

SEPARABLE INTEGER PARTITION CLASSES

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ABSTRACT. A classical method for partition generating function is developed into a tool with wide applications. New expansions of well-known theorems are derived, and new results for partitions with n copies of n are presented.

1. INTRODUCTION

The object of this paper is to systematize a process in the theory of integer partition that really dates back to Euler. It is epitomized in the partition-theoretic interpretation of three classical identities:

$$(1.1) \quad \sum_{n \geq 0} \frac{q^n}{(q; q)_n} = \frac{1}{(q; q)_\infty},$$

$$(1.2) \quad \sum_{n \geq 0} \frac{q^{n(n+1)/2}}{(q; q)_n} = (-q; q)_\infty,$$

and

$$(1.3) \quad \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty},$$

where

$$(1.4) \quad (A; q)_N = (1 - A)(1 - Aq) \cdots (1 - Aq^{N-1}).$$

Equations (1.1) and (1.2) are Euler's [10, p. 19] while (1.3) is the first of the celebrated Rogers-Ramanujan identities [10, Ch. 7].

In section 2, we will analyze (1.1)-(1.3) from the point of view of separable integer partition classes.

A separable integer partition class (SIP), \mathcal{P} , with modulus k , is a subset of all the integer partitions satisfying the following:

There is a subset $\mathcal{B} \subset \mathcal{P}$ (\mathcal{B} is called the basis of \mathcal{P}) such that for each integer $n \geq 1$, the number of elements of \mathcal{B} with n parts is finite and every element of \mathcal{P} with n parts is uniquely of the form

$$(1.5) \quad (b_1 + \pi_1) + (b_2 + \pi_2) + \cdots + (b_n + \pi_n),$$

where $0 < b_1 \leq b_2 \leq \cdots \leq b_n$ is a partition in \mathcal{B} and $0 \leq \pi_1 \leq \pi_2 \leq \cdots \leq \pi_n$ is a partition into n nonnegative parts, whose only restriction is that each part is divisible by k . Furthermore, all partitions of the form (1.5) are in \mathcal{P} .

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As we will see in section 2, each of (1.1)-(1.3) can be developed from this point of view with modulus $k = 1$. However, this setting allows a similar examination of the first Göllnitz-Gordon identity [16] in section 3:

$$(1.6) \quad \sum_{n \geq 0} \frac{q^{n^2} (-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q; q^8)_\infty (q^4; q^8)_\infty (q^7; q^8)_\infty}.$$

More surprising is an analysis of Schur’s 1926 partition theorem [24] in section 4. It is interesting to note that this analysis leads naturally to full proofs of both (1.6) and Schur’s theorem.

In section 5, we examine a new application. In section 6, we extend the concept of SIP classes to partitions with n copies of n [1]. Section 7 extends SIP classes to overpartitions. In the final section, we discuss open questions and note that the idea first arose in dynamical systems concerning billiard trajectories [13].

2. GENERAL THEORY AND CLASSICAL IDENTITIES

The infinite series in each of (1.1)-(1.3) fit neatly into the SIP program with modulus $k = 1$. We begin with (1.1). In this case, we let $\mathcal{P}_\mathbb{N}$ be the set of all integer partitions. Now for each n , there is only one element of $\mathcal{B}_\mathbb{N}$ with n parts, namely

$$1 + 1 + 1 + \cdots + 1,$$

and every element of $\mathcal{P}_\mathbb{N}$ with n parts, say $\pi_1 + \pi_2 + \cdots + \pi_n$ can be written

$$(1 + (\pi_1 - 1)) + (1 + (\pi_2 - 1)) + \cdots + (1 + (\pi_n - 1)).$$

The generating function for the elements of $\mathcal{P}_\mathbb{N}$ with n parts is therefore

$$\frac{q^{1+1+\cdots+1}}{(1-q)(1-q^2)\cdots(1-q^n)} = \frac{q^n}{(q; q)_n}.$$

Summing over all n yields

$$\sum_{n \geq 0} \frac{q^n}{(q; q)_n},$$

the left-hand side of (1.1).

Next we consider $\mathcal{P}_\mathcal{D}$, the integer partitions that have distinct parts. Here for each n there is again exactly one partition in $\mathcal{B}_\mathcal{D}$ with n parts, namely

$$1 + 2 + 3 + \cdots + n,$$

and every element of $\mathcal{P}_\mathcal{D}$, say

$$\pi_1 + \pi_2 + \cdots + \pi_n \quad (0 \leq \pi_1 \leq \pi_2 \leq \cdots \leq \pi_n),$$

can be written

$$(1 + \pi_1) + (2 + \pi_2) + \cdots + (n + \pi_n).$$

Furthermore

$$\pi_1 + \pi_2 + \cdots + \pi_n$$

constitutes an ordinary partition into n nonnegative parts.

The generating function is therefore

$$\sum_{n \geq 0} \frac{q^{1+2+\cdots+n}}{(q; q)_n} = \sum_{n \geq 0} \frac{q^{n(n+1)/2}}{(q; q)_n}.$$

Finally, if \mathcal{P}_R (R for Rogers and Ramanujan) is the set of integer partitions where the difference between parts is ≥ 2 , the only element of \mathcal{B}_R with n parts is

$$1 + 3 + 5 + \dots + (2n - 1),$$

and, as with (1.2), we obtain the generating function for the partitions in \mathcal{P}_R as

$$\sum_{n \geq 0} \frac{q^{1+3+5+\dots+(2n-1)}}{(q; q)_n} = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n}.$$

The above analysis of the series in (1.1)-(1.3) is far from new. Indeed these are proofs whose ideas date back to Euler and were discussed fully in centuries old number theory and combinatorics books (cf. [23, Sec. 7, Ch. III], [21, Ch. 19]) for this way of looking at (1.1)-(1.3).

Perhaps the reason that this type of study has not gone farther is the fact that in each of the classical cases there was only one element of \mathcal{B} with n parts.

As we will see in the remaining sections, there are many SIP classes with a number of elements of \mathcal{B} with n parts. The real challenge in each instance will be to determine the generating function for \mathcal{B} . Obviously if we denote by $b_{\mathcal{B}}(n)$ the generating function for these elements of \mathcal{B} , then the generating function for all the partitions in \mathcal{P} is given by

$$\sum_{n \geq 0} \frac{b_{\mathcal{B}}(n)}{(q^k; q^k)_n},$$

where k is the modulus associated with \mathcal{P} . In the three cases just considered, we hardly need to think about \mathcal{B} since in each case, there is only one element of \mathcal{B} with n parts.

So how does one determine $b_{\mathcal{B}}(n)$? The idea is to refine one's consideration of $b_{\mathcal{B}}(n, h)$ where $b_{\mathcal{B}}(n, h)$ is the generating function for those elements of \mathcal{B} with n parts and largest part h . Clearly

$$b_{\mathcal{B}}(n) = \sum_{h \geq 0} b_{\mathcal{B}}(n, h).$$

In practice we shall obtain recurrences for the $b_{\mathcal{B}}(n, h)$. The recurrences will arise by noting the parts in the partitions in \mathcal{B} can't get too far apart. Namely if h is too far from the next part then k can be subtracted from h yielding another partition in \mathcal{B} and contradicting the uniqueness of the decomposition (1.5).

The previous paragraph is vague because each individual SIP class provides different meaning for "too far from."

Theorem 1 provides a large number of SIP classes and will facilitate the subsequent theorems in sections 3-5.

Theorem 1. *Let $\{c_1, \dots, c_k\}$ be a set of positive integers with $c_r \equiv r \pmod k$ and $\{d_1, \dots, d_k\}$ be a set of nonnegative integers. Let \mathcal{P} be the set of all integer partitions*

$$b_1 + b_2 + \dots + b_j,$$

where $0 < b_1 \leq b_2 \leq \dots \leq b_j$, and for $1 \leq r \leq k$ and each b_i if $b_i \equiv r \pmod k$, then $b_i \geq c_r$, and if $i > 1$, $b_i - b_{i-1} \geq d_r$.

Then \mathcal{P} is an SIP class with modulus k , and \mathcal{B} consists of all those partitions

$$\beta_1 + \beta_2 + \dots + \beta_j$$

where if $\beta_1 \equiv r \pmod k$ then $\beta_1 = c_r$, and for $q \leq i \leq j$, if $\beta_i \equiv r \pmod k$, then

$$d_r \leq \beta_i - \beta_{i-1} < d_r + k.$$

Proof. We proceed by induction on the number of parts N in the partition of \mathcal{B} .

Clearly, if $N = 1$, then we see that the single part partitions in \mathcal{B} are $\{c_1, c_2, \dots, c_k\}$. Furthermore if b_1 is a one part partition in \mathcal{P} with $b_1 \equiv c_r \pmod k$, then

$$b_1 = kq + c_r$$

with $q \geq 0$.

Now suppose that our theorem holds for all partitions with fewer than N parts. Let us consider an arbitrary partition π in \mathcal{P} with N parts

$$\pi_1 + \pi_2 + \dots + \pi_N.$$

From the definition of \mathcal{P} , we see that

$$\pi_1 + \pi_2 + \dots + \pi_{N-1} = (\beta_1 + q_1k) + (\beta_2 + q_2k) + \dots + (\beta_{N-1} + q_{N-1}k),$$

where $\beta_1 + \beta_2 + \dots + \beta_{N-1}$ is in \mathcal{B} and $(q_1k) + (q_2k) + \dots + (q_{N-1}k)$ is a partition whose parts are multiples of k and $0 \leq q_1 \leq q_2 \leq \dots \leq q_{N-1}$.

Now we know from the definition of \mathcal{P} that if $\pi_N \equiv c_r \equiv r \pmod k$, then

$$(2.1) \quad \begin{aligned} \pi_N - \pi_{N-1} &= \pi_N - (\beta_{N-1} + q_{N-1}k) \\ &\geq d_r. \end{aligned}$$

Now define β_N to be the unique integer congruent to r modulo k in the interval

$$\beta_{N-1} + d_r \leq \beta_N \leq \beta_{N-1} + d_r + k.$$

Clearly $\beta_1 + \dots + \beta_N$ is in \mathcal{B} . It remains to show that $q_N \geq 0$ exists so that

$$(2.2) \quad \pi_N = \beta_N + q_Nk.$$

Since $\pi_N \equiv r \equiv \beta_N \pmod k$, we need only show that q_N in (2.2) is $\geq q_{N-1}$.

Now

$$\begin{aligned} \beta_N + q_Nk &= \pi_N \\ &\geq d_r + \beta_{N-1} + q_{N-1}k \\ &> \beta_N - k + q_{N-1}k, \end{aligned}$$

thus

$$q_Nk > (q_{N-1} - 1)k,$$

or

$$q_N > q_{N-1} - 1,$$

i.e.

$$q_N \geq q_{N-1}.$$

Thus we have completed the induction step and the theorem follows. □

Corollary 2. *As before, $b_{\mathcal{B}}(n)$ denotes the generating function for partitions in \mathcal{B} with exactly n parts, and $P_{\mathcal{P}}(q)$ denotes the generating function for the partition in \mathcal{P} , where \mathcal{P} is an SIP class of modulus k . Then*

$$P_{\mathcal{P}}(q) = \sum_{n \geq 0} \frac{b_{\mathcal{B}}(n)}{(q^k; q^k)_n}.$$

Proof. It is clear from Theorem 1 that

$$\frac{b_{\mathcal{B}}(n)}{(q^k; q^k)_n}$$

is the generating function for all partitions in \mathcal{P} with exactly n parts. Summing over all n proves the corollary. \square

3. GÖLLNITZ-GORDON

Identity (1.6) is the perfect prototype to reveal how SIPs truly generalize the classical series that appear in (1.1)-(1.3).

First let us give the well-known partition-theoretic interpretation in [16], [17], [19], [20]:

First Göllnitz-Gordon Theorem. *The number of partitions of n in which the difference between parts is at least 2 and at least 4 between even parts equals the number of partitions of n into parts congruent to 1, 4 or 7 modulo 8.*

Let $\mathcal{P}_{\mathcal{G}}$ denote the set of all partitions in which the difference between parts is at least 2 and at least 4 between multiples of 2.

Lemma 3. *$\mathcal{P}_{\mathcal{G}}$ is an SIP class of modulus 2.*

Proof. $\mathcal{P}_{\mathcal{G}}$ is an instance of Theorem 1 with $c_1 = 1, c_2 = 2, d_1 = 2, d_2 = 3$. \square

Lemma 4. *Let $b_{\mathcal{G}}(n, h)$ be the generating function for the partitions in $\mathcal{B}_{\mathcal{G}}$ with n parts and largest part equal to n . Then*

$$(3.1) \quad b_{\mathcal{G}}(1, h) = \begin{cases} q & \text{if } h = 1 \\ q^2 & \text{if } h = 2 \\ 0 & \text{otherwise} \end{cases},$$

and for $n > 1, h > 0$,

$$(3.2) \quad b_{\mathcal{G}}(n, 2h) = qb_{\mathcal{G}}(n, 2h - 1),$$

and

$$(3.3) \quad b_{\mathcal{G}}(n, 2n + 2h - 1) = q^{n^2+h^2+2h} \begin{bmatrix} n-1 \\ h \end{bmatrix}_2,$$

where

$$(3.4) \quad \begin{bmatrix} A \\ B \end{bmatrix}_n = \frac{(q^n; q^n)_A}{(q^n; q^n)_B (q^n; q^n)_{A-B}},$$

and

$$(3.5) \quad (A; q)_n = \prod_{j=0}^{n-1} (1 - Aq^j).$$

Proof. We refer to Theorem 1 to determine recurrences for $b_{\mathcal{G}}(n, h)$. Clearly (3.1) is immediate.

By the conditions requiring closeness of parts as stated in Theorem 1, we see that for $n > 1$,

$$(3.6) \quad b_{\mathcal{G}}(n, h) = \begin{cases} q^h(b_{\mathcal{G}}(n-1, h-2) + b_{\mathcal{G}}(n-1, h-3)) & \text{if } h \text{ is odd,} \\ q^h(b_{\mathcal{G}}(n-1, h-3) + b_{\mathcal{G}}(n-1, h-4)) & \text{if } h \text{ is even.} \end{cases}$$

This clearly implies

$$(3.7) \quad b_G(n, 2h) = qb_G(n, 2h - 1),$$

which establishes (3.2).

As for (3.3), we see by (3.6) that

$$(3.8) \quad \begin{aligned} b_G(n, 2n + 2h - 1) &= q^{2n+2h-1} (b_G(n - 1, 2n + 2h - 3) + b_G(n - 1, 2n + 2h - 4)) \\ &= q^{2n+2h-1} (b_G(n - 1, 2n + 2h - 3) + qb_G(n - 1, 2n + 2h - 5)) \quad (\text{by (3.7)}). \end{aligned}$$

Now the standard recurrence for the q -binomial coefficients (defined in (3.4)), namely [10, p. 35]

$$(3.9) \quad \begin{bmatrix} A \\ B \end{bmatrix}_n = \begin{bmatrix} A - 1 \\ B - 1 \end{bmatrix}_n + q^{nB} \begin{bmatrix} A - 1 \\ B \end{bmatrix}_n,$$

establishes that the right-hand side of (3.3) also satisfies the recurrence (3.8). In addition the right-hand side of (3.3) fulfills (3.1) when $n = 1$. Thus (3.3) follows by a straightforward mathematical induction on n . \square

Theorem 5. *First Göllnitz-Gordon identity*

$$(3.10) \quad P_G(q) = \sum_{n \geq 0} \frac{(-q; q^2)_n q^{n^2}}{(q^2; q^2)_n}$$

$$(3.11) \quad = \frac{1}{(q; q^8)_\infty (q^4; q^8)_\infty (q^7; q^8)_\infty}.$$

Remark 6. This celebrated theorem and its history are presented in [3] and [10, sec. 7.4]. Indeed the proof of (3.11) given below follows essentially that given in [6, pp. 40-41]. We include it here to reveal that it is naturally suggested by the work here.

Proof. Let us first treat (3.10). By Corollary 2,

$$(3.12) \quad \begin{aligned} P_G(q) &= \sum_{n \geq 0} \frac{\sum_{j \geq 0} b_G(n, j)}{(q^2; q^2)_n} \\ &= \sum_{n \geq 0} \frac{\sum_{j \geq 0} (b_G(n; 2j - 1) + b_G(n; 2j))}{(q^2; q^2)_n} \\ &= 1 + \sum_{n \geq 1} \frac{\sum_{j \geq 0} (1 + q)q^{n^2+j^2+2j} \begin{bmatrix} n - 1 \\ j \end{bmatrix}_2}{(q^2; q^2)_n}. \end{aligned}$$

Now sum the j -sum using the q -binomial theorem [10, p. 36]. Hence

$$\begin{aligned} P_G(q) &= 1 + \sum_{n \geq 1} \frac{q^{n^2} (1 + q) (-q^3; q^2)_{n-1}}{(q^2; q^2)_n} \\ &= \sum_{n \geq 0} \frac{q^{n^2} (-q; q^2)_n}{(q^2; q^2)_n}, \end{aligned}$$

and (3.10) is proved.

Suppose in (3.12), we summed on n rather than j . Thus

$$\begin{aligned}
 P_{\mathcal{G}}(q) &= 1 + \sum_{n \geq 1} \frac{q^{n^2} \left(\sum_{j \geq 0} q^{j^2+2j} \begin{bmatrix} n-1 \\ j \end{bmatrix}_2 + \sum_{j \geq 1} q^{j^2} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_2 \right)}{(q^2; q^2)_n} \\
 &= \sum_{n \geq 0} \frac{q^{n^2} \sum_{j \geq 0} q^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_2}{(q^2; q^2)_n} \text{ by [10, p. 36]} \\
 &= \sum_{j \geq 0} \frac{q^{j^2}}{(q^2; q^2)_j} \sum_{n \geq j} \frac{q^{n^2}}{(q^2; q^2)_{n-j}} \\
 &= \sum_{j \geq 0} \frac{q^{j^2}}{(q^2; q^2)_j} \sum_{n \geq 0} \frac{q^{(n+j)^2}}{(q^2; q^2)_n} \\
 &= \sum_{j \geq 0} \frac{q^{2j^2}}{(q^2; q^2)_j} (-q^{2j+1}; q^2)_{\infty} \text{ by [10, p. 36]} \\
 &= (-q; q^2)_{\infty} \sum_{j \geq 0} \frac{q^{2j^2}}{(-q; -q)_{2j}}.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 P_{\mathcal{G}}(-q^2) &= (q^2; q^4)_{\infty} \cdot \frac{1}{2} \sum_{j \geq 0} \frac{q^{j^2}}{(q^2; q^2)_j} (1 + (-1)^j) \\
 &= \frac{1}{2} (q^2; q^4)_{\infty} ((-q; q^2)_{\infty} + (q; q^2)_{\infty}) \\
 &= \frac{1}{2} \frac{(q^2; q^4)_{\infty}}{(q^4; q^4)_{\infty}} ((q^4; q^4)_{\infty} (-q; q^4)_{\infty} (-q^3; q^4)_{\infty} \\
 &\quad + (q^4; q^4)_{\infty} (q; q^4)_{\infty} (q^3; q^4)_{\infty}) \\
 &= \frac{1}{2} \frac{(q^2; q^4)_{\infty}}{(q^4; q^4)_{\infty}} \left(\sum_{n=-\infty}^{\infty} q^{2n^2-n} + \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2-n} \right) \text{ by [10, p. 22]} \\
 &= \frac{(q^2; q^4)_{\infty}}{(q^4; q^4)_{\infty}} \sum_{n=-\infty}^{\infty} q^{8n^2-2n} \\
 &= \frac{(q^2; q^4)_{\infty}}{(q^4; q^4)_{\infty}} (q^{16}; q^{16})_{\infty} (-q^6; q^{16})_{\infty} (-q^{10}; q^{16})_{\infty} \text{ by [10, p. 22]}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 P_{\mathcal{G}}(q) &= \frac{(-q; q^2)_{\infty} (q^8; q^8)_{\infty} (q^3; q^8)_{\infty} (q^5; q^8)_{\infty}}{(q^2; q^2)_{\infty}} \\
 &= \frac{(q^2; q^4)_{\infty} (q^8; q^8)_{\infty} (q^3; q^8)_{\infty} (q^5; q^8)_{\infty}}{(q; q)_{\infty}} \\
 &= \frac{1}{(q; q^8)_{\infty} (q^4; q^8)_{\infty} (q^7; q^8)_{\infty}}
 \end{aligned}$$

as desired. \square

4. SCHUR'S 1926 THEOREM

Here is the theorem in question [24].

Schur's Theorem. *The number of partitions of n in which the parts are $\equiv \pm 1 \pmod{6}$ equals the number of partitions of n in which the parts differ by at least 3 and at least 4 if one of the parts in question is divisible by 3.*

In light of the fact that

$$\begin{aligned}
 (4.1) \quad \prod_{n=1}^{\infty} (1 + q^{3n-1})(1 + q^{3n-2}) &= \prod_{n=1}^{\infty} \frac{(1 - q^{6n-2})(1 - q^{6n-4})}{(1 - q^{3n-1})(1 - q^{3n-2})} \\
 &= \prod_{n=1}^{\infty} \frac{1}{(1 - q^{6n-1})(1 - q^{6n-5})},
 \end{aligned}$$

we see that the first class of partitions in Schur's theorem may be replaced by partitions into distinct nonmultiples of 3.

Indeed, this revision of Schur may be refined as follows (an idea first effectively considered in [2] and amplified in [11]):

Refinement of Schur's Theorem. *The generating function for partitions in which there are m parts $\equiv 0, 1 \pmod{3}$ and n parts $\equiv 0, 2 \pmod{3}$ and the difference conditions in Schur's original theorem hold is the coefficient of $u^m v^n$ in*

$$\prod_{j=1}^{\infty} (1 + uq^{3j-2})(1 + vq^{3j-1}).$$

Let \mathcal{P}_S denote the class of all partitions satisfying the difference conditions in Schur's theorem.

Theorem 7. *\mathcal{P}_S is an SIP class.*

Proof. \mathcal{P}_S is the instance of Theorem 1 with $k = 3, \{c_1, c_2, c_3\} = \{1, 2, 3\}$, and $\{d_1, d_2, d_3\} = \{3, 3, 4\}$. □

In the remainder of this section, we shall first determine an explicit formula for the generating functions associated with \mathcal{B}_S . Then we will apply Corollary 2 to prove the Refinement of Schur's Theorem.

Theorem 8. *Let $b_S(u, v, n, h)$ ($= b_S(n, h)$) be the generating function for the partitions in \mathcal{B}_S (with u marking parts $\equiv 0, 1 \pmod{3}$ and v marking parts $\equiv 0, 2$*

(mod 3). Then for $n > 1$ and $h \geq 0$,

(4.2)

$$b_S(n, 3n + 3h - 1) = \sum_{j \geq 0} \sum_{i=0}^{n-h} v^{n-j} u^{j+h-i} q^{n(3n+1)/2+h(3h+5)/2+i(3i+1)/2} \\ \times q^{-j} \begin{bmatrix} n-j-1 \\ h \end{bmatrix}_3 \begin{bmatrix} j+h-i \\ h \end{bmatrix}_3 \begin{bmatrix} h \\ i \end{bmatrix}_3,$$

(4.3)

$$b_S(n, 3n + 3h - 2) = (1 + uq) \sum_{j \geq 0} \sum_{i=0}^{n-h} v^{n-j} u^{j+h-i} q^{n(3n+1)/2+h(3h+5)/2+i(3i-5)/2} \\ \times q^{-j} \begin{bmatrix} n-j-1 \\ h \end{bmatrix}_3 \begin{bmatrix} j+h-i \\ h \end{bmatrix}_3 \begin{bmatrix} h-1 \\ i-1 \end{bmatrix}_3,$$

(4.4) $b_S(n, 3n + 3h) = uqb_S(n, 3n - 3h - 1)$,

and for $n = 1$,

(4.5)
$$b_S(1, h) = \begin{cases} uq & \text{if } h = 1, \\ vq^2 & \text{if } h = 2, \\ uvq^3 & \text{if } h = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{bmatrix} A \\ B \end{bmatrix}_j = \begin{cases} 0 & \text{if } B > A \text{ or } B < 0, \\ (q^j; q^j)_A / (q^j; q^j)_B (q^j; q^j)_{A-B} & \text{otherwise.} \end{cases}$$

Proof. We let

(4.6)
$$\chi_3(n) = \begin{cases} 1 & \text{if } 3 \mid n, \\ 0 & \text{otherwise,} \end{cases}$$

and

(4.7)
$$s(h) = \begin{cases} uv & \text{if } 3 \mid h, \\ u & \text{if } h \equiv 1 \pmod{3}, \\ v & \text{if } h \equiv 2 \pmod{3}. \end{cases}$$

By the definition of \mathcal{B}_S , we see that

(4.8)
$$b_S(1, h) = \begin{cases} uq & \text{if } h = 1, \\ vq^2 & \text{if } h = 2, \\ uvq^3 & \text{if } h = 3, \\ 0 & \text{otherwise,} \end{cases}$$

and for $n = 1$,

(4.9)
$$b_S(n, h) = s(h)q^h \sum_{j=3}^5 b_S(n-1, h-j-\chi_3(h)).$$

Now (4.4) follows directly by comparing (4.9) with $h \rightarrow 3n + 3h - 1$, and (4.5) is a restatement of (4.8).

To prove (4.2) and (4.3), we need only show that the coefficient of $v^{n-j} u^{j+h-i}$ on both sides of (4.9) is identical, when the $b_S(n, h)$ are replaced by the corresponding right hand sides of (4.2) and (4.3).

We begin with (4.9) when $n \rightarrow 3n + 3h - 1$. To make clear what we are doing, we write the right hand side of (4.2) as

$$(4.10) \quad \sum_{j=0}^{n-h} \sum_{i=0}^h b_1(n, h, i, j) v^{n-j} u^{j+h-i}$$

and the right hand side of (4.3) as

$$(4.11) \quad (1 + uq) \sum_{j=0}^{n-h} \sum_{i=0}^h b_2(n, h, i, j) v^{n-j} u^{j+h-i}.$$

Subtracting these expressions into the right hand side of (4.9) with h replaced by $3n + 3h - 1$, we have

$$(4.12) \quad vq^{3n+3h-1} \left(\sum_{j=0}^{n-1-h} \sum_{i=0}^h b_1(n-1, h, i, j) v^{n-1-j} u^{j+h-i} + (1 + uq) \sum_{j=0}^{n-1-h} \sum_{i=0}^h b_2(n-1, h, i, j) v^{n-1-j} u^{j+h-i} + uq \sum_{j=0}^{n-h} \sum_{i=0}^{h-1} b_1(n-1, h, i, j) v^{n-1-j} u^{j+h-i} \right),$$

the coefficient of $v^{n-j} u^{j+h-i}$ in (4.12) is

$$(4.13) \quad q^{3n+3h+1} (b_1(n-1, h, i, j) + b_2(n-1, h, i, j) + qb_2(n-1, h, i+1, j) + qb_1(n-1, h-1, i, j))$$

and this last expression simplifies to $b_1(n-1, h, i, j)$ through three applications of one or the other of the standard q -binomial recurrences [10, p. 35] reiterated here:

$$(4.14) \quad \begin{bmatrix} A \\ B \end{bmatrix}_j = \begin{bmatrix} A-1 \\ B-1 \end{bmatrix}_j + q^{jB} \begin{bmatrix} A-1 \\ B \end{bmatrix}_j,$$

and

$$(4.15) \quad \begin{bmatrix} A \\ B \end{bmatrix}_j = \begin{bmatrix} A-1 \\ B \end{bmatrix}_j + q^{j(A-B)} \begin{bmatrix} A-1 \\ B-1 \end{bmatrix}_j.$$

Thus (4.9) is established for $h \rightarrow 3n + 3n - 1$.

Finally we consider (4.9) with $h \rightarrow 3n + 3h + 1$ in which case the assertion becomes

$$(4.16) \quad (1 + uq) \sum_{i=0}^{n-h-1} \sum_{j=0}^{h+1} b_2(n, h+1, i, j) v^{n-j} u^{j+h-i+1} = uq^{3n+3h+1} \left((1 + uq) \sum_{i=0}^{n-h-1} \sum_{j=0}^h b_2(n-1, h+1, i, j) v^{n-1-j} u^{j+h-i+1} + (1 + uq) \sum_{i=0}^{n-h-1} \sum_{j=0}^h b_1(n-1, h, i, j) v^{n-1-j} u^{j+h-i} \right).$$

Cancelling the $u(1+uq)$ from both sides of (4.16), we see that we need to establish that the coefficients of $v^{n-j}u^{j+h-i}$ are identical on each side.

On the right hand side the coefficient is

$$\begin{aligned} & q^{3n+3h+1}(b_2(n-1, h+1, i, j) + b_1(n-1, h, i-1, j-1)) \\ &= q^{n(3n+1)/2+(h+1)(3h+8)/2+i(3i-5)/2} \\ & \times \left(q^{3h+3} \begin{bmatrix} n-1-j \\ h \end{bmatrix}_3 \begin{bmatrix} j+h-i \\ h+1 \end{bmatrix}_3 \begin{bmatrix} h \\ i-1 \end{bmatrix}_3 \right. \\ & \left. + \begin{bmatrix} n-1-j \\ h \end{bmatrix}_3 \begin{bmatrix} j+h-i \\ h \end{bmatrix}_3 \begin{bmatrix} h \\ i-1 \end{bmatrix}_3 \right) \\ &= b_2(n, h+1, i, j) \end{aligned}$$

by (4.14).

Thus we have fulfilled the defining recurrence and initial conditions to establish the formulas (4.2) and (4.3). This concludes the proof of Theorem 8. \square

We shall conclude this section by proving the Refinement of Schur's Theorem combining Corollary 2 with Theorem 8.

For the sake of brevity, we write

$$(4.17) \quad S_1(n, h) = b_S(n, 3n + 3h - 1)$$

and

$$(4.18) \quad S_2(n, h) = b_S(n, 3n + 3h - 2).$$

We note that by (4.4),

$$(4.19) \quad uqS_2(n, h) = b_S(n, 3n + 3h).$$

To make the final theorem of this section readable, we first prove Lemmas 9–11.

Lemma 9.

$$\begin{aligned} (4.20) \quad & (1 + uq)S_1(n, h) + S_2(n, h + 1) \\ &= \sum_{j=0}^{n-h} \sum_{i=-1}^h v^{n-j}u^{j+h-i} q^{n(3n+1)/2+h(3h+8)/2+i(3i+1)/2-j} \\ & \times \begin{bmatrix} n-1-j \\ h \end{bmatrix}_3 \begin{bmatrix} j+h-i \\ j \end{bmatrix}_3 \begin{bmatrix} j+1 \\ i+1 \end{bmatrix}_3. \end{aligned}$$

Proof. The coefficient of $v^{n-j}u^{j+h-i}$ in

$$(1 + uq)S_1(n, h) + S_2(n, h + 1)$$

is

$$\begin{aligned}
 & q^{n(3n+1)/2+h(3h+5)/2+i(3i+1)/2-j} \\
 & \times \left(\begin{matrix} [n-1-j] \\ h \end{matrix} \right)_3 \begin{matrix} [j+h-i] \\ h \end{matrix} \begin{matrix} [h] \\ i \end{matrix} \Big|_3 \\
 & + q^{3i+3} \begin{matrix} [n-1-j] \\ h \end{matrix} \begin{matrix} [j+h-i-1] \\ h \end{matrix} \begin{matrix} [h] \\ i+1 \end{matrix} \Big|_3 \\
 & + q^{3h+3} \begin{matrix} [n-1-j] \\ h \end{matrix} \begin{matrix} [j+h-i] \\ h+1 \end{matrix} \begin{matrix} [h] \\ i \end{matrix} \Big|_3 \\
 & + q^{3i+3h+6} \begin{matrix} [n-1-j] \\ h \end{matrix} \begin{matrix} [j+h-i-1] \\ h+1 \end{matrix} \begin{matrix} [h] \\ i+1 \end{matrix} \Big|_3 \Big) \\
 & = q^{n(3n+1)/2+h(3h+5)/2+i(3i+1)/2-j} \\
 & \times \left(\begin{matrix} [n-1-j] \\ h \end{matrix} \right)_3 \begin{matrix} [j+h-i+1] \\ h+1 \end{matrix} \begin{matrix} [h] \\ i \end{matrix} \Big|_3 \\
 & + q^{3i+3} \begin{matrix} [n-1-j] \\ h \end{matrix} \begin{matrix} [j+h-i] \\ h+1 \end{matrix} \begin{matrix} [h] \\ i+1 \end{matrix} \Big|_3 \Big) \text{ (by (4.14) twice)} \\
 & = q^{n(3n+1)/2+h(3h+5)/2+i(3i+1)/2-j} \begin{matrix} [n-1-j] \\ h \end{matrix} \Big|_3 \\
 & \times \frac{(q^3; q^3)_{j+h-i} (q^3; q^3)_h}{(q^3; q^3)_{h+1} (q^3; q^3)_{j-i} (q^3; q^3)_{i+1} (q^3; q^3)_{h-i}} \\
 & \times \left((1 - q^{3(j+h-i+1)})(1 - q^{3(i+1)}) + q^{3i+3}(1 - q^{3(j-i)})(1 - q^{3(h-i)}) \right) \\
 & = q^{n(3n+1)/2+h(3h+5)/2+i(3i+1)/2-j} \times \begin{matrix} [n-1-j] \\ h \end{matrix} \begin{matrix} [j+h-i] \\ j \end{matrix} \begin{matrix} [j+1] \\ i+1 \end{matrix} \Big|_3,
 \end{aligned}$$

which is the coefficient asserted in (4.20). □

Lemma 10.

$$(4.21) \quad \sum_{h=0}^r \begin{matrix} [s-1] \\ h \end{matrix} \begin{matrix} [n+1] \\ r-h \end{matrix} \Big|_3 q^{3h^2+3h(n+1-r)} = \begin{matrix} [n+s] \\ r \end{matrix} \Big|_3.$$

Proof. This is an instance of the q -Chu-Vandermonde summation [4, p. 37, eq. (3,(3.10)], $q \rightarrow q^3, n \rightarrow n+1, m \rightarrow s-1, h \rightarrow r$, and $k \rightarrow r-k$. □

Lemma 11.

$$(4.22) \quad \sum_{n \geq 0} \begin{matrix} [r] \\ n \end{matrix} \begin{matrix} [n+s] \\ r \end{matrix} \Big|_3 \frac{q^{3n^2+3n(s-r)}}{(q^3; q^3)_{n+s}} = \frac{1}{(q^3; q^3)_r (q^3; q^3)_s}.$$

Proof. The left-hand side may be transformed into q -Pochhammer symbols as is done in detail in the proof of Theorem 3.3 from [9, p. 36].

$$\begin{aligned} & \sum_{n \geq 0} \begin{bmatrix} r \\ n \end{bmatrix}_3 \begin{bmatrix} n+s \\ r \end{bmatrix}_3 \frac{q^{3n^2+3n(s-r)}}{(q^3; q^3)_{n+s}} \\ &= \frac{1}{(q^3; q^3)_r (q^3; q^3)_{s-r}} \lim_{\tau \rightarrow 0} \sum_{n=0}^r \frac{(q^{-3r}; q^3)_n (q^{3s+3}/\tau; q^3)_n \tau^n}{(q^3; q^3)_n (q^{3(s-r+1)}; q^3)_n} \\ &= \frac{1}{(q^3; q^3)_r (q^3; q^3)_{s-r}} \cdot \frac{1}{(q^{3(s-r+1)}; q^3)_r} \\ & \text{(by [10, p. 38 eq. (3.3,12)]), } q \rightarrow q^3, N = r, a \rightarrow \infty, b = q^{3s+3}/\tau, \\ & c = q^{3(s-r+1)} \\ &= \frac{1}{(q^3; q^3)_r (q^3; q^3)_s}. \end{aligned}$$

□

Theorem 12.

$$(4.23) \quad \sum_{n \geq 0} \frac{b_S(n; q)}{(q^3; q^3)_n} = (-uq; q^3)_\infty (-vq; q^3)_\infty.$$

Remark 13. This is the restatement of the refinement of Schur’s theorem. It will appear in the following proof that the $b_S(n; q)$ are infinite sums, but this is only a convenience of notation. Thus

$$\begin{aligned} \sum_{n \geq 0} \frac{b_S(n; q)}{(q^3; q^3)_n} &= 1 + \frac{uq + vq^2 + uvq^3}{1 - q^3} \\ &+ \frac{u^2q^5 + uvq^6 + (u^2v + v^2)q^7 + uv^2q^8 + uvq^9 + u^2vq^{10} + v^2uq^{11} + u^2v^2q^{12}}{(1 - q^3)(1 - q^6)} \\ &+ \dots \end{aligned}$$

Proof. First we note that

$$(-uq; q^3)_\infty (-vq^2; q^3)_\infty = \sum_{r=0}^\infty \frac{u^r q^{\frac{1}{2}r(3r-1)}}{(q^3; q^3)_r} \sum_{s=0}^\infty \frac{v^s q^{\frac{1}{2}s(3s+1)}}{(q^3; q^3)_s}.$$

Hence the coefficient of $u^r v^s$ on the right hand of (4.23) is

$$(4.24) \quad \frac{q^{r(3r-1)/2+s(3s+1)/2}}{(q^3; q^3)_r (q^3; q^3)_s}.$$

To complete the proof we must evaluate the coefficient of $u^r v^s$ on the left side of (4.23).

Now

$$\begin{aligned} \sum_{n \geq 0} \frac{b_S(n, q)}{(q^3; q^3)_n} &= \sum_{\substack{n \geq 0 \\ h \geq -1}} \frac{(1 + uq)S_1(n, h) + S_2(n, h + 1)}{(q^3; q^3)_n} \\ &= 1 + \sum_{\substack{n \geq 0 \\ h \geq -1}} \frac{1}{(q^3; q^3)_n} \sum_{j=0}^{n-h} \left(\sum_{i=-1}^h v^{n-j} u^{j+h-i} q^{f(n, h, i, j)} \right. \\ &\quad \left. \times \begin{bmatrix} n-j-1 \\ h \end{bmatrix}_3 \begin{bmatrix} j+h-i \\ j \end{bmatrix}_3 \begin{bmatrix} j+1 \\ i+1 \end{bmatrix}_3 \right), \end{aligned}$$

by Lemma 9, with

$$f(n, h, i, j) = n(3n + 1)/2 + h(3h + 5)/2 + i(3i + 1)/2 - j.$$

So to get the coefficient of $u^r v^s$, we need $j = n - s, i = n - s - r + h$. Thus the coefficient of $u^r v^s$ on the left side of (4.23) is

$$\sum_{\substack{n \geq 0 \\ h \geq -1}} \begin{bmatrix} s-1 \\ h \end{bmatrix}_3 \begin{bmatrix} r \\ n-s \end{bmatrix}_3 \begin{bmatrix} n-s+1 \\ r-h \end{bmatrix}_3 \times \frac{q^{f(n, h, n-s-r+h, n-s)}}{(q^3; q^3)_n}.$$

Now the sum on h turns out to be the sum on the left side of (4.21). Hence by Lemma 10, the above sum reduces to

$$\sum_{n \geq 0} \begin{bmatrix} r \\ n \end{bmatrix}_3 \begin{bmatrix} n+s \\ r \end{bmatrix}_3 \frac{q^{s(3s+1)/2+r(3r-1)/2+3n^2+3n(s-r)}}{(q^3; q^3)_{n+s}} = \frac{q^{s(3s+1)/2+r(3r-1)/2}}{(q^3; q^3)_t (q^3; q^3)_s}$$

by Lemma 11 which is exactly the expression in (4.24). Thus Theorem 12 is proved. □

5. GLASGOW MOD 8

H. Göllnitz [16], [17] provided four partition identities related to partitions whose parts are restricted to certain residue classes modulo 8. Two of these theorems were independently discovered by B. Gordon [19], [20] and have been given the name Göllnitz-Gordon, as mentioned previously.

Lesser known is the following theorem which first appeared in the *Glasgow Mathematics Journal* in 1967 [4, p. 127]:

Glasgow Mod 8 Theorem. *Let $A(n)$ denote the number of partitions of n into parts congruent to 0, 2, 3, 4, or 7 (mod 8). Let $B(n)$ denote the number of partitions of n in which all parts are ≥ 2 and each odd part is at least 3 larger than any part not exceeding it. Then for $n \geq 0$,*

$$A(n) = B(n).$$

For example, $A(10) = 8$ enumerating

$$10, 8 + 2, 7 + 3, 4 + 3 + 3, 4 + 4 + 2, 4 + 2 + 2 + 2, 3 + 3 + 2 + 2, 2 + 2 + 2 + 2 + 2,$$

and $B(10) = 8$ enumerating

$$10, 8 + 2, 7 + 3, 6 + 4, 6 + 2 + 2, 4 + 4 + 2, 4 + 2 + 2 + 2, 2 + 2 + 2 + 2 + 2.$$

A natural bijective proof appears in [5].

We have chosen to consider this theorem owing to the fact that it has never appeared as a direct consequence of a series-product identity. Indeed, the relevant identity turns out to be

$$(5.1) \quad 1 + \frac{q^2 + q^3}{1 - q^2} + \sum_{n=2}^{\infty} \frac{(-q^3; q^4)_{n-1} q^{2n} (1 + q^{2n-1})}{(q^2; q^2)_n} = \prod_{\substack{n=1 \\ n \neq 1,5,6 \pmod{8}}} \frac{1}{1 - q^n}.$$

We shall first prove that (5.1) is valid. We shall then prove that the left side of (5.1) is an instance of Corollary 2.

Theorem 14. *Equation (5.1) is valid.*

Proof. For $N \geq 1$,

$$(5.2) \quad 1 + \frac{q^2 + q^3}{1 - q^2} + \sum_{n=2}^N \frac{(-q^3; q^4)_{n-1} q^{2n} (1 + q^{2n-1})}{(q^2; q^2)_n} = \frac{(-q^3; q^4)_N}{(q^2; q^2)_N}.$$

This follows by mathematical induction. For $N = 1$,

$$1 + \frac{q^2 + q^3}{1 - q^2} = \frac{1 + q^3}{1 - q^2}.$$

Generally,

$$\begin{aligned} \frac{(-q^3; q^4)_N}{(q^2; q^2)_N} - \frac{(-q^3; q^4)_{N-1}}{(q^2; q^2)_{N-1}} &= \frac{(-q^3; q^4)_{N-1}}{(q^2; q^2)_N} ((1 + q^{4N-1}) - (1 - q^{2N})) \\ &= \frac{(-q^3; q^4)_{N-1} q^{2N} (1 + q^{2N-1})}{(q^2; q^2)_N}, \end{aligned}$$

which is the N^{th} term of the left-hand side. The result then follows by induction.

Now let $N \rightarrow \infty$ in (5.2). The left side converges to the left side of (5.1), and

$$\begin{aligned} \frac{(-q^3; q^4)_{\infty}}{(q^2; q^2)_{\infty}} &= \frac{(q^6; q^8)_{\infty}}{(q^2; q^2)_{\infty} (q^3; q^4)_{\infty}} \\ &= \frac{1}{(q^2, q^3, q^4, q^7, q^8; q^8)_{\infty}} \left(= \sum_{n \geq 0} A(n) q^n \right). \end{aligned}$$

□

Lemma 15. *Let \mathcal{E} denote the class of partitions related to $B(n)$. Then \mathcal{E} is an SIP class of modulus 2.*

Proof. This follows immediately from Theorem 1 with $k = 2, d_1 = 3, d_2 = 0, c_1 = 3, c_2 = 2$. □

Lemma 16. *Let $b_{\mathcal{E}}(n, h)$ be the generating function for the partitions in $B_{\mathcal{E}}$ with n parts and largest part equal to h . Then*

$$(5.3) \quad b_{\mathcal{E}}(1, h) = \begin{cases} q^2 & \text{if } h = 2 \\ q^3 & \text{if } h = 3 \\ 0 & \text{otherwise} \end{cases},$$

and for $n > 1, h > 0$,

$$(5.4) \quad b_{\mathcal{E}}(n, 4h + 1) = q^{2n+2h^2+h} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4,$$

$$(5.5) \quad b_{\mathcal{E}}(n, 4h) = q^{2n+2h^2+h} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4,$$

$$(5.6) \quad b_{\mathcal{E}}(n, 4h - 1) = q^{4n+2h^2-3h} \begin{bmatrix} n-2 \\ h-2 \end{bmatrix}_4,$$

$$(5.7) \quad b_{\mathcal{E}}(n, 4h - 2) = q^{2n-3+2h^2+h} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4.$$

Proof. First we see that (5.3) is immediate by inspection. Next we note that the two part partitions in $B_{\mathcal{E}}$ are $2 + 2, 2 + 5, 3 + 4, 3 + 7$. Thus

$$(5.8) \quad b_{\mathcal{E}}(2, h) = \begin{cases} q^4 & \text{if } h = 2 \\ q^7 & \text{if } h = 4 \\ q^7 & \text{if } h = 5 \\ q^{10} & \text{if } h = 7 \\ 0 & \text{otherwise} \end{cases},$$

and inspection reveals that (5.4)-(5.7) are valid for $n = 2$.

Now as in the previous sections, we see that

$$(5.9) \quad b_{\mathcal{E}}(n, h) = \begin{cases} q^h(b_{\mathcal{E}}(n-1, h) + b_{\mathcal{E}}(n-1, h-1)) & \text{if } h \text{ is even,} \\ q^h(b_{\mathcal{E}}(n-1, h-3) + b_{\mathcal{E}}(n-1, h-4)) & \text{if } h \text{ is odd.} \end{cases}$$

All that remains is to show that the right hand sides of (5.4)-(5.7) satisfy the defining recurrence (5.9). Each is proved using instances of (4.14) or (4.15). We shall do one case which is typical. When $h \equiv 1 \pmod{4}$, equation (5.9) asserts

$$(5.10) \quad b_{\mathcal{E}}(n, 4h + 1) = q^{4h+1} (b_{\mathcal{E}}(n-1, 4h-2) + b_{\mathcal{E}}(n-1, 4h-3)).$$

Do the asserted formulas for the $b_{\mathcal{E}}(n, -)$ satisfy the recurrence (5.10)? Yes, as is clear from the following:

$$\begin{aligned} & q^{4h+1} \left(q^{2n-5+2h^2+h} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4 + q^{2n-2+2(h-1)^2+h-1} \begin{bmatrix} n-2 \\ h-2 \end{bmatrix}_4 \right) \\ &= q^{2n+2h^2+h} \left(q^{4h-4} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4 + \begin{bmatrix} n \\ - \end{bmatrix} 2h-2_4 \right) \\ &= q^{2n+2h^2+h} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4 \quad (\text{by (4.14)}), \end{aligned}$$

and this is exactly the recurrence (5.10). □

Lemma 17.

$$(5.11) \quad \sum_{h \geq 0} b_{\mathcal{E}}(n, 4h + 1) = q^{2n+3}(-q^7; q^4)_{n-2},$$

$$(5.12) \quad \sum_{h \geq 0} b_{\mathcal{E}}(n, 4h) = q^{4n-1}(-q^7; q^4)_{n-2},$$

$$(5.13) \quad \sum_{h \geq 0} b_{\mathcal{E}}(n, 4h - 1) = q^{4n+2}(-q^7; q^4)_{n-2},$$

$$(5.14) \quad \sum_{h \geq 0} b_{\mathcal{E}}(n, 4h - 2) = q^{2n}(-q^7; q^4)_{n-2}.$$

Proof. Each of these four assertions is an instance of the q -binomial theorem [10, p. 36] applied to the corresponding equation in Lemma 16. We prove (5.10) as typical.

By (5.4) and the q -binomial theorem,

$$\begin{aligned} \sum_{h \geq 0} b_{\mathcal{E}}(n, 4h + 1) &= \sum_{h \geq 0} q^{2n+2h^2+h} \begin{bmatrix} n-2 \\ h-1 \end{bmatrix}_4 \\ &= \sum_{h \geq 0} q^{2n+2(h+1)^2+h+1} \begin{bmatrix} n-2 \\ h \end{bmatrix}_4 \\ &= q^{2n+3}(-q^7; q^4)_{n-2}. \end{aligned}$$

□

Theorem 18.

$$(5.15) \quad \sum_{n \geq 0} \frac{b_{\mathcal{E}}(n)}{(q^2; q^2)_n} = 1 + \frac{q^2 + q^3}{1 - q^2} + \sum_{n \geq 2} \frac{(-q^3; q^4)_{n-1} q^{2n} (1 + q^{2n-1})}{(q^2; q^2)_n}.$$

Proof.

$$\begin{aligned} \sum_{n \geq 0} \frac{b_{\mathcal{E}}(n)}{(q^2; q^2)_n} &= 1 + \frac{q^2 + q^3}{1 - q^2} + \sum_{n \geq 2} \sum_{h \geq 1} \frac{b_{\mathcal{E}}(n, h)}{(q^2; q^2)_n} \\ &= 1 + \frac{q^2 + q^3}{1 - q^2} \\ &\quad + \sum_{n \geq 2} \sum_{h \geq 1} \frac{b_{\mathcal{E}}(n, 4h + 1) + b_{\mathcal{E}}(n, 4h) + b_{\mathcal{E}}(n, 4h - 1) + b_{\mathcal{E}}(n, 4h - 2)}{(q^2; q^2)_n} \\ &= + \sum_{n \geq 2} \frac{(-q^7; q^4)_{n-2} (q^{2n+3} + q^{4n-1} + q^{4n+2} + q^{2n})}{(q^2; q^2)_n} \text{ (by Lemma 17)} \\ &= 1 + \frac{q^2 + q^3}{1 - q^2} + \sum_{n \geq 2} \frac{(-q^7; q^4)_{n-2} q^{2n} (1 + q^3) (1 + q^{2n-1})}{(q^2; q^2)_n} \\ &= 1 + \frac{q^2 + q^3}{1 - q^2} + \sum_{n \geq 2} \frac{(-q^3; q^4)_{n-1} q^{2n} (1 + q^{2n-1})}{(q^2; q^2)_n}. \end{aligned}$$

□

Corollary 19. *The Glasgow Mod 8 Theorem is true.*

Proof. This follows by comparing (5.1) with (5.14) and invoking Corollary 2 with $\mathcal{P} = \mathcal{E}$. □

6. PARTITIONS WITH n COPIES OF n

The basic idea epitomized by Theorem 1 is actually applicable in a broader context. In this section we shall describe its application to partitions with “ n copies of n ” [1].

This subject considers partitions taken from the set M of ordered pairs of positive integers with the second entry not exceeding the first entry. A partition with n copies of n of the positive integer v is a finite collection of elements of M wherein the first elements of the ordered pairs sum to v . For example, there are six partitions of 3 with n copies of n :

$$3_1, 3_2, 3_3, 2_2 + 1_1, 2_1 + 1_1, 1_1 + 1_1 + 1_1.$$

As was noted in [1], there is a bijection between partitions with n copies of n and plane partitions.

Most important for our current considerations is the weighted difference between two elements of M . Namely, we define $((m_i - n_j))$, the weighted difference of m_i and n_j , as follows:

$$((m_i - n_j)) = m - n - i - j.$$

The main point of [1] was to prove the following two results.

Theorem 20 ([1, p. 41]). *The partitions of v with n copies of n wherein each pair of parts has positive weighted difference are equinumerous with the ordinary partitions of v into parts $\equiv 0, \pm 4 \pmod{10}$.*

Theorem 21 ([1, p. 41]). *The partitions of v with n copies of n wherein each pair of parts has nonnegative weighted difference are equinumerous with the ordinary partitions of v into parts $\equiv 0, \pm 6 \pmod{14}$.*

These two theorems are special cases of a general theorem proved in [1, Th. 3, p. 42]. The proofs relied on bijection between the partitions in question and the results which provide several Rogers-Ramanujan type theorems concerning partitions with specified hook differences.

Our object here is to reveal a completely different path to proof by using an adaptation of Theorem 1.

Let $\beta_r(m; q)$ denote the generating function for partitions with n copies of n where the weighted difference between successive parts (written in lexicographic ascending order, i.e. $m_i > n_j$ if $m > n$ or $m = n$ and $i > j$) is exactly r , there are exactly m parts, and the smallest part is of the form i_i .

Theorem 22.

$$(6.1) \quad \beta_r(m, q) = \frac{q^{m^2+r\binom{m}{2}}}{(q; q^2)_m}.$$

As we will see, Theorems 20 and 21 follow from Theorem 22 plus Lemma 24 via two identities given in L. J. Slater’s compendium [25, eqs (46) and (61)]:

$$(6.2) \quad \sum_{n \geq 0} \frac{q^{n(3n-1)/2}}{(q; q)_n (q; q^2)_n} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 4 \pmod{10}}} \frac{1}{1 - q^n},$$

and

$$(6.3) \quad \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n (q; q^2)_n} = \prod_{\substack{n=1 \\ n \neq 0, \pm 6 \pmod{14}}} \frac{1}{1 - q^n}.$$

In addition, the case $r = -1$ is related to the Slater identity [25, p. 160, eq. (81)]

$$(6.4) \quad \sum_{n \geq 0} \frac{q^{\binom{n+1}{2}}}{(q; q)_n (q; q^2)_n} = \prod_{n=1}^{\infty} (1 + q^{7n}) \prod_{m=1}^{\infty} \frac{1}{(1 - q^{14m-3})(1 - q^{14m-11})} \times \prod_{\substack{n=1 \\ n \equiv \pm 2, 3, 4 \pmod{14}}}^{\infty} \frac{1}{1 - q^n}.$$

The right hand side of (6.4) is easily seen to be the generating function for $C(n)$, the number of partitions in which multiples of 7 are not repeated, all other parts are $\equiv \pm 2, \pm 3, \pm 4 \pmod{14}$ and parts $\equiv \pm 3 \pmod{14}$ appear in two colors. This observation together with (6.4) establishes the following result.

Theorem 23. *The number of partitions of v with n copies of n wherein successive parts have weighted difference ≥ -1 equals $C(n)$.*

We shall not require the full generality of Theorem 1 for our application of the SIP idea to partitions with n copies of n . Indeed we only need something analogous to the three classical examples provided initially in section 2.

Lemma 24. *Let $r \geq -1$. Suppose π is a partition with n copies of n with the parts written in ascending lexicographic order (i.e. $m_i > n_j$ if $m > n$ or $m = n$ and $i > j$). Assume that the weighted difference between successive parts is $\geq r$. Then if π has h parts*

$$m_{i_1} + n_{i_2} + o_{i_3} + \cdots + t_{i_n},$$

there is a unique ordinary partition with h nonnegative parts in nondecreasing order

$$\psi_1 + \psi_2 + \cdots + \psi_n$$

and a unique partition $\bar{\pi}$ with n copies of n

$$\bar{m}_{i_1} + \bar{n}_{i_2} + \bar{o}_{i_3} + \cdots + \bar{t}_{i_n},$$

where $\bar{m}_i = i_i$ and the successive weighted differences are all equal to r , and

$$\begin{aligned} m_{i_1} &= \bar{m}_{i_1} + \psi_1 = (\bar{m} + \psi_1)_{i_1}, \\ n_{i_2} &= \bar{n}_{i_2} + \psi_2 = (\bar{n} + \psi_2)_{i_2}, \\ &\vdots \\ t_{i_n} &= \bar{t}_{i_n} + \psi_n = (\bar{t} + \psi_n)_{i_n}. \end{aligned}$$

Remark 25. Note that the subscripts for the original π are identical with the subscript set for $\bar{\pi}$.

Proof. We begin by noting that the subscript tuple (i_1, i_2, \dots, i_n) uniquely defines the $\bar{m}_{i_1}, \bar{n}_{i_2}, \dots$ as follows:

$$\begin{aligned} \bar{m}_i &= i_i \quad (\text{where } i = i_1), \\ \bar{n}_{i_2} &= (i_2 + 2i_1 + r)_{i_2}, \\ \bar{o}_{i_3} &= (i_3 + 2i_2 + 2i_1 + 2r)_{i_3}, \\ &\vdots \\ \bar{t}_{i_n} &= (t + \dots + 2i_3 + 2i_2 + 2i_1 + (p - 1)r)_{i_n}, \end{aligned}$$

where p is the number of parts of the partition. Note that the weighted difference between successive terms in this sequence is always r .

Now we uniquely construct the ψ as follows. We begin with ψ_1 :

$$\psi_1 = m - i,$$

and since the first subscript is i , we know that m must be $\geq i$, so $\psi_1 \geq 0$. Next we define

$$\psi_2 = n - j - 2i - r.$$

Clearly ψ_2 is unique. Is $\psi_2 \geq \psi_1$? Yes, because

$$\psi_2 - \psi_1 = (n - j - 2i - r) - (m - i) = n - m - i - j - r = ((n_i - m_j)) - r \geq r - r \geq 0.$$

Next we define

$$\psi_3 = o - h - 2i - 2j - 2r$$

and again

$$\begin{aligned} \psi_3 - \psi_2 &= (o - h - 2i - 2j - 2r) - (n - j - 2i - r) = o - h - n - j - r \\ &= ((o_h - n_j)) - r \geq r - r \geq 0. \end{aligned}$$

This continues for all the parts of $\bar{\pi}$, and this concludes the proof of the lemma. \square

Thus we have established the analogous paradigm for n -copies of n partitions that we considered for ordinary partitions.

The next step is to consider the generating function for the partitions

$$\bar{m}_i + \bar{n}_j + \dots + \bar{t}_k,$$

where $\bar{m}_i = i_i$ and all weighted differences between successive parts in ascending order equal r . Call this generating function $g_r(n, m, j)$ for such partitions where the number of parts is n and the largest part is m_j .

Lemma 26.

$$(6.5) \quad g_r(n, m, j) = \begin{cases} 0 & \text{if } m < 1, \\ q^m & \text{if } n = 1 \text{ and } m = j, \\ 0 & \text{if } n = 1 \text{ and } m \neq j, \\ q^m \sum_{i=1}^m g_r(n - 1, m - j - i - r, i) & \text{otherwise.} \end{cases}$$

Proof. The first three lines of (6.5) are immediate because the smallest part must be of the form m_m .

For the last line, we see that the part m_j must have directly below it a part that produces a weighted difference of r . Thus if the subscript is i , the part must be

$$(m - j - i - r)_i$$

because

$$m - j - (m - j - i - r) - i = r.$$

Thus summing over i we obtain the fourth line of (6.5). □

Lemma 27. For $r \geq 0$,

(6.6)

$$\begin{aligned} g_{2r-1}(2n, 2m, 2j - 1) &= q^2 g_{2r-1}(2n, 2m - 1, 2j) \\ &= q^{3m-j+(4r+2)n^2-(8r+2)n+3r+1} \begin{bmatrix} m - (2r-1)n - j + r - 1 \\ 2n - 2 \end{bmatrix}_2, \end{aligned}$$

(6.7)

$$\begin{aligned} g_{2r-1}(2n - 1, 2m, 2j) &= q g_{2r-1}(2n - 1, 2m - 1, 2j - 1) \\ &= q^{3m-j+(4r+2)n^2-(12r+4)n+8r+2} \begin{bmatrix} m - (2r-1)n - j + 2r - 2 \\ 2n - 3 \end{bmatrix}_2, \end{aligned}$$

(6.8) $g_{2r}(2n, 2m, 2j) = q g_{2r}(2n, 2m - 1, 2j - 1)$

$$= q^{3m-j+(4r+4)n^2-(8r+6)n+3r+2} \begin{bmatrix} m - 2rn - j + r - 1 \\ 2n - 2 \end{bmatrix}_2,$$

(6.9)

$$\begin{aligned} g_{2r}(2n - 1, 2m, 2j) &= q g_{2r}(2n - 1, 2m - 1, 2j - 1) \\ &= q^{3m-j+(4r+4)n^2-(12r+10)n+8r+6} \begin{bmatrix} m - 2rn - j + 2r - 1 \\ 2n - 3 \end{bmatrix}_2. \end{aligned}$$

All instances of $g_r(n, m, j)$ apart from those listed in (6.5)-(6.9) are identically zero.

Proof. Note that (6.6)-(6.9) are asserting 8 identities between the $g_r(n, m, j)$ and a power of q times a q -binomial coefficient. We denote by $\gamma_r(n, m, j)$ that asserted expression that is to be shown equal to $g_r(n, m, j)$. For example, from (6.6) we are asserting

$$\gamma_{2r-1}(2n, 2m, 2j - 1) = q^{3m-j+(4r+2)n^2-(8r+2)n+3r+1} \begin{bmatrix} m - (2r - 1)n - j + r - 1 \\ 2n - 2 \end{bmatrix}_2.$$

It is easy to check directly that the two top lines of (6.5) hold for $\gamma_r(n, m, j)$. It is also a simple matter to verify that all of the instances of $\gamma_r(n, m, j)$ that should be identically zero are indeed that via the given recurrence.

The heart of the proof is to show that each of 8 instances for $\gamma_r(n, m, j)$ required by (6.6)-(6.9) actually fulfills the recurrence. Each one is very similar to the others, so we will do (6.6) for $\gamma_{2r-1}(2, 2m, 2j)$. First the case $n = 1$, which asserts

$$\gamma_{2r-1}(2, 2m, 2j - 1) = q^{3m-j-r+1}.$$

This is true because if there are just two parts where the larger lexicographically is $(2m)_{2j-1}$ and the smaller is some M_M with the requirement that

$$(((2m)_{2j-1} - M_M)) = 2r - 1,$$

then $2m - (2j - 1) - M - M = 2r - 1$. So $M = m - j - r + 1$, and $2m + M = 3m - j - r + 1$, as required.

Next we must treat the recurrence step. We evaluate the last line in (6.5) for the γ 's,

$$\begin{aligned}
 & q^{2m} \sum_{i=1}^{2m} \gamma_{2r-1}(2n-1, 2m-(2j-1)-i-(2r-1), i) \\
 &= q^{2m} \left(\sum_{i=1}^m \gamma_{2r-1}(2n-1, 2m-2j-2i-2r+2, 2i) \right. \\
 & \quad \left. + \sum_{i=1}^m \gamma_{2r-1}(2n-1, (2m-2j-2i-2r+4)-1, 2i-1) \right) \\
 &= q^{2m} \sum_{i=1}^m \left(q^{3(m-j-i-r+1)-i+(4r+2)n^2-(12r+4)n+8r+2} \right. \\
 & \quad \times \left[\begin{matrix} (m-j-i-r+1)-(2r-1)n-i+2r-2 \\ 2n-3 \end{matrix} \right]_2 \\
 & \quad \left. + q^{3(m-j-i-r+2)-i+(4r+2)n^2-(12r+4)n+8r+1} \right. \\
 & \quad \left. \times \left[\begin{matrix} (m-j-i-r+2)-(2r-1)n-i+2r-2 \\ 2n-3 \end{matrix} \right]_2 \right) \\
 &= q^{3m-3j+5r+(4r+2)n^2-(12r+4)n+5} \sum_{i=1}^m \left(q^{2m-4i} \left[\begin{matrix} m-j+r-1-(2r-1)n-2i \\ 2n-3 \end{matrix} \right]_2 \right. \\
 & \quad \left. + q^{2m-4i+2} \left[\begin{matrix} m-j+r-1-(2r-1)n-(2i-1) \\ 2n-3 \end{matrix} \right]_2 \right) \\
 &= q^{3m-3j+5r+(4r+2)n^2-(12r+4)n+5} \sum_{i=1}^{2m} q^{2m-2i} \left[\begin{matrix} m-i-j+r-1-(2r-1)n \\ 2n-3 \end{matrix} \right]_2 \\
 &= q^{3m-3j+5r+(4r+2)n^2-(12r+4)n+8r+5} \times q^{2(j-r+1+(2r-1)n+2n-3)} \\
 & \quad \times \sum_{i=1}^{2m} q^{2(m-i-j+r-1-(2r-1)n-2n+3)} \left[\begin{matrix} m-i-j+r-1-(2r-1)n \\ 2n-3 \end{matrix} \right]_2 \\
 &= q^{3m-j+(4r+2)n^2-(8r+2)n+3r+1} \times \left[\begin{matrix} m-1-j+r-(2r-1)n \\ 2n-2 \end{matrix} \right]_2 \\
 & \text{(by [10, p. 37, eq. (3.3.9)]} \\
 &= \gamma_{2r-1}(2n, 2m, 2j-1).
 \end{aligned}$$

The other seven recurrences and initial conditions are proved in exactly this way. □

We are now in a position to prove Theorem 22.

Proof. By the definition of $g_r(n, m, j)$ we see that $\beta_r(n, q) = \sum_{m \geq 1} \sum_{j=1}^m g_r(n, m, j)$. There are four cases to treat: n even or odd and r even or odd. The cases are entirely similar, so we consider only r odd and n odd.

$$\begin{aligned}
 \beta_{2r-1}(2n-1, q) &= \sum_{m \geq 1} \sum_{j=1}^m g_{2r-1}(2n-1, m, j) \\
 &= \sum_{m \geq 1} \sum_{j=1}^m (g_{2r-1}(2n-1, 2m, 2j) + g_{2r-1}(2n-1, 2m-1, 2j-1)) \\
 &= (1+q) \sum_{m \geq 1} \sum_{j=1}^m q^{3m-j+(4r+2)n^2-(12r+4)n+8r+1} \\
 &\quad \times \left[\begin{matrix} m - (2r-1)n - j + 2r - 2 \\ 2n - 3 \end{matrix} \right]_2 \\
 &= (1+q) \sum_{m \geq 0} \sum_{j \geq 1} q^{3(m+(2r-1)n+j-2r+2+2n-3)} \\
 &\quad \times q^{-j+(4r+2)n^2-(12r+4)n+8r+1} \left[\begin{matrix} m + 2n - 3 \\ 2n - 3 \end{matrix} \right]_2 \\
 &= (1+q) \frac{q^2}{1-q^2} \frac{1}{(q^3; q^2)_{2n-2}} \\
 &\quad \times q^{(4r+2)n^2-(6r+1)n+2r-2} \text{ by [10, p. 36, eq. (3.3.8)]} \\
 &= \frac{q^{(2n+1)^2+(2r-1)\binom{2n-1}{2}}}{(q; q^2)_{2n-1}},
 \end{aligned}$$

as desired. The other three cases, as noted previously, are perfectly analogous to this case. □

Corollary 28. *For $r \geq -1$, the generating function for partitions with n copies of n in which the weighted difference between parts is at least r is given by*

$$\sum_{m \geq 0} \frac{q^{m^2+r\binom{m}{2}}}{(q; q)_m (q; q^2)_m}.$$

Proof. By Lemma 24 and Theorem 22, we see that the generating function for all partitions with n copies of n and having exactly m parts is given by

$$\frac{\beta_r(m, q)}{(q; q)_m} = \frac{q^{m^2+r\binom{m}{2}}}{(q; q)_m (q; q^2)_m},$$

and summing over all m , we obtain the result. □

Proof of Theorem 20. This follows directly from Corollary 28 with $r = 1$ and the identity (6.2). □

Proof of Theorem 21. This follows directly from Corollary 28 with $r = 0$ and the identity (6.3). □

Proof of Theorem 22. This follows directly from Corollary 28 with $r = -1$ and the identity (6.4). □

In addition, we can now interpret a couple of Ramanujan’s mock theta functions with partitions with n copies of n .

Theorem 29. *The tenth order mock theta function [12, p. 149, eq. (8.1.2)]*

$$\psi_{10}(q) := \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(q; q^2)_n}$$

is the generating function for partitions with n copies of n where the weighted difference between parts is -1 , and the smallest part is of the form j_j

Proof. We note by Theorem 21 that

$$\beta_{-1}(m, q) = \frac{q^{\binom{m+1}{2}}}{(q; q^2)_m}$$

and $\beta_{-1}(m, q)$ is the generating function for partitions with n copies of n where the weighted difference between parts is -1 and the smallest part is of the form j_j . Summing over all m , we obtain the result. □

Theorem 30. *The third order mock theta function [12, p. 5, eq. (2.1.3)]*

$$\psi_3(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}$$

is the generating function for partitions with n copies of n where the weighted difference between parts is 0 and the smallest part is of the form j_j .

Proof. The argument here is exactly that of the proof of Theorem 29 with the only change being that

$$\beta_0(m; q) = \frac{q^{m^2}}{(q; q^2)_m}.$$

□

Corollary 31. *Let $M_1(m)$ denote the number of ordinary partitions of m in which the largest part is unique and every other part occurs exactly twice. Let $M_2(m)$ denote the number of partitions of m with n copies of n where the weighted difference between successive parts is 0 and the smallest part is of the form j_j . Then*

$$M_1(m) = M_2(m).$$

Remark 32. We shall show that $\psi_3(q)$ is the generating function for both $M_1(m)$ and $M_2(m)$. As an example, consider $m = 9$. $M_1(9) = 4$, the partitions in question being $9, 7 + 1 + 1, 5 + 2 + 2, 3 + 2 + 2 + 1 + 1$. $M_2(9) = 4$, the partitions in question being $9_9, 8_6 + 1_1, 7_3 + 2_2, 5_1 + 3_1 + 1_1$.

Proof. N. Fine [15, p 57] has observed that $\psi_3(q)$ is the generating function for partitions into odd parts without gaps. The conjugates of these partitions are the partitions enumerated by $M_1(m)$.

The result now follows from Corollary 31 which shows that $\psi_3(q)$ is also the generating function for $M_2(m)$. □

7. OVERPARTITIONS

Overpartitions were introduced by Corteel and Lovejoy [14] as the natural combinatorial object counted by the coefficients of

$$\prod_{n=1}^{\infty} \frac{1 + q^n}{1 - q^n}.$$

Namely these are the partitions of n wherein each part size can have (or not) one summand overlined. Thus the 8 overpartitions of 3 are $3, \bar{3}, 2 + 1, \bar{2} + 1, 2 + \bar{1}, \bar{2} + \bar{1}, 1 + 1 + 1$, and $1 + 1 + \bar{1}$.

Among the most appealing theorems on overpartitions is Lovejoy’s extension of Schur’s theorem to overpartitions [22].

Theorem 33. *The number of overpartitions of n in which no part is divisible by 3 equals the number of overpartitions of n wherein adjacent parts differ by at least 3 if the smaller is overlined or divisible by 3 and by at least 6 if the smaller is overlined and divisible by 3.*

Now the generating function for overpartitions in which no part is divisible by 3 is:

$$\prod_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{1 + q^n}{1 - q^n},$$

and this function appears in the sixth identity in L. J. Slater’s list [25, p. 152, eq. (6) corrected].

$$(7.1) \quad \prod_{\substack{n=1 \\ 3 \nmid n}}^{\infty} \frac{1 + q^n}{1 - q^n} = \sum_{n \geq 0} \frac{(-1; q)_n q^{n^2}}{(q; q)_n (q; q^2)_n}.$$

It is completely unclear how exactly the right-hand side of (7.1) fits in with Lovejoy’s theorem. It turns out that (7.1) naturally fits into overpartitions with n copies of n . Here the same principle as before applies where now only one instance of n_j ($1 \leq j \leq n$) may be overlined in any partition. The generating function is:

$$\prod_{n=1}^{\infty} \frac{(1 + q^n)^n}{(1 - q^n)^n} = 1 + 2q + 6q^2 + 16q^3 + 38q^4 + \dots$$

Thus the 16 overpartitions of 3 using n copies of n are $3_3, \bar{3}_3, 3_2, \bar{3}_2, 3_1, \bar{3}_1, 2_2 + 1_1, \bar{2}_2 + 1_1, 2_2 + \bar{1}_1, \bar{2}_2 + \bar{1}_1, 1_1 + 1_1 + 1_1 + 1_1, 1_1 + 1_1 + 1_1 + \bar{1}_1$.

Theorem 34. *Let $J(m)$ denote the number of overpartitions of m whose parts are not divisible by 3. Let $L(m)$ denote the number of overpartitions with n copies of n in which (i) the weighted differences between adjacent parts is ≥ 0 (ii). If the weighted difference of two or more successive parts is zero, then only the smallest part in the sequence (ignoring the subscript) can be overlined. Then for $m \geq 1$,*

$$J(m) = L(m).$$

As an example, when $m = 4$, $J(4) = 10$, and the overpartitions in question are $4, \bar{4}, 2 + 2, 2 + \bar{2}, 2 + 1, 2 + \bar{1}, \bar{2} + 1, \bar{2} + \bar{1}, 1 + 1 + 1 + 1, 1 + 1 + 1 + \bar{1}$. $L(4) = 10$, and the overpartitions with n copies of n are $4_4, \bar{4}_4, 4_3, \bar{4}_3, 4_2, \bar{4}_2, 4_1, \bar{4}_1, 3_1 + 1_1, 3_1 + \bar{1}_1$.

Proof of Theorem 34. This result relies heavily on the discoveries chronicled in section 6. First let us consider

$$(7.2) \quad \frac{(-1; q)_m}{(q; q)_m} = \frac{(1+q)(1+q^2)\cdots(1+q^{m-1})(q^m+q^n)}{(1-q)(1-q^2)\cdots(1-q^{n-1})(1-q^n)} + \frac{(1+1)(1+q)(1+q^2)\cdots(1+q^{m-1})}{(1-q)(1-q^2)\cdots(1-q^{m-1})}.$$

The first term on the right in (7.2) generates overpartitions with exactly m positive parts. The second term generates overpartitions with exactly m nonnegative parts (including exactly one zero).

This dissection of (7.2) then leads directly to the desired conclusion. Namely

$$\sum_{m \geq 0} \frac{(-1; q)_m q^{m^2}}{(q; q)_m (q; q^2)_m} = 1 + \sum_{m \geq 1} \left(\frac{(-q; q)_{m-1} (q^m + q^m)}{(q; q)_m} + \frac{(-1; q)_m}{(q; q)_{m-1}} \right) \frac{q^{m^2}}{(q; q^2)_m}.$$

Now we recall from Theorem 22 with $r = 0$ that

$$\frac{q^{m^2}}{(q; q^2)_m}$$

is the generating function for partitions with n copies of n having m parts with smallest part of the form j_j .

Now instead of attaching an ordinary partition to this basic partition (as is done in Corollary 28), we attach overpartitions as generated in (7.2).

First note that these are overpartitions that are being attached. Consequently this means that if there are several identical parts being attached the result will be a sequence of parts with successive differences still 0 and with only the smallest part in the chain possibly being overlined.

Second, the first term (as noted after (7.2)) produces those partitions where the smallest summand is not of the form j_j , and the second term accounts for those partitions where the smallest summand is of the form j_j . □

8. PARTITIONS WITH n COPIES OF n AND EVEN SUBSCRIPTS

It may at first appear rather artificial to restrict ourselves to only those n_j with j even ($1 < j \leq n$). However, this restriction leads to a new interpretation of one of the more striking results in L. J. Slater's compendium [25, p. 161, eq. (86)]

$$(8.1) \quad \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q; q)_{2n}} = \prod_{\substack{n=1 \\ n \equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}}}^{\infty} \frac{1}{1 - q^n}.$$

The right hand side of (8.1) is clearly the generating function for $G(n)$, the number of partitions of n into parts $\equiv \pm 2, \pm 3, \pm 4, \pm 5 \pmod{16}$. On the other hand, noting that

$$2n^2 = 1 + 1 + 3 + 3 + \cdots + (2n - 1) + (2n - 1),$$

we may use the argument used to produce (1.3) to see that the left hand side produces partitions

$$b_1 + b_2 + b_3 + b_4 + \cdots + b_{2j-1} + b_{2j}$$

into nondecreasing parts with $b_3 - b_2 \geq 2, b_5 - b_4 \geq 2, b_7 - b_6 \geq 2, \dots$ (cf. [19]).

The reason that we renew our study of (8.1) is that it fits perfectly into the theme of the last two sections.

Theorem 35. *Let $H(n)$ denote the number of partitions of n using n copies of n but (i) restricted to even subscripts, (ii) the weighted differences between successive parts is ≥ 0 , and (iii) excluding adjacent pairs n_i, m_j with n and m both odd and $((n_i - m_j)) = 0$. Then for $n \geq 1$,*

$$G(n) = H(n).$$

As an example, $G(10) = 7$ where the relevant partitions are $5 + 5, 5 + 3 + 2, 4 + 4 + 2, 4 + 3 + 3, 4 + 2 + 2 + 2, 3 + 3 + 2 + 2, 2 + 2 + 2 + 2 + 2$. $H(10) = 7$ where the relevant partitions are $10_{10}, 10_8, 10_6, 10_4, 10_2, 8_2 + 2_2, 8_4 + 2_2$. Note that $7_2 + 3_2$ is disallowed because $((7_2 - 3_2)) = 0$ with both 7 and 3 odd.

Proof of Theorem 35. We rewrite the left side of (8.1) as

$$\sum_{m \geq 0} \frac{q^{2m^2}}{(q; q)_{2m}} = \sum_{m \geq 0} \frac{(-q; q^2)_m}{(q^2; q^2)_m} \frac{q^{2m^2}}{(q^2; q^4)_m}.$$

Now by conjugation of the 2 modular representations of partitions without repeated odd parts, we see that

$$(8.2) \quad \frac{(-q; q^2)_m}{(q^2; q^2)_m}$$

is the generating function for partitions with exactly n nonnegative parts with no repeated odd parts. On the other hand,

$$(8.3) \quad \frac{q^{2m^2}}{(q^2; q^4)_m}$$

is merely the dilation ($q \rightarrow q^2$) of $\beta_0(m, q)$ from Theorem 22. Thus $\beta_0(m, q^2)$ is the generating function for partitions into n copies of n where summands are of the form $(2r)_{2s}$ ($1 \leq s \leq r$).

Now we proceed here exactly as before in Lemma 27 and Corollary 28. The attachment of the ordinary partitions generated by (8.2) to the partitions with n copies of n as generated by (8.3) yields partitions into n copies of n with m parts subject to the requirement that the weighted difference between parts is nonnegative, and that two successive parts r_i and s_j cannot have $((r_i - s_j)) = 0$ with both r and s odd. \square

9. CONCLUSION

There are several points to be made in summary.

First, we have chosen a sampling of possible applications of this method to make clear its widespread utility. Thus there are many instances of Theorem 1 that have yet to be considered. We have treated only a few of these.

Second, there are other examples of SIP classes. Indeed, the inspiration for this paper arose from [13]. The SIP class in [13] is the set of integer partitions in which the parts are distinct, the smallest is even, and there are no consecutive odd parts. This SIP class is not an instance of Theorem 1. Consequently, it is surely valuable to explore SIPs not included in Theorem 1.

Finally, there are other theorems that cry out for an analogous theory. For example, the mod 7 instance of the generalization of the Rogers–Ramanujan identities given in [7] may be stated:

$$(9.1) \quad \sum_{N \geq 0} \frac{q^{N^2}}{(q; q)_N} \sum_{m=0}^N \left[\begin{matrix} N \\ m \end{matrix} \right]_1 q^{m^2} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 3 \pmod{7}}}^{\infty} \frac{1}{1 - q^n}.$$

There are several interpretations of the left hand side of (6.1) [8], [9], [18]; however none seems to lend itself to an SIP-style interpretation. If such an interpretation could be found, this would open many further possibilities.

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