' be placed between γ_i and γ_i''. Let

$$\gamma' := \bigcup_i (\gamma_i\cup\gamma_i'\cup\gamma_i'').$$

Let U be an unknot in M. The proof of [KM10a, Theorem 3.1] implies that

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma') = 2^m\cdot \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma) \tag{3.3}$$

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma',K) = 2^m\cdot \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K). \tag{3.4}$$

Decompose $A(\gamma')$ as $A(\gamma)=\bigcup_i(A(\gamma_i)\cup A(\gamma_i')\cup A(\gamma_i''))$. For each i, let D_i be a disk, let φ_i be an orientation-reversing diffeomorphism from $[-1,1]\times\partial D_i$ to $A(\gamma_i')$, and let

$$\hat{M} = M\bigcup_{\{\varphi_i\}}\Big(\bigcup_i [-1,1]\times D_i\Big).$$

Let $\hat{\gamma}$ be the image of $\bigcup_i(\gamma_i\cup\gamma_i'')$ on $\partial\hat{M}$. Then every component of $\partial\hat{M}$ contains at least one component of $\hat{\gamma}$. Moreover, it is straightforward to verify that $(\hat{M},\hat{\gamma})$ is a balanced sutured manifold. Let $T_i=[-1,1]\times\{0\}\subset[-1,1]\times D_i$ be a tangle, we can view $T=T_1\cup\cdots\cup T_m$ as a vertical tangle inside $(\hat{M},\hat{\gamma})$. Let $(\hat{M}_T,\hat{\gamma}_T)$ be the balanced sutured manifold obtained from $(\hat{M},\hat{\gamma})$ by removing a neighborhood of T and adding a meridian of every component of T to the suture. It is straightforward to check that $(\hat{M}_T,\hat{\gamma}_T)$ is simply diffeomorphic to (M,γ'). Hence by [XZ19b, Lemma 7.10], we have

$$\mathrm{SHI}(M,\gamma')\cong \mathrm{SHI}(\hat{M},\hat{\gamma},T).$$

In fact, the proof of [XZ19b, Lemma 7.10] also applies verbatim to give

$$\mathrm{SHI}(M,\gamma',K)\cong \mathrm{SHI}(\hat{M},\hat{\gamma},T\cup K),$$
$$\mathrm{SHI}(M,\gamma',U)\cong \mathrm{SHI}(\hat{M},\hat{\gamma},T\cup U).$$

By Lemma 2.23, we have

$$\dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},T\cup U;\Gamma_U(h)) = 2\cdot \dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},T)$$

for every $h\in\mathbb{C}$. Since K represents the zero element in $\pi_1'(M)$, it is homotopic to U in $\hat{M}$. By Lemma 2.22, when $h\notin\{\pm1\}$, we have

$$\dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},K\cup T;\Gamma_K(h)) = \dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},K\cup U;\Gamma_U(h)).$$

Hence

$$\dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},K\cup T;\Gamma_K(h)) = 2\cdot \dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},T)$$

for all $h \notin \{\pm 1\}$. Therefore, by Lemma 2.21,

$$\begin{aligned}\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma',K) &= \dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},T\cup K)\\ &\geq 2\cdot \dim_{\mathbb{C}} \mathrm{SHI}(\hat{M},\hat{\gamma},T)\\ &= 2\cdot \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma').\end{aligned}$$

Thus by (3.3) and (3.4), we have

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K) \geq 2\cdot \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma),$$

and the lemma is proved. □

Now we can prove Theorem 3.2.

Proof of Theorem 3.2. We define a new instanton Floer homology group $\mathrm{SHI}(M,\gamma,K^{\natural})$ as follows. Regard K as a tangle inside (M,γ). Pick a closure (Y,R,L,u) of (M,γ,K). Recall that from the construction of closures in Section 2.3, L has the form

$$L = K \cup L_1,$$

where L_1 is the image of the arcs $[-1,1]\times\{p_1,\ldots,p_n\}\subset[-1,1]\times F$ in the closure. Now define $L' = K\cup\mu\cup L_1$ to be a link in Y, where μ is a meridian of K, and define $u' = u\cup v$, where v is an arc connecting K with μ. Define

$$\mathrm{SHI}(M,\gamma,K^{\natural}) := \mathrm{I}(Y,L',u'|R).$$

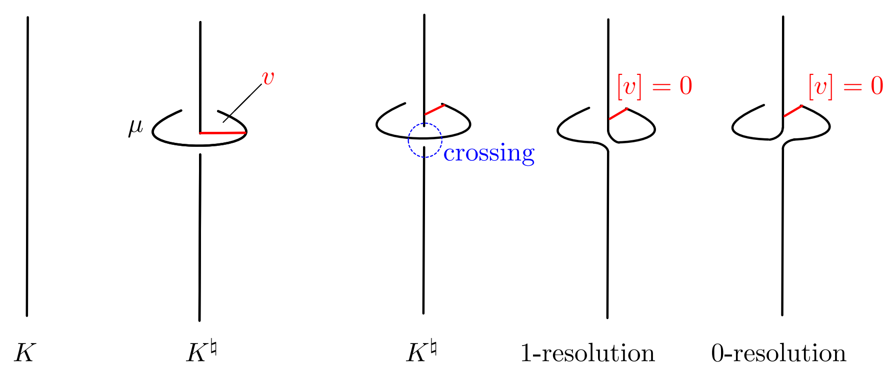

FIGURE 1. The knot K, $K^{\natural}$, and the unoriented skein triangle

Remark 3.4. If we replace (M,γ) by a closed 3-manifold Y and disregard R, L_1, and u, then the above construction is the same as the definition of $\mathrm{I}^{\natural}(Y,K)$ introduced by Kronheimer and Mrowka in [KM11].

Now apply the un-oriented skein triangle from [KM11, Section 6 and 7] to a crossing between K and μ, as shown in Figure 1. The result is the following exact triangle

$$\begin{tikzcd}
\mathrm{SHI}(M,\gamma,K) \arrow[rr, "f"] & & \mathrm{SHI}(M,\gamma,K) \arrow[dl] \\
& \mathrm{SHI}(M,\gamma,K^{\natural}) \arrow[ul] &
\end{tikzcd}$$

The map f has been computed explicitly in [Xie21a, Proposition 3.4] (also cf. [DS19, Theorem 8.8]), and we have

$$f = \pm\sigma(q) : \mathrm{SHI}(M,\gamma,K) \to \mathrm{SHI}(M,\gamma,K)$$

for a base point $q \in K$. Since $\sigma(q)^2 = 0$, we have

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K) \le \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K^{\natural}), \tag{3.5}$$

and the equality holds if and only if

$$\mathrm{rank}(\sigma(q)) = \frac{1}{2}\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K). \tag{3.6}$$

Let γ_K be the union of γ with a pair of oppositely oriented meridians of K on $\partial N(K)$. The proof of [KM11, Proposition 1.4] (also [XZ19b, Lemma 7.10]) applies verbatim to give the following isomorphism

$$\mathrm{SHI}(M,\gamma,K^{\natural}) \cong \mathrm{SHI}(M\backslash N(K),\gamma_K). \tag{3.7}$$

Recall that K is assumed to be instanton Floer minimal, namely

$$\dim_{\mathbb{C}} \mathrm{SHI}(M\backslash N(K),\gamma_K) = 2\cdot\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma). \tag{3.8}$$

By Lemma 3.3, we have

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K) \ge 2\cdot\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma). \tag{3.9}$$

By (3.5), (3.7), (3.8), (3.9), we conclude that

$$\dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K) = \dim_{\mathbb{C}} \mathrm{SHI}(M,\gamma,K^{\natural}),$$

therefore the equality holds for (3.5), and hence (3.6) holds. Recall that the action of X equals $\sigma(q)$ in the $\mathbb{C}[X]/(X^2)$–module structure of $\mathrm{SHI}(M,\gamma,K)$. Hence by (3.6), $\mathrm{SHI}(M,\gamma,K)$ is a free $\mathbb{C}[X]/(X^2)$–module. □

3.3. **Proofs of the main results.** This subsection proves Theorems 1.4, 1.6, and 1.7.

Proof of Theorem 1.4. By Theorem 3.2, $\mathrm{SHI}(M,\gamma,K)$ is a non-trivial free module over $\mathbb{C}[X]/(X^2)$. Hence by Theorem 3.1, $M\backslash K$ is reducible.

Therefore, M decomposes as $M = M_1\# M_2$, where $K \subset M_2$ after isotopy. If M_2 is a closed manifold, then the desired result holds.

If M_2 is a manifold with boundary, we apply [KM10b, Corollary 7.2] to show that $(M_2, \gamma\cap\partial M_2)$ is also a balanced sutured manifold. In fact, [KM10b, Corollary 7.2] implies that for an admissible triple (Y,L,ω) and an embedded closed oriented surface $R \subset Y - L$, if the intersection number of R and ω is odd and the genus of R is positive, then the eigenvalues of $\mu(R)$ on $\mathrm{I}(Y,L,\omega)$ are at most $2g-2$, where g is the genus of R. (The statement of [KM10b, Corollary 7.2] is only given for the case when $L = \emptyset$, but the same proof applies verbatim as long as $R \cap L = \emptyset$.)

The statement above gives a lower bound for the genus of R by the eigenvalues of $\mu(R)$. Notice that $\mu(R)$ only depends on the fundamental class of R in $H_2(Y;\mathbb{Z})$.

Applying [KM10b, Corollary 7.2] to the closure of (M,γ) and invoking the assumption that $\mathrm{SHI}(M,\gamma)\neq 0$, we obtain the following result: if R' is an embedded surface without closed component in (M,γ) such that $\partial R'=\partial R_+(\gamma)$ and

$$[R',\partial R']=[R_+(\gamma),\partial R_+(\gamma)]\in H_2(M,A(\gamma)),$$

then

$$\chi(R')\leq\chi(R_+(\gamma)).$$

As a consequence,

$$\begin{aligned}&\chi(R_-(\gamma)\cap M_2)+\chi(R_+(\gamma)\cap M_1)\\ \leq&\,\chi(R_+(\gamma))\\ =&\,\chi(R_+(\gamma)\cap M_2)+\chi(R_+(\gamma)\cap M_1),\end{aligned}$$

therefore

$$\chi(R_-(\gamma)\cap M_2)\leq\chi(R_+(\gamma)\cap M_2).$$

Similarly,

$$\chi(R_+(\gamma)\cap M_2)\leq\chi(R_-(\gamma)\cap M_2).$$

In conclusion, $(M_2,\gamma\cap\partial M_2)$ is a balanced sutured manifold.

Since M_2 is a balanced sutured manifold, by the connected sum formula for SHI and the assumption that K is instanton Floer minimal in (M,γ), we have

$$\dim_{\mathbb{C}}\mathrm{SHI}(M_2,\gamma,K)=2\cdot\dim_{\mathbb{C}}\mathrm{SHI}(M_2,\gamma).$$

Hence the desired result follows from induction on the number of factors in the prime decomposition of M. $\square$

Proof of Theorem 1.6. By the connected sum formula for SHI and the non-vanishing theorem for taut sutured manifolds [KM10b, Theorem 7.12], we have $\mathrm{SHI}(M,\gamma)\neq 0$.

By Theorem 1.4, there exists a 3–ball $B^3\subset M$ such that $K\subset B^3$. Hence we can view K as a knot in the northern hemisphere of S^3. Define $S^3(K)=S^3\backslash N(K)$, and let $\gamma_\mu\subset\partial S^3(K)$ be a pair of oppositely oriented meridians of K. Let $\gamma_K\subset\partial(M\backslash N(K))$ be the union of γ and a pair of oppositely oriented meridians on $\partial N(K)$. The connected sum formula for SHI implies

$$\dim_{\mathbb{C}}\mathrm{SHI}(M\backslash N(K),\gamma_K)=2\cdot\dim_{\mathbb{C}}\mathrm{SHI}(M,\gamma)\cdot\dim_{\mathbb{C}}\mathrm{SHI}(S^3(K),\gamma_\mu).$$

Therefore K is instanton Floer minimal if and only if

$$\mathrm{I}^\natural(S^3,K)\cong\mathrm{SHI}(S^3(K),\gamma_\mu)\cong\mathbb{C}.$$

By the unknot detection result for $\mathrm{KHI}(S^3,K)$ [KM10b, Proposition 7.16] and the fact that $\mathrm{KHI}(S^3,K)\cong\mathrm{I}^\natural(S^3,K)$ [KM11, Proposition 1.4], the above equation holds if and only if K is the unknot. $\square$

Proof of Theorem 1.7. Let $M=S^3\backslash N(L)$, then the prime decomposition of M does not contain any non-trivial closed component. Let $\gamma\subset\partial M$ be the suture consisting of a pair of oppositely oriented meridians for each component of L, then $\pi_1'(M)=0$. By [KM10b, Proposition 7.16], we have $\mathrm{SHI}(M,\gamma)\neq 0$, therefore M satisfies the conditions of Theorem 1.6, and hence (1.8) holds if and only if K is the unknot in M. $\square$

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