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COMPACT DIFFERENCE OF COMPOSITION OPERATORS ON THE HARDY SPACES

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ABSTRACT. Answering to a long-standing question raised by Shapiro and Sundberg in 1990, Choe et al. have recently obtained a characterization for compact differences of composition operators acting on the Hilbert-Hardy space over the unit disk. Their characterization is described in terms of certain Bergman-Carleson measures involving derivatives of the inducing maps. In this paper, based on such results, we take one step further to obtain a completely new characterization, which is more intuitive and much simpler. In particular, our new characterization does not involve derivatives of the inducing maps and includes the Reproducing Kernel Thesis characterization. Moreover, our proofs are constructive enough to yield optimal estimates for the essential norms.

1. INTRODUCTION

Let $\mathcal{S}(\mathbf{D})$ be the class of all holomorphic self-maps of the unit disk $\mathbf{D}$ of the complex plane. Each $\varphi \in \mathcal{S}(\mathbf{D})$ induces a composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for functions f holomorphic on $\mathbf{D}$. It is clear that C_φ takes the space of holomorphic functions on $\mathbf{D}$ into itself. An extensive study on the theory of composition operators has been established during the past few decades on various settings. We refer to standard references [6] and [17] for various aspects on the theory of composition operators acting on classical holomorphic function spaces.

For $0 < p < \infty$, the Hardy space $H^p(\mathbf{D})$ is the space of all holomorphic functions f on $\mathbf{D}$ such that

$$\|f\|_{H^p} := \sup_{0<r<1} \left\{ \int_{\mathbf{T}} |f(r\zeta)|^p \, dm(\zeta) \right\}^{1/p} < \infty,$$

where m is the normalized arc-length measure on the unit circle $\mathbf{T} := \partial\mathbf{D}$. It is a classical result that to each function f in a Hardy space corresponds its boundary function f^* defined by the radial limits almost everywhere on $\mathbf{T}$. Also is well known that each space $H^p(\mathbf{D})$ is isometrically embedded in $L^p(\mathbf{T})$ via the boundary functions.

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Throughout the paper there will be many statements involving a pair of inducing maps. In order to make the notation as simple as possible, those inducing maps will always be denoted by φ and ψ. In addition, for such a pair, we will use the notation

$$\rho = \rho_{\varphi,\psi} := d(\varphi, \psi),$$

where d is the pseudohyperbolic distance; see Section 2.2. Since φ^* and ψ^* can coincide on a set of positive measure only when $\varphi = \psi$, we note that ρ also has its boundary function ρ^* defined by the radial limits almost everywhere on $\mathbf{T}$.

In 1981 Berkson [1] first found the isolation phenomenon for composition operators acting on $H^2(\mathbf{D})$. Berkson's isolation result was refined later by Shapiro and Sundberg [18], and also by MacCluer [11]. In the course of their study in [18], Shapiro and Sundberg noticed that if two composition operators on $H^2(\mathbf{D})$ form a compact difference, then they belong to the same path component in the space of composition operators. Based on such an observation, they were eventually led to the question of *whether two composition operators form a compact difference if they belong to the same path component*. Later their question was answered negatively; see [2], [7] and [13]. On the other hand, by such a negative result, the problem of characterizing compact differences of composition operators became more interesting and had been open until quite recently. In fact Choe et al. [3] have recently obtained a characterization (see Theorem 3.8), in terms of Carleson measures, for compact differences of composition operators on $H^2(\mathbf{D})$. Earlier Saukko found in [15, Theorem C] a sufficient condition for compact differences of composition operators acting from a Hardy space into a smaller one. In case the domain space and the target space are the same, Saukko's result can be rephrased (in terms of Carleson measures) as follows: *If the pullback measures* $(\rho^{*p}\, dm) \circ \varphi^{*-1}$ *and* $(\rho^{*p}\, dm) \circ \psi^{*-1}$ *are compact Hardy-Carleson measures, then* $C_\varphi - C_\psi$ *is compact on* $H^p(\mathbf{D})$ *for each* $p > 1$; see Section 2.4 for the notion of compact Hardy-Carleson measures. While this Saukko's sufficient condition should imply the equivalent conditions in Theorem 3.8, it is not clear at all to see any direct implication. As for us, we believe that Saukko's sufficient condition is *not* necessary in general, but do not have an explicit example. The purpose of the current paper is to obtain a new characterization for compact differences of composition operators on $H^2(\mathbf{D})$ in terms of integrals which are closely related to Saukko's sufficient condition. Our new characterization seems quite different from and much simpler than those in Theorem 3.8. In particular, our new characterization does not involve derivatives of the inducing maps.

In order to state our main result, we first set some notation to be used for the rest of the paper. Given $\varphi, \psi \in \mathcal{S}(\mathbf{D})$, $0 < p < \infty$ and $s \geq 0$, put

$$Q_s(a) := \{z \in \overline{\mathbf{D}} : |z - a| < 2^{s+1}(1 - |a|)\} \tag{1.1}$$

and define

$$\begin{aligned}\lambda_{p,s}(a) &= \lambda_{p,s}^{\varphi,\psi}(a) \\ &:= \int_{\varphi^{*-1}[Q_s(a)]} \left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\psi^*} \right|^p dm + \int_{\psi^{*-1}[Q_s(a)]} \left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\varphi^*} \right|^p dm\end{aligned}$$

for $a \in \mathbf{D}$.

Our main result is Theorem 1.1. In Assertion (c) below $k_{a,t}^{2,-1}$ denotes the H^2-normalized test function defined in (2.8).

Theorem 1.1. *Let* $\varphi, \psi \in \mathcal{S}(\mathbf{D})$. *Then the following assertions are equivalent:*

(a) $C_\varphi - C_\psi$ *is compact on* $H^2(\mathbf{D})$*;*
(b) $\lim_{|a|\to 1} \frac{\lambda_{2,s}(a)}{1-|a|} = 0$ *for some/any* $s \geq 0$*;*
(c) $\lim_{|a|\to 1} \left\|(C_\varphi - C_\psi)k_{a,t}^{2,-1}\right\|_{H^2} = 0$ *for some/any* $t > \frac{1}{2}$.

When $t = 1$, Assertion (c) is known as the Reproducing Kernel Thesis for compactness on $H^2(\mathbf{D})$ of differences of composition operators, which has been conjectured by many experts. As for the equivalence of Assertions (b) and (c), we actually obtain the H^p-version for arbitrary $0 < p < \infty$; see Theorem 3.1. Moreover, it turns out that Assertions (b) and (c) can be replaced by their p-analogues; see (4.2). Finally, one may combine Theorem 1.1 with a result of Nieminen and Saksman [14] (see Section 4.2) to obtain the H^p-version for $1 \leq p < \infty$; see Corollary 4.1.

It seems worth mentioning the special case $\psi \equiv 0$. In such a case, one may easily check that $\lambda_{p,s}(a)$ is comparable to $(m \circ \varphi^{*-1})[Q_s(a)]$ for all a sufficiently close to the boundary. So, Theorem 1.1 might be regarded as an extension of the well-known characterization for compact composition operators on $H^2(\mathbf{D})$; see Section 2.4. We also note that Theorem 1.1 has a clear connection with Saukko's sufficient condition mentioned above. To see it, note that Saukko's sufficient condition can be explicitly written as

$$(1.2) \qquad \int_{\varphi^{*-1}[Q_s(a)]} \left|\frac{\varphi^* - \psi^*}{1 - \overline{\varphi^*}\psi^*}\right|^p dm + \int_{\psi^{*-1}[Q_s(a)]} \left|\frac{\varphi^* - \psi^*}{1 - \overline{\psi^*}\varphi^*}\right|^p dm = o(1 - |a|)$$

as $|a| \to 1$; see Section 2.4. It is easily verified (see Section 4.1) that $\lambda_{p,s}(a)$ is dominated (up to a constant factor) by the left-hand side of the above. Thus Saukko's sufficient condition implies Assertion (b) in Theorem 1.1. We also remark in passing that, in case of the standard weighted Bergman spaces over $\mathbf{D}$, the corresponding analogues of (1.2) characterize the compactness of $C_\varphi - C_\psi$; see [9] or [15].

In view of Theorem 1.1 one may naturally expect that the essential norm of $C_\varphi - C_\psi$ (= its distance, in the operator norm, from the space of all compact operators on $H^2(\mathbf{D})$), denoted by $\|C_\varphi - C_\psi\|_e$, should be closely connected with the behavior of $\frac{\lambda_{2,s}(a)}{1-|a|}$ or/and $\left\|(C_\varphi - C_\psi)k_{a,t}^{2,-1}\right\|_{H^2}$ near the boundary. The next result asserts that this intuition is correct.

Theorem 1.2. *Let* $s \geq 0$*,* $t > \frac{1}{2}$ *and* $\varphi, \psi \in S(\mathbf{D})$*. Then*

$$\|C_\varphi - C_\psi\|_e^2 \approx \limsup_{|a|\to 1} \frac{\lambda_{2,s}(a)}{1 - |a|} \approx \limsup_{|a|\to 1} \left\|(C_\varphi - C_\psi)k_{a,t}^{2,-1}\right\|_{H^2}^2;$$

the constants suppressed here depend only on s*,* t*,* φ *and* ψ*.*

In the course of the proof of the second estimate in Theorem 1.2 we actually obtain its H^p-version for arbitrary $0 < p < \infty$; see Corollary 3.7.

In Section 2 we recall and collect some basic facts to be used in later sections. Section 3 is devoted to the proofs of Theorems 1.1 and 1.2. In Section 4 we collect remarks, which are related to our results, on the following topics: (i) Saukko's sufficient condition; (ii) Dependency on the parameter p of compact differences on $H^p(\mathbf{D})$; (iii) Compact differences on $H^2(\mathbf{D})$ of composition operators induced by maps of bounded multiplicity; and (iv) Equivalent representations of the integral $\lambda_{p,s}(a)$.

Constants. Throughout the paper we use the same letter C to denote various positive constants which may vary at each occurrence but do not depend on the essential parameters. Variables indicating the dependency of constants C will be often specified in parentheses. For nonnegative quantities X and Y the notation $X \lesssim Y$ or $Y \gtrsim X$ means $X \le CY$ for some inessential constant C. Similarly, we write $X \approx Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold.

2. Preliminaries

In this section we collect well-known basic facts to be used in later sections. One may find details in standard references such as [6] and [17], unless otherwise specified.

2.1. **Compact operator.** We clarify the notion of compact operators, since the spaces under consideration are not Banach spaces when $0 < p < 1$. Let X and Y be topological vector spaces whose topologies are induced by complete metrics. A continuous linear operator $L : X \to Y$ is said to be compact if the image of every bounded sequence in X has a convergent subsequence in Y.

For a linear combination of composition operators acting on the Hardy spaces, we have the following convenient compactness criterion.

Lemma 2.1. *Let $X = H^p(\mathbf{D})$ or $A^p_\alpha(\mathbf{D})$ for $\alpha > -1$ and $0 < p < \infty$. Let T be a linear combination of composition operators acting on X. Then T is compact on X if and only if $Tf_n \to 0$ in X for any bounded sequence $\{f_n\}$ in X such that $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$.*

The proof below, which is basically the same as the proof of [6, Proposition 3.11] for single composition, is included for completeness.

Proof. Assume that T is compact on X and let $\{f_n\}$ be a bounded sequence in X with $f_n \to 0$ uniformly on compact subsets of $\mathbf{D}$. Then $\{Tf_n\}$ has a subsequence which converges in X by compactness of T. Since $f_n \to 0$ on any compact subset of $\mathbf{D}$, we see that $Tf_n(z) \to 0$ at every $z \in \mathbf{D}$. Since this is true for any subsequence of $\{f_n\}$, we conclude that $Tf_n \to 0$ in X.

Conversely, let $\{g_n\}$ be any bounded sequence in X. By normality we may find a subsequence $\{g_{n_k}\}$ converging uniformly on compact subsets of $\mathbf{D}$ to some holomorphic function g. Note $g \in X$ by Fatou's Lemma. Note also that the sequence $\{g_{n_k} - g\}$ is bounded in X and converges to 0 uniformly on compact subsets of $\mathbf{D}$. Thus the hypothesis guarantees that $Tg_{n_k} \to Tg$ in X. The proof is complete. $\square$

2.2. **Pseudohyperbolic distance.** The pseudohyperbolic distance between $z, w \in \mathbf{D}$ is given by

$$d(z,w) := |\gamma_w(z)|,$$

where $\gamma_w(z) := \frac{w-z}{1-z\overline{w}}$ is the involutive automorphism of $\mathbf{D}$ that exchanges 0 and w. The explicit expression of $d(z,w)$ is given by the identity

$$1 - d^2(z,w) = \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}. \tag{2.1}$$

Since

$$\frac{1-|z|^2}{1-|w|^2} = \frac{1-|\gamma_w(z)|^2}{|1-\gamma_w(z)\overline{w}|^2} \le \frac{1+d(z,w)}{1-d(z,w)},$$

we note

$$\begin{aligned}\left|1-\left(\frac{1-\overline{a}z}{1-\overline{a}w}\right)\right| &= d(z,w)\frac{|a|}{|1-\overline{a}w|}\left[\frac{(1-|z|^2)(1-|w|^2)}{1-d^2(z,w)}\right]^{1/2} \qquad \text{by (2.1)}\\ &\le d(z,w)\left[\frac{1-|z|^2}{1-|w|^2}\cdot\frac{1}{1-d^2(z,w)}\right]^{1/2}\frac{1-|w|^2}{1-|w|}\\ &\le \frac{2d(z,w)}{1-d(z,w)}\end{aligned}$$

for all $a, z, w \in \mathbf{D}$. In particular, we obtain the following inequality, which is useful for our purpose:

$$\left|\frac{1-\overline{a}z}{1-\overline{a}w}\right| \le \frac{1+d(z,w)}{1-d(z,w)} \tag{2.2}$$

for all $a, z, w \in \mathbf{D}$.

2.3. **Littlewood-Paley Identity.** By Green's Theorem the norm of a function $f \in H^2(\mathbf{D})$ can be computed through an area integral as follows:

$$\|f\|_{H^2}^2 = |f(0)|^2 + \int_{\mathbf{D}} |f'(z)|^2 \log\frac{1}{|z|^2}\,dA(z),$$

where A is the normalized area measure on $\mathbf{D}$. It is this formula, known as the Littlewood-Paley Identity, which provides a useful connection between two Hilbert spaces $H^2(\mathbf{D})$ and $A_1^2(\mathbf{D})$ described below.

Given $\alpha > -1$, let A_α be the weighted area measure on $\mathbf{D}$ given by

$$dA_\alpha(z) := (\alpha+1)(1-|z|^2)^\alpha dA(z).$$

The α-weighted Bergman space $A_\alpha^p(\mathbf{D})$ is then the space of all holomorphic functions f on $\mathbf{D}$ such that

$$\|f\|_{A_\alpha^p} := \left\{\int_{\mathbf{D}} |f|^p\,dA_\alpha\right\}^{1/p} < \infty.$$

The space $A_\alpha^p(\mathbf{D})$ is a Banach space equipped with the norm above for $1 \le p < \infty$ and a complete metric space for $0 < p < 1$ with respect to the translation-invariant metric $(f,g) \mapsto \|f-g\|_{A_\alpha^p}^p$. In this paper we will need only the case $p = 2$ and $\alpha = 1$, but have included the general case for convenience later in some unified statements.

Now, since $\log|z|^{-2}$ is integrable near 0 and comparable to $1-|z|^2$ near boundary, the Littlewood-Paley Identity yields

$$\|f\|_{H^2} \approx |f(0)| + \|f'\|_{A_1^2}; \tag{2.3}$$

the constants suppressed here are independent of $f \in H^2(\mathbf{D})$.

2.4. **Hardy-Carleson measure.** Let μ be a positive finite Borel measure on $\overline{\mathbf{D}}$. For $0 < p < \infty$, we say that μ is a *(compact, resp.) H^p-Carleson* measure if the embedding $H^p(\mathbf{D}) \subset L^p(\mu)$ is (compact, resp.) bounded. Here, functions in $H^p(\mathbf{D})$ are identified with their radial extensions on $\overline{\mathbf{D}}$.

The (compact, resp.) H^p-Carleson measures are characterized by the bounded (vanishing, resp.) property of their averages over Carleson sets $S_\delta(\zeta)$ defined by

$$S_\delta(\zeta) := \{z \in \overline{\mathbf{D}} : |z - \zeta| < \delta\}$$

for $\delta > 0$ and $\zeta \in \mathbf{T}$. More explicitly, it is known that μ is a (compact, resp.) H^p-Carleson measure if and only if

$$\mu[S_\delta(\zeta)] = \mathcal{O}(\delta) \quad (o(\delta) \ \text{ resp.}) \qquad \text{uniformly in } \zeta \in \mathbf{T} \tag{2.4}$$

as $\delta \to 0$. So, the notion of (compact) H^p-Carleson measures is independent of the parameter p. Accordingly, H^p-Carleson measures will be simply called *Hardy-Carleson* measures.

We note that the Carleson sets mentioned above can be replaced by the sets $Q_s(a)$ introduced in (1.1). In fact we have

$$S_\delta(\zeta) \subset Q_s\big((1-\delta)\zeta\big) \subset S_{c\delta}(\zeta) \qquad \text{where } \ c = 2^{s+1} + 1$$

for all $\delta \in (0,1)$ and $\zeta \in \mathbf{T}$. We thus see that the (compact, resp.) Hardy-Carleson condition (2.4) is equivalent to

$$\sup_{a \in \mathbf{D}} \frac{\mu[Q_s(a)]}{1 - |a|} < \infty \quad \left(\lim_{|a| \to 1} \frac{\mu[Q_s(a)]}{1 - |a|} = 0 \ \text{ resp.} \right) \tag{2.5}$$

for some/any $s \geq 0$.

The connection between composition operators and Carleson measures comes from the measure theoretic change-of-variable formula (see [8, p. 163])

$$\int_{\mathbf{T}} (h \circ \varphi^*) \, dm = \int_{\overline{\mathbf{D}}} h \, d(m \circ \varphi^{*-1}) \tag{2.6}$$

valid for $\varphi \in \mathcal{S}(\mathbf{D})$ and positive Borel functions h on $\overline{\mathbf{D}}$. For example, since C_φ is bounded on the Hardy spaces, one may verify via the above formula that $m \circ \varphi^{*-1}$ is always a Hardy-Carleson measure.

2.5. **Bergman-Carleson measure.** Let μ be a positive finite Borel measure on $\mathbf{D}$. For $\alpha > -1$ and $0 < p < \infty$, as in the case of the Hardy-Carleson measures, we say that μ is a *(compact, resp.) A^p_α-Carleson* measure if the embedding $A^p_\alpha(\mathbf{D}) \subset L^p(d\mu)$ is (compact, resp.) bounded. The notion of (compact) A^p_α-Carleson measures is also independent of the parameter p. In fact, it is known that (compact) A^p_α-Carleson measures are characterized by the condition which is obtained by replacing δ by $\delta^{\alpha+2}$ in the right-hand side of (2.4). So, A^p_α-Carleson measures will be simply called *α-Bergman-Carleson* measures.

Here we recall another characterization for (compact) α-Bergman-Carleson measures, which is useful for our purpose. Given $\delta \in (0,1)$, put

$$\widehat{\mu}_{\alpha,\delta}(a) = \frac{\mu[E_\delta(a)]}{(1 - |a|)^{\alpha+2}}$$

for $a \in \mathbf{D}$ where $E_\delta(a)$ denotes the pseudohyperbolic disk with center a and radius δ. It is known by Luecking [10] that μ is a (compact, resp.) α-Bergman-Carleson measure if and only if

$$\sup_{a\in\mathbf{D}} \widehat{\mu}_{\alpha,\delta}(a) < \infty \quad \left(\lim_{|a|\to 1} \widehat{\mu}_{\alpha,\delta}(a) = 0 \ \text{ resp.}\right) \tag{2.7}$$

for some/any $\delta \in (0,1)$.

2.6. **Test functions.** Let $0 < p < \infty$ and $\alpha \geq -1$. Here, we put $A^p_{-1}(\mathbf{D}) := H^p(\mathbf{D})$ for unified statements below.

As is well known for the Hilbert space $A^2_\alpha(\mathbf{D})$, to each $a \in \mathbf{D}$ corresponds a unique reproducing kernel whose explicit formula is known as $z \mapsto K_a^{\alpha+2}(z)$ where

$$K_a(z) := \frac{1}{1-\bar{a}z}.$$

Powers of these functions will be the source of test functions in conjunction with Lemma 2.1. The norms of such kernel-type functions are well known. Namely, when $t > \frac{\alpha+2}{p}$, we have

$$\|K_a^t\|_{A^p_\alpha} \approx (1-|a|)^{-t+\frac{\alpha+2}{p}}, \qquad a \in \mathbf{D};$$

the constants suppressed in this estimate depend only on α, p and t; see, for example, [19, Theorem 1.12]. In view of this norm estimate, we put

$$k_{a,t}^{p,\alpha} := (1-|a|)^{t-\frac{\alpha+2}{p}} K_a^t, \qquad t > \frac{\alpha+2}{p} \tag{2.8}$$

for test functions normalized in $A^p_\alpha(\mathbf{D})$. We note in conjunction with Lemma 2.1 that

$$k_{a,t}^{p,\alpha} \to 0 \ \text{ uniformly on compact subsets of } \ \mathbf{D} \tag{2.9}$$

as $|a| \to 1$.

3. Proofs

In this section we prove Theorems 1.1 and 1.2. We first prove Theorem 3.1 of which the special case $p = 2$ is the equivalence of Assertions (b) and (c) in Theorem 1.1.

Theorem 3.1. *Let $0 < p < \infty$ and $\varphi, \psi \in \mathcal{S}(\mathbf{D})$. Then*

$$\lim_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|} = 0 \quad \textit{for some/any } s \geq 0 \tag{3.1}$$

if and only if

$$\lim_{|a|\to 1} \|(C_\varphi - C_\psi)k_{a,t}^{p,-1}\|_{H^p} = 0 \quad \textit{for some/any } t > \frac{1}{p}. \tag{3.2}$$

Before proceeding to the proof, we pause to notice some preliminary lemmas. For that purpose we introduce some auxiliary points associated with points in $\mathbf{D}$. Put

$$\check{a}_s := [1 - 2^s(1-|a|)]\frac{a}{|a|} \tag{3.3}$$

for $a \in \mathbf{D}$ and $s \geq 0$ with $1 - |a| < 2^{-s}$. Also, put

$$a_N := ae^{-iN(1-|a|)} \tag{3.4}$$

for $a \in \mathbf{D}$ and $N > 0$ with $N(1-|a|) < \pi$. These auxiliary points will play essential roles in our proofs.

Lemma 3.2, concerning the property of the auxiliary points $\check{a}_s$, will be used in our proof of the necessity in Theorem 3.1.

Lemma 3.2. *Let $t \ge s \ge 0$. Then there is a constant $r = r(s,t) \in (0,1)$ such that*

$$Q_t(a) \subset Q_s(\check{a}_{t-s+1}) \quad \text{and} \quad d(a, \check{a}_{t-s+1}) < r$$

for all $a \in \mathbf{D}$ with $1-|a| < 2^{-(t-s+1)}$.

Proof. Let $a \in \mathbf{D}$ and assume $1-|a| < 2^{-(t-s+1)}$. Note

$$a - \check{a}_{t-s+1} = (2^{t-s+1}-1)(1-|a|)\frac{a}{|a|}.$$

Thus, for $z \in Q_t(a)$, we have

$$\begin{aligned}
|z - \check{a}_{t-s+1}| &\le |z-a| + |a - \check{a}_{t-s+1}| \\
&< 2^{t+1}(1-|a|) + (2^{t-s+1}-1)(1-|a|) \\
&= (2^{t+1} + 2^{t-s+1} - 1)(1-|a|) \\
&< 2^{t+2}(1-|a|) \\
&= 2^{s+1}(1-|\check{a}_{t-s+1}|).
\end{aligned}$$

So, we conclude the first assertion of the lemma.

Meanwhile, note

$$1 - \overline{a}\check{a}_{t-s+1} = (1 + 2^{t-s+1}|a|)(1-|a|) \le (1+2^{t-s+1})(1-|a|).$$

We thus have

$$\begin{aligned}
1 - d^2(a, \check{a}_{t-s+1}) &= \frac{(1-|a|^2)(1-|\check{a}_{t-s+1}|^2)}{|1-\overline{a}\check{a}_{t-s+1}|^2} \\
&\ge \frac{(1-|a|)(1-|\check{a}_{t-s+1}|)}{(1+2^{t-s+1})^2(1-|a|)^2} \\
&= \frac{2^{t-s+1}}{(1+2^{t-s+1})^2}.
\end{aligned}$$

This implies the second assertion of the lemma. The proof is complete. □

Lemmas 3.3 and 3.4 will be used in the proof of Lemma 3.5, which is the main property of the auxiliary points a_N.

Lemma 3.3. *Let $s \ge 0$. Then there is a constant $C = C(s) > 0$ such that*

$$\left|\frac{z-w}{1-\overline{a}w}\right| \le Cd(z,w)$$

for $a \in \mathbf{D}$, $z \in Q_s(a)$ and $w \in \overline{\mathbf{D}}$.

Proof. Consider an arbitrary $a \in \mathbf{D}$. Let $z \in Q_s(a)$. Since

$$|1-\overline{a}z| \le 1 - |a|^2 + |\overline{a}||a-z| \le (2+2^{s+1})(1-|a|), \tag{3.5}$$

we have

$$
\begin{aligned}
|1-\overline{z}w|^{1/2} &\le |1-\overline{a}z|^{1/2} + |1-\overline{a}w|^{1/2} \\
&\le \big(2+2^{s+1}\big)^{1/2}\big(1-|a|\big)^{1/2} + |1-\overline{a}w|^{1/2} \\
&\le \Big[\big(2+2^{s+1}\big)^{1/2}+1\Big]|1-\overline{a}w|^{1/2}
\end{aligned}
$$

and thus

$$\left|\frac{z-w}{1-\overline{a}w}\right| \le \big[(2+2^{s+1})^{1/2}+1\big]^2 d(z,w)$$

for all $w \in \overline{\mathbf{D}}$. The proof is complete. □

Lemma 3.4. *Let $s \ge 0$ and $N > 0$. Then there is a constant $C = C(s,N) > 0$ such that*

$$\left|\frac{1-\overline{a_N}w}{1-\overline{a}w}\right| \le C$$

for all $a \in \mathbf{D}$ with $N(1-|a|) < \pi$ and $w \in \overline{\mathbf{D}}$.

Proof. Let $a \in \mathbf{D}$ and assume $N(1-|a|) < \pi$. Note

$$
\begin{aligned}
|1-\overline{a_N}a| &= \left|1-|a|^2 e^{iN(1-|a|)}\right| \\
&\le 1-|a|^2 + |a|^2\left|1-e^{iN(1-|a|)}\right| \\
&\le (N+2)(1-|a|).
\end{aligned}
$$

We thus have

$$
\begin{aligned}
|1-\overline{a_N}w|^{1/2} &\le |1-\overline{a_N}a|^{1/2} + |1-\overline{a}w|^{1/2} \\
&\le (\sqrt{N+2}+1)|1-\overline{a}w|^{1/2}
\end{aligned}
$$

for all $w \in \overline{\mathbf{D}}$. So, the lemma holds with $C = (\sqrt{N+2}+1)^2$. The proof is complete. □

Lemma 3.5 is the key to our proof of the sufficiency in Theorem 3.1.

Lemma 3.5. *Let $s \ge 0$ and $t > 0$. Then there are constants $N = N(s,t) > 0$ and $C = C(s,t) > 0$ such that*

$$\left|\frac{z-w}{1-\overline{a}w}\right| \le C\left(\left|1-\left(\frac{1-\overline{a}z}{1-\overline{a}w}\right)^t\right| + \left|1-\left(\frac{1-\overline{a_N}z}{1-\overline{a_N}w}\right)^t\right|\right)$$

for all $a \in \mathbf{D}$ with $|a| \ge \frac{1}{2}$ and $N(1-|a|) < \pi$, $z \in Q_s(a)$ and $w \in \overline{\mathbf{D}}$.

Proof. For $a \in \mathbf{D}$ with $|a| \ge \frac{1}{2}$, let $z \in Q_s(a)$ be arbitrary point throughout the proof. Let $w \in \overline{\mathbf{D}}$. We consider two cases separately: (i) $w \notin Q_{s+2}(a)$ and (ii) $w \in Q_{s+2}(a)$.

First, in Case (i), since

$$\left|\frac{1-\overline{a}z}{1-\overline{a}w}\right| \le \left|\frac{1-\overline{a}z}{w-a}\right| \le \frac{2+2^{s+1}}{2^{s+3}} \le \frac{1}{2}$$

by (3.5), we have by Lemma 3.3

$$\left|\frac{z-w}{1-\overline{a}w}\right| \le C \le C\left(1-\frac{1}{2^t}\right)^{-1}\left|1-\left(\frac{1-\overline{a}z}{1-\overline{a}w}\right)^t\right|,$$

where $C = C(s) > 0$ is the constant provided by Lemma 3.3. Thus the asserted inequality holds for Case (i).

Next, consider Case (ii). Note

$$|a_N - a| = |a| \left| e^{-iN(1-|a|)} - 1 \right| \ge \sin\left(\frac{N(1-|a|)}{2}\right) \ge \frac{N}{\pi}(1-|a|)$$

whenever $N > 0$ and $N(1-|a|) < \pi$; recall $|a| \ge \frac{1}{2}$ for the first inequality. Thus, taking

$$N := \pi\Big(10 \cdot 2^{s+M} + 2^{s+3}\Big),$$

where M is a positive number to be chosen later, we have

$$\begin{aligned}
|1 - \overline{a_N} w| &\ge |a_N - w| \\
&\ge |a_N - a| - |a - w| \\
&> |a| \left| e^{-iN(1-|a|)} - 1 \right| - 2^{s+3}(1-|a|) \\
&\ge \Big(\frac{N}{\pi} - 2^{s+3}\Big)(1-|a|) \\
&= 10 \cdot 2^{s+M}(1-|a|) \qquad (3.6)
\end{aligned}$$

for all $a \in \mathbf{D}$ with $N(1-|a|) < \pi$. Since

$$|z - w| \le |z - a| + |w - a| < 10 \cdot 2^s (1-|a|),$$

we obtain by (3.6)

$$\left| 1 - \left(\frac{1 - \overline{a_N} z}{1 - \overline{a_N} w} \right) \right| = |a| \left| \frac{z - w}{1 - \overline{a_N} w} \right| < \frac{1}{2^M}.$$

Now, choosing $M = M(t) > 0$ so large that

$$1 - b^t \neq 0 \quad \text{whenever} \;\; 0 < |1 - b| \le \frac{1}{2^M}$$

and setting

$$C_t := \sup_{|1-b| \le \frac{1}{2^M}} \left| \frac{1 - b}{1 - b^t} \right| < \infty,$$

we obtain

$$\left| \frac{z - w}{1 - \overline{a_N} w} \right| \le 2 \left| 1 - \left(\frac{1 - \overline{a_N} z}{1 - \overline{a_N} w} \right) \right| \le 2C_t \left| 1 - \left(\frac{1 - \overline{a_N} z}{1 - \overline{a_N} w} \right)^t \right|$$

and thus conclude the lemma by Lemma 3.4. The proof is complete. □

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. To begin with, we set some notation. Given $a \in \mathbf{D}$, put

$$\Gamma_1(a) := \left\{ \zeta \in \mathbf{T} : \left| \frac{1 - \overline{a} \varphi^*(\zeta)}{1 - \overline{a} \psi^*(\zeta)} \right| \le 2 \right\}$$

and

$$\Gamma_2(a) := \left\{ \zeta \in \mathbf{T} : \left| \frac{1 - \overline{a} \psi^*(\zeta)}{1 - \overline{a} \varphi^*(\zeta)} \right| \le 2 \right\}.$$

Since φ^* and ψ^* are defined almost everywhere on $\mathbf{T}$, we note that the set $\Gamma_1(a) \cup \Gamma_2(a)$ is of full measure in $\mathbf{T}$. Also, put

$$T := C_\varphi - C_\psi \quad \text{and} \quad f_a := k_{a,t}^{p,-1}$$

for short.

First, we prove the necessity. So, assume (3.1) and consider an arbitrary $t > \frac{1}{p}$. Setting

$$I_j(a) := \int_{\Gamma_j(a)} \left| \frac{1}{(1-\overline{a}\varphi^*)^t} - \frac{1}{(1-\overline{a}\psi^*)^t} \right|^p dm, \qquad j = 1, 2,$$

we note

$$\|T k_{a,t}^{p,-1}\|_{H^p}^p \leq (1-|a|)^{tp-1}[I_1(a) + I_2(a)]$$

for $a \in \mathbf{D}$. We will complete the proof by showing that the right-hand side of the above tends to 0 as $|a| \to 1$. For that purpose it suffices to establish

$$\lim_{|a|\to 1} (1-|a|)^{tp-1} I_1(a) = 0 \tag{3.7}$$

by symmetry.

To begin with, we note

$$I_1(a) := \int_{\Gamma_1(a)} \frac{1}{|1-\overline{a}\varphi^*|^{tp}} \left| 1 - \left(\frac{1-\overline{a}\varphi^*}{1-\overline{a}\psi^*} \right)^t \right|^p dm. \tag{3.8}$$

Now, we fix an arbitrary positive integer $J \geq s$ and consider $a \in \mathbf{D}$ sufficiently close to the boundary so that

$$1 - |a| < 2^{-(J-s+1)}.$$

To each a corresponds a unique positive integer $N_a \geq J$ such that

$$2^{N_a - s+1}(1-|a|) < 1 \leq 2^{N_a - s+2}(1-|a|). \tag{3.9}$$

Using J and N_a, we now decompose the integral over $\Gamma_1(a)$ into three pieces as follows:

$$\begin{aligned} \int_{\Gamma_1(a)} &= \int_{\Gamma_1(a)\cap\varphi^{*-1}[Q_J(a)]} + \int_{\Gamma_1(a)\setminus\varphi^{*-1}[Q_{N_a}(a)]} \\ &\qquad\qquad + \int_{\Gamma_1(a)\cap\varphi^{*-1}[Q_{N_a}(a)\setminus Q_J(a)]} \\ &=: I_{1,1}(a) + I_{1,2}(a) + I_{1,3}(a) \end{aligned} \tag{3.10}$$

for each a. We will estimate these three integrals separately.

For the first integral in (3.10), setting

$$M_t := \sup_{0<|z|\leq 2} \left| \frac{1-z^t}{1-z} \right| < \infty,$$

we note

$$\left| 1 - \left(\frac{1-\overline{a}\varphi^*}{1-\overline{a}\psi^*} \right)^t \right| \leq M_t \left| 1 - \left(\frac{1-\overline{a}\varphi^*}{1-\overline{a}\psi^*} \right) \right| = M_t |a| \left| \frac{\varphi^* - \psi^*}{1-\overline{a}\psi^*} \right| \quad \text{on} \quad \Gamma_1(a)$$

and thus

$$I_{1,1}(a) \leq \frac{M_t^p}{(1-|a|)^{tp}} \int_{\varphi^{*-1}[Q_J(a)]} \left| \frac{\varphi^* - \psi^*}{1-\overline{a}\psi^*} \right|^p dm. \tag{3.11}$$

In conjunction with this, we note $Q_J(a) \subset Q_s(\check{a}_{J-s+1})$ by Lemma 3.2 and, moreover, we have by Lemma 3.2 and (2.2)

$$\left|\frac{1-\overline{\check{a}_{J-s+1}}\psi^*}{1-\overline{a}\psi^*}\right| \le C$$

for some constant $C = C(s, J) > 0$. Accordingly, we obtain by (3.11)

$$\begin{aligned}(1-|a|)^{tp-1} I_{1,1}(a) &\lesssim \frac{1}{1-|a|}\int_{\varphi^{*-1}[Q_s(\check{a}_{J-s+1})]}\left|\frac{\varphi^*-\psi^*}{1-\overline{\check{a}_{J-s+1}}\psi^*}\right|^p dm \\ &\le \frac{\lambda_{p,s}(\check{a}_{J-s+1})}{1-|a|} \\ &= 2^{J-s+1}\cdot\frac{\lambda_{p,s}(\check{a}_{J-s+1})}{1-|\check{a}_{J-s+1}|} \qquad (3.12)\end{aligned}$$

for all a; the constant suppressed above depends only on t, s, p and J.

For the second integral in (3.10), note

$$\left|1-\left(\frac{1-\overline{a}\varphi^*}{1-\overline{a}\psi^*}\right)^t\right| \le 1+2^t \quad \text{on} \quad \Gamma_1(a).$$

Also, note from (3.9)

$$|1-\overline{a}z| \ge |a-z| \ge 2^{N_a+1}(1-|a|) \ge 2^{s-1}$$

for all $z \in \overline{\mathbf{D}} \setminus Q_{N_a}(a)$. Thus, denoting by m_φ the pullback measure $m \circ \varphi^{*-1}$, we obtain

$$\begin{aligned}I_{1,2}(a) &\lesssim \int_{\varphi^{*-1}[\overline{\mathbf{D}}\setminus Q_{N_a}(a)]}\frac{dm}{|1-\overline{a}\varphi^*|^{tp}} \\ &= \int_{\overline{\mathbf{D}}\setminus Q_{N_a}(a)}\frac{dm_\varphi(z)}{|1-\overline{a}z|^{tp}} \qquad \text{by (2.6)} \\ &\le \frac{1}{2^{tp(s-1)}} \qquad (3.13)\end{aligned}$$

for all a; the constant suppressed above depends only on t and p.

For the last integral in (3.10), as in the estimate of $I_{1,2}(a)$, we have

$$I_{1,3}(a) \lesssim \int_{Q_{N_a}(a)\setminus Q_J(a)}\frac{dm_\varphi(z)}{|a-z|^{tp}} \qquad (3.14)$$

for all a. Recall that m_φ is a Hardy-Carleson measure (see the remark at the end of Section 2.4) and thus by (2.5)

$$\|m_\varphi\|_s := \sup_{z\in\mathbf{D}}\frac{m_\varphi[Q_s(z)]}{1-|z|} < \infty. \qquad (3.15)$$

So, we obtain by Lemma 3.2

$$m_\varphi[Q_u(a)] \le m_\varphi[Q_s(\check{a}_{u-s+1})] \le 2^{u-s+1}\|m_\varphi\|_s(1-|a|)$$

for $u \ge s$. It follows that

$$\int_{Q_j(a)\setminus Q_{j-1}(a)}\frac{dm_\varphi(z)}{|a-z|^{tp}} \le \frac{m_\varphi[Q_j(a)]}{2^{jtp}(1-|a|)^{tp}} \le \frac{\|m_\varphi\|_s}{(1-|a|)^{tp-1}}\cdot\frac{2^{j-s+1}}{2^{jtp}}$$

for each $j = J+1, \ldots, N_a$. Accordingly, taking the sum on such j's, we obtain by (3.14)

$$(1-|a|)^{tp-1} I_{1,3}(a) \lesssim \|m_\varphi\|_s \sum_{j=J+1}^{\infty} \frac{2^{j-s+1}}{2^{jtp}} \approx \frac{\|m_\varphi\|_s}{2^{J(tp-1)}}; \tag{3.16}$$

the constants suppressed above depend only on s, t and p.

We now see from (3.12), (3.13) and (3.16) that there are constants $C_1 = C_1(s,t,p,J) > 0$ and $C_2 = C_2(t,s,p) > 0$ such that

$$(1-|a|)^{pt-1} I_1(a) \le C_1 \frac{\lambda_{p,s}(\check{a}_{J-s+1})}{1-|\check{a}_{J-s+1}|} + C_2 \left[(1-|a|)^{tp-1} + \frac{\|m_\varphi\|_s}{2^{J(tp-1)}}\right] \tag{3.17}$$

for all a sufficiently close to the boundary. Note $|\check{a}_{J-s+1}| \to 1$ as $|a| \to 1$ (with J fixed). It follows from (3.1) that the first term on the right-hand side of (3.17) tends to 0 as $|a| \to 1$. So, taking first the limit $|a| \to 1$ in (3.17) and then the limit $J \to \infty$, we conclude (3.7), as required. This completes the proof of the necessity.

We now turn to the proof of the sufficiency. So, assume (3.2) for some $t > \frac{1}{p}$ and fix an arbitrary $s \ge 0$. Put

$$I_{\varphi,\psi}(a) := \frac{1}{1-|a|} \int_{\varphi^{*-1}[Q_s(a)]} \left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\psi^*} \right|^p dm.$$

Note from (3.5)

$$|1 - \overline{a}\varphi^*| \le (2 + 2^{s+1})(1-|a|) \qquad \text{on} \quad \varphi^{*-1}[Q_s(a)] \tag{3.18}$$

and thus

$$\begin{aligned} \|Tf_a\|_{H^p}^p &= \int_{\mathbf{T}} \frac{(1-|a|)^{tp-1}}{|1-\overline{a}\varphi^*|^{tp}} \left| 1 - \left(\frac{1-\overline{a}\varphi^*}{1-\overline{a}\psi^*} \right)^t \right|^p dm \\ &\gtrsim \frac{1}{1-|a|} \int_{\varphi^{*-1}[Q_s(a)]} \left| 1 - \left(\frac{1-\overline{a}\varphi^*}{1-\overline{a}\psi^*} \right)^t \right|^p dm. \end{aligned}$$

Also, using a constant $N = N(s,t) > 0$ provided by Lemma 3.5, we see from Lemma 3.4 and (3.18)

$$|1 - \overline{a_N}\varphi^*| \le C(1-|a|) \qquad \text{on} \quad \varphi^{*-1}[Q_s(a)]$$

for some constant $C = C(s,N) > 0$. Thus, since $|a_N| = |a|$, we have by the same argument

$$\|Tf_{a_N}\|_{H^p}^p \gtrsim \frac{1}{1-|a|} \int_{\varphi^{*-1}[Q_s(a)]} \left| 1 - \left(\frac{1-\overline{a_N}\varphi^*}{1-\overline{a_N}\psi^*} \right)^t \right|^p dm.$$

Now, it follows from Lemma 3.5 that

$$I_{\varphi,\psi}(a) \lesssim \|Tf_a\|_{H^p}^p + \|Tf_{a_N}\|_{H^p}^p$$

for all $a \in \mathbf{D}$ sufficiently close to the boundary. This estimate remains valid by symmetry when the roles of φ and ψ are exchanged. Consequently, we obtain

$$\frac{\lambda_{p,s}(a)}{1-|a|} = I_{\varphi,\psi}(a) + I_{\psi,\varphi}(a) \lesssim \left[\|Tf_a\|_{H^p}^p + \|Tf_{a_N}\|_{H^p}^p \right] \tag{3.19}$$

for all $a \in \mathbf{D}$ sufficiently close to the boundary; the constant suppressed in this estimate depends only on s, t and p. Thus, taking the limit $|a| \to 1$, we conclude (3.1). The proof is complete. $\square$

In conjunction with Theorem 3.1 we notice a couple of observations. First, as an immediate consequence of Theorem 3.1 and Lemma 2.1, we obtain a necessary condition for the compactness of $C_\varphi - C_\psi$ on $H^p(\mathbf{D})$. Note that the special case $p = 2$ is the implication (a) $\Longrightarrow$ (b) in Theorem 1.1.

Corollary 3.6. *Let $0 < p < \infty$ and $\varphi, \psi \in \mathcal{S}(\mathbf{D})$. If $C_\varphi - C_\psi$ is compact on $H^p(\mathbf{D})$, then*

$$\lim_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|} = 0$$

for all $s \geq 0$.

Next, closely examining the proof of Theorem 3.1, one actually obtains the following estimates of which the special case $p = 2$ implies the second estimate in Theorem 1.2. Here, $\|m_\varphi\|_s$ denotes the quantity specified in (3.15).

Corollary 3.7. *Let $0 < p < \infty$, $s \geq 0$ and $t > \frac{1}{p}$. Then there is a constant $C = C(s,t,p) > 0$ such that*

$$\begin{aligned} C^{-1} \limsup_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|} &\leq \limsup_{|a|\to 1} \|(C_\varphi - C_\psi) k_{a,t}^{p,-1}\|_{H^p}^p \\ &\leq C(1 + \|m_\varphi\|_s + \|m_\psi\|_s) \limsup_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|} \end{aligned}$$

for $\varphi, \psi \in \mathcal{S}(\mathbf{D})$.

Proof. The first inequality is clear by (3.19). In order to prove the second inequality we may assume by Theorem 3.1

$$\limsup_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|} > 0.$$

Given a positive integer $J \geq s$, we note from (3.17) that there are constants $C_1 = C_1(s,t,p,J) > 0$ and $C_2 = C_2(s,t,p) > 0$ such that

$$\limsup_{|a|\to 1} \|T k_{a,t}^{p,-1}\|_{H^p}^p \leq C_1 \limsup_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|} + C_2 \frac{\|m_\varphi\|_s + \|m_\psi\|_s}{2^{J(tp-1)}},$$

where $T := C_\varphi - C_\psi$. Now, choosing $J = J(s,t,p)$ so large that

$$\frac{1}{2^{J(tp-1)}} \leq \limsup_{|a|\to 1} \frac{\lambda_{p,s}(a)}{1-|a|},$$

we obtain the second inequality. The proof is complete. □

For the proof of the implication (c) $\Longrightarrow$ (a) in Theorem 1.1 we need to recall some results from [3]. Before proceeding further, we first recall a connection between ordinary composition operators and weighted composition operators. To be more precise, given $\varphi \in \mathcal{S}(\mathbf{D})$, denote by W_φ the weighted composition operator defined by

$$W_\varphi f := \varphi'(f \circ \varphi)$$

for functions f holomorphic on $\mathbf{D}$. It is then not hard to verify via the Littlewood-Paley Identity that C_φ is compact on $H^2(\mathbf{D})$ if and only if W_φ is compact on $A_1^2(\mathbf{D})$. Similarly, given $\varphi, \psi \in \mathcal{S}(\mathbf{D})$, one may verify that $C_\varphi - C_\psi$ is compact on $H^2(\mathbf{D})$ if and only if $W_\varphi - W_\psi$ is compact on $A_1^2(\mathbf{D})$; see [3, Section 5] for details.

In [3] Choe et al. have recently obtained characterizations for compact differences of general weighted composition operators acting on the weighted Bergman spaces. As a consequence based on the observation mentioned in the preceding paragraph, they provided an answer to a long-standing problem of characterizing compact differences of composition operators on $H^2(\mathbf{D})$. Such a characterization was in fact the starting point of our investigation into Theorem 1.1. In order to state it, we need several pullback measures described below.

Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and recall that ρ denotes the pseudohyperbolic distance between φ and ψ. Now, given $\beta > 0$ and $0 < r < 1$, we introduce pullback measures $\omega = \omega_{\varphi,\psi}$, $\sigma^\beta = \sigma^\beta_{\varphi,\psi}$ and $\sigma_r = \sigma_r^{\varphi,\psi}$ as follows:

$$\begin{aligned}\omega &:= (|\rho\varphi'|^2\, dA_1) \circ \varphi^{-1} + (|\rho\psi'|^2\, dA_1) \circ \psi^{-1},\\ \sigma^\beta &:= [(1-\rho)^\beta |\varphi' - \psi'|^2\, dA_1] \circ \varphi^{-1} + [(1-\rho)^\beta |\varphi' - \psi'|^2\, dA_1] \circ \psi^{-1},\\ \sigma_r &:= (\chi_{G_r} |\varphi' - \psi'|^2\, dA_1) \circ \varphi^{-1} + (\chi_{G_r} |\varphi' - \psi'|^2\, dA_1) \circ \psi^{-1},\end{aligned}$$

where χ_{G_r} denotes the characteristic function of the set $G_r := \{z \in \mathbf{D} : \rho(z) < r\}$. Note that these are finite measures, because $\varphi', \psi' \in A^2_1$ by the Littlewood-Paley Identity.

We now recall a couple of results from [3]. First, our proof of Theorem 1.1 will utilize Theorem 3.8 taken from a special case of [3, Theorem 1.2].

Theorem 3.8 ([3])**.** *Let $\beta > 3$, $0 < r < 1$ and $\varphi, \psi \in \mathcal{S}(\mathbf{D})$. Then the following assertions are equivalent:*

(a) *$C_\varphi - C_\psi$ is compact on $H^2(\mathbf{D})$;*
(b) *$\omega + \sigma^\beta$ is a compact 1-Bergman-Carleson measure;*
(c) *$\omega + \sigma_r$ is a compact 1-Bergman-Carleson measure.*

Next, Lemma 3.9 is a special case of [3, Eq. (4.6)]. Recall that $k^{p,\alpha}_{a,t}$ denotes the normalized test function introduced in (2.8). Also, recall that a_N denotes the auxiliary point introduced in (3.4).

Lemma 3.9. *Let $0 < \delta < 1$, $\beta > 3$ and $\frac{3}{2} < t < \frac{\beta}{2}$. Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and put $\mu := \omega + \sigma^\beta$ and $S := W_\varphi - W_\psi$. Then there exist constants $N = N(\beta, \delta, t) > 0$ and $C = C(\beta, \delta, t) > 0$ such that*

$$\begin{aligned}\widehat{\mu}_{1,\delta}(a) \le C\Big[&\big\|Sk^{2,1}_{a_N,t}\big\|^2_{A^2_1} + \big\|Sk^{2,1}_{a_N,2t}\big\|^2_{A^2_1} + \big\|Sk^{2,1}_{\overline{a_N},t}\big\|^2_{A^2_1}\\ &+ \big\|Sk^{2,1}_{\overline{a_N},2t}\big\|^2_{A^2_1} + \big\|Sk^{2,1}_{a,t}\big\|^2_{A^2_1} + \big\|Sk^{2,1}_{a,t+1}\big\|^2_{A^2_1}\Big]\end{aligned}$$

for all a sufficiently close to the boundary.

We now proceed to the proof of our main result Theorem 1.1.

Proof of Theorem 1.1. As is mentioned before, the implication (a) $\Longrightarrow$ (b) is contained in Corollary 3.6 and the equivalence (b) $\Longleftrightarrow$ (c) is contained in Theorem 3.1.

We will complete the proof by establishing the implication (c) $\Longrightarrow$ (a). To this end we note from the aforementioned equivalence (b) $\Longleftrightarrow$ (c) that if (c) holds for some $t > \frac{1}{2}$, then it holds for any $t > \frac{1}{2}$. So, in order to complete the proof, we assume (c) for any $t > \frac{1}{2}$ and show (a). We now fix an arbitrary $t > \frac{1}{2}$ and pick $\beta > 2(t+1)$. By Theorem 3.8, it suffices to show that the measure μ specified in Lemma 3.9 is a compact 1-Bergman-Carleson measure.

Note $\left(k_{a,t}^{2,-1}\right)' = t\overline{a}k_{a,t+1}^{2,1}$. Thus, setting

$$T := C_\varphi - C_\psi \quad \text{and} \quad S := W_\varphi - W_\psi$$

for short, we have by (2.3)

$$\left\|Tk_{a,t}^{2,-1}\right\|_{H^2} \approx \left|k_{a,t}^{2,-1}(\varphi(0)) - k_{a,t}^{2,-1}(\psi(0))\right| + t|a|\left\|Sk_{a,t+1}^{2,1}\right\|_{A_1^2} \tag{3.20}$$

for all $a \in \mathbf{D}$. Note that the first term on the right-hand side of the above is dominated by

$$(1-|a|)^{t-\frac{1}{2}}\left[\frac{1}{(1-|\varphi(0)|)^t} + \frac{1}{(1-|\psi(0)|)^t}\right],$$

which tends to 0 as $|a| \to 1$. Consequently, we obtain

$$\lim_{|a|\to 1}\left\|Sk_{a,t+1}^{2,1}\right\|_{A_1^2} = \lim_{|a|\to 1}\left\|Tk_{a,t}^{2,-1}\right\|_{H^2} = 0.$$

Since this holds for any $t > \frac{1}{2}$, we conclude by Lemma 3.9 and (2.7) that μ is a compact 1-Bergman-Carleson measure, as required. This completes the proof. □

Recall that the essential norm of a bounded linear operator on $H^2(\mathbf{D})$, denoted by $\|\cdot\|_e$, is its distance, in the operator norm, from the space of all compact operators on $H^2(\mathbf{D})$. So, given a bounded linear operator T on $H^2(\mathbf{D})$, we have

$$\|T\|_e := \inf\left\{\|T-L\| : L \text{ is compact on } H^2(\mathbf{D})\right\},$$

where $\|\cdot\|$ denotes the operator norm.

Now, we estimate the essential norm of $C_\varphi - C_\psi$ viewed as an operator on $H^2(\mathbf{D})$. We need Lemma 3.10 taken from a special case of [3, Eq. (4.1)].

Lemma 3.10. *Let $0 < \delta < 1$ and $0 < r < 1$. Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and put $\nu := \omega + \sigma_r$. Then there is a constant $C = C(r, \delta) > 0$ such that*

$$\|(W_\varphi - W_\psi)f\|_{A_1^2}^2 \le C\int_{\mathbf{D}} |f|^2 \widehat{\nu}_{1,\delta}\, dA_1$$

for $f \in A_1^2(\mathbf{D})$.

Proof of Theorem 1.2. The second estimate being contained in Corollary 3.7, we only need to establish the first estimate.

As before, we put $T := C_\varphi - C_\psi$ throughout the proof. For the lower estimate, consider H^2-normalized test functions $f_a := k_{a,1}^{2,-1}$ for $a \in \mathbf{D}$. Given an arbitrary compact operator L on $H^2(\mathbf{D})$, we have

$$\|T\|_e \ge \|T-L\| \gtrsim \|(T-L)f_a\|_{H^2} \ge \|Tf_a\|_{H^2} - \|Lf_a\|_{H^2} \tag{3.21}$$

for all $a \in \mathbf{D}$. Meanwhile, using the polarized Littlewood-Paley Identity and (2.9), one may verify that $f_a \to 0$ weakly in $H^2(\mathbf{D})$ as $|a| \to 1$ and thus that $\lim_{|a|\to 1}\|Lf_a\|_{H^2} = 0$. Consequently, using (3.19) and taking the limit $|a| \to 1$ in (3.21), we obtain the lower estimate

$$\|T\|_e^2 \ge C\limsup_{|a|\to 1}\frac{\lambda_{2,s}(a)}{1-|a|}$$

for some constant $C = C(s) > 0$.

We now proceed to establish the upper estimate. Given a positive integer n, denote by P_n the projection on $H^2(\mathbf{D})$ defined by

$$P_n g(z) := \sum_{k=n+1}^{\infty} a_k z^k$$

for $g \in H^2(\mathbf{D})$ with power series representation $g(z) = \sum_{k=0}^{\infty} a_k z^k$. It is known that

$$\|T\|_e = \lim_{n\to\infty} \|TP_n\|;$$

see [16, Proposition 5.1]. We will complete the proof by establishing

$$\lim_{n\to\infty} \|TP_n\|^2 \le C(1 + \|m_\varphi\|_s + \|m_\psi\|_s) \limsup_{|a|\to 1} \frac{\lambda_{2,s}(a)}{1-|a|} \tag{3.22}$$

for some constant $C = C(s) > 0$. Here, $\|m_\varphi\|_s$ and $\|m_\psi\|_s$ are the quantities used in Corollary 3.7.

First, we estimate the operator norms $\|TP_n\|$. So, consider an arbitrary $g \in H^2(\mathbf{D})$ and put $g_n := P_n g$. Let ν be the measure specified in Lemma 3.10 with any fixed $r \in (0,1)$. Also, fix any $\delta \in (0,1)$. Since $g_n(0) = 0$, we have by (2.3) and Lemma 3.10

$$\|TP_n g\|_{H^2}^2 \approx \|(W_\varphi - W_\psi) g_n'\|_{A_1^2}^2 \lesssim \int_{\mathbf{D}} |g_n'|^2 \widehat{\nu}_{1,\delta}\, dA_1 \tag{3.23}$$

for all n. In conjunction with the above integral, we note $\|g_n'\|_{A_1^2} \approx \|g_n\|_{H^2} \le \|g\|_{H^2}$ for all n by (2.3). Thus, given $t \in (0,1)$, we have

$$\int_{\mathbf{D}} |g_n'|^2 \widehat{\nu}_{1,\delta}\, dA_1 \le \left(\sup_{t\mathbf{D}} \widehat{\nu}_{1,\delta}\right) \int_{t\mathbf{D}} |g_n'|^2\, dA_1 + \left(\sup_{\mathbf{D}\setminus t\mathbf{D}} \widehat{\nu}_{1,\delta}\right) \|g\|_{H^2}^2 \tag{3.24}$$

for all n. Meanwhile, using the power series representation $g(z) = \sum_{k=0}^{\infty} a_k z^k$ and setting

$$I_k(t) := k^2 \int_{t\mathbf{D}} |z|^{2(k-1)}\, dA_1(z),$$

we have

$$\begin{aligned}
\int_{t\mathbf{D}} |g_n'|^2\, dA_1 &= \int_{t\mathbf{D}} \left| \sum_{k=n+1}^{\infty} k a_k z^{k-1} \right|^2 dA_1(z) \\
&= \sum_{k=n+1}^{\infty} |a_k|^2 I_k(t) \\
&\le \left(\sup_{k\ge n+1} I_k(t) \right) \|g\|_{H^2}^2.
\end{aligned}$$

This, together with (3.23) and (3.24), yields

$$\|TP_n\|^2 \lesssim \left(\sup_{t\mathbf{D}} \widehat{\nu}_{1,\delta}\right) \left(\sup_{k\ge n+1} I_k(t) \right) + \sup_{\mathbf{D}\setminus t\mathbf{D}} \widehat{\nu}_{1,\delta} \tag{3.25}$$

for all n and $t \in (0,1)$; the constant suppressed in this estimate is independent of n and t. When t is fixed, note

$$I_{n+1}(t) \to 0$$

as $n \to \infty$ by the Dominated Convergence Theorem. Thus, taking first the limit $n \to \infty$ and then $t \to 1$ in (3.25), we obtain

$$\lim_{n\to\infty} \|TP_n\|^2 \lesssim \lim_{t\to 1} \left(\sup_{\mathbf{D}\setminus t\mathbf{D}} \widehat{\nu}_{1,\delta} \right) = \limsup_{|a|\to 1} \widehat{\nu}_{1,\delta}(a); \tag{3.26}$$

recall for the first inequality that ν is a finite measure.

Next, we estimate the limsup on the right-hand side of (3.26). Let μ be the measure specified in Lemma 3.9 corresponding to any (but fixed) $\beta > 3$. Since $1 - r \le 1 - \rho$ on G_r, we have $\nu \le (1-r)^{-\beta}\mu$ and hence

$$\limsup_{|a|\to 1} \widehat{\nu}_{1,\delta}(a) \le \frac{1}{(1-r)^\beta} \limsup_{|a|\to 1} \widehat{\mu}_{1,\delta}(a). \tag{3.27}$$

Meanwhile, fixing $t \in (\frac{1}{2}, \frac{\beta}{2} - 1)$ and using a constant $N > 0$ provided by Lemma 3.9 (with $t+1$ in place of t), we have by Lemma 3.9 and (3.20)

$$\begin{aligned}\widehat{\mu}_{1,\delta}(a) \lesssim \|Tf_{a_N,t}\|^2_{H^2} + \|Tf_{a_N,2t+1}\|^2_{H^2} + \|Tf_{\overline{a_N},t}\|^2_{H^2} \\ + \|Tf_{\overline{a_N},2t+1}\|^2_{H^2} + \|Tf_{a,t}\|^2_{H^2} + \|Tf_{a,t+1}\|^2_{H^2}\end{aligned}$$

for all a sufficiently close to the boundary. So, one may deduce from Corollary 3.7 (with $p = 2$) that

$$\limsup_{|a|\to 1} \widehat{\mu}_{1,\delta}(a) \le C(1 + \|m_\varphi\|_s + \|m_\psi\|_s) \limsup_{|a|\to 1} \frac{\lambda_{2,s}(a)}{1-|a|} \tag{3.28}$$

for some constant $C = C(s, \delta, \beta, t) = C(s) > 0$.

Finally, combining (3.26), (3.27) and (3.28), we conclude (3.22), as required. The proof is complete. □

4. Remarks

In this section we collect some remarks connected with our results of the current paper.

4.1. **Saukko's sufficient condition.** When $p = 2$, we observe that Saukko's result mentioned in Section 1 is easily recovered by Theorem 1.1.

Given $s \ge 0$, note from Lemma 3.3

$$\left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\psi^*} \right| \lesssim \rho^* \qquad \text{a.e. on } \varphi^{*-1}[Q_s(a)] \tag{4.1}$$

for all a sufficiently close to the boundary. This estimate remains valid by symmetry when the roles of φ and ψ are exchanged. Accordingly, given $0 < p < \infty$, using the joint pullback measure

$$\xi_p := (\rho^{*p}\, dm) \circ \varphi^{*-1} + (\rho^{*p}\, dm) \circ \psi^{*-1} \quad \text{on } \overline{\mathbf{D}},$$

we have

$$\lambda_{p,s}(a) \lesssim \xi_p[Q_s(a)]$$

for all a sufficiently close to the boundary. So, in case $p = 2$, Saukko's sufficient condition is immediate from Theorem 1.1, as is mentioned in Section 1.

4.2. **Dependency on the parameter** p. Note that our proof of Theorem 1.1, relying strongly on the Littlewood-Paley Identity, which is a very special property for the case $p = 2$, does not extend to general p at all. So, we are naturally led to a question: *What happens if* $p \neq 2$*?* In this regard, we recall a result of Nieminen and Saksman [14] asserting that the compactness of $C_\varphi - C_\psi$ on $H^p(\mathbf{D})$ is independent of the parameter $p \in [1, \infty)$. In addition, we also remark that

(4.2) • *Condition* (3.1) *(and thus Condition* (3.2) *as well) is independent of* p.

Proof. Suppose (3.1) holds for some $p \in (0, \infty)$. Let $q \in (0, \infty)$ and $s \geq 0$. In case $p < q < \infty$, we obtain by (4.1)

$$\lambda_{q,s}(a) \lesssim \lambda_{p,s}(a)$$

for all a sufficiently close to the boundary, which yields

$$\lim_{|a| \to 1} \frac{\lambda_{q,s}(a)}{1 - |a|} = 0. \tag{4.3}$$

Meanwhile, in case $0 < q \leq p$, applying Jensen's Inequality, we note

$$\left(\frac{1}{1 - |a|} \int_{\varphi^{*-1}[Q_s(a)]} \left| \frac{\varphi^* - \psi^*}{1 - \overline{a}\psi^*} \right|^q dm \right)^{\frac{p}{q}} \leq \|m_\varphi\|_s^{\frac{p}{q} - 1} \frac{\lambda_{p,s}(a)}{1 - |a|}$$

for $a \in \mathbf{D}$ where $||m_\varphi||_s$ is the quantity specified in (3.15). In this estimate the roles of φ and ψ can be exchanged by symmetry. So, we see that (4.3) also holds for $q \leq p$. □

As a consequence of the observations in the preceding paragraph, we may extend Theorem 1.1 as in Corollary 4.1.

Corollary 4.1. *Let* $1 \leq p < \infty$ *and* $\varphi, \psi \in \mathcal{S}(\mathbf{D})$. *Then the following assertions are equivalent:*

(a) $C_\varphi - C_\psi$ *is compact on* $H^p(\mathbf{D})$*;*
(b) $\lim_{|a| \to 1} \frac{\lambda_{p,s}(a)}{1 - |a|} = 0$ *for some/any* $s \geq 0$*;*
(c) $\lim_{|a| \to 1} \left\| (C_\varphi - C_\psi) k_{a,t}^{p,-1} \right\|_{H^p} = 0$ *for some/any* $t > \frac{1}{p}$.

Proof. As in the proof of Theorem 1.1, the implication (a) $\Longrightarrow$ (b) is contained in Corollary 3.6 and the equivalence (b) $\Longleftrightarrow$ (c) is contained in Theorem 3.1. The implication (b) $\Longrightarrow$ (a) holds by (4.2), Theorem 1.1 and the aforementioned result of Nieminen and Saksman. □

Corollary 4.2 is another consequence.

Corollary 4.2. *Let* $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ *and assume that* $C_\varphi - C_\psi$ *is compact on* $H^p(\mathbf{D})$ *for some* $p \in (0, \infty)$. *Then* $C_\varphi - C_\psi$ *is compact on* $H^q(\mathbf{D})$ *for any* $q \in [1, \infty)$.

Proof. We may assume $0 < p < 1$ by the result of Nieminen and Saksman. So, we have $p < q$ for any $q \in [1, \infty)$ and thus see that (4.3) holds. Accordingly, we conclude the corollary by Corollary 4.1. □

In many cases the compactness of a composition operator on a smaller space implies that of the operator on lager spaces; see, for example, [5, Theorem 1.3]. In this regard, when $p < 1$, Corollary 4.2 seems somewhat interesting, because it goes the other way round. In connection with this remark and the observations above, we wonder if the restriction to the range of p in the result of Nieminen and Saksman is essential. So, we propose Conjecture 4.3.

Conjecture 4.3. *The compactness of $C_\varphi - C_\psi$ on $H^p(\mathbf{D})$ is independent of the parameter $p \in (0, \infty)$.*

In connection with this conjecture we also conjecture that Property (3.1), or equivalently Property (3.2), is independent of p. In addition, we remark that the Bergman space analogue of this conjecture is known; see [4].

4.3. **Inducing maps of bounded multiplicity.** When the inducing maps are univalent, Choe et al. [3] noticed a much simpler characterization for compact differences of composition operators on $H^2(\mathbf{D})$. To be more explicit, put

$$R_\varphi(z) := \frac{1 - |z|^2}{1 - |\varphi(z)|^2}$$

for $\varphi \in \mathcal{S}(\mathbf{D})$. It has been known by Moorhouse [12] that compactness of $C_\varphi - C_\psi$ on the weighted Bergman spaces (described in Section 2.3) is characterized by the condition

$$\lim_{|z|\to 1} [R_\varphi(z) + R_\psi(z)]\rho(z) = 0. \tag{4.4}$$

It turns out that this condition also characterizes compactness of $C_\varphi - C_\psi$ on $H^2(\mathbf{D})$, when φ and ψ are univalent; see [3, Theorem 1.3]. In the proof of such a result in [3], the key tool was Lemma 4.4 under the additional assumption that inducing maps φ and ψ are univalent. After [3] was published, we realized through a simpler argument that the lemma holds in general as follows.

Lemma 4.4. *Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ and assume* (4.4)*. Let $\Omega \subset \mathbf{D}$ and assume*

$$\inf_\Omega R_\varphi > 0. \tag{4.5}$$

Then

$$\lim_{|z|\to 1, z\in\Omega} |\varphi'(z) - \psi'(z)| = 0.$$

Proof. Given $z \in \mathbf{D}$ and $\delta \in (0, 1)$, recall that $E_\delta(z)$ denotes the pseudohyperbolic disk with center z and radius δ. A straightforward calculation shows that $E_\delta(z)$ is the Euclidean disk with

$$(\text{center}) = \frac{(1-\delta^2)z}{1-|z|^2\delta^2} \quad \text{and} \quad (\text{radius}) = \frac{(1-|z|^2)\delta}{1-|z|^2\delta^2}. \tag{4.6}$$

So, another straightforward calculation shows that $E_\delta(z)$ contains the Euclidean disk with center z and radius $\frac{(1-|z|^2)\delta}{1+|z|\delta}$. Taking $\delta = \frac{1}{4}$, we thus have by the Cauchy Estimates

$$|\varphi'(z) - \psi'(z)| \le \frac{5}{1-|z|^2} \left(\sup_{E_{1/4}(z)} |\varphi - \psi| \right) \tag{4.7}$$

for $z \in \mathbf{D}$.

We now assume $z \in \Omega$ and proceed to estimate the supremum on the right-hand side of (4.7). Consider an arbitrary $w \in E_{1/4}(z)$. Note from (4.4) and (4.5) that $\rho(z) \to 0$ as $|z| \to 1$ inside Ω. Thus we may assume $\rho < \frac{1}{4}$ on Ω (after shrinking the set Ω to a smaller one if necessary). Note $d\big(\varphi(w), \varphi(z)\big) < 1/4$ and

$d\big(\psi(w),\psi(z)\big) < 1/4$ by the Schwarz-Pick Lemma. We thus have $\rho < \frac{3}{4}$ on $E_{1/4}(z)$ and thus by (2.2)

$$\big|1-\varphi(w)\overline{\psi(w)}\big| \approx 1-|\varphi(w)|^2 \tag{4.8}$$

for all $w \in E_{1/4}(z)$. We also have by (2.2)

$$1-|w|^2 \approx 1-|z|^2 \quad \text{and} \quad 1-|\varphi(w)|^2 \approx 1-|\varphi(z)|^2 \tag{4.9}$$

for all $w \in E_{1/4}(z)$. It follows from (4.8) and (4.9) that

$$|\varphi(w)-\psi(w)| = \rho(w)\big|1-\varphi(w)\overline{\psi(w)}\big| \approx \rho(w)(1-|\varphi(z)|^2)$$

and

$$R_\varphi(w) \approx R_\varphi(z)$$

for all $w \in E_{1/4}(z)$. Accordingly, we obtain

$$\frac{|\varphi(w)-\psi(w)|}{1-|z|^2} \approx \frac{\rho(w)}{R_\varphi(z)} \lesssim M^2\rho(w)R_\varphi(w), \qquad w \in E_{1/4}(z),$$

where $M := \sup_\Omega R_\varphi^{-1} < \infty$. This, together with (4.7), yields

$$|\varphi'(z)-\psi'(z)| \lesssim M^2 \left(\sup_{E_{1/4}(z)} \rho R_\varphi \right);$$

the constant suppressed in this estimate is independent of $z \in \Omega$. Moreover, the right-hand side of the above tends to 0 as $|z| \to 1$ inside Ω by (4.6) and (4.4). This completes the proof. □

Now, having Lemma 4.4, one may repeat the same proof of [3, Theorem 1.3], except for three spots where the univalent change-of-variable formula needs to be replaced by the Area Formula (see [6, Theorem 2.32]), to conclude Theorem 4.5.

Theorem 4.5. *Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$ be of bounded multiplicity. Then $C_\varphi - C_\psi$ is compact on $H^2(\mathbf{D})$ if and only if* (4.4) *holds.*

4.4. **Equivalent representations of $\lambda_{p,s}$.** We observe that defining integral of $\lambda_{p,s}(a)$ can be replaced by some other integrals. We need Lemma 4.6.

Lemma 4.6. *Let $t > s \geq 0$. Then the estimate*

$$\left| \frac{z-w}{1-\overline{a}w} \right| \approx d(z,w) \approx 1$$

holds for all $a \in \mathbf{D}$, $z \in Q_s(a)$ and $w \in \overline{\mathbf{D}} \setminus Q_t(a)$; the constants suppressed in these estimates depend only on s and t.

Proof. Let $a \in \mathbf{D}$, $z \in Q_s(a)$ and $w \in \overline{\mathbf{D}} \setminus Q_t(a)$ throughout the proof. By Lemma 3.3 we only need to prove

$$\left| \frac{z-w}{1-\overline{a}w} \right| \geq C \tag{4.10}$$

for some constant $C = C(s,t) > 0$.

We have

$$|1-\overline{a}w| \leq |1-\overline{a}z| + |z-w| \leq (2+2^{s+1})(1-|a|) + |z-w|;$$

the second inequality holds by (3.5). Meanwhile, we have

$$|z-w| \geq |a-w| - |a-z| \geq (2^{t+1} - 2^{s+1})(1-|a|).$$

Combining these estimates, we obtain

$$|1-\overline{a}w| \leq \left(\frac{1+2^s}{2^t - 2^s} + 1\right)|z-w|,$$

which yields (4.10) with $C = \frac{2^t - 2^s}{1+2^t}$. The proof is complete. $\square$

Now, consider functions $\tau^{\varphi,\psi}_{p,s,t}$ and $\eta^{\varphi,\psi}_{p,s,t}$ defined by

$$\tau^{\varphi,\psi}_{p,s,t}(a) := \int_{\varphi^{*-1}[Q_s(a)]\setminus\psi^{*-1}[Q_t(a)]} \rho^{*p}\, dm$$

and

$$\eta^{\varphi,\psi}_{p,s,t}(a) := \frac{1}{(1-|a|)^p} \int_{\varphi^{*-1}[Q_s(a)]\cap\psi^{*-1}[Q_t(a)]} |\varphi^* - \psi^*|^p\, dm$$

for $a \in \mathbf{D}$. Put

$$\tau_{p,s,t} := \tau^{\varphi,\psi}_{p,s,t} + \tau^{\psi,\varphi}_{p,s,t} \quad \text{and} \quad \eta_{p,s,t} := \eta^{\varphi,\psi}_{p,s,t} + \eta^{\psi,\varphi}_{p,s,t}.$$

We now see more clearly from the next result a difference between our integral $\lambda_{p,s}(a)$ and the integral in Saukko's sufficient condition (4.1). However, we do not have an explicit example demonstrating that they are indeed different.

Proposition 4.7. *Let $0 < p < \infty$ and $t > s \geq 0$. Let $\varphi, \psi \in \mathcal{S}(\mathbf{D})$. Then the estimate*

$$\lambda_{p,s}(a) \approx \tau_{p,s,t}(a) + \eta_{p,s,t}(a)$$

holds for $a \in \mathbf{D}$; the constants suppressed in this estimate depend only on p, s and t.

Proof. By Lemma 4.6 we have

$$\left|\frac{\varphi^* - \psi^*}{1-\overline{a}\psi^*}\right| \approx \rho^* \qquad \text{a.e. on } \varphi^{*-1}[Q_s(a)] \setminus \psi^{*-1}[Q_t(a)].$$

Also, we have by (3.5)

$$|1-\overline{a}\psi^*| \approx 1-|a| \qquad \text{a.e. on } \psi^{*-1}[Q_t(a)].$$

It follows that

$$\int_{\varphi^{*-1}[Q_s(a)]} \left|\frac{\varphi^* - \psi^*}{1-\overline{a}\psi^*}\right|^p dm \approx \tau^{\varphi,\psi}_{p,s,t}(a) + \eta^{\varphi,\psi}_{p,s,t}(a)$$

for $a \in \mathbf{D}$; the constants suppressed in this estimate depend only on p, s and t. We thus conclude the proposition by symmetry. The proof is complete. $\square$

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References

[1] Earl Berkson, *Composition operators isolated in the uniform operator topology*, Proc. Amer. Math. Soc. **81** (1981), no. 2, 230–232, DOI 10.2307/2044200. MR593463

[2] Paul S. Bourdon, *Components of linear-fractional composition operators*, J. Math. Anal. Appl. **279** (2003), no. 1, 228–245, DOI 10.1016/S0022-247X(03)00004-0. MR1970503

[3] Boo Rim Choe, Koeun Choi, Hyungwoon Koo, and Jongho Yang, *Difference of weighted composition operators*, J. Funct. Anal. **278** (2020), no. 5, 108401, 38, DOI 10.1016/j.jfa.2019.108401. MR4046212

[4] Boo Rim Choe, Hyungwoon Koo, and Inyoung Park, *Compact differences of composition operators on the Bergman spaces over the ball*, Potential Anal. **40** (2014), no. 1, 81–102, DOI 10.1007/s11118-013-9343-z. MR3146510

[5] Boo Rim Choe, Hyungwoon Koo, and Wayne Smith, *Composition operators acting on holomorphic Sobolev spaces*, Trans. Amer. Math. Soc. **355** (2003), no. 7, 2829–2855, DOI 10.1090/S0002-9947-03-03273-2. MR1975402

[6] Carl C. Cowen and Barbara D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995. MR1397026

[7] Eva A. Gallardo-Gutiérrez, María J. González, Pekka J. Nieminen, and Eero Saksman, *On the connected component of compact composition operators on the Hardy space*, Adv. Math. **219** (2008), no. 3, 986–1001, DOI 10.1016/j.aim.2008.06.005. MR2442059

[8] Paul R. Halmos, *Measure Theory*, D. Van Nostrand Co., Inc., New York, N. Y., 1950. MR0033869

[9] Hyungwoon Koo and Maofa Wang, *Joint Carleson measure and the difference of composition operators on* $A^p_\alpha(\mathbf{B}_n)$, J. Math. Anal. Appl. **419** (2014), no. 2, 1119–1142, DOI 10.1016/j.jmaa.2014.05.037. MR3225424

[10] Daniel H. Luecking, *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. **107** (1985), no. 1, 85–111, DOI 10.2307/2374458. MR778090

[11] Barbara D. MacCluer, *Components in the space of composition operators*, Integral Equations Operator Theory **12** (1989), no. 5, 725–738, DOI 10.1007/BF01194560. MR1009027

[12] Jennifer Moorhouse, *Compact differences of composition operators*, J. Funct. Anal. **219** (2005), no. 1, 70–92, DOI 10.1016/j.jfa.2004.01.012. MR2108359

[13] Jennifer Moorhouse and Carl Toews, *Differences of composition operators*, Trends in Banach spaces and operator theory (Memphis, TN, 2001), Contemp. Math., vol. 321, Amer. Math. Soc., Providence, RI, 2003, pp. 207–213, DOI 10.1090/conm/321/05645. MR1978818

[14] Pekka J. Nieminen and Eero Saksman, *On compactness of the difference of composition operators*, J. Math. Anal. Appl. **298** (2004), no. 2, 501–522, DOI 10.1016/j.jmaa.2004.05.024. MR2086972

[15] Erno Saukko, *Difference of composition operators between standard weighted Bergman spaces*, J. Math. Anal. Appl. **381** (2011), no. 2, 789–798, DOI 10.1016/j.jmaa.2011.03.058. MR2802114

[16] Joel H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. (2) **125** (1987), no. 2, 375–404, DOI 10.2307/1971314. MR881273

[17] Joel H. Shapiro, *Composition operators and classical function theory*, Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993, DOI 10.1007/978-1-4612-0887-7. MR1237406

[18] Joel H. Shapiro and Carl Sundberg, *Isolation amongst the composition operators*, Pacific J. Math. **145** (1990), no. 1, 117–152. MR1066401

[19] Kehe Zhu, *Spaces of holomorphic functions in the unit ball*, Graduate Texts in Mathematics, vol. 226, Springer-Verlag, New York, 2005. MR2115155

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