

## ON REALIZATIONS OF THE SUBALGEBRA $\mathcal{A}^{\mathbb{R}}(1)$ OF THE $\mathbb{R}$ -MOTIVIC STEENROD ALGEBRA

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*The first author would like to dedicate this work to his baba Jayanta Bhattacharya whose life was lost to COVID-19 on May 2, 2021. He always said, whatever you do do it in depth, and leave no stone unturned. You will be deeply missed!*

ABSTRACT. In this paper, we show that the finite subalgebra  $\mathcal{A}^{\mathbb{R}}(1)$ , generated by  $Sq^1$  and  $Sq^2$ , of the  $\mathbb{R}$ -motivic Steenrod algebra  $\mathcal{A}^{\mathbb{R}}$  can be given 128 different  $\mathcal{A}^{\mathbb{R}}$ -module structures. We also show that all of these  $\mathcal{A}$ -modules can be realized as the cohomology of a 2-local finite  $\mathbb{R}$ -motivic spectrum. The realization results are obtained using an  $\mathbb{R}$ -motivic analogue of the Toda realization theorem. We notice that each realization of  $\mathcal{A}^{\mathbb{R}}(1)$  can be expressed as a cofiber of an  $\mathbb{R}$ -motivic  $v_1$ -self-map. The  $C_2$ -equivariant analogue of the above results then follows because of the Betti realization functor. We identify a relationship between the  $RO(C_2)$ -graded Steenrod operations on a  $C_2$ -equivariant space and the classical Steenrod operations on both its underlying space and its fixed-points. This technique is then used to identify the geometric fixed-point spectra of the  $C_2$ -equivariant realizations of  $\mathcal{A}^{C_2}(1)$ . We find another application of the  $\mathbb{R}$ -motivic Toda realization theorem: we produce an  $\mathbb{R}$ -motivic, and consequently a  $C_2$ -equivariant, analogue of the Bhattacharya-Egger spectrum  $\mathcal{Z}$ , which could be of independent interest.

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### 1. INTRODUCTION

This paper is a continuation of the work that began in [BGL], where we studied periodic self-maps of a certain finite  $\mathbb{R}$ -motivic spectrum  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$ . There, we proved

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that  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$  supports a 1-periodic  $v_{(1,\text{nil})}$ -self-map

$$(1.1) \quad v : \Sigma^{2,1}\mathcal{Y}_{(h,1)}^{\mathbb{R}} \longrightarrow \mathcal{Y}_{(h,1)}^{\mathbb{R}},$$

whose cofiber realizes the subalgebra  $\mathcal{A}^{\mathbb{R}}(1)$  of the  $\mathbb{R}$ -motivic Steenrod algebra  $\mathcal{A}^{\mathbb{R}}$  generated by  $\text{Sq}^1$  and  $\text{Sq}^2$ . Recall from [BGL, Definition 1.7] that a  $v_{(1,\text{nil})}$ -self-map is a  $v_1$ -self-map which is nilpotent on the geometric fixed-points.

The spectrum  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$  is an  $\mathbb{R}$ -motivic lift of the classical spectrum

$$\mathcal{Y} := \Sigma^{-3}\mathbb{C}\mathbb{P}^2 \wedge \mathbb{R}\mathbb{P}^2.$$

From the chromatic point of view, the spectrum  $\mathcal{Y}$  is extremely useful because it supports a  $v_1$ -self-map of lowest possible periodicity, that is, one. Famously, Mark Mahowald used the spectrum  $\mathcal{Y}$  and the low periodicity of its  $v_1$ -self-map to prove the height 1 telescope conjecture at the prime 2 [M1, M2]. However, 1-periodic  $v_1$ -self-maps of  $\mathcal{Y}$  are not unique. In fact, up to homotopy, there are eight different  $v_1$ -self-maps supported by  $\mathcal{Y}$ , all of whose cofibers are realizations of  $\mathcal{A}(1)$  (see [DM]). Up to weak equivalence, there are four different finite spectra realizing  $\mathcal{A}(1)$ , and all of them can be obtained as the cofiber of some  $v_1$ -self-map of  $\mathcal{Y}$ . These four different realizations can be distinguished by their  $\mathcal{A}$ -module structures. Therefore, it is natural to ask if all of the  $v_1$ -self-maps of  $\mathcal{Y}$  can be lifted to  $\mathbb{R}$ -motivic analogues, and whether all of the  $\mathbb{R}$ -motivic realizations of  $\mathcal{A}^{\mathbb{R}}(1)$  can be obtained as the cofiber of such a lift.

The answer to the above question is complicated by the fact that there are multiple  $\mathbb{R}$ -motivic lifts of the spectrum  $\mathcal{Y}$  (see [BGL]). Even if we insist on those lifts which can potentially realize  $\mathcal{A}^{\mathbb{R}}(1)$  as a cofiber of a periodic self-map, we are left with two choices:  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$  and  $\mathcal{Y}_{(2,1)}^{\mathbb{R}}$ . We state our first result towards answering these questions after establishing some notations. Further, we shall see that some realizations of  $\mathcal{A}^{\mathbb{R}}(1)$  must be given as the cofiber of a map between  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$  and  $\mathcal{Y}_{(2,1)}^{\mathbb{R}}$  rather than as the cofiber of a self-map of either.

Before describing the results of this article, we present some notation that will be used throughout.

*Notation 1.2.* Throughout this paper, we use the following notations:

- $\mathbf{Sp}^{\mathbb{R}}$  – the  $\infty$ -category of  $\mathbb{R}$ -motivic spectra.
- $\mathbf{Sp}^{C_2}$  – the  $\infty$ -category of genuine  $C_2$ -equivariant spectra.
- $\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$  – the  $\mathbb{R}$ -motivic Eilenberg-Mac Lane spectrum with  $\mathbb{F}_2$ -coefficients.
- $\mathbf{H}\underline{\mathbb{F}}_2$  – the  $C_2$ -equivariant Eilenberg-Mac Lane spectrum at the constant Mackey functor  $\underline{\mathbb{F}}_2$ .
- $\mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  – the category of cellular  $\mathbf{H}_{\mathbb{R}}\mathbb{F}_2$ -complete  $\mathbb{R}$ -motivic spectra with finitely many cells.
- We denote the 1-dimensional trivial  $\mathbb{R}$ -representation of  $C_2$  by  $\epsilon$ , the sign representation by  $\sigma$ , and the regular representation by  $\rho$ .
- $\mathbf{H}_{\mathbb{R}}^{n,m}(\mathbf{E}) := [\mathbf{E}, \Sigma^{n,m}\mathbf{H}_{\mathbb{R}}\mathbb{F}_2]$  – the  $\mathbb{R}$ -motivic cohomology of  $\mathbf{E} \in \mathbf{Sp}^{\mathbb{R}}$  with constant sheaf  $\mathbb{F}_2$ , where  $n$  is the topological degree and  $m$  is the motivic weight.
- $\mathbf{Sp}_{2,\text{fin}}^{C_2}$  – the category of cellular  $\mathbf{H}\underline{\mathbb{F}}_2$ -complete  $C_2$ -equivariant spectra with finitely many cells.
- $\mathbf{H}_{C_2}^*(\mathbf{E}) = [\mathbf{E}, \mathbf{H}\underline{\mathbb{F}}_2]_{-}^{C_2}$  – the  $\text{RO}(C_2)$ -graded cohomology of  $\mathbf{E} \in \mathbf{Sp}^{C_2}$  with coefficients in the constant Mackey functor. We will often use motivic

bigrading for  $H_{C_2}^*(E)$  under the identification

$$(n, m) \rightsquigarrow (n - m)\epsilon + m\sigma.$$

- $M_2^{\mathbb{R}} := \pi_{*,*} H_{\mathbb{R}} F_2$ . By a calculation of Voevodsky [V]

$$M_2^{\mathbb{R}} \cong \mathbb{F}_2[\tau, \rho]$$

with  $|\tau| = (0, -1)$  and  $|\rho| = (-1, -1)$ .

- $M_2^{C_2} := \pi_* H_{\mathbb{F}_2}$ . This computation can be found in [C, Appendix] or [HK, Proposition 6.2] and is given by

$$M_2^{C_2} \cong \mathbb{F}_2[u_\sigma, a_\sigma] \oplus \Theta\{u_\sigma^{-i} a_\sigma^{-j} : i, j \geq 0\},$$

where  $|u_\sigma| = (0, -1)$ ,  $|a_\sigma| = (-1, -1)$  and  $|\Theta| = (0, 2)$ .

- We follow [DI, BI, BGL] in grading  $\text{Ext}_{\mathcal{A}^{\mathbb{R}}}^s$  as  $\text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}$ , where  $s$  is the stem,  $f$  is the Adams filtration, and  $w$  is the motivic weight.

Our first result concerns realizations of  $\mathcal{A}^{\mathbb{R}}(1)$ .

**Theorem 1.3.** *There exists 128 different  $\mathcal{A}^{\mathbb{R}}$ -modules whose underlying  $\mathcal{A}^{\mathbb{R}}(1)$ -module structures are free on one generator, all of which can be realized as  $H_{\mathbb{R}}^{*,*}(X)$  for some  $X \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$ .*

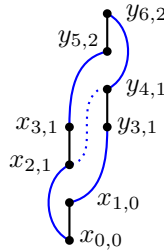


FIGURE 1.4. We depict a singly-generated free  $\mathcal{A}^{\mathbb{R}}(1)$ -module, where each  $\bullet$  represents a  $M_2^{\mathbb{R}}$ -generator. The black and blue lines represent the action of motivic  $Sq^1$  and  $Sq^2$ , respectively. A dotted line represents that the action hits the  $\tau$ -multiple of the given  $M_2^{\mathbb{R}}$ -generator.

*Notation 1.5.* For the rest of the paper, we fix an  $M_2^{\mathbb{R}}$ -basis

$$\{x_{0,0}, x_{1,0}, x_{2,1}, x_{3,1}, y_{3,1}, y_{4,1}, y_{5,2}, y_{6,2}\}$$

of  $\mathcal{A}^{\mathbb{R}}(1)$  as in Figure 1.4, so that

- $Sq^1(x_{0,0}) = x_{1,0}$
- $Sq^1(x_{2,1}) = x_{3,1}$
- $Sq^1(y_{3,1}) = y_{4,1}$
- $Sq^1(y_{5,2}) = y_{6,2}$
- $Sq^2(x_{0,0}) = x_{2,1}$
- $Sq^2(x_{1,0}) = y_{3,1}$
- $Sq^2(x_{2,1}) = \tau y_{4,1}$
- $Sq^2(x_{3,1}) = y_{5,2}$
- $Sq^2(y_{4,1}) = y_{6,2}$ .

We now record all 128  $\mathcal{A}^{\mathbb{R}}$ -modules of Theorem 1.3 using the basis above.

**Theorem 1.6.** For every vector  $(\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V} = \mathbb{F}_2^7$  and

$$j_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25} + \beta_{26}),$$

there exists a unique isomorphism class of  $\mathcal{A}^{\mathbb{R}}$ -module structures on  $\mathcal{A}^{\mathbb{R}}(1)$  determined by the formulas

- (i)  $\text{Sq}^4(x_{0,0}) = \beta_{03}(\rho \cdot y_{3,1}) + (1 + \beta_{03} + \beta_{14})(\tau \cdot y_{4,1}) + \alpha_{03}(\rho \cdot x_{3,1})$
- (ii)  $\text{Sq}^4(x_{1,0}) = y_{5,2} + \beta_{14}(\rho \cdot y_{4,1})$
- (iii)  $\text{Sq}^4(x_{2,1}) = \beta_{26}(\tau \cdot y_{6,2}) + \beta_{25}(\rho \cdot y_{5,2}) + j_{24}(\rho^2 \cdot y_{4,1})$
- (iv)  $\text{Sq}^4(x_{3,1}) = (\beta_{25} + \beta_{26})(\rho \cdot y_{6,2})$
- (v)  $\text{Sq}^4(y_{3,1}) = \gamma_{36}(\rho \cdot y_{6,2})$
- (vi)  $\text{Sq}^8(x_{0,0}) = \beta_{06}(\rho^2 \cdot y_{6,2})$ .

Further, any  $\mathcal{A}^{\mathbb{R}}$ -module whose underlying  $\mathcal{A}^{\mathbb{R}}(1)$ -module is free on one generator is isomorphic to one listed above.

*Notation 1.7.* For any vector  $\bar{v} \in \mathcal{V}$ , we denote the corresponding  $\mathcal{A}^{\mathbb{R}}$ -module in Theorem 1.6 by  $\mathcal{A}_{\bar{v}}^{\mathbb{R}}(1)$ . By  $\mathcal{A}_1^{\mathbb{R}}[\bar{v}]$ , we denote an object of  $\mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$ , whose cohomology is isomorphic to  $\mathcal{A}_{\bar{v}}^{\mathbb{R}}(1)$  as an  $\mathcal{A}^{\mathbb{R}}$ -module. We let

$$\mathcal{A}_1^{\mathbb{R}} := \{\mathcal{A}_1^{\mathbb{R}}[\bar{v}] : \bar{v} \in \mathcal{V}\} / (\text{weak equivalence})$$

denote the set of equivalence classes of finite  $\mathbb{R}$ -motivic spectra whose cohomology are free of rank 1 over  $\mathcal{A}^{\mathbb{R}}(1)$ .

Let  $\mathcal{B}_h^{\mathbb{R}}(1)$  and  $\mathcal{B}_2^{\mathbb{R}}(1)$  denote the  $\mathcal{A}^{\mathbb{R}}$ -modules  $H_{\mathbb{R}}^{*,*}(\mathcal{Y}_{(h,1)}^{\mathbb{R}})$  and  $H_{\mathbb{R}}^{*,*}(\mathcal{Y}_{(2,1)}^{\mathbb{R}})$ , respectively. As shown in [BGL, Lemma 4.4], these differ in that the bottom cell of  $\mathcal{Y}_{(2,1)}^{\mathbb{R}}$  supports a  $\text{Sq}^4$ , whereas the bottom cell of  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$  does not. In [BGL], we used a method due to Smith ([R1, Appendix C]) to produce the  $\mathcal{A}^{\mathbb{R}}$ -module  $\mathcal{A}_0^{\mathbb{R}}(1)$ . Then we observed that  $\mathcal{A}_0^{\mathbb{R}}(1)$  fits into a short exact sequence

$$\Sigma^{3,1}\mathcal{B}_h^{\mathbb{R}}(1) \hookrightarrow \mathcal{A}_0^{\mathbb{R}}(1) \twoheadrightarrow \mathcal{B}_h^{\mathbb{R}}(1)$$

that can be realized as a cofiber sequence of finite spectra. The connecting map of this cofiber sequence is the map (1.1). In this paper, we extend the above result of [BGL] to prove the following.

**Theorem 1.8.** Given  $\bar{v} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$ , define

$$\epsilon = \begin{cases} h & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 0 \\ 2 & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 1, \end{cases} \quad \text{and} \quad \delta = \begin{cases} h & \text{if } \alpha_{03} + \beta_{03} = 0 \\ 2 & \text{if } \alpha_{03} + \beta_{03} = 1. \end{cases}$$

Then there exists a short exact sequence

$$(1.9) \quad \Sigma^{3,1}\mathcal{B}_{\epsilon}^{\mathbb{R}}(1) \hookrightarrow \mathcal{A}_{\bar{v}}^{\mathbb{R}}(1) \twoheadrightarrow \mathcal{B}_{\delta}^{\mathbb{R}}(1)$$

of  $\mathcal{A}^{\mathbb{R}}$ -modules. Moreover, this exact sequence can be realized as the cohomology of a cofiber sequence

$$(1.10) \quad \mathcal{Y}_{(\delta,1)}^{\mathbb{R}} \longrightarrow \mathcal{A}_1^{\mathbb{R}}[\bar{v}] \longrightarrow \Sigma^{3,1}\mathcal{Y}_{(\epsilon,1)}^{\mathbb{R}}$$

in the category  $\mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$ .

The map of spectra that underlies the connecting map

$$(1.11) \quad v: \Sigma^{2,1} \mathcal{Y}_{(\epsilon,1)}^{\mathbb{R}} \longrightarrow \mathcal{Y}_{(\delta,1)}^{\mathbb{R}}$$

of (1.10) is a  $v_1$ -self-map of  $\mathcal{Y}$  of periodicity 1.

The algebraic part of Theorem 1.8 is a straightforward consequence of Theorem 1.6 once we identify the  $\mathcal{A}^{\mathbb{R}}$ -modules  $\mathcal{B}_h^{\mathbb{R}}(1)$  and  $\mathcal{B}_2^{\mathbb{R}}(1)$  (we refer to [BGL, Figure 7] for a complete description). However, the topological assertions in Theorem 1.8, as well as in Theorem 1.3, require a technical tool, which we refer to as the  $\mathbb{R}$ -motivic Toda realization theorem.

**Theorem 1.12** ( $\mathbb{R}$ -motivic Toda realization theorem). *Let  $M$  be an  $\mathcal{A}^{\mathbb{R}}$ -module whose underlying  $\mathbb{M}_2^{\mathbb{R}}$ -module is free and finite. There exists  $X \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  such that  $H_{\mathbb{R}}^{*,*}(X) \cong M$  as  $\mathcal{A}^{\mathbb{R}}$ -modules if*

$$(1.13) \quad \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{-2,f,0}(M, M) = 0$$

for all  $f \geq 3$ .

In this paper, we also prove various weaker versions of the  $\mathbb{R}$ -motivic Toda realization theorem (see Theorem 2.4, Theorem 2.6 and Theorem 2.11), which are perhaps more convenient for application purposes.

A realization theorem is often accompanied by a uniqueness theorem, as is the case with Toda's classical result (see [BE, Proposition 5.1]). The  $\mathbb{R}$ -motivic analogue can be stated as follows:

**Theorem 1.14** ( $\mathbb{R}$ -motivic unique realization theorem). *Let  $X \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  such that*

$$\text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{-1,f,0}(H_{\mathbb{R}}^{*,*}(X), H_{\mathbb{R}}^{*,*}(X)) = 0$$

for any  $f \geq 2$ . Then any spectrum  $X' \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  for which there exists an  $\mathcal{A}^{\mathbb{R}}$ -module isomorphism  $H_{\mathbb{R}}^{*,*}(X') \cong H_{\mathbb{R}}^{*,*}(X)$ , is weakly equivalent to  $X$ . In other words, the  $\mathcal{A}^{\mathbb{R}}$ -module  $H_{\mathbb{R}}^{*,*}(X)$  is uniquely realized in  $\mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  up to a weak equivalence.

*Proof.* The result follows from the fact that the nonzero element

$$\iota \in \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{0,0,0}(H_{\mathbb{R}}^{*,*}(X'), H_{\mathbb{R}}^{*,*}(X))$$

representing the isomorphism  $H_{\mathbb{R}}^{*,*}(X') \cong H_{\mathbb{R}}^{*,*}(X)$  survives the Adams spectral sequence converging to  $[X, X']$ .  $\square$

The uniqueness theorem applies to the  $\mathcal{A}^{\mathbb{R}}$ -modules  $\mathcal{B}_h^{\mathbb{R}}(1)$  and  $\mathcal{B}_2^{\mathbb{R}}(1)$  (see Lemma 4.4). However, it does not apply to  $\mathcal{A}_1^{\mathbb{R}}[\bar{v}]$  for any  $\bar{v} \in \mathcal{V}$ . Potentially, there can be multiple different finite spectra realizing  $\mathcal{A}_{\bar{v}}^{\mathbb{R}}(1)$  up to a weak equivalence (see Remark 4.3), making it difficult to get a precise count of the number of 1-periodic  $v_1$ -self-maps on  $\mathcal{Y}_{(h,1)}^{\mathbb{R}}$  and  $\mathcal{Y}_{(2,1)}^{\mathbb{R}}$  from Theorem 1.8.

Upon applying the Betti realization functor

$$\beta: \mathbf{Sp}^{\mathbb{R}} \longrightarrow \mathbf{Sp}^{C_2}$$

we get various  $C_2$ -equivariant maps  $\beta(v): \Sigma^{2,1} \mathcal{Y}_{(\epsilon,1)}^{C_2} \longrightarrow \mathcal{Y}_{(\delta,1)}^{C_2}$  (where  $\epsilon, \delta \in \{2, h\}$ ) whose underlying maps are  $v_1$ -self-maps of  $\mathcal{Y}$ . We also get Corollary 1.15 of Theorem 1.3 (see Remark 4.8).

**Corollary 1.15.** *There exists 128 different  $\mathcal{A}^{C_2}$ -modules whose underlying  $\mathcal{A}^{C_2}(1)$ -module structures are free on one generator, all of which can be realized as the  $\text{RO}(C_2)$ -graded cohomology of a 2-local finite  $C_2$ -spectrum.*

*Notation 1.16.* Let  $\mathcal{A}_1^{C_2}[\bar{\nu}]$  denote the Betti realization  $\beta(\mathcal{A}_1^{\mathbb{R}}[\bar{\nu}])$ , where  $\bar{\nu} \in \mathcal{V}$ .

Let  $\mathfrak{R} : \mathbf{Sp}^{C_2} \rightarrow \mathbf{Sp}$  denote the restriction functor (restricting the  $C_2$ -action to the trivial group) and  $\Phi : \mathbf{Sp}^{C_2} \rightarrow \mathbf{Sp}$  denote the geometric fixed-point functor (see Section 3). Note that the geometric fixed-point spectra

$$\Phi(\mathcal{Y}_{(h,1)}^{C_2}) := \Phi(C^{C_2}(h) \wedge C^{C_2}(\eta_{1,1})) \simeq M_2(1) \vee \Sigma M_2(1)$$

and

$$\Phi(\mathcal{Y}_{(2,1)}^{C_2}) := \Phi(C^{C_2}(2) \wedge C^{C_2}(\eta_{1,1})) \simeq M_2(1) \wedge M_2(1)$$

are both of type 1 (see [BGL]). Further, for degree reasons

$$\Phi(\beta(v)) : \Sigma\Phi(\mathcal{Y}_{(\epsilon,1)}^{C_2}) \longrightarrow \Phi(\mathcal{Y}_{(\delta,1)}^{C_2})$$

cannot be a  $v_1$ -self-map, and hence must be nilpotent, using [HS]. Therefore, the fiber  $\Phi(\mathcal{A}_1^{C_2}[\bar{\nu}])$  is a type 1 spectrum, i.e.  $\mathcal{A}_1^{C_2}[\bar{\nu}]$  is of type (2, 1) in the sense of [BGL]. Much more can be said about  $\Phi(\mathcal{A}_1^{C_2}[\bar{\nu}])$  than just its type. In this paper, we give a complete description of the  $\mathcal{A}$ -module structure of  $H^*(\Phi(\mathcal{A}_1^{C_2}[\bar{\nu}]))$  for all  $\bar{\nu} \in \mathcal{V}$  by developing a general method that compares the  $\text{RO}(C_2)$ -graded squaring operations of a  $C_2$  spectrum with the ordinary squaring operations of its underlying spectrum as well as its geometric fixed-point spectrum (compare [BW, §3]).

Since  $\mathfrak{R}(H\underline{\mathbb{F}}_2) \simeq H\underline{\mathbb{F}}_2$  we have a natural map

$$\mathfrak{R}_* : H_{C_2}^{n,m}(\mathbb{E}) \simeq [E, \Sigma^{n,m}H\underline{\mathbb{F}}_2] \longrightarrow [\mathfrak{R}(E), \Sigma^n\mathfrak{R}(H\underline{\mathbb{F}}_2)] \simeq H^n(\mathfrak{R}(E))$$

for any  $E \in \mathbf{Sp}^{C_2}$ . We use Theorem 1.17 to identify the spectrum underlying  $\mathcal{A}_1^{C_2}[\bar{\nu}]$  (see Theorem 4.9). Here and in Theorem 1.19, we write  $\underline{\text{Sq}}^n$  for the  $C_2$ -equivariant squaring operations, as in Section 3.2.

**Theorem 1.17.** *For  $E \in \mathbf{Sp}_{2,\text{fin}}^{C_2}$  and any class  $u \in H_{C_2}^*(E)$ ,  $\mathfrak{R}_*(\underline{\text{Sq}}^n(u)) = \text{Sq}^n(\mathfrak{R}_*(u))$ .*

Using the fact that the projection map

$$\pi_{\mathbb{F}_2}^{(0)} : \Phi(H\underline{\mathbb{F}}_2) \simeq H\underline{\mathbb{F}}_2[t] \longrightarrow H\underline{\mathbb{F}}_2$$

is an  $\mathbb{E}_\infty$ -ring map, as in the discussion preceding (3.3), one defines (also see [BW, (2.7)]) the map

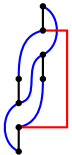
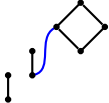
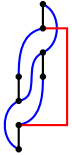
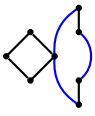
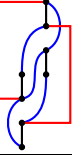
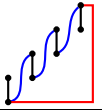
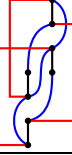
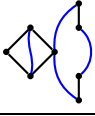
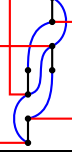
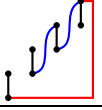
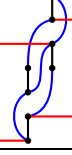
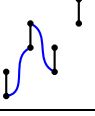
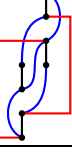
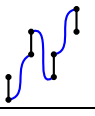
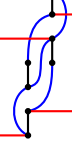
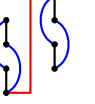
$$(1.18) \quad \hat{\Phi}_* : H_{C_2}^{n,m}(\mathbb{E}) \longrightarrow H^{n-m}(\Phi(\mathbb{E}))$$

which compares the  $\text{RO}(C_2)$ -graded cohomology of a  $C_2$ -spectrum  $E$  with the ordinary cohomology of its geometric fixed-point spectrum. We show:

**Theorem 1.19.** *For  $E \in \mathbf{Sp}_{2,\text{fin}}^{C_2}$  and any class  $u \in H_{C_2}^*(E)$ ,*

$$\hat{\Phi}_*(\underline{\text{Sq}}^{2n}(u)) = \text{Sq}^n(\hat{\Phi}_*(u)).$$

FIGURE 1.20. Some underlying and fixed  $\mathcal{A}$ -modules of  $\mathcal{A}_1^{C_2}$

$\bar{v} \in \mathcal{V}$	$H^*(\mathfrak{R}(\mathcal{A}_1^{C_2}[\bar{v}])))$	$H^*(\Phi(\mathcal{A}_1^{C_2}[\bar{v}])))$	Cofiber of
$(0, 0, 1, 0, 0, 0, 0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(h,1)} \rightarrow \mathcal{Y}_{(h,1)}$
$(1, 1, 0, 0, 0, 0, 0, 1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \rightarrow \mathcal{Y}_{(h,1)}$
$(0, 1, 0, 1, 0, 1, 0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \rightarrow \mathcal{Y}_{(2,1)}$
$(1, 0, 0, 0, 0, 0, 1, 1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(h,1)} \rightarrow \mathcal{Y}_{(2,1)}$
$(0, 0, 0, 1, 0, 1, 0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \rightarrow \mathcal{Y}_{(h,1)}$
$(1, 0, 0, 0, 0, 0, 0, 0)$			$v : \Sigma^{2,1}\mathcal{Y}_{(h,1)} \rightarrow \mathcal{Y}_{(2,1)}$
$(1, 0, 0, 0, 0, 0, 0, 1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \rightarrow \mathcal{Y}_{(2,1)}$
$(1, 1, 1, 1, 1, 0, 0, 1)$			$v : \Sigma^{2,1}\mathcal{Y}_{(2,1)} \rightarrow \mathcal{Y}_{(2,1)}$

We find Theorem 1.17 and Theorem 1.19 very handy for computational purposes. These results can be applied to understand the  $RO(C_2)$ -graded squaring

operations on the cohomology of a wide variety of  $C_2$ -spectra whose underlying and geometric fixed-point spectra are known. Alternatively, one can identify the action of the classical Steenrod algebra on the cohomology of the underlying as well as the geometric fixed-points of a  $C_2$ -spectrum from the knowledge of  $RO(C_2)$ -graded Steenrod operations. We apply Theorem 1.17 and Theorem 1.19 to identify the  $\mathcal{A}$ -module structure of the underlying and the geometric fixed-points of  $\mathcal{A}_1^{C_2}[\bar{v}]$  (see Theorem 4.9 and Theorem 4.11).

In Figure 1.20, we provide the  $\mathcal{A}$ -module structure of the underlying and the geometric fixed-points of  $\mathcal{A}_1^{C_2}[\bar{v}]$  for selected values of  $\bar{v} \in \mathcal{V}$ . We express  $Sq^1$ ,  $Sq^2$  and  $Sq^4$  using black, blue, and red lines respectively.

*Remark 1.21* (Appearance of the Joker). We note that the  $\mathcal{A}(1)$ -module



often called the Joker, is a subcomplex of the geometric fixed-point of  $\mathcal{A}_1^{\mathbb{R}}[\bar{v}]$  if and only if  $j_{24} = 1$ . Further, when  $j_{24} = 1$  then in (1.9),  $\epsilon$  and  $\delta$  cannot both equal  $h$ . This can easily be derived from Theorem 1.6 and Theorem 1.19.

*Remark 1.22.* In [BGL], the authors construct  $\mathcal{A}_1^{\mathbb{R}}[\mathbf{0}]$  as a split summand of  $\mathcal{Q}_{\mathbb{R}}^{\wedge 3}$  using a certain idempotent of  $\mathbb{Z}_{(2)}[\Sigma_3]$ . Let  $\tilde{\mathcal{Q}}_{\mathbb{R}} \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  be such that its cohomology as an  $\mathcal{A}^{\mathbb{R}}(1)$ -module is isomorphic to  $\mathcal{Q}_{\mathbb{R}}$ , but has the additional relation

$$Sq^4(a) = \rho \cdot c$$

(in the notation of [BGL, Figure 2]), as an  $\mathcal{A}^{\mathbb{R}}$ -module. If we replace  $\mathcal{Q}_{\mathbb{R}}$  by a complex  $\tilde{\mathcal{Q}}_{\mathbb{R}}$  in [BGL], we get  $\mathcal{A}_1^{\mathbb{R}}[\bar{v}]$ , where  $\bar{v} = (1, 1, 1, 1, 1, 0, 1)$  (see the last diagram in Figure 1.20).

*Remark 1.23.* The classical spectrum  $\mathcal{A}_1$  is a type 2 spectrum and supports a  $v_2^{32}$ -self-map [BEM]. It remains to be seen if this  $v_2$ -self-map can be lifted to  $\mathcal{A}_1^{\mathbb{R}}[\bar{v}]$  for various  $\bar{v} \in \mathcal{V}$ .

Recently in [BE], the authors introduced a new type 2 spectrum  $\mathcal{Z}$  which is notable for admitting a  $v_2$ -self-map of lowest possible periodicity, that is 1. The low periodicity of the  $v_2$ -self-map makes the spectrum  $\mathcal{Z}$  suitable for the analysis of the telescope conjecture which, if true, would imply that the natural map from the telescope of  $\mathcal{Z}$  to the  $K(2)$ -localization of  $\mathcal{Z}$  is a weak equivalence. While the telescope conjecture is true for finite spectra of type 1 [M1, M2, M], it is expected to be false for finite spectra of type  $\geq 2$  (see [MRS]). In fact, in [BBB<sup>+</sup>], the authors study the prime 2, height 2 telescope conjecture using the spectrum  $\mathcal{Z}$  and lay down several conjectures (see [BBB<sup>+</sup>, §9]), whose validity would lead to a disproof of the telescope conjecture. In this paper, we also construct an  $\mathbb{R}$ -motivic analogue of  $\mathcal{Z}$  which is likely to shed light on some of these conjectures.

**Theorem 1.24.** *There exists  $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  such that the underlying  $\mathcal{A}^{\mathbb{R}}(2)$ -module structure of its cohomology is isomorphic to*

$$H_{\mathbb{R}}^{*,*}(\mathcal{Z}_{\mathbb{R}}) \cong_{\mathcal{A}^{\mathbb{R}}(2)} \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(\tilde{\mathcal{Q}}_2^{\mathbb{R}})} M_2^{\mathbb{R}},$$

where  $\tilde{\mathcal{Q}}_2^{\mathbb{R}} := [Sq^4, Q_1^{\mathbb{R}}]$ .



See Subsection 5.2 for a discussion of the element  $\tilde{Q}_2^{\mathbb{R}}$ . In future work, we intend to study the properties of  $\mathcal{Z}_{\mathbb{R}}$  extensively and hope to prove, among other things, Conjecture 1.25.

**Conjecture 1.25.** *The spectrum  $\mathcal{Z}_{\mathbb{R}}$  is of type  $(2, 2)$  and admits a  $v_{(2, \text{nil})}$ -self-map*

$$v : \Sigma^{6,3} \mathcal{Z}_{\mathbb{R}} \longrightarrow \mathcal{Z}_{\mathbb{R}}$$

*of periodicity 1.*

**Organization of the paper.** In Section 2, we discuss the  $\mathbb{R}$ -motivic Toda realization Theorem 1.12 and derive various weak forms that are suitable for applications. In Section 3, we construct the equivariant Steenrod operations using the equivariant extended power construction and prove Theorem 1.17 and Theorem 1.19, which establish comparisons with the classical Steenrod operations. In Section 4, we apply the discussion in Section 2 to obtain the  $\mathbb{R}$ -motivic topological realizations of  $\mathcal{A}^{\mathbb{R}}(1)$  and analyze the properties of their Betti realizations using results from Section 3. In Section 5, we construct the  $\mathbb{R}$ -motivic spectrum  $\mathcal{Z}_{\mathbb{R}}$  using a method of Smith. Finally, the short Section A lists the Adem relations in the  $\mathbb{R}$ -motivic Steenrod algebra.

## 2. $\mathbb{R}$ -MOTIVIC TODA REALIZATION THEOREM

The classical Toda realization theorem [T] (see also [BE, Theorem 3.1]), is recast in the modern literature as a special case of Goerss-Hopkins obstruction theory [GH] (when the chosen operad is trivial). This obstruction theory can be generalized to the  $\mathbb{R}$ -motivic setting [MG], and Theorem 1.12 would then be a special case of such a generalization.

More recent work of [PV] conceptualizes Goerss-Hopkins obstruction theory in the general setup of stable  $\infty$ -categories with  $t$ -structures. If we set  $\mathcal{C} = \mathbf{Sp}_{2, \text{fin}}^{\mathbb{R}}$ ,  $\mathcal{A} = \mathbb{S}_{\mathbb{H}_{\mathbb{R}\mathbb{F}_2}}$ , and let  $\mathbb{K}$  to be a finite  $\mathcal{A}_*^{\mathbb{R}}$ -comodule in [PV, Corollary 4.10], then we get a sequence of obstruction classes

$$(2.1) \quad \theta_n \in \text{Ext}_{\mathcal{A}_*^{\mathbb{R}}}^{-2, n+2, 0}(\mathbb{K}, \mathbb{K})$$

for each  $n \geq 0$ , the vanishing of which guarantees the existence of an  $\mathbb{S}_{\mathbb{H}_{\mathbb{R}\mathbb{F}_2}}$ -module whose homology is isomorphic to  $\mathbb{K}$  as an  $\mathcal{A}_*^{\mathbb{R}}$ -comodule. Since the  $t$ -structure in  $\mathbf{Sp}^{\mathbb{R}}$  does not change the motivic weight, the obstruction classes in (2.1) lie in the Ext-groups of motivic weight 0.

If  $M$  is a finite  $\mathbb{M}_2^{\mathbb{R}}$ -free  $\mathcal{A}^{\mathbb{R}}$ -module then  $\mathbb{K} := \text{hom}_{\mathbb{M}_2^{\mathbb{R}}}(M, \mathbb{M}_2^{\mathbb{R}})$  is a finite  $\mathcal{A}_*^{\mathbb{R}}$ -comodule,

$$\text{Ext}_{\mathcal{A}_*^{\mathbb{R}}}^{*,*,*}(\mathbb{K}, \mathbb{K}) \cong \text{Ext}_{\mathcal{A}_*^{\mathbb{R}}}^{*,*,*}(M, M),$$

and therefore, Theorem 1.12 follows. Alternatively, one can prove Theorem 1.12 simply by emulating the classical proof (as exposed in [BE, §3]).

The purpose of this section is to deduce, from the  $\mathbb{R}$ -motivic Toda realization theorem (Theorem 1.12), various weaker forms which are perhaps more convenient for application purposes. Explicit calculation of  $\text{Ext}_{\mathcal{A}_*^{\mathbb{R}}}^{*,*,*}(M, M)$  can often be difficult, and one can use a sequence of spectral sequences to approximate these ext groups. Each such approximation leads to a corresponding weaker form.

**2.1. Weak  $\mathbb{R}$ -motivic Toda realization – version (I).** Let  $M$  be an  $\mathcal{A}^{\mathbb{R}}$ -module whose underlying  $\mathbb{M}_2^{\mathbb{R}}$ -module is free and finitely generated. Let  $\mathcal{B}_M$  denote its  $\mathbb{M}_2^{\mathbb{R}}$ -basis and  $\mathcal{D}_M$  denote the collection of bidegrees in which there is an element in  $\mathcal{B}_M$ . For any element  $x \in M^{s,w}$ , we let  $t(x) = s + w$  and define

$$M_{\geq n} := \mathbb{M}_2^{\mathbb{R}} \cdot \{b \in \mathcal{B}_M : t(b) \geq n\}$$

as the free sub  $\mathbb{M}_2^{\mathbb{R}}$ -module of  $M$  generated by  $\{b \in \mathcal{B}_M : t(b) \geq n\}$ .

Note that the  $\mathcal{A}^{\mathbb{R}}$ -module structure of  $M$  is determined by the action of  $\mathcal{A}^{\mathbb{R}}$  on the elements of  $\mathcal{B}_M$  and the Cartan formula. This, along with the fact that  $t(a) \geq 0$  for all  $a \in \mathcal{A}^{\mathbb{R}}$ , implies that  $M_{\geq n}$  are also a sub  $\mathcal{A}^{\mathbb{R}}$ -module of  $M$ . Therefore, we get an  $\mathcal{A}^{\mathbb{R}}$ -module filtration of  $M$

$$M = M_{\geq k} \supset M_{\geq k+1} \supset \cdots \supset M_{\geq k+l} = \mathbf{0}$$

such that we for each  $i$  there is a short exact sequences

$$(2.2) \quad 0 \longrightarrow M_{\geq i+1} \longrightarrow M_{\geq i} \longrightarrow \bigoplus_{\{b \in \mathcal{B}_M : t(b)=i\}} \Sigma^{|b|} \mathbb{M}_2^{\mathbb{R}} \longrightarrow 0$$

of  $\mathcal{A}^{\mathbb{R}}$ -modules.

A short exact sequence of  $\mathcal{A}^{\mathbb{R}}$ -modules gives a long exact sequence in  $\text{Ext}$ . By splicing the long exact sequences induced by (2.2), we get an “algebraic” Atiyah-Hirzebruch spectral sequence

$$(2.3) \quad E_2^{s',w',s,f,w} := \mathcal{B}_M^{s',w'} \otimes \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(M, \mathbb{M}_2^{\mathbb{R}}) \Rightarrow \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-s',f,w-w'}(M, M)$$

and a corresponding weak version of Theorem 1.12, along with a uniqueness criterion, which is a weak form of Theorem 1.14.

**Theorem 2.4.** *Let  $M$  denote an  $\mathcal{A}^{\mathbb{R}}$ -module whose underlying  $\mathbb{M}_2^{\mathbb{R}}$ -module is free and finite. Suppose*

$$\text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-2,f,w}(M, \mathbb{M}_2^{\mathbb{R}}) = 0$$

for  $f \geq 3$  whenever  $(s,w) \in \mathcal{D}_M$ . Then there exists an  $X \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  such that  $H_{\mathbb{R}}^{*,*}(X) \cong M$  as an  $\mathcal{A}^{\mathbb{R}}$ -module. Further, such a realization is unique if

$$\text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s-1,f,w}(M, \mathbb{M}_2^{\mathbb{R}}) = 0$$

for all  $f \geq 2$  and  $(s,w) \in \mathcal{D}_M$ .

**2.2. Weak  $\mathbb{R}$ -motivic Toda realization – version (II).** For any  $\mathcal{A}^{\mathbb{R}}$ -module  $M$  which is  $\mathbb{M}_2^{\mathbb{R}}$ -free, the quotient  $M/(\rho)$  is an  $\mathcal{A}^{\mathbb{C}}$ -module. In particular,

$$\mathcal{A}^{\mathbb{R}}/(\rho) \cong \mathcal{A}^{\mathbb{C}}$$

as a graded Hopf-algebra. Therefore, we have a spectral sequence

$$(2.5) \quad {}^{\rho}E_2^{s,f,w,i} := \bigoplus_{i \geq 0} \text{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s+i,f,w+i}(M/(\rho), \mathbb{M}_2^{\mathbb{C}}) \implies \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(M, \mathbb{M}_2^{\mathbb{R}})$$

which is often called the (algebraic)  $\rho$ -Bockstein spectral sequence. Thus we get the following version of the  $\mathbb{R}$ -motivic Toda realization and uniqueness theorem which is weaker than Theorem 2.4.

**Theorem 2.6.** *Let  $M$  denote an  $\mathcal{A}^{\mathbb{R}}$ -module whose underlying  $\mathbb{M}_2^{\mathbb{R}}$ -module is free and finite. Suppose*

$$\text{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s-2+i,f,w+i}(M/(\rho), \mathbb{M}_2^{\mathbb{C}}) = 0$$

for  $f \geq 3$  and all  $i \geq 0$  whenever  $(s, w) \in \mathcal{D}_M$ , then there exists an  $X \in \mathbf{Sp}_{2, \text{fin}}^{\mathbb{R}}$  such that  $H_{\mathbb{R}}^{*,*}(X) \cong M$  as an  $\mathcal{A}^{\mathbb{R}}$ -module. Further, such a realization is unique if

$$\text{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s-1+i, f, w+i}(M/(\rho), \mathbb{M}_2^{\mathbb{C}}) = 0$$

for all  $f \geq 2, i \geq 0$  and  $(s, w) \in \mathcal{D}_M$ .

**2.3. Weak  $\mathbb{R}$ -motivic Toda realization – version (III).** Similarly to the classical case, the  $\mathbb{C}$ -motivic Steenrod algebra enjoys an increasing filtration called the May filtration (see [DI]), which is easier to express on its dual. On  $\mathcal{A}_{*}^{\mathbb{C}}$ , the May filtration is induced by assigning the May weights

$$m(\tau_{i-1}) = m(\xi_i^{2^j}) = 2i - 1$$

and extending it multiplicatively. The associated graded is an exterior algebra

$$(2.7) \quad \text{gr}(\mathcal{A}^{\mathbb{C}}) \cong \Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{i,j} : i \geq 1, j \geq 0),$$

where  $\xi_{i,0}$  represents  $(\tau_{i-1})_*$  and  $(\xi_{i,j+1})_*$  represents  $(\xi_i^{2^j})_*$  in the associated graded. When  $M = \mathbb{M}_2^{\mathbb{R}}$  in (2.10), then

$$(2.8) \quad \text{May } E_{1, \mathbb{M}_2^{\mathbb{C}}}^{*,*,*,*} \cong \mathbb{M}_2^{\mathbb{C}}[\mathbf{h}_{i,j} : i \geq 1, j \geq 0],$$

where  $\mathbf{h}_{i,j}$  represents the class  $\xi_{i,j}$ . The  $(s, f, w, m)$ -degrees of these generators are given by

$$|\mathbf{h}_{i,j}| = \begin{cases} (2^i - 2, 1, 2^{i-1} - 1, 2i - 1) & \text{if } j = 0, \text{ and,} \\ (2^j(2^i - 1) - 1, 1, 2^{j-1}(2^i - 1), 2i - 1) & \text{otherwise.} \end{cases}$$

*Remark 2.9.* After reindexing the May filtration of (2.8) by setting the May weight of  $\mathbf{h}_{i,j}$  equal to  $i$ , it is consistent with the indexing used in [DI].

When  $M$  is a cyclic  $\mathcal{A}^{\mathbb{R}}$ -module,  $M/(\rho)$  is also cyclic as an  $\mathcal{A}^{\mathbb{C}}$ -module, thus the May filtration induces a filtration on  $M/(\rho)$ . Thus, we get a corresponding May spectral sequence

$$(2.10) \quad \text{May } E_{1, M/(\rho)}^{s, f, w, m} := \text{Ext}_{\text{gr}(\mathcal{A}^{\mathbb{C}})}^{s, f, w, m}(\text{gr}(M/(\rho)), \mathbb{M}_2^{\mathbb{C}}) \Rightarrow \text{Ext}_{\mathcal{A}^{\mathbb{C}}}^{s, f, w}(M/(\rho), \mathbb{M}_2^{\mathbb{C}})$$

computing the input of the  $\rho$ -Bockstein spectral sequence (2.5). Thus we can formulate a version of  $\mathbb{R}$ -motivic Toda realization theorem which is even weaker than Theorem 2.6.

**Theorem 2.11.** *Let  $M$  denote an cyclic  $\mathcal{A}^{\mathbb{R}}$ -module whose underlying  $\mathbb{M}_2^{\mathbb{R}}$ -module is free and finite. Suppose*

$$\text{May } E_{1, M/(\rho)}^{s-2+i, f, w+i, *} = 0.$$

for  $f \geq 3$  and all  $i \geq 0$  whenever  $(s, w) \in \mathcal{D}_M$ . Then there exists an  $X \in \mathbf{Sp}_{2, \text{fin}}^{\mathbb{R}}$  such that  $H_{\mathbb{R}}^{*,*}(X) \cong M$  as an  $\mathcal{A}^{\mathbb{R}}$ -module. Further, such a realization is unique if

$$\text{May } E_{1, M/(\rho)}^{s-1+i, f, w+i, *} = 0$$

for  $f \geq 2, i \geq 0$  and  $(s, w) \in \mathcal{D}_M$ .

3. A COMPARISON BETWEEN  $C_2$ -EQUIVARIANT AND CLASSICAL SQUARING OPERATIONS

For any  $C_2$ -equivariant space  $X \in \mathbf{Top}_*^{C_2}$  we can functorially assign two non-equivariant spaces – the underlying space  $\mathfrak{R}(X)$ , which is obtained by restricting the action of  $C_2$  to the trivial group, and the space of  $C_2$ -fixed-points  $X^{C_2}$ . For a  $C_2$ -equivariant spectrum  $E \in \mathbf{Sp}^{C_2}$ , restricting the action to the trivial subgroup results in a monoidal functor

$$\mathfrak{R} : \mathbf{Sp}^{C_2} \longrightarrow \mathbf{Sp}$$

that identifies the underlying spectrum. However, there are two different notions of fixed-point spectrum – the categorical fixed-points and the geometric fixed-points.

The categorical fixed-point functor is a lax monoidal functor

$$(-)^{C_2} : \mathbf{Sp}^{C_2} \longrightarrow \mathbf{Sp},$$

which is defined so that  $\pi_k(E^{C_2}) \cong \pi_k^{C_2}(E)$ , but it does not interact well with infinite suspensions. The correction term is explained by the tom Dieck splitting [LMSM, Theorem V.11.1]:

$$(3.1) \quad (\Sigma_{C_2}^\infty X)^{C_2} \simeq \Sigma^\infty(X^{C_2}) \vee \Sigma^\infty(X_{hC_2}),$$

where  $X_{hC_2}$  is the homotopy orbit space. Let  $\widetilde{EC}_2 := \text{Cof}(EC_{2+} \rightarrow \mathbb{S})$ . The geometric fixed-point functor

$$\Phi : \mathbf{Sp}^{C_2} \longrightarrow \mathbf{Sp},$$

is a symmetric monoidal functor given by  $\Phi(E) := (E \wedge \widetilde{EC}_2)^{C_2}$ . When  $X \in \text{Top}_*^{C_2}$ ,

$$(3.2) \quad \Phi(\Sigma_{C_2}^\infty X) \simeq \Sigma^\infty X^{C_2}$$

is the first component in (3.1). For any  $E \in \mathbf{Sp}^{C_2}$ , there is a natural map of spectra

$$\iota_E : E^{C_2} \longrightarrow \Phi(E)$$

induced by the map  $\mathbb{S} \rightarrow \widetilde{EC}_2$ .

The Eilenberg-Mac Lane spectrum  $H\mathbb{F}_2$  is an  $\mathbb{E}_\infty^{C_2}$ -ring ([LMSM, VII]), i.e. a commutative monoid as a genuine  $C_2$ -spectrum. The restriction  $\mathfrak{R}(H\mathbb{F}_2) \simeq H\mathbb{F}_2$ , the categorical fixed-points  $H\mathbb{F}_2^{C_2} \simeq H\mathbb{F}_2$  and the geometric fixed-points  $\Phi(H\mathbb{F}_2) \simeq H\mathbb{F}_2[t]$  are  $\mathbb{E}_\infty$ -rings. It follows from the knowledge of  $\mathbb{M}_2^{C_2} := \pi_*^{C_2} H\mathbb{F}_2$  that for  $n \geq 0$

$$(\Sigma^{n\sigma} H\mathbb{F}_2)^{C_2} \simeq \bigvee_{i=0}^n \Sigma^i H\mathbb{F}_2 \longleftarrow \Phi(H\mathbb{F}_2) \simeq \text{colim}_{n \rightarrow \infty} (\Sigma^{n\sigma} H\mathbb{F}_2)^{C_2} \simeq H\mathbb{F}_2[t]$$

is the inclusion of the first  $(n + 1)$  components. The above map clearly splits. One can endow  $(\Sigma^{n\sigma} H\mathbb{F}_2)^{C_2}$  with an  $\mathbb{E}_\infty$ -structure isomorphic to the truncated polynomial algebra  $H\mathbb{F}_2[t]/(t^{n+1})$  so that the splitting map

$$\pi_{\mathbb{F}_2}^{(n)} : \Phi(H\mathbb{F}_2) \simeq H\mathbb{F}_2[t] \longrightarrow (\Sigma^{n\sigma} H\mathbb{F}_2)^{C_2} \simeq H\mathbb{F}_2[t]/(t^{n+1})$$

is an  $\mathbb{E}_\infty$ -map. Alternatively,  $\pi_{\mathbb{F}_2}^{(n)}$  can be obtained as an  $\mathbb{E}_\infty$ -map by applying the functor  $H$  to the map of commutative rings  $\mathbb{F}_2[t] \rightarrow \mathbb{F}_2[t]/(t^{n+1})$ . The composition

$$(3.3) \quad \mathbb{H}\mathbb{F}_2^{C_2} \xrightarrow{\iota_{\mathbb{F}_2}} \Phi(\mathbb{H}\mathbb{F}_2) \xrightarrow{\pi_{\mathbb{F}_2}^{(0)}} \mathbb{H}\mathbb{F}_2^{C_2}$$

is the identity and exhibits  $\Phi(\mathbb{H}\mathbb{F}_2)$  as an augmented  $\mathbb{H}\mathbb{F}_2$ -algebra.

For any  $C_2$ -space  $X \in \mathbf{Top}_*^{C_2}$ , the restriction functor induces a natural transformation

$$\mathfrak{R}_* : H_{C_2}^{i,j}(X_+) \longrightarrow H^i(\mathfrak{R}(X)_+).$$

To compare the cohomology of  $X^{C_2}$  with the  $\mathrm{RO}(C_2)$ -graded cohomology of  $X$ , we make use of the splitting (3.3) to define the natural ring map

$$\hat{\Phi}_* : H_{C_2}^{i,j}(X_+) \longrightarrow H^{i-j}(X_+^{C_2}),$$

which sends  $u \in H_{C_2}^{i,j}(X_+)$  to the composite (as defined in [BW, (2.7)])

$$\Sigma^\infty X^{C_2} \xrightarrow{\Phi(u)} \Sigma^{i-j} \Phi(\mathbb{H}\mathbb{F}_2) \xrightarrow{\pi_{\mathbb{F}_2}^{(0)}} \Sigma^{i-j} \mathbb{H}\mathbb{F}_2.$$

The purpose of this section is to compare the  $\mathrm{RO}(C_2)$ -graded squaring operations with the classical squaring operations along the maps  $\mathfrak{R}_*$  and  $\hat{\Phi}_*$ . We begin with a brief recollection of the construction of the classical and  $C_2$ -equivariant squaring operations.

**3.1. A construction of Steenrod squaring operations.** The construction of the classical mod 2 Steenrod algebra, which is the algebra of stable cohomology operations for ordinary cohomology with  $\mathbb{F}_2$ -coefficients, involves the  $\mathbb{E}_\infty$ -structure<sup>1</sup> of  $\mathbb{H}\mathbb{F}_2$  and the fact that the tautological line bundle  $\gamma$  over  $\mathbb{R}\mathbb{P}^\infty$  is  $\mathbb{H}\mathbb{F}_2$ -orientable. We review here how the mod 2 Steenrod operations are derived from that structure. A similar discussion can be found in [BMMS, Section VIII.2].

*Notation 3.4.* For any space or spectrum  $X$  and  $n \geq 1$ , we let

$$D_n(X) := (E\Sigma_n)_+ \wedge_{\Sigma_n} (X^{\wedge n}),$$

where  $\Sigma_n$  acts by permuting the factors of  $X^{\wedge n}$ . By convention,  $D_0(X) = \mathbb{S}$ .

An  $\mathbb{E}_\infty$ -ring structure on a spectrum  $R$  gives a collection of maps of the form

$$\Theta_n^R : D_n(R) \longrightarrow R$$

for each  $n \geq 0$ , which satisfy the usual coherence conditions (see [Ma2]). By assumption,  $\Theta_0^R$  is the unit map of  $R$  and  $\Theta_1^R$  is the identity map.

The  $\mathbb{H}\mathbb{F}_2$ -orientability of  $\gamma$  implies the existence of an  $\mathbb{H}\mathbb{F}_2$ -Thom class

$$(3.5) \quad u_n : \mathrm{Th}(\gamma^{\oplus n}) \simeq \mathbb{R}\mathbb{P}_n^\infty \longrightarrow \Sigma^n \mathbb{H}\mathbb{F}_2$$

---

<sup>1</sup>Technically, we only make use of the  $\mathbb{H}_\infty$ -ring structure that underlies the  $\mathbb{E}_\infty$ -structure of  $\mathbb{H}\mathbb{F}_2$ .

for each  $n \geq 0$ . These are compatible as  $n$  varies, in the sense that the following diagram commutes:

$$(3.6) \quad \begin{array}{ccc} \mathrm{Th}(\gamma^{\oplus(m+n)}) & \longrightarrow & \mathrm{Th}(\gamma^{\oplus m}) \wedge \mathrm{Th}(\gamma^{\oplus n}) \xrightarrow{u_m \wedge u_n} \Sigma^m \mathbb{H}\mathbb{F}_2 \wedge \Sigma^n \mathbb{H}\mathbb{F}_2 \\ & \searrow & \downarrow \mu_{\mathbb{F}_2} \\ & & \mathbb{H}\mathbb{F}_2. \end{array}$$

$u_{m+n}$

For any spectra  $E$  and  $F$ , there is a natural map

$$\delta_n : D_n(E \wedge F) \longrightarrow D_n(E) \wedge D_n(F)$$

induced by the diagonal on  $E\Sigma_n$  and the isomorphism  $(E \wedge F)^{\wedge n} \cong E^{\wedge n} \wedge F^{\wedge n}$ . Thus, we may define the map  $\tau_n$  as the composition

$$(3.7) \quad \begin{array}{ccc} D_2(\Sigma^n \mathbb{H}\mathbb{F}_2) & \xrightarrow{\tau_n} & \Sigma^{2n} \mathbb{H}\mathbb{F}_2 \\ \downarrow \delta_2 & & \uparrow \mu_{\mathbb{F}_2} \\ D_2(S^n) \wedge D_2(\mathbb{H}\mathbb{F}_2) & \xrightarrow{\cong} \Sigma^n \mathbb{R}\mathbb{P}_n^\infty \wedge D_2(\mathbb{H}\mathbb{F}_2) \xrightarrow{\Sigma^n u_n \wedge \Theta_{\mathbb{F}_2}^2} & \Sigma^{2n} \mathbb{H}\mathbb{F}_2 \wedge \mathbb{H}\mathbb{F}_2. \end{array}$$

**Definition 3.8.** The power operation is the natural transformation

$$\mathcal{P}_2 : H^n(-) \longrightarrow H^{2n}(D_2(-))$$

which takes a class  $u \in H^n(E)$  to the composite class

$$\mathcal{P}_2(u) : D_2(E) \xrightarrow{D_2(u)} D_2(\Sigma^n \mathbb{H}\mathbb{F}_2) \xrightarrow{\tau_n} \Sigma^n \mathbb{H}\mathbb{F}_2$$

for any  $E \in \mathbf{Sp}$ .

From (3.6), we deduce the commutativity of the diagram

$$(3.9) \quad \begin{array}{ccc} D_2(\Sigma^n \mathbb{H}\mathbb{F}_2 \wedge \Sigma^m \mathbb{H}\mathbb{F}_2) & \longrightarrow & D_2(\Sigma^n \mathbb{H}\mathbb{F}_2) \wedge D_2(\Sigma^m \mathbb{H}\mathbb{F}_2) \xrightarrow{\tau_n \wedge \tau_m} \Sigma^{2n} \mathbb{H}\mathbb{F}_2 \wedge \Sigma^{2m} \mathbb{H}\mathbb{F}_2 \\ \downarrow D_2(\mu_{\mathbb{F}_2}) & & \downarrow \mu_{\mathbb{F}_2} \\ D_2(\Sigma^{n+m} \mathbb{H}\mathbb{F}_2) & \xrightarrow{\tau_{n+m}} & \Sigma^{2n+2m} \mathbb{H}\mathbb{F}_2. \end{array}$$

As a result, we have

$$\delta_2^*(\mathcal{P}_2(u) \otimes \mathcal{P}_2(v)) = \mathcal{P}_2(u \otimes v)$$

which leads to the Cartan formula for the Steenrod algebra.

If  $X \in \mathbf{Top}_*$  is given the trivial  $\Sigma_2$ -action and  $X \wedge X$  the permutation action, the diagonal map  $X \rightarrow X \wedge X$  is  $\Sigma_2$ -equivariant. Consequently, we have an induced map

$$\Delta_X : (B\Sigma_2)_+ \wedge X \simeq (E\Sigma_2)_+ \wedge_{\Sigma_2} X \longrightarrow D_2(X).$$

Since  $H^*(B\Sigma_2) \cong \mathbb{F}_2[t]$ , we may write (using the Kunnetth isomorphism)

$$(3.10) \quad \Delta_X^*(\mathcal{P}_2(u)) = \sum_{i=0}^n t^{n-i} \otimes \mathrm{Sq}^i(u),$$

which defines the natural transformations  $\mathrm{Sq}^i : H^n(-) \longrightarrow H^{n+i}(-)$ .

*Remark 3.11.* The squaring operation  $Sq^i(u)$  for any class  $u \in H^n(X)$  is determined by  $Sq^i(\iota_n)$ , where  $\iota_n \in H^*(K(\mathbb{F}_2, n))$  is the fundamental class, because of the universal property of  $K(\mathbb{F}_2, n)$ . A priori,  $Sq^i(u)$  depends on the cohomological degree of  $u$ . However, this dependence is eradicated by the fact that the squaring operations are stable, i.e. for any  $u \in H^*(X)$

$$Sq^i(\sigma_*(u)) = \sigma_*(Sq^i(u)),$$

where  $\sigma_* : H^*(X) \cong H^{*+1}(\Sigma X)$  is the suspension isomorphism. The  $H\mathbb{F}_2$ -orientability of  $\gamma$  implies  $Sq^0(\iota) = \iota$  for the generator  $\iota \in H^1(S^1)$ , which, along with Cartan formula, implies stability.

**3.2. The  $C_2$ -equivariant squaring operations.** The construction of the classical squaring operations can be adapted to construct squaring operations on the  $RO(C_2)$ -graded cohomology of a  $C_2$ -space.

*Remark 3.12.* Our ideas are closely related to the construction of the  $\mathbb{R}$ -motivic squaring operations due to Voevodsky [V]. Certain parts, such as the construction of the power operation Definition 3.21, though different, can be compared to [W1, W2], where the author studies  $C_2$ -equivariant power operations on the homology of spaces.

*Notation 3.13.* For any group  $G$  and a family of subgroups  $\mathcal{F}$  closed under subconjugacy, there exists a space  $E\mathcal{F}$  determined up to a  $G$ -weak equivalence by its universal property

$$E\mathcal{F}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{F}, \\ \emptyset & \text{otherwise.} \end{cases}$$

When  $G = C_2 \times \Sigma_n$  and  $\mathcal{F}_n = \{H \subset G : H \cap \Sigma_n = \mathbb{1}\}$ , we denote  $E\mathcal{F}_n$  by  $E_{C_2}\Sigma_n$ . Note that there is a natural  $C_2$ -equivariant map  $E\Sigma_n \rightarrow E_{C_2}\Sigma_n$ .

*Notation 3.14.* For a based  $C_2$ -space or a  $C_2$ -spectrum  $X$ , we let

$$D_n^{C_2}(X) := (E_{C_2}\Sigma_n)_+ \wedge_{\Sigma_n} (X^{\wedge n})$$

the  $n$ -th equivariant extended power construction on  $X$ . There is a natural  $C_2$ -equivariant map

$$\delta_n^{C_2} : D_n^{C_2}(X \wedge Y) \longrightarrow D_n^{C_2}(X) \wedge D_n^{C_2}(Y)$$

induced by the diagonal map of  $E_{C_2}\Sigma_n$  for any pair  $X$  and  $Y$  of  $C_2$  space or spectra.

For a  $C_2$ -equivariant space  $X \in \mathbf{Top}_*^{C_2}$ , the inclusions  $X^{C_2} \hookrightarrow X$  and  $E\Sigma_n \rightarrow E_{C_2}\Sigma_n$  together induce a natural map

$$(3.15) \quad \lambda_X : D_2(X^{C_2}) \longrightarrow D_2^{C_2}(X)^{C_2},$$

which is usually not an equivalence.

**Example 3.16.** When  $X \simeq S^0$ ,  $\lambda_{S^0} : (B\Sigma_2)_+ \longrightarrow (B_{C_2}\Sigma_2)_+^{C_2} \simeq (B\Sigma_2)_+ \wedge S_+^0$  is the inclusion of a summand.

Likewise, when  $E \in \mathbf{Sp}^{C_2}$ , the map  $E^{C_2} \hookrightarrow E$  induces a natural map

$$\lambda_E : D_2(E^{C_2}) \longrightarrow D_2^{C_2}(E)^{C_2}.$$

Using the fact that  $\widetilde{\mathbb{E}C}_2$  is an  $\mathbb{E}_\infty$ -ring  $C_2$ -spectrum we define a map  $\lambda_{\mathbb{E}}^\Phi$  as the composition

(3.17)

$$\begin{array}{ccc} D_2(\Phi(E)) & \xrightarrow{\lambda_{\mathbb{E}}^\Phi} & \Phi(D_2^{C_2}(E)) \\ \downarrow \lambda_{\mathbb{E}\widetilde{\mathbb{E}C}_2 \wedge E} & & \uparrow \\ (D_2(\widetilde{\mathbb{E}C}_2 \wedge E))^{C_2} & \longrightarrow & (D_2(\widetilde{\mathbb{E}C}_2) \wedge D_2(E))^{C_2} \longrightarrow (\widetilde{\mathbb{E}C}_2 \wedge D_2(E))^{C_2} \cong \Phi(D_2(E)) \end{array}$$

By definition, an  $\mathbb{E}_\infty^{C_2}$ -ring structure on a spectrum  $R$  consists of a system of maps

$$\Theta_n^R : D_n^{C_2}(R) \longrightarrow R$$

for each  $n \geq 0$ , which satisfy certain compatibility criteria [LMSM, §VII.2]. The categorical fixed-point spectrum  $R^{C_2}$  as well as the geometric fixed-point spectrum  $\Phi(R)$  of an  $\mathbb{E}_\infty^{C_2}$ -ring spectrum  $R$  are  $\mathbb{E}_\infty$ -ring spectra with structure maps

$$\Theta_n^{R^{C_2}} : D_2(R^{C_2}) \xrightarrow{\lambda_R} D_2^{C_2}(R)^{C_2} \xrightarrow{(\Theta_n^R)^{C_2}} R^{C_2}$$

and

$$\Theta_n^{\Phi(R)} : D_2(\Phi(R)) \xrightarrow{\lambda_R^\Phi} \Phi(D_2^{C_2}(R)) \xrightarrow{\Phi(\Theta_n^R)} \Phi(R),$$

respectively. Further, the natural map

$$\iota_R : R^{C_2} \longrightarrow \Phi(R)$$

is an  $\mathbb{E}_\infty$ -ring map.

Let  $\omega$  denote the sign representation of  $\Sigma_2$ . The equivariant Eilenberg-Mac Lane spectrum  $H\mathbb{F}_2$  does not distinguish between the  $C_2$ -equivariant bundles

$$\bar{\epsilon} : E_{C_2}\Sigma_2 \times_{\Sigma_2} (\rho) \longrightarrow B_{C_2}\Sigma_2$$

$$\bar{\gamma} : E_{C_2}\Sigma_2 \times_{\Sigma_2} (\rho \otimes \omega) \longrightarrow B_{C_2}\Sigma_2,$$

i.e. there exists a  $C_2$ -equivariant Thom isomorphism (see Remark 3.20)

$$(3.18) \quad \text{Th}(\bar{\gamma}) \wedge H\mathbb{F}_2 \simeq \text{Th}(\bar{\epsilon}) \wedge H\mathbb{F}_2 \simeq \Sigma^\rho(B_{C_2}\Sigma_2)_+ \wedge H\mathbb{F}_2.$$

The above Thom isomorphism results in an  $H\mathbb{F}_2$ -Thom class

$$\underline{u}_n : \text{Th}(\bar{\gamma}^{\oplus n}) \longrightarrow \Sigma^{n\rho} H\mathbb{F}_2$$

for each  $n \geq 0$ , and these Thom classes can be used to define the  $C_2$ -equivariant power operations. Since

$$D_2^{C_2}(S^{n\rho}) \simeq \text{Th}(n\rho \oplus n(\rho \otimes \omega)) \simeq \Sigma^{n\rho} \text{Th}(\bar{\gamma}^{\oplus n}),$$

we define the map  $\tau_n$  as the composition

(3.19)

$$\begin{array}{ccc} D_2^{C_2}(\Sigma^{n\rho} H\mathbb{F}_2) & \xrightarrow{\tau_n} & \Sigma^{2n\rho} H\mathbb{F}_2 \\ \downarrow & & \uparrow \mu_{\mathbb{F}_2} \\ D_2^{C_2}(S^{n\rho}) \wedge D_2^{C_2}(H\mathbb{F}_2) & \xrightarrow{\simeq} & \Sigma^{n\rho} \text{Th}(\bar{\gamma}^{\oplus n}) \wedge D_2^{C_2}(H\mathbb{F}_2) \xrightarrow{\Sigma^n \underline{u}_n \wedge \Theta_2^{\mathbb{F}_2}} \Sigma^{2n\rho} H\mathbb{F}_2 \wedge H\mathbb{F}_2. \end{array}$$



*Remark 3.20.* The Thom isomorphism (3.18) does not follow immediately from the general theory of equivariant Thom isomorphisms [CW] because the base space  $B_{C_2}\Sigma_2$  is *not*  $C_2$ -connected. In such cases, there is no guarantee of an  $H\mathbb{F}_2$ -Thom class; instead an  $H\mathbb{F}_2$ -orientation is encoded by a family of classes (see [Ma1]). However, there is a map of  $C_2$ -equivariant  $\mathbb{R}$ -vector bundles from  $\overline{\gamma}$  to the tautological Atiyah Real line bundle over  $BS^\sigma = BU_{\mathbb{R}}(1)$ , which is  $C_2$ -connected. Further, the tautological Atiyah Real line bundle admits a single  $H\mathbb{F}_2$ -Thom class as  $H\mathbb{F}_2^\times$  is  $C_2$ -equivariantly contractible. Therefore,  $\overline{\gamma}$  also admits a single  $H\mathbb{F}_2$ -Thom class which leads to (3.18) using a standard argument involving the Thom diagonal map.

**Definition 3.21.** The equivariant power operation is the natural transformation

$$\mathcal{P}_2^{C_2} : H_{C_2}^{n\rho}(-) \longrightarrow H_{C_2}^{2n\rho}(D_2^{C_2}(-)),$$

which takes a class  $u \in H_{C_2}^{n\rho}(E)$  to the composite class

$$\mathcal{P}_2^{C_2}(u) : D_2^{C_2}(E) \xrightarrow{D_2^{C_2}(u)} D_2^{C_2}(\Sigma^{n\rho}H\mathbb{F}_2) \xrightarrow{\tau_n} \Sigma^{2n\rho}H\mathbb{F}_2$$

for any  $E \in \mathbf{Sp}^{C_2}$ .

When  $X \in \mathbf{Top}_*^{C_2}$  is given the trivial  $\Sigma_2$ -action and  $X \wedge X$  is given the permutation action, the diagonal map  $X \rightarrow X \wedge X$  is a  $C_2 \times \Sigma_2$ -equivariant map. Consequently, we have a  $C_2$ -equivariant map

$$\Delta_X^{C_2} : (B_{C_2}\Sigma_2)_+ \wedge X \simeq (E_{C_2}\Sigma_2)_+ \wedge_{\Sigma_2} X \longrightarrow D_2^{C_2}(X).$$

By [HK, Lemma 6.27] (also see [W1, Proposition 3.2]),

$$H_{C_2}^*((B_{C_2}\Sigma_2)_+) \cong \mathbb{M}_2^{C_2}[y, x]/(y^2 = a_\sigma y + u_\sigma x),$$

where  $|y| = (1, 1)$  and  $|x| = (2, 1)$ . Since  $H_{C_2}^*((B_{C_2}\Sigma_2)_+)$  is  $\mathbb{M}_2^{C_2}$ -free, we also have a Kunnetth isomorphism

$$H_{C_2}^*((B_{C_2}\Sigma_2)_+ \wedge X) \cong H_{C_2}^*((B_{C_2}\Sigma_2)_+) \otimes_{\mathbb{M}_2^{C_2}} H_{C_2}^*(X).$$

Thus, for any  $u \in H_{C_2}^{n\rho}(X)$ , we may write  $(\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u))$  using the formula

$$(3.22) \quad (\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u)) = \sum_{i=0}^n x^{n-i} \otimes \underline{\text{Sq}}^{2i}(u) + \sum_{i=0}^n yx^{n-i-1} \otimes \underline{\text{Sq}}^{2i+1}(u),$$

which defines the equivariant squaring operations  $\underline{\text{Sq}}^i$  for all  $i \geq 0$ . These can be extended to operations on the entire  $\text{RO}(C_2)$ -graded cohomology ring as in [V, Prop 2.6].

*Remark 3.23.* Just like the classical case, one can easily deduce that the  $\text{RO}(C_2)$ -graded squaring operations defined this way are natural, stable and obey the Cartan formula. In fact, Voevodsky [V] uses a similar approach to establish these properties for the  $\mathbb{R}$ -motivic Steenrod algebra, which can be emulated in the  $C_2$ -equivariant case using the Betti realization functor.

**3.3. Comparison theorems.** Since the restriction functor is monoidal, it induces a ring map

$$\mathfrak{R}_* : H_{C_2}^*(X_+) \longrightarrow H^*(\mathfrak{R}(X)_+)$$

for any  $X \in \mathbf{Top}_*^{C_2}$ .

**Example 3.24.** When  $X = *$ , the map

$$\mathfrak{R}_* : \pi_*^{C_2} \underline{HF}_2 \longrightarrow \pi_* \underline{HF}_2 \cong \mathbb{F}_2$$

sends  $u_\sigma \mapsto 1$ ,  $a_\sigma \mapsto 0$ , and  $\Theta \mapsto 0$ . This follows from the fact that the cofiber sequence  $C_{2+} \longrightarrow S^0 \xrightarrow{a_\sigma} S^\sigma$  shows that the kernel of  $\mathfrak{R}_*$  consists of precisely the  $a_\sigma$ -divisible elements.

**Proposition 3.25.** For any  $X \in \mathbf{Top}_*^{C_2}$  and a class  $u \in H_{C_2}^{n\rho}(X)$

$$\mathfrak{R}_*(\mathcal{P}_2^{C_2}(u)) = \mathcal{P}_2(\mathfrak{R}_*(u)).$$

*Proof.* Since,  $\mathfrak{R}(\overline{\gamma}) = 2\gamma$ , it follows that  $\mathfrak{R}_*(\underline{u}_n) = \underline{u}_{2n}$ . This, along with the fact that  $\mathfrak{R}(\Theta_2^{\mathbb{F}_2}) = \Theta_2^{\mathbb{F}_2}$  shows  $\mathfrak{R}(\underline{\tau}_n) = \tau_{2n}$ , and the result follows.  $\square$

*Proof of Theorem 1.17.* Let  $X \in \mathbf{Top}_*^{C_2}$  and  $u \in H_{C_2}^{n\rho}(X)$ . Since  $\mathfrak{R}(B_{C_2}\Sigma_2) \simeq B\Sigma_2$ ,  $\mathfrak{R}(\Delta^{C_2}) = \Delta$ ,  $\mathfrak{R}_*(y) = \mathfrak{t}$  and  $\mathfrak{R}_*(x) = \mathfrak{t}^2$ , it follows that

$$\Delta_{\mathfrak{R}(X)}^*(\mathcal{P}_2(\mathfrak{R}_*(u))) = \sum_{i=-n}^n \mathfrak{t}^{n-i} \otimes \text{Sq}^i(\mathfrak{R}_*(u))$$

must equal

$$\begin{aligned} \mathfrak{R}_*((\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u))) &= \mathfrak{R}_*\left(\sum_{i=-n}^n x^{n-i} \otimes \underline{\text{Sq}}^{2i}(u) + \sum_{i=-n}^n yx^{n-i-1} \otimes \underline{\text{Sq}}^{2i+1}(u)\right) \\ &= \sum_{i=-n}^n \mathfrak{t}^{2n-2i} \otimes \mathfrak{R}_*(\underline{\text{Sq}}^{2i}(u)) \\ &\quad + \sum_{i=-n}^n \mathfrak{t}^{2n-2i-1} \otimes \mathfrak{R}_*(\underline{\text{Sq}}^{2i+1}(u)). \end{aligned}$$

Thus, the result is true for cohomology classes  $u \in H_{C_2}^{n\rho}(X)$  for any space  $X \in \mathbf{Top}_*^{C_2}$ .

Since the squaring operations are stable, the result extends to arbitrary  $\text{RO}(C_2)$ -graded cohomology classes. Moreover, since  $\mathfrak{R}$  commutes with suspensions, in the sense that  $\mathfrak{R} \circ \Sigma_{C_2}^\infty \simeq \Sigma^\infty \circ \mathfrak{R}$ , and any  $E \in \mathbf{Sp}_{2,\text{fin}}^{C_2}$  is equivalent to  $\Sigma^{-n}\Sigma_{C_2}^\infty X$  for some  $n$  and  $X \in \mathbf{Top}_*^{C_2}$ , we conclude the same for any  $u \in H_{C_2}^*(E)$ .  $\square$

Now we draw our attention towards comparing the action of the  $C_2$ -equivariant Steenrod algebra  $\mathcal{A}^{C_2}$  on  $H^*(X_+)$  to the action of the classical Steenrod algebra  $\mathcal{A}$  on  $H^*(X_+^{C_2})$ , where  $X \in \mathbf{Top}^{C_2}$ . Note that

$$\hat{\Phi}_* : H_{C_2}^*(X_+) \longrightarrow H^*(X_+^{C_2})$$

is a ring map.

**Example 3.26.** When  $X = *$ , the map

$$\hat{\Phi}_* : \pi_*^{C_2} \mathbb{H}\mathbb{F}_2 \cong \mathbb{F}_2[u_\sigma, a_\sigma] \oplus \Theta\{u_\sigma^{-i} a_\sigma^{-j}\} \longrightarrow \pi_* \mathbb{H}\mathbb{F}_2 \cong \mathbb{F}_2$$

sends  $a_\sigma \mapsto 1$ ,  $u_\sigma \mapsto 0$ , and  $\Theta \mapsto 0$ . This is essentially because smashing with

$$\widetilde{EC}_2 \simeq \operatorname{colim}\{S^0 \xrightarrow{a_\sigma} S^\sigma \xrightarrow{a_\sigma} S^{2\sigma} \longrightarrow \dots\}$$

amounts to inverting  $a_\sigma$  and the projection  $\pi_{\mathbb{F}_2}^{(0)}$  kills  $u_\sigma$ .

*Remark 3.27.* One can deduce from Example 3.16 that in cohomology, the map

$$\lambda_{S^0}^* : H^*(B_{C_2}\Sigma_2^{C_2}) \cong \mathbb{F}_2[t][\iota]/(\iota^2 - \iota) \longrightarrow H^*((B\Sigma_2)_+) \cong \mathbb{F}_2[t],$$

is the quotient map sending  $\iota \mapsto 0$ .

**Example 3.28.** The map  $\hat{\Phi}_* : H_{C_2}^*((B_{C_2}\Sigma_2)_+) \rightarrow H^*(B_{C_2}\Sigma_2^{C_2})$  sends  $x \mapsto t$  and  $y \rightarrow \iota$ ,  $a_\sigma \mapsto 1$  and  $u_\sigma \mapsto 0$ .

**Lemma 3.29.** *The composition*

$$H_{C_2}^*(\operatorname{Th}(\overline{\gamma}^{\oplus n})) \xrightarrow{\hat{\Phi}_*} H^*(\operatorname{Th}(\overline{\gamma}^{\oplus n})^{C_2}) \xrightarrow{\lambda_{S^0 \otimes \omega}^*} H^*(\operatorname{Th}(\gamma^{\oplus n}))$$

sends  $\underline{u}_n \mapsto u_n$ .

*Proof.* Let  $\zeta_{C_2} : (B_{C_2}\Sigma_2)_+ \rightarrow \operatorname{Th}(\overline{\gamma}^{\oplus n})$  denote the zero-section. Under the zero-section map the Thom class is mapped to the Euler class, and therefore  $\zeta_{C_2}^*(\underline{u}_n) = x^n$ . Likewise, the zero-section for the nonequivariant bundle  $\zeta : (B\Sigma_2)_+ \rightarrow \operatorname{Th}(\gamma^{\oplus n})$  sends  $u_n \mapsto t^n$ . By naturality of  $\hat{\Phi}_*$  and  $\lambda$ , we get a commutative diagram

$$\begin{array}{ccccc} H_{C_2}^*(\operatorname{Th}(\overline{\gamma}^{\oplus n})) & \xrightarrow{\hat{\Phi}_*} & H^*(\operatorname{Th}(\overline{\gamma}^{\oplus n})^{C_2}) & \xrightarrow{\lambda_{S^0 \otimes \omega}^*} & H^*(\operatorname{Th}(\gamma^{\oplus n})) \\ \downarrow \zeta_{C_2}^* & & \downarrow (\zeta_{C_2}^{C_2})^* & & \downarrow \zeta^* \\ H_{C_2}^*((B_{C_2}\Sigma_2)_+) & \xrightarrow{\hat{\Phi}_*} & H^*(B_{C_2}\Sigma_2^{C_2}) & \xrightarrow{\lambda_{S^0}^*} & H^*((B\Sigma_2)_+). \end{array}$$

which along with Remark 3.27 and injectivity of  $\zeta^*$  implies the result. □

**Corollary 3.30.** *For any space  $X \in \mathbf{Top}_*^{C_2}$  and a class  $u \in H_{C_2}^{n\rho}(X)$ ,*

$$(3.31) \quad \mathcal{P}_2(\hat{\Phi}_*(u)) = \lambda_X^*(\hat{\Phi}_*(\mathcal{P}_2^{C_2}(u))).$$

*Proof.* It is enough to show that the following diagram commutes, as the blue path and the red path indicates the left-hand side and the right-hand side of (3.31), respectively.

$$\begin{array}{ccccc}
 D_2(X^{C_2}) & \xrightarrow{\lambda_X} & D_2^{C_2}(X)^{C_2} & & \\
 \downarrow D_2(\Phi(u)) & & \downarrow \Phi(D_2^{C_2}(u)) & & \\
 D_2(\Sigma^n \mathbb{H}\mathbb{F}_2) & \xleftarrow{D_2(\Sigma^n \pi_{\mathbb{F}_2}^{(0)})} & D_2(\Phi(\Sigma^{n\rho} \mathbb{H}\mathbb{F}_2)) & \xrightarrow{\lambda_{\Sigma^{n\rho} \mathbb{H}\mathbb{F}_2}^\Phi} & \Phi(D_2^{C_2}(\Sigma^{n\rho} \mathbb{H}\mathbb{F}_2)) \\
 \downarrow \delta_2 & & \downarrow \delta_2 & & \downarrow \Phi(\delta_2^{C_2}) \\
 D_2(S^n) \wedge D_2(\mathbb{H}\mathbb{F}_2) & \xleftarrow{1 \wedge D_2(\pi_{\mathbb{F}_2}^{(0)})} & D_2(S^n) \wedge D_2(\Phi(\mathbb{H}\mathbb{F}_2)) & \longrightarrow & \Phi(D_2^{C_2}(S^{n\rho})) \wedge \Phi(D_2^{C_2}(\mathbb{H}\mathbb{F}_2)) \\
 \downarrow u_n \wedge \Theta_2^{\mathbb{F}_2} & & \downarrow u_n \wedge \Theta_2^{\Phi \mathbb{F}_2} & & \downarrow \underline{u}_n \wedge \Theta_2^{\mathbb{F}_2} \\
 \Sigma^{2n} \mathbb{H}\mathbb{F}_2 \wedge \mathbb{H}\mathbb{F}_2 & \xleftarrow{1 \wedge \pi_{\mathbb{F}_2}^{(0)}} & \Sigma^{2n} \mathbb{H}\mathbb{F}_2 \wedge \Phi(\mathbb{H}\mathbb{F}_2) & \xleftarrow{\Sigma^{2n} \pi_{\mathbb{F}_2}^{(0)} \wedge 1} & \Phi(\Sigma^{2n\rho} \mathbb{H}\mathbb{F}_2) \wedge \Phi(\mathbb{H}\mathbb{F}_2) \\
 \downarrow \Sigma^{2n} \mu_{\mathbb{F}_2} & & \downarrow & & \downarrow \Sigma^{2n} \Phi(\mu_{\mathbb{F}_2}) \\
 \Sigma^{2n} \mathbb{H}\mathbb{F}_2 & \xleftarrow{\Sigma^{2n} \pi_{\mathbb{F}_2}^{(0)}} & \Sigma^{2n} \Phi(\mathbb{H}\mathbb{F}_2) & & \Sigma^{2n} \Phi(\mathbb{H}\mathbb{F}_2)
 \end{array}$$

The squares (A), (B) and (C) commute naturally, the squares (E) and (G) commute because  $\pi_{\mathbb{F}_2}^{(0)}$  is an  $\mathbb{E}_\infty$ -ring map, and (F) commutes because of Lemma 3.29.  $\square$

*Proof of Theorem 1.19.* For any space  $X \in \mathbf{Top}_*^{C_2}$  and a class  $u \in H_{C_2}^{n\rho}(X)$ , we have a commutative diagram

$$\begin{array}{ccc}
 (B\Sigma_2)_+ \wedge X^{C_2} & \xrightarrow{\Delta_X^{C_2}} & D_2(X^{C_2}) \\
 \lambda_{S^0} \wedge 1_{X^{C_2}} \downarrow & & \downarrow \lambda_X \\
 (B_{C_2} \Sigma_2)_+^{C_2} \wedge X^{C_2} & \xrightarrow{(\Delta_X^{C_2})^{C_2}} & D_2^{C_2}(X)^{C_2}.
 \end{array}$$

Therefore,

$$\begin{aligned}
 \sum_{i=0}^n \mathbf{t}^{n-i} \otimes \text{Sq}^i(\hat{\Phi}_*(u)) &= \Delta_{X^{C_2}}^*(\mathcal{P}_2(\hat{\Phi}_*(u))) \\
 &= \Delta_{X^{C_2}}^*(\lambda_X^*(\hat{\Phi}_*(\mathcal{P}_2^{C_2}(u)))) \\
 &= (\lambda_{S^0}^* \otimes \mathbb{1}_{X^{C_2}}^*)((\Delta_X^{C_2})^{C_2})^*(\hat{\Phi}_*(\mathcal{P}_2^{C_2}(u))) \\
 &= (\lambda_{S^0}^* \otimes \mathbb{1}_{X^{C_2}}^*)\hat{\Phi}_*((\Delta_X^{C_2})^*(\mathcal{P}_2^{C_2}(u))) \\
 &= (\lambda_{S^0}^* \otimes \mathbb{1}_{X^{C_2}}^*)\hat{\Phi}_*\left(\sum_{i=0}^n x^{n-i} \otimes \text{Sq}^{2i,i}(u) \right. \\
 &\qquad\qquad\qquad \left. + \sum_{i=0}^n yx^{n-i-1} \otimes \underline{\text{Sq}}^{2i+1}(u)\right) \\
 &= (\lambda_{S^0}^* \otimes \mathbb{1}_{X^{C_2}}^*)\left(\sum_{i=0}^n \mathbf{t}^{n-i} \otimes \hat{\Phi}_*(\underline{\text{Sq}}^{2i}(u)) \right. \\
 &\qquad\qquad\qquad \left. + \sum_{i=0}^n t\mathbf{t}^{n-i-1} \otimes \hat{\Phi}_*(\underline{\text{Sq}}^{2i+1}(u))\right) \\
 &= \sum_{i=0}^n \mathbf{t}^{n-i} \otimes \hat{\Phi}_*(\underline{\text{Sq}}^{2i}(u)),
 \end{aligned}$$

and hence, the result is true for all  $u \in H_{C_2}^{n\rho}(X)$  for any  $X \in \mathbf{Top}_*^{C_2}$ .

Since the squaring operations are stable, the result extends to arbitrary  $\text{RO}(C_2)$ -graded cohomology classes. Moreover, since the geometric fixed-point functor  $\Phi$  commutes with suspensions (3.2), and any  $E \in \mathbf{Sp}_{2,\text{fin}}^{C_2}$  is equivalent to  $\Sigma^{-n}\Sigma_{C_2}^\infty X$  for some  $n$  and  $X \in \mathbf{Top}_*^{C_2}$ , we conclude the same for any  $u \in H_{C_2}^*(E)$ .  $\square$

#### 4. TOPOLOGICAL REALIZATION OF $\mathcal{A}^{\mathbb{R}}(1)$

We begin by proving Theorem 1.6, which identifies all possible  $\mathcal{A}^{\mathbb{R}}$ -module structures on  $\mathcal{A}^{\mathbb{R}}(1)$  up to isomorphism.

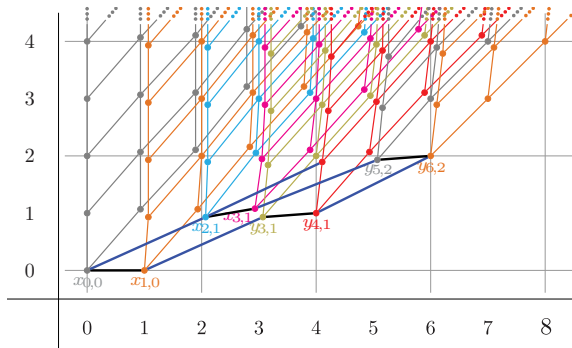


FIGURE 4.1. This figure displays the free  $\mathbb{M}_2^{\mathbb{R}}$ -module  $\mathcal{A}^{\mathbb{R}}(1)$ , with  $\text{Sq}^1$  and  $\text{Sq}^2$ -multiplications drawn in only on the  $\mathbb{M}_2^{\mathbb{R}}$ -module generators.

*Proof of Theorem 1.6.* Note that the Cartan formula of  $\mathcal{A}^{\mathbb{R}}$  and finiteness of  $\mathcal{A}^{\mathbb{R}}(1)$  imply that the  $\mathcal{A}^{\mathbb{R}}$ -module structure on  $\mathcal{A}^{\mathbb{R}}(1)$  is determined once the action of  $\text{Sq}^4$  and  $\text{Sq}^8$  are specified on its  $\mathbb{M}_2^{\mathbb{R}}$ -generators. The following are possible  $\text{Sq}^4$  and  $\text{Sq}^8$ -actions on the  $\mathbb{M}_2^{\mathbb{R}}$ -module generators. As can be seen in Figure 4.1, there is no room for other possible actions.

$$\begin{aligned} \text{Sq}^4(x_{0,0}) &= \beta_{03}(\rho \cdot y_{3,1}) + \beta_{04}(\tau \cdot y_{4,1}) + \alpha_{03}(\rho \cdot x_{3,1}) \\ \text{Sq}^4(x_{1,0}) &= \beta_{14}(\rho \cdot y_{4,1}) + \beta_{15}(y_{5,2}) \\ \text{Sq}^4(x_{2,1}) &= j_{24}(\rho^2 \cdot y_{4,1}) + \beta_{25}(\rho \cdot y_{5,2}) + \beta_{26}(\tau \cdot y_{6,2}) \\ \text{Sq}^4(x_{3,1}) &= \beta_{36}(\rho \cdot y_{6,2}) \\ \text{Sq}^4(y_{3,1}) &= \gamma_{36}(\rho \cdot y_{6,2}) \\ \text{Sq}^8(x_{0,0}) &= \beta_{06}(\rho^2 \cdot y_{6,2}) \end{aligned}$$

The Adem relation  $\text{Sq}^2\text{Sq}^3 = \text{Sq}^5 + \text{Sq}^4\text{Sq}^1 + \rho\text{Sq}^3\text{Sq}^1$  (see Proposition A.1), when applied to  $x_{0,0}$  and  $x_{2,1}$ , yields  $\beta_{15} = 1$ ,  $\beta_{03} + \beta_{04} + \beta_{14} = 1$  and  $\beta_{25} + \beta_{26} = \beta_{36}$ . The equation

$$j_{24} = \beta_{03}\gamma_{36} + \alpha_{03}\beta_{36},$$

is forced by the Adem relation  $\text{Sq}^4\text{Sq}^4 = \text{Sq}^2\text{Sq}^4\text{Sq}^2 + \tau\text{Sq}^3\text{Sq}^4\text{Sq}^1$  when applied to  $x_{0,0}$ . This exhausts all constraints imposed by Adem relations in these dimensions.  $\square$

In Theorem 1.6, there are exactly seven free variables taking values in  $\mathbb{F}_2$ , and therefore, there are exactly 128 different  $\mathcal{A}^{\mathbb{R}}$ -module structure on  $\mathcal{A}^{\mathbb{R}}(1)$ . Thus, in order to complete the proof of Theorem 1.3, we realize these  $\mathcal{A}^{\mathbb{R}}$ -modules as spectra using Theorem 2.11, which is a weak form of the  $\mathbb{R}$ -motivic Toda realization theorem.

*Proof of Theorem 1.3.* Firstly, note that  $\mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{R}}(1)$  is a cyclic  $\mathcal{A}^{\mathbb{R}}$ -module for all  $\overline{\mathbb{V}} \in \mathcal{V}$ , therefore  $\mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{C}}(1) := \mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{R}}(1)/(\rho)$  admits a May filtration. Secondly, note that

$$\text{gr}(\mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{C}}(1)) \cong \Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{1,0}, \xi_{1,1}, \xi_{2,0})$$

as an  $\text{gr}(\mathcal{A}^{\mathbb{C}})$ -module (see (2.7) for notation). Consequently,

$$(4.2) \quad \text{May} E_{1, \mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{C}}(1)}^{*,*,*} \cong \text{May} E_{1, \mathbb{M}_2^{\mathbb{C}}}^{*,*,*} / (\mathbf{h}_{1,0}, \mathbf{h}_{1,1}, \mathbf{h}_{2,0}) \cong \frac{\mathbb{M}_2^{\mathbb{C}}[\mathbf{h}_{i,j} : i \geq 1, j \geq 0]}{(\mathbf{h}_{1,0}, \mathbf{h}_{1,1}, \mathbf{h}_{2,0})}.$$

In the notation of Section 2.3

$$\mathcal{D}_{\mathcal{A}^{\mathbb{R}}(1)} = \{(0, 0), (1, 0), (2, 1), (3, 1), (4, 1), (5, 2), (6, 2)\}.$$

By directly inspecting the  $(s, f, w)$ -degree of  $\text{May} E_{1, \mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{C}}(1)}^{*,*,*}$ , we see that the condition necessary for existence in Theorem 2.11 is satisfied. Hence, the result.  $\square$

*Remark 4.3.* The vanishing region of  $\text{May} E_{1, \mathcal{A}_{\overline{\mathbb{V}}}^{\mathbb{C}}(1)}^{*,*,*}$  does not preclude the possibility of having a nonzero element in  $\text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{-1,2,0}(M, M)$ . We suspect (even after running the differentials in (2.3) and (2.5)), that the above group is nonzero for a given  $\mathcal{A}^{\mathbb{R}}$ -module structure on  $\mathcal{A}^{\mathbb{R}}(1)$ , and that there are, up to homotopy, multiple realizations as  $\mathbb{R}$ -motivic spectra.

Our next goal is to prove Theorem 1.8. We begin with the following observation.

**Lemma 4.4.** *The  $\mathcal{A}^{\mathbb{R}}$ -modules  $\mathcal{B}_2^{\mathbb{R}}(1)$  and  $\mathcal{B}_h^{\mathbb{R}}(1)$  are uniquely realizable as objects in  $\mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$ .*

*Proof.* Both  $\mathcal{B}_2^{\mathbb{R}}(1)$  and  $\mathcal{B}_h^{\mathbb{R}}(1)$  are cyclic  $\mathbb{M}_2^{\mathbb{R}}$ -free finite  $\mathcal{A}^{\mathbb{R}}$ -module and

$$\mathcal{B}_2^{\mathbb{R}}(1)/(\rho) \cong \mathcal{B}_h^{\mathbb{R}}(1)/(\rho)$$

as  $\mathcal{A}^{\mathbb{C}}$ -module. Let  $\mathcal{B}^{\mathbb{C}}(1) := \mathcal{B}_2^{\mathbb{R}}(1)/(\rho)$ . It is easy to see that  $\text{gr}(\mathcal{B}^{\mathbb{C}}(1))$  is isomorphic to  $\Lambda_{\mathbb{M}_2^{\mathbb{C}}}(\xi_{1,0}, \xi_{1,1})$  as an  $\text{gr}(\mathcal{A}^{\mathbb{C}})$ -module, and therefore

$$(4.5) \quad \text{May } E_{1,\mathcal{B}^{\mathbb{C}}(1)}^{*,*,*} \cong \mathbb{F}_2[\tau][h_{i,j} : i \geq 1, j \geq 0]/(h_{1,0}, h_{1,1}).$$

Using this, along with the fact that

$$\mathcal{D}_{\mathcal{B}^{\mathbb{C}}(1)} = \{(0, 0), (1, 0), (2, 1), (3, 1), \}$$

shows that the condition necessary for existence as well as uniqueness in Theorem 2.11 is satisfied. Hence, the result.  $\square$

*Proof of Theorem 1.8.* Consider the injective  $\mathcal{A}^{\mathbb{R}}$ -module map  $\Sigma^{3,1}\mathcal{B}_\epsilon^{\mathbb{R}}(1) \rightarrow \mathcal{A}_{\mathbb{V}}^{\mathbb{R}}$  sending the  $\mathbb{M}_2^{\mathbb{R}}$  generator in degree  $(3, 1)$  to  $x_{3,1} + y_{3,1}$ . It follows from Theorem 1.6 that the quotient is isomorphic to  $\mathcal{B}_\delta^{\mathbb{R}}(1)$ . Thus, we have the exact sequence (1.9).

The topological realization of (1.9), i.e. the cofiber sequence (1.10), would follow immediately from Lemma 4.4 once we show that any one of the  $\mathcal{A}^{\mathbb{R}}$ -module maps in (1.9) can be realized as a map in  $\mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$ . Thus, it is enough to show that the nonzero class in the  $E_2$ -page represented by the projection map  $\mathcal{A}_{\mathbb{V}}^{\mathbb{R}}(1) \rightarrow \mathcal{B}_\delta^{\mathbb{R}}(1)$  in degree  $(0, 0, 0)$  of the  $\mathbb{R}$ -motivic Adams spectral sequence

$$(4.6) \quad E_2^{s,f,w} := \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(\mathcal{A}_{\mathbb{V}}^{\mathbb{R}}(1), \mathcal{B}_\delta^{\mathbb{R}}(1)) \Rightarrow [\mathcal{Y}_{(\delta,1)}^{\mathbb{R}}, \mathcal{A}_1^{\mathbb{R}}[\overline{\mathbb{V}}]]_{s,w}$$

is a nonzero permanent cycle.

Using (4.2), the  $\rho$ -Bockstein spectral sequence (2.5) for  $\mathcal{A}_1^{\mathbb{R}}[\overline{\mathbb{V}}]$  and the Atiyah-Hirzebruch spectral sequence

$$\mathcal{B}_{\mathcal{B}_\delta^{\mathbb{R}}(1)}^{s',w'} \otimes \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s,f,w}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathbb{V}}], \mathbb{M}_2^{\mathbb{R}}) \Rightarrow \text{Ext}_{\mathcal{A}^{\mathbb{R}}}^{s+s',f,w+w'}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathbb{V}}], \mathcal{B}_\delta^{\mathbb{R}}(1)),$$

one can easily check  $E_2^{-1,f,0} = 0$  for all  $f \geq 2$ , and thus any nonzero element in degree  $(0, 0, 0)$  of the  $E_2$ -page in (4.6) is, in fact, a nonzero permanent cycle.  $\square$

Our next goal is to analyze the underlying spectrum and geometric fixed-points spectrum of  $\mathcal{A}_1^{\mathbb{C}^2}[\overline{\mathbb{V}}]$ , the Betti realization of  $\mathcal{A}_1^{\mathbb{R}}[\overline{\mathbb{V}}]$ .

**4.1. The Betti realization of  $\mathcal{A}_1^{\mathbb{R}}$ .** Under the Betti-realization map

$$(4.7) \quad \beta_* : \pi_{*,*} \mathbf{H}_{\mathbb{R}} \mathbb{F}_2 \cong \mathbb{F}_2[\rho, \tau] \longrightarrow \pi_* \mathbf{H} \mathbb{F}_2$$

$\rho \mapsto a_\sigma$  and  $\tau \mapsto u_\sigma$ . Since the functor  $\beta$  is symmetric monoidal and  $\beta(\mathbf{H}_{\mathbb{R}} \mathbb{F}_2) = \mathbf{H} \mathbb{F}_2$ , the  $i$ -th  $\mathbb{R}$ -motivic squaring operations maps to the  $i$ -th  $\text{RO}(C_2)$ -graded squaring operations under the map

$$\beta_* : \mathcal{A}^{\mathbb{R}} \longrightarrow \mathcal{A}^{C_2}.$$

Hence,  $\mathbf{H}_{C_2}^*(\mathcal{A}_1^{\mathbb{C}^2}[\overline{\mathbb{V}}])$  is  $\mathbb{M}_2^{C_2}$ -free (as  $\mathbf{H}_{\mathbb{R}}^{*,*}(\mathcal{A}_1^{\mathbb{R}}[\overline{\mathbb{V}}])$  is  $\mathbb{M}_2^{\mathbb{R}}$ -free) and its  $\mathcal{A}^{C_2}$ -module structure is essentially given by Theorem 1.6 (after replacing  $\text{Sq}^i$  with  $\underline{\text{Sq}}^i$  and  $\mathbb{M}_2^{\mathbb{R}}$ -basis elements by its image under  $\beta_*$ ).

*Remark 4.8.* The map  $\beta_*$  of (4.7) is only an injection with cokernel the summand  $\Theta\{u_{\sigma}^{-i}a_{\sigma}^{-j} : i, j \geq 0\}$  of  $\mathbb{M}_2^{\mathbb{C}^2}$ . In general, for an  $\mathcal{A}^{\mathbb{R}}(1)$ -module  $M_{\mathbb{R}}$ , the number of  $\mathcal{A}^{\mathbb{C}^2}$ -module structures on  $\beta(M_{\mathbb{R}})$  can be strictly larger than the number of  $\mathcal{A}^{\mathbb{R}}$ -module structures on  $M_{\mathbb{R}}$ . But this is not the case when  $M_{\mathbb{R}} = \mathcal{A}_{\nabla}^{\mathbb{R}}(1)$  simply for degree reasons, therefore Corollary 1.15 holds.

As discussed in Example 3.24, the restriction map

$$\mathfrak{R}_* : \mathbb{M}_2^{\mathbb{C}^2} \longrightarrow \mathbb{F}_2$$

sends  $a_{\sigma} \mapsto 0$ ,  $u_{\sigma} \mapsto 1$ , and  $\Theta \mapsto 0$ . Thus, when  $H_{\mathbb{C}_2}^*(E)$  is  $\mathbb{M}_2^{\mathbb{C}^2}$ -free,  $\mathfrak{R}_*$  is simply “setting  $a_{\sigma} = 0$ ,  $u_{\sigma} = 1$ , and  $\Theta = 0$ ”. This observation, along with Theorem 1.17, allows us to completely deduce the  $\mathcal{A}$ -module structure of  $H^*(\mathfrak{R}(\mathcal{A}_{\nabla}^{\mathbb{R}}(1)))$  from Theorem 1.6. Together with the fact that the  $\mathcal{A}$ -module structures on  $\mathcal{A}(1)$  are uniquely-realized, our observations yield Theorem 4.9, where the notation  $A_1[i, j]$  is adopted from [BEM].

**Theorem 4.9.** For  $\bar{v} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$  (as in Theorem 1.6),

$$\mathfrak{R}(\mathcal{A}_1^{\mathbb{C}^2}[\bar{v}]) \simeq A_1[1 + \beta_{03} + \beta_{14}, \beta_{26}].$$

Now we shift our attention towards understand the geometric fixed-points of  $\mathcal{A}_1^{\mathbb{C}^2}[\bar{v}]$ . As discussed in Example 3.26, the modified geometric fixed-points functor

$$\hat{\Phi}_* : \mathbb{M}_2^{\mathbb{C}^2} \longrightarrow \mathbb{F}_2$$

sends  $a_{\sigma} \mapsto 1$ ,  $u_{\sigma} \mapsto 0$ , and  $\Theta \mapsto 0$ . Thus, when  $H_{\mathbb{C}_2}^*(E)$  is  $\mathbb{M}_2^{\mathbb{C}^2}$ -free,  $\hat{\Phi}_*$  is simply “setting  $a_{\sigma} = 1$ ,  $u_{\sigma} = 0$ , and  $\Theta = 0$ ”. This, along with Theorem 1.6 and Theorem 1.19, gives the following.

*Notation 4.10.* Because  $H_{\mathbb{C}_2}^*(\mathcal{A}_1^{\mathbb{C}^2}[\bar{v}])$  is  $\mathbb{M}_2^{\mathbb{C}^2}$ -free, the  $\mathrm{HF}_2$ -cohomology of  $\Phi(\mathcal{A}_1^{\mathbb{C}^2}[\bar{v}])$  consists of eight  $\mathbb{F}_2$ -generators, all of which are in the image of  $\hat{\Phi}_*$ . We let

$$\begin{aligned} s_0 &:= \hat{\Phi}_*(x_{0,0}), & s_{1a} &:= \hat{\Phi}_*(x_{2,1}), & s_{1b} &:= \hat{\Phi}_*(x_{1,0}), & s_2 &:= \hat{\Phi}_*(y_{3,1}) \\ t_2 &:= \hat{\Phi}_*(x_{3,1}), & t_{3a} &:= \hat{\Phi}_*(y_{5,2}), & t_{3b} &:= \hat{\Phi}_*(y_{4,1}), & t_4 &:= \hat{\Phi}_*(y_{6,2}). \end{aligned}$$

Note that  $|s_{i(-)}| = |t_{i(-)}| = i$ .

**Theorem 4.11.** Let  $\bar{v} = (\alpha_{03}, \beta_{03}, \beta_{14}, \beta_{06}, \beta_{25}, \beta_{26}, \gamma_{36}) \in \mathcal{V}$ , and let

$$j_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25} + \beta_{26})$$

as in Theorem 1.6. The  $\mathcal{A}$ -module structure on  $H^*(\Phi(\mathcal{A}_1^{\mathbb{C}^2}[\bar{v}]))$  is determined by the following relations, as depicted in Figure 4.12:

- $\mathrm{Sq}^1(s_0) = s_{1a}$
- $\mathrm{Sq}^1(s_{1b}) = s_2$
- $\mathrm{Sq}^1(t_2) = t_{3a}$
- $\mathrm{Sq}^1(t_{3b}) = t_4$
- $\mathrm{Sq}^2(s_0) = \beta_{03}s_2 + \alpha_{03}t_2$
- $\mathrm{Sq}^2(s_{1a}) = \beta_{25}t_{3a} + j_{24}t_{3b}$
- $\mathrm{Sq}^2(s_{1b}) = t_{3a} + \beta_{14}t_{3b}$
- $\mathrm{Sq}^2(s_2) = \gamma_{36}t_4$
- $\mathrm{Sq}^2(t_2) = (\beta_{25} + \beta_{26})t_4$
- $\mathrm{Sq}^4(s_0) = \beta_{06}t_4$ .



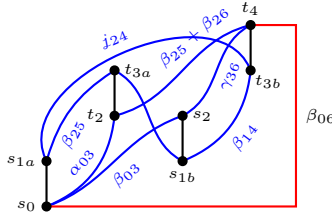


FIGURE 4.12. The  $\mathcal{A}$ -module  $H^*(\Phi(\mathcal{A}_1^{C_2}[\bar{v}]))$

5. AN  $\mathbb{R}$ -MOTIVIC ANALOGUE OF THE SPECTRUM  $\mathcal{Z}$

The type 2 spectrum  $\mathcal{Z} \in \mathbf{Sp}_{2,\text{fin}}$ , introduced in [BE], is defined by the property that its cohomology as an  $\mathcal{A}(2)$ -module is

$$\mathcal{B}(2) := \mathcal{A}(2) \otimes_{\Lambda(Q_2)} \mathbb{F}_2,$$

where  $Q_2 = [Sq^4, Q_1]$  is dual to the Milnor generator  $\xi_3$  of the dual Steenrod algebra. They first show that an  $\mathcal{A}$ -module structure on  $\mathcal{A}(2)$  satisfying the criteria in [BE, Lemma 2.7] leads to an  $\mathcal{A}$ -module structure on  $\mathcal{B}(2)$ . In [BE], the authors show that among the 1600 possible  $\mathcal{A}$ -module structures on  $\mathcal{A}(2)$  [R2], there are some  $\mathcal{A}$ -modules that satisfy [BE, Lemma 2.7]. Then they use the classical Toda realization theorem to show that any  $\mathcal{A}$ -module whose underlying  $\mathcal{A}(2)$ -module structure is  $\mathcal{B}(2)$  can be realized as a 2-local finite spectrum, which they call  $\mathcal{Z}$ .

We construct  $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  by emulating the construction of the classical  $\mathcal{Z}$  (as in [BE]) in the  $\mathbb{R}$ -motivic context. Since there is no a priori  $\mathcal{A}^{\mathbb{R}}$ -module structure on  $\mathcal{A}^{\mathbb{R}}(2)$ , we produce one in Section 5.1. In fact, we construct an  $\mathbb{R}$ -motivic spectrum whose cohomology is the desired  $\mathcal{A}^{\mathbb{R}}$ -module.

**5.1. A topological realization of  $\mathcal{A}^{\mathbb{R}}(2)$ .** Let  $\mathcal{A}^{\mathbb{R}}(2)$  denote the sub- $M_2^{\mathbb{R}}$ -algebra of the  $\mathbb{R}$ -motivic Steenrod algebra  $\mathcal{A}^{\mathbb{R}}$  generated by  $Sq^1, Sq^2$ , and  $Sq^4$ . We will use a method of Smith (exposed in [R1, Appendix C]) to construct an  $\mathbb{R}$ -motivic spectrum  $\mathcal{A}_2^{\mathbb{R}} \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  such that its cohomology as an  $\mathcal{A}^{\mathbb{R}}(2)$ -module is free on one generator.

Let  $h, \eta_{1,1}$  and  $\nu_{3,2}$  denote the first three  $\mathbb{R}$ -motivic Hopf-elements (these are denoted  $\omega, \eta$ , and  $\nu$  in [DI, Section 8]).

**Lemma 5.1.** *The  $\mathbb{R}$ -motivic Toda-bracket  $\langle h, \eta_{1,1}, \nu_{3,2} \rangle$  contains 0.*

*Proof.* In this argument, it will be convenient to refer to the “coweight”, by which we mean the difference  $s - w$ , as in [GI].

Since  $h$  and  $\eta_{1,1}$  have coweight 0 while  $\nu_{3,2}$  has coweight 1, it follows that the bracket  $\langle h, \eta_{1,1}, \nu_{3,2} \rangle$  is comprised of elements in stem 5 with coweight 2. The only element in stem 5 with coweight 2 is  $\rho \cdot \nu_{3,2}^2$  [BI]. Since this element is a  $\nu_{3,2}$  multiple, it lies in the indeterminacy, which means that the  $\mathbb{R}$ -motivic Toda-bracket does contain zero. □

Lemma 5.1 implies that we can construct a 4-cell complex  $\mathcal{K}$  whose cohomology as an  $\mathcal{A}^{\mathbb{R}}$ -module has the structure described in Corollary 5.2 and displayed in Figure 5.3.

**Corollary 5.2.** *There exists  $\mathcal{K} \in \mathbf{Sp}_{2,\text{fin}}^{\mathbb{R}}$  such that  $H_{\mathbb{R}}^{*,*}(\mathcal{K})$  is  $\mathbb{M}_2^{\mathbb{R}}$ -free on four generators  $x_0, x_1, x_3$  and  $x_7$ , such that  $\text{Sq}^{i+1}(x_i) = x_{2i+1}$  for  $i \in \{0, 1, 3\}$ .*

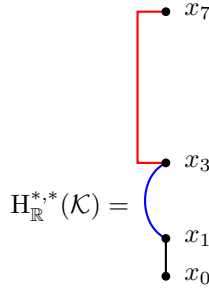


FIGURE 5.3. We depict the  $\mathcal{A}^{\mathbb{R}}$ -structure of  $H_{\mathbb{R}}^{*,*}(\mathcal{K})$  by marking the  $\text{Sq}^1$ -action by black straight lines, the  $\text{Sq}^2$ -action by blue curved lines, and the  $\text{Sq}^4$ -action by red lines between the  $\mathbb{M}_2^{\mathbb{R}}$ -generators.

Let  $e \in \mathbb{Z}_{(2)}[\Sigma_6]$  denote the idempotent corresponding to the Young tableaux

1	2	3
4	5	
6		

which is constructed as follows. Let  $\Sigma_{\text{Row}} \subset \Sigma_6$  denote the subgroup comprised of permutations that preserve each row. Likewise, let  $\Sigma_{\text{Col}}$  denote the subgroup comprised of column-preserving permutations. Let

$$(5.4) \quad R = \sum_{r \in \Sigma_{\text{Row}}} r \quad \text{and} \quad C = \sum_{c \in \Sigma_{\text{Col}}} (-1)^{\text{sign}(c)} c$$

and define

$$e = \frac{1}{\mu} R \cdot C,$$

where  $\mu$  is an odd integer defined in [R1, Theorem C.1.3]. We let  $\bar{e}$  denote the resulting idempotent in  $\mathbb{F}_2[\Sigma_6]$ .

**Proposition 5.5.** *The idempotent  $\bar{e} \in \mathbb{F}_2[\Sigma_6]$  has the property that  $\bar{e}(V^{\otimes 6}) = 0$  if  $\dim_{\mathbb{F}_2} V < 3$  and*

$$\dim_{\mathbb{F}_2} \bar{e}(V^{\otimes 6}) = \begin{cases} 8 & \text{if } \dim_{\mathbb{F}_2} V = 3, \\ 64 & \text{if } \dim_{\mathbb{F}_2} V = 4. \end{cases}$$

*Proof.* Let  $\bar{R}$  and  $\bar{C}$  denote the images of  $R$  and  $C$  in  $\mathbb{F}_2[\Sigma_6]$ , respectively. Then  $\bar{e} = \bar{R} \cdot \bar{C}$ . It is straightforward that  $\bar{C}$  vanishes on  $V^{\otimes 6}$  if  $\dim V \leq 2$ .

Now suppose that  $V$  has basis  $\{a, b, c\}$ . Then a basis for  $\bar{e}(V^{\otimes 6})$  is given by

$$\left\{ \bar{e} \left( \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \end{array} \right), \bar{e} \left( \begin{array}{ccc} b & a & c \\ a & c & b \\ c & b & a \end{array} \right), \bar{e} \left( \begin{array}{ccc} a & c & b \\ c & b & a \\ b & a & c \end{array} \right), \bar{e} \left( \begin{array}{ccc} c & a & b \\ b & c & a \\ a & b & c \end{array} \right), \bar{e} \left( \begin{array}{ccc} b & c & a \\ c & a & b \\ a & b & c \end{array} \right), \bar{e} \left( \begin{array}{ccc} a & b & c \\ b & a & c \\ c & a & b \end{array} \right), \bar{e} \left( \begin{array}{ccc} a & c & b \\ c & a & b \\ b & a & c \end{array} \right) \right\}.$$

Finally, suppose that  $\dim V = 4$  with basis  $\{a, b, c, d\}$ . For any subspace  $W \subset V$  spanned by three of these basis elements, the space  $\bar{e}(W^{\otimes 6})$  has dimension 8, as we have just seen. There are 4 choices of  $W$ , which together yield a 32-dimensional subspace of  $V^{\otimes 6}$ . Now consider Young tableaux in which all four basis elements appear and only one is repeated. In the case that  $d$  is repeated, we generate only two independent elements:

$$\bar{e} \left( \begin{array}{|c|c|c|} \hline a & c & d \\ \hline b & d & d \\ \hline d & & \\ \hline \end{array} \right) \quad \text{and} \quad \bar{e} \left( \begin{array}{|c|c|c|} \hline c & a & d \\ \hline b & d & d \\ \hline d & & \\ \hline \end{array} \right).$$

Allowing any basis element to be the repeating one, this gives an 8-dimensional subspace. Finally, we consider Young tableaux in which all four basis elements appear and two are repeated. In the case that  $c$  and  $d$  are repeated, we have the four elements

$$\bar{e} \left( \begin{array}{|c|c|c|} \hline a & c & d \\ \hline b & d & d \\ \hline c & & \\ \hline \end{array} \right), \bar{e} \left( \begin{array}{|c|c|c|} \hline a & c & d \\ \hline c & b & d \\ \hline d & & \\ \hline \end{array} \right), \bar{e} \left( \begin{array}{|c|c|c|} \hline b & d & c \\ \hline b & c & d \\ \hline d & & \\ \hline \end{array} \right), \quad \text{and} \quad \bar{e} \left( \begin{array}{|c|c|c|} \hline a & d & c \\ \hline d & b & d \\ \hline c & & \\ \hline \end{array} \right).$$

As there are  $\binom{4}{2} = 6$  such choices, this contributes another subspace of dimension  $4 \cdot 6 = 24$ . □

We define

$$\mathcal{A}_2^{\mathbb{R}} := \Sigma^{-5,-1}e(\mathcal{K}^{\wedge 6}) = \Sigma^{-5,-1}(\text{hocolim} \{ \mathcal{K}^{\wedge 6} \xrightarrow{e} \mathcal{K}^{\wedge 6} \xrightarrow{e} \dots \}),$$

which is a split summand of  $\Sigma^{-5,-1}\mathcal{K}^{\wedge 6}$  as  $e$  is an idempotent. We shift the grading by  $(-5, -1)$  to make sure that the  $\mathcal{A}^{\mathbb{R}}(2)$ -module generator of  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is in  $(0, 0)$  (see Remark 5.12).

**Theorem 5.6.**  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}}) \cong \mathcal{A}^{\mathbb{R}}(2)$  as an  $\mathcal{A}^{\mathbb{R}}(2)$ -module.

*Proof.* By [BGL, Corollary 2.2],  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is a free  $\mathcal{A}^{\mathbb{R}}(2)$ -module if and only if  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is free as an  $\mathbb{M}_2^{\mathbb{R}}$ -module and the Margolis homology  $\mathcal{M}(H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2, x)$  vanishes for  $x \in \{Q_0^{\mathbb{R}}, Q_1^{\mathbb{R}}, \bar{P}_1^1, Q_2^{\mathbb{R}}, \bar{P}_2^1\}$ , where  $\bar{P}_1^1$  and  $\bar{P}_2^1$  are the elements in  $\mathcal{A}^{\mathbb{R}}$  dual to  $\xi_1$  and  $\xi_2$ , respectively.

Let  $K_{\mathbb{R}} := H_{\mathbb{R}}^{*,*}(\mathcal{K})$ . The  $\mathcal{A}^{\mathbb{R}}$ -module  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is  $\mathbb{M}_2^{\mathbb{R}}$ -projective as it is a summand of

$$H_{\mathbb{R}}^{*,*}(\Sigma^{-5}\mathcal{K}^{\wedge 6}) \cong \Sigma^{-5}K_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6},$$

which is  $\mathbb{M}_2^{\mathbb{R}}$ -free. However,  $\mathbb{M}_2^{\mathbb{R}}$  is a graded local ring, and over a local ring, being projective is equivalent to being free. Hence,  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is  $\mathbb{M}_2^{\mathbb{R}}$ -free. Since  $Q_0^{\mathbb{R}}, Q_1^{\mathbb{R}}, Q_2^{\mathbb{R}}, \bar{P}_1^1$ , and  $\bar{P}_2^1$  are primitive modulo  $(\rho, \tau)$ , and for  $K := K_{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2$ ,  $i \in \{0, 1, 2\}$  and  $t \in \{1, 2\}$

$$\dim_{\mathbb{F}_2} \mathcal{M}(K, Q_i^{\mathbb{R}}) = 2 = \dim_{\mathbb{F}_2} \mathcal{M}(K, \bar{P}_t^1),$$

it follows from Proposition 5.5 that

$$\mathcal{M}(H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{F}_2, x) \cong \mathcal{M}(\bar{e}(K^{\otimes 6}), x) \cong \bar{e}(\mathcal{M}(K, x)^{\otimes 6}) = 0$$

for  $x \in \{Q_0^{\mathbb{R}}, Q_1^{\mathbb{R}}, \bar{P}_1^1, Q_2^{\mathbb{R}}, \bar{P}_2^1\}$ . Thus,  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is free over  $\mathcal{A}^{\mathbb{R}}(2)$ . Proposition 5.5 also implies that the  $\mathbb{M}_2^{\mathbb{R}}$ -rank of  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  is 64, and therefore  $H_{\mathbb{R}}^{*,*}(\mathcal{A}_2^{\mathbb{R}})$  has rank 1 over  $\mathcal{A}^{\mathbb{R}}(2)$ . □

5.2. **An  $\mathbb{R}$ -motivic lift of  $\mathcal{B}(2)$ .** Let  $\tilde{Q}_2^{\mathbb{R}} = [\mathrm{Sq}^4, Q_1^{\mathbb{R}}]$ . Unlike the classical Steenrod algebra,  $Q_2^{\mathbb{R}}$  does not agree with  $\tilde{Q}_2^{\mathbb{R}}$ . Instead, as in [V, Example 13.7], these are related by the formula

$$Q_2^{\mathbb{R}} = [\mathrm{Sq}^4, Q_1^{\mathbb{R}}] + \rho \mathrm{Sq}^5 \mathrm{Sq}^1.$$

However, one can check that both  $Q_2^{\mathbb{R}}$  and  $\tilde{Q}_2^{\mathbb{R}}$  square to zero, hence generate exterior algebras. We define (left)  $\mathcal{A}^{\mathbb{R}}(2)$ -modules

$$\mathcal{B}^{\mathbb{R}}(2) := \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(Q_2^{\mathbb{R}})} \mathbb{M}_2^{\mathbb{R}}$$

and

$$(5.7) \quad \tilde{\mathcal{B}}^{\mathbb{R}}(2) := \mathcal{A}^{\mathbb{R}}(2) \otimes_{\Lambda(\tilde{Q}_2^{\mathbb{R}})} \mathbb{M}_2^{\mathbb{R}}.$$

Let  $A_2^{\mathbb{R}}$  denote  $H_{\mathbb{R}}^{*,*}(A_2^{\mathbb{R}})$ . It is easy to check that the left ideal generated by  $Q_2^{\mathbb{R}}$  (likewise  $\tilde{Q}_2^{\mathbb{R}}$ ) in  $\mathcal{A}^{\mathbb{R}}(2)$  is isomorphic to  $\Sigma^{7,3}\mathcal{B}^{\mathbb{R}}(2)$  (likewise  $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ ). It follows that there is an exact sequence of  $\mathcal{A}^{\mathbb{R}}(2)$ -modules

$$(5.8) \quad 0 \longrightarrow \Sigma^{7,3}B_{\mathbb{R}} \xleftarrow{\iota} A_2^{\mathbb{R}} \xrightarrow{\pi} B_{\mathbb{R}} \longrightarrow 0,$$

where  $B_{\mathbb{R}}$  is either  $\mathcal{B}^{\mathbb{R}}(2)$  or  $\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ . The main purpose of this section is to show that:

**Lemma 5.9.** *There exists an exact sequence of  $\mathcal{A}^{\mathbb{R}}$ -modules whose underlying  $\mathcal{A}^{\mathbb{R}}(2)$ -module exact sequence is isomorphic to (5.8) with  $B_{\mathbb{R}} \cong \tilde{\mathcal{B}}^{\mathbb{R}}(2)$ .*

*Remark 5.10.* In the case of  $B_{\mathbb{R}} = \mathcal{B}^{\mathbb{R}}(2)$ , the image of  $\Sigma^{7,1}\mathcal{B}^{\mathbb{R}}(2) \rightarrow A_2^{\mathbb{R}}$  is a sub- $\mathcal{A}^{\mathbb{R}}(2)$ -module, but not a sub- $\mathcal{A}^{\mathbb{R}}$ -module. See Remark 5.15 for more details.

Lemma 5.9 and Remark 5.10 are direct consequences of the  $\mathcal{A}^{\mathbb{R}}$ -module structure of  $A_2^{\mathbb{R}}$  which can be deduced from the injection

$$\Sigma^{5,1}A_2^{\mathbb{R}} \hookrightarrow K_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6},$$

where  $K_{\mathbb{R}} = H_{\mathbb{R}}^{*,*}(K_{\mathbb{R}})$ . We do not want to entirely leave this calculation to the reader because, without a few tricks, this calculation is likely to require computer assistance as  $e$  has 144 elements in its expression (in terms of the standard  $\mathbb{F}_2$ -generators of  $\mathbb{F}_2[\Sigma_3]$ ) and  $K_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6}$  has  $2^{12}$  elements in its  $\mathbb{M}_2^{\mathbb{R}}$ -basis. We begin after setting the following notation.

*Notation 5.11.* Let  $x_i$  denote the  $\mathbb{M}_2^{\mathbb{R}}$ -generators of  $K_{\mathbb{R}}$  in degree  $i$  as in Corollary 5.2. We use the numbered Young diagram (abbrev. NYD)

$$\begin{array}{|c|c|c|} \hline i_1 & i_2 & i_3 \\ \hline i_4 & i_5 & \\ \hline i_6 & & \\ \hline \end{array}$$

to denote the  $\mathbb{M}_2^{\mathbb{R}}$ -basis element  $x_{i_1} \otimes \cdots \otimes x_{i_6} \in K_{\mathbb{R}}^{\otimes_{\mathbb{M}_2^{\mathbb{R}}} 6}$ , where  $i_j \in \{0, 1, 3, 7\}$ .

As in Proposition 5.5, let  $\bar{R}$  and  $\bar{C}$  denote the images of  $R$  and  $C$  (see (5.4)) in  $\mathbb{F}_2[\Sigma_6]$ , respectively. Since  $\bar{e} = \bar{R} \cdot \bar{C}$ , we record a few properties of  $\bar{R}$  and  $\bar{C}$ . Note that  $\bar{R}$  annihilates an NYD if it has repeating digits in a row. Likewise,  $\bar{C}$  annihilates an NYD if there are repeating digits in a column. For instance,

$$\bar{R}\left(\begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 3 & 7 & \\ \hline 3 & & \\ \hline \end{array}\right) = 0 = \bar{C}\left(\begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 3 & 7 & \\ \hline 3 & & \\ \hline \end{array}\right).$$

*Remark 5.12.* The lowest degree NYD which is not annihilated by  $\bar{e}$  is

$$\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 1 & \\ \hline 3 & & \\ \hline \end{array}$$

which lives in degree  $(5, 1)$ . Of course, there are multiple NYD's in bidegree  $(5, 1)$  not annihilated by  $\bar{e}$  but their images are the same. Likewise, the NYD of the highest degree not annihilated by  $\bar{e}$  is

$$\begin{array}{|c|c|c|} \hline 7 & 7 & 7 \\ \hline 3 & 3 & \\ \hline 1 & & \\ \hline \end{array}$$

which lives in bidegree  $(28, 11)$ .

The lowest degree element  $\iota := \bar{e}(\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 1 & \\ \hline 3 & & \\ \hline \end{array})$ , which serves as the  $\mathcal{A}^{\mathbb{R}}$ -module generator of  $\Sigma^{5,1}A_2^{\mathbb{R}}$ , can also be expressed as

$$\iota = \bar{R}(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array})$$

because the other NYDs present in the expression  $\bar{C}(\begin{array}{|c|c|c|} \hline 0 & 0 & 0 \\ \hline 1 & 1 & \\ \hline 3 & & \\ \hline \end{array})$  are annihilated by  $\bar{R}$ .

Since the  $\mathbb{R}$ -motivic Steenrod algebra is cocommutative we get

$$\bar{R}(\bar{C}(\text{Sq}^i(-))) = \bar{R}(\text{Sq}^i(\bar{C}(-))) = \text{Sq}^i(\bar{R}(\bar{C}(-))).$$

This, along with the Cartan formula, allows us to calculate  $a \cdot \iota$  for any  $a \in \mathcal{A}^{\mathbb{R}}$ , fairly easily. For example,

$$\begin{aligned} \text{Sq}^1 \cdot \iota &= \bar{R}(\text{Sq}^1(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array})) \\ &= \bar{R}(\begin{array}{|c|c|c|} \hline \cancel{3} & \cancel{1} & 1 \\ \hline \cancel{1} & \cancel{0} & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & 1 & 0 \\ \hline \cancel{1} & 1 & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 1 & & \\ \hline \end{array}) \\ &= \bar{R}(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 1 & & \\ \hline \end{array}), \\ \text{Sq}^2 \cdot \iota &= \bar{R}(\text{Sq}^2(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array})) \\ &= \bar{R}(\begin{array}{|c|c|c|} \hline \cancel{3} & \cancel{3} & 0 \\ \hline \cancel{1} & \cancel{0} & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 3 & 0 & \\ \hline 0 & & \\ \hline \end{array} + \tau(\begin{array}{|c|c|c|} \hline \cancel{3} & \cancel{1} & 1 \\ \hline \cancel{1} & \cancel{1} & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & 1 & 1 \\ \hline \cancel{1} & 0 & \\ \hline 1 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & 1 & 0 \\ \hline \cancel{1} & 1 & \\ \hline 1 & & \\ \hline \end{array})) \\ &= \bar{R}(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 3 & 0 & \\ \hline 0 & & \\ \hline \end{array}), \\ \text{Sq}^4 \cdot \iota &= \bar{R}(\text{Sq}^4(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array})) \\ &= \bar{R}(\begin{array}{|c|c|c|} \hline \cancel{7} & \cancel{1} & 0 \\ \hline \cancel{1} & \cancel{0} & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & \cancel{3} & 0 \\ \hline \cancel{3} & 0 & \\ \hline 0 & & \\ \hline \end{array} + \tau(\begin{array}{|c|c|c|} \hline \cancel{3} & \cancel{3} & 1 \\ \hline \cancel{1} & \cancel{1} & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & 3 & 1 \\ \hline \cancel{1} & 0 & \\ \hline 1 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & 3 & 0 \\ \hline \cancel{1} & 1 & \\ \hline 1 & & \\ \hline \end{array})) \\ &\quad + \tau(\begin{array}{|c|c|c|} \hline \cancel{3} & \cancel{1} & 1 \\ \hline \cancel{3} & \cancel{1} & \\ \hline 0 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline \cancel{3} & 1 & 1 \\ \hline \cancel{3} & 0 & \\ \hline 1 & & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 3 & 1 & \\ \hline 1 & & \\ \hline \end{array})) \\ &= \bar{R}(\begin{array}{|c|c|c|} \hline 7 & 1 & 0 \\ \hline 1 & 0 & \\ \hline 0 & & \\ \hline \end{array} + \tau(\begin{array}{|c|c|c|} \hline 3 & 1 & 0 \\ \hline 3 & 1 & \\ \hline 1 & & \\ \hline \end{array})). \end{aligned}$$

In this way, we calculate

$$\begin{aligned} \tilde{Q}_2^{\mathbb{R}} \cdot \iota &= \overline{R} \left( \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 0 \\ 7 & 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 1 & 0 \\ 0 \end{bmatrix} \right) \\ Q_2^{\mathbb{R}} \cdot \iota &= \overline{R} \left( \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 0 \\ 7 & 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 1 \\ 1 & 0 \\ 0 \end{bmatrix} + \rho \begin{bmatrix} 3 & 1 & 0 \\ 3 & 1 \\ 3 \end{bmatrix} \right), \end{aligned}$$

where the details are left to the reader.

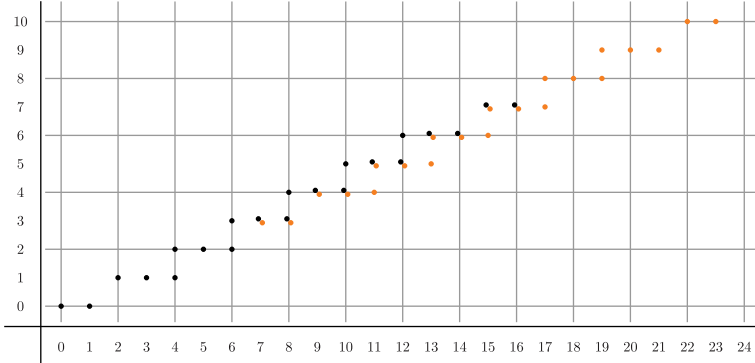


FIGURE 5.13.  $M_2^{\mathbb{R}}$ -module generators of  $\mathcal{A}^{\mathbb{R}}(2)$ . Black dots correspond to generators of  $\tilde{\mathcal{B}}^{\mathbb{R}}(2)$  and orange dots to  $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$ .

*Remark 5.14.* We record (see Figure 5.13), in the notation introduced in Section 2.1, that

$$\begin{aligned} \mathcal{D}_{\tilde{\mathcal{B}}^{\mathbb{R}}(2)} &= \{(0, 0), (1, 0), (2, 1), (3, 1), (4, 1), (4, 2), (5, 2), (6, 2), (6, 3), (7, 3), (8, 3), \\ &\quad (8, 4), (9, 4), (10, 4), (10, 5), (11, 5), (12, 5), (12, 6), (13, 6), (14, 6), (15, 7), (16, 7)\} \\ \text{and } \mathcal{D}_{A_2^{\mathbb{R}}} &= \{(i + 7\epsilon, j + 3\epsilon) : (i, j) \in \mathcal{D}_{\tilde{\mathcal{B}}^{\mathbb{R}}(2)} \text{ and } \epsilon \in \{0, 1\}\}. \end{aligned}$$

*Proof of Lemma 5.9.* Recall that the image of  $\Sigma^{7,3}\tilde{\mathcal{B}}^{\mathbb{R}}(2)$  in (5.8) is the (left)  $\mathcal{A}^{\mathbb{R}}(2)$ -submodule of  $A_2^{\mathbb{R}}$  generated by  $\tilde{Q}_2^{\mathbb{R}}$ . We must check that this is closed under the action of  $\mathcal{A}^{\mathbb{R}}$ . Since  $Sq^1, Sq^2, Sq^4$  are in  $\mathcal{A}^{\mathbb{R}}(2)$ , it remains to check that for all  $i \geq 3$  and  $a \in \mathcal{A}^{\mathbb{R}}(2)$

$$Sq^{2^i} \cdot (a\tilde{Q}_2^{\mathbb{R}} \cdot \iota) = b\tilde{Q}_2^{\mathbb{R}} \cdot \iota$$

for some  $b \in \mathcal{A}^{\mathbb{R}}(2)$ . For degree reasons (see Remark 5.14), we only need to consider the case when  $i = 3$  and  $a \in \{1, Sq^1, Sq^2\}$ . We check

$$\begin{aligned} Sq^8 \cdot (\tilde{Q}_2^{\mathbb{R}} \cdot \iota) &= (Sq^4Sq^4 + Sq^4Sq^2Sq^2)\tilde{Q}_2^{\mathbb{R}} \cdot \iota \\ Sq^8 \cdot (Sq^1\tilde{Q}_2^{\mathbb{R}} \cdot \iota) &= (Sq^7Sq^2 + Sq^2Sq^7)Sq^1\tilde{Q}_2^{\mathbb{R}} \cdot \iota \\ Sq^8 \cdot (Sq^2\tilde{Q}_2^{\mathbb{R}} \cdot \iota) &= (Sq^4Sq^4Sq^2 + Sq^4Sq^2Sq^4 + \tau Sq^5Sq^4Sq^1)Sq^2\tilde{Q}_2^{\mathbb{R}} \cdot \iota \end{aligned}$$

and thus the result holds. □

*Remark 5.15.* We notice that

$$Sq^8 \cdot (Q_2^{\mathbb{R}} \cdot \iota) = \overline{R} \left( \begin{bmatrix} 7 & 3 & 0 \\ 3 & 0 \\ 7 \end{bmatrix} + \begin{bmatrix} 7 & 3 & 0 \\ 7 & 3 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} 7 & 3 & 1 \\ 7 & 1 \\ 1 \end{bmatrix} + \rho \left( \begin{bmatrix} 7 & 1 & 0 \\ 7 & 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 7 & 1 & 0 \\ 3 & 1 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 0 \\ 7 & 1 \\ 7 \end{bmatrix} \right) \right)$$

cannot be equal to  $bQ_2^{\mathbb{R}} \cdot \iota$  for any  $b \in \mathcal{A}^{\mathbb{R}}(2)$ . This is an easy but tedious calculation. For the convenience of the reader, we note that an  $\mathbb{F}_2$ -basis for the elements in degree  $|\mathrm{Sq}^8| = (8, 4)$  of  $\mathcal{A}^{\mathbb{R}}(2)$  is given by

$$\{\mathrm{Sq}^6\mathrm{Sq}^2, \tau\mathrm{Sq}^7\mathrm{Sq}^1, \tau\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1, \rho\mathrm{Sq}^7, \rho\mathrm{Sq}^6\mathrm{Sq}^1, \rho\mathrm{Sq}^5\mathrm{Sq}^2, \rho\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1, \rho^2\mathrm{Sq}^5\mathrm{Sq}^1\}.$$

**5.3. The construction of  $\mathcal{Z}_{\mathbb{R}}$ .** Recall the  $\mathcal{A}^{\mathbb{R}}$ -module  $\tilde{B}_2^{\mathbb{R}}$ , as given in (5.7), and let

$$B_2^{\mathbb{C}} := \tilde{B}_2^{\mathbb{R}}/(\rho).$$

*Proof of Theorem 1.24.* Since  $B_2^{\mathbb{C}}$  is cyclic as an  $\mathcal{A}^{\mathbb{C}}$ -module, it admits a May filtration, whose associated graded is isomorphic to

$$\mathrm{gr}(B_2^{\mathbb{C}}) \cong \Lambda(\xi_{1,0}, \xi_{1,1}, \xi_{1,2}, \xi_{2,0}, \xi_{2,1})$$

and whose  $E_2$ -page of the corresponding May spectral sequence is isomorphic to

$$(5.16) \quad \mathrm{May}E_{1, B_2^{\mathbb{C}}}^{*,*,*,*} \cong \frac{\mathbb{M}_2^{\mathbb{C}}[h_{i,j} : i \geq 1, j \geq 0]}{(h_{1,0}, h_{1,1}, h_{1,2}, h_{2,0}, h_{2,1})}.$$

From this and Remark 5.14, one easily checks that the condition for Theorem 2.11 is satisfied. Thus, there exists  $\mathcal{Z}_{\mathbb{R}} \in \mathbf{Sp}_{2,\mathrm{fin}}^{\mathbb{R}}$  such that  $H_{\mathbb{R}}^{*,*}(\mathcal{Z}_{\mathbb{R}}) \cong \tilde{B}_2^{\mathbb{R}}$ .  $\square$

*Remark 5.17.* Since, as an  $\mathcal{A}(2)$ -module

$$H^*(\mathfrak{R}(\beta(\mathcal{Z}_{\mathbb{R}}))) \cong \mathfrak{R}_*(\beta_*(H_{\mathbb{R}}^{*,*}(\mathcal{Z}_{\mathbb{R}}))) \cong \mathcal{B}(2),$$

the underlying spectrum of  $\beta(\mathcal{Z}_{\mathbb{R}})$  is indeed one of the spectra  $\mathcal{Z}$  considered in [BE], and therefore of type 2.

APPENDIX A. THE  $\mathbb{R}$ -MOTIVIC ADEM RELATIONS

Voevodsky established the motivic version of the Adem relations [V, Section 10]. However, his formulas contain some typos, so for the convenience of the reader, we here present the Adem relations, in the  $\mathbb{R}$ -motivic case.

**Proposition A.1.** *In the  $\mathbb{R}$ -motivic Steenrod algebra  $\mathcal{A}^{\mathbb{R}}$ , the product  $\mathrm{Sq}^a\mathrm{Sq}^b$  is equal to*

(1) *(a and b both even)*

$$\sum_{j=0}^{a/2} \tau^{j \bmod 2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

(2) *(a odd and b even)*

$$\sum_{j=0}^{(a-1)/2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \rho \binom{b-j}{a-2j} \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j.$$

(3) *(a even and b odd)*

$$\sum_{j=0}^{a/2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j + \rho \binom{b-1-j}{a+1-2j} \mathrm{Sq}^{a+b-j-1} \mathrm{Sq}^j.$$

(4) *(a and b both odd)*

$$\sum_{j=0}^{(a-1)/2} \binom{b-1-j}{a-2j} \mathrm{Sq}^{a+b-j} \mathrm{Sq}^j$$

*Remark A.2.* Given that  $\mathrm{Sq}^a = \mathrm{Sq}^1 \mathrm{Sq}^{a-1}$  if  $a$  is odd and also that  $\mathrm{Sq}^1(\tau) = \rho$ , cases (2) and (4) follow from (1) and (3), respectively. Note also that (1) is the classical formula, but with  $\tau$  thrown in whenever needed to balance the weights. In formula (2), the left term appears only when  $j$  is even, while the second appears only when  $j$  is odd. In formula (3), the second term appears only when  $j$  is odd.

**Example A.3.** Some examples of the  $\mathbb{R}$ -motivic Adem relation in low degrees are

$$\mathrm{Sq}^2 \mathrm{Sq}^2 = \tau \mathrm{Sq}^3 \mathrm{Sq}^1, \quad \mathrm{Sq}^3 \mathrm{Sq}^2 = \rho \mathrm{Sq}^3 \mathrm{Sq}^1,$$

and

$$\mathrm{Sq}^2 \mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1 + \rho \mathrm{Sq}^3 \mathrm{Sq}^1.$$

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