

THREE TOPOLOGICAL REDUCIBILITIES FOR DISCONTINUOUS FUNCTIONS

ADAM R. DAY, ROD DOWNEY, AND LINDA WESTRICK

ABSTRACT. We define a family of three related reducibilities, $\leq_{\mathbf{T}}$, $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$, for arbitrary functions $f, g : X \rightarrow \mathbb{R}$, where X is a compact separable metric space. The $\equiv_{\mathbf{T}}$ -equivalence classes mostly coincide with the proper Baire classes. We show that certain α -jump functions $j_{\alpha} : 2^{\omega} \rightarrow \mathbb{R}$ are $\leq_{\mathbf{m}}$ -minimal in their Baire class. Within the Baire 1 functions, we completely characterize the degree structure associated to $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$, finding an exact match to the α hierarchy introduced by Bourgain [Bull. Soc. Math. Belg. Sér. B 32 (1980), pp. 235–249] and analyzed in Kechris & Louveau [Trans. Amer. Math. Soc. 318 (1990), pp. 209–236].

1. INTRODUCTION

1.1. Reducibilities. Computability theory seeks to understand the effective content of mathematics. Ever since its beginnings in the work of Gödel, Turing, Post, Kleene, Church and others, the idea of a reduction has been a central notion in this area. Turing [Tur39] formalized what we now call *Turing reducibility* which can be viewed as the most general way of allowing computation of one set of natural numbers from another using oracle queries.

In the last 60 years, we have seen the introduction of a large number of reducibilities $A \leq B$. These different reducibilities reflect different oracle access mechanisms for the computation of A from B . Different oracle access mechanisms give different equivalence classes calibrating computation. The measure of the efficacy of such reductions is the extent to which

- (i) they give insight into computation, and
- (ii) they are useful in mathematics.

Examples of (ii) above include the use of polynomial time reductions to enable the theory of NP -completeness, but also include the use of Π_1^1 -completeness to demonstrate that classical isomorphism problems like the classification of countable abelian groups cannot have reasonable invariants (Downey-Montalbán [DM08]), Ziegler reducibility to classify algebraic closures of finitely presented groups (see e.g. Higman-Scott [HS88]), truth-table reducibility to analyze algorithmic randomness for continuous measures (Reimann-Slaman [RS]), and enumeration reducibility for the relativized Higman embedding theorem (see [HS88]). There are many other examples.

Received by the editors September 21, 2020, and, in revised form, August 2, 2021, and March 8, 2022.

2020 *Mathematics Subject Classification.* Primary 03D30, 03D78.

This work was supported in part by the Marsden Fund of New Zealand. The third author was supported in part by Noam Greenberg’s Rutherford Discovery Fellowship as a postdoctoral fellow.

©2022 by the author(s) under Creative Commons
Attribution-NonCommercial-NoDerivatives 4.0 License (CC BY NC ND 4.0)

1.2. Reducibilities in type II computation. The narrative above really only refers to notions of relative computability for infinite bit sequences (or objects, such as real numbers, which can be coded by such sequences). That is, the objects whose information content is being compared have function type $A : \omega \rightarrow \omega$ or similar.

What if instead we wanted to compare the information content of functions $f : [0, 1] \rightarrow \mathbb{R}$? The collection $\mathcal{F}([0, 1])$ of all such functions has cardinality greater than the continuum, so it is not possible to use infinite bit sequences to code all these objects. In Section 2 we will say a bit more about some approaches to the problem of relative computability for higher type objects, the most prominent of which is the Weihrauch computable reducibility framework.

In this paper, we introduce and analyze three notions of reduction for $\mathcal{F}(X)$, where X is a compact Polish space. Two of our notions are completely new and one has had little previous attention. We argue that that they meet the criteria (i) and (ii) above, and provide computational insight into the hierarchies previously introduced in classical analysis for the classification of the Baire classes of functions.¹

We first concentrate upon what we define to be $f \leq_{\mathbf{T}} g$. This reduction is interpreted to mean that f is continuously Weihrauch reducible to the *parallelization* of g . In Section 2, we define what we mean by this, and argue that this is the most natural (continuous) analog of Turing reducibility for higher type objects. We introduce the new notions of $f \leq_{\mathbf{tt}} g$ and $f \leq_{\mathbf{m}} g$ by restricting the oracle use of the functionals in the Weihrauch reduction in an appropriate way described in Section 5.

It seems to be folklore that the $\leq_{\mathbf{T}}$ degrees of the Baire functions are linearly ordered, and these degrees correspond to the proper Baire classes. Our main results concern the $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ degrees. We show that the α th jump operator² j_{α} is $\leq_{\mathbf{m}}$ -minimal in its Baire class.

Theorem 1.1. *If a Baire function f is not Baire α , then $f \geq_{\mathbf{m}} j_{\alpha+1}$ or $f \geq_{\mathbf{m}} -j_{\alpha+1}$.*

Then we restrict attention to the Baire 1 functions. In [KL90], Kechris and Louveau consider three ranking functions α , β and γ , which take Baire 1 functions to countable ordinals. These ranks are especially robust at levels of the form ω^{ξ} . Letting $\xi(f)$ denote the least ξ such that $\alpha(f) \leq \omega^{\xi}$, in our main theorem we characterize the $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ degrees of the Baire 1 functions as follows.

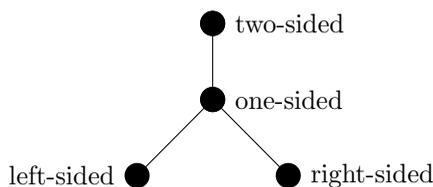
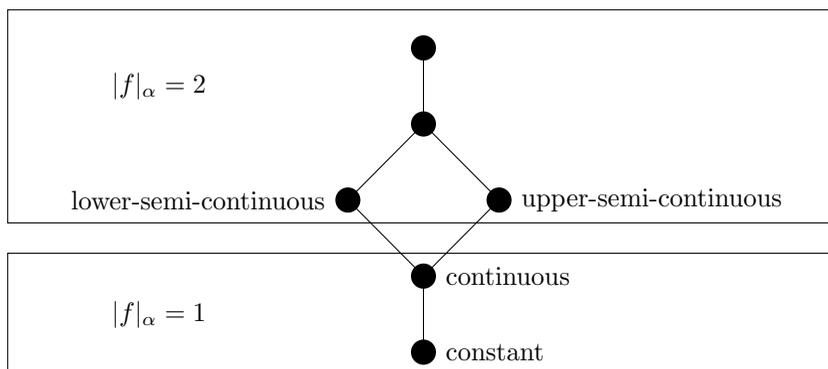
Theorem 1.2. *For f and g discontinuous Baire 1 functions,*

- (1) $f \leq_{\mathbf{tt}} g$ if and only if $\xi(f) \leq \xi(g)$.
- (2) If $|f|_{\alpha} < |g|_{\alpha}$, then $f \leq_{\mathbf{m}} g$.
- (3) If ν is a limit ordinal, $\{f : |f|_{\alpha} = \nu\}$ is an $\leq_{\mathbf{m}}$ -degree.
- (4) If ν is a successor, $\{f : |f|_{\alpha} = \nu\}$ contains exactly four $\leq_{\mathbf{m}}$ -degrees arranged as in Figure 1.

The smallest $\leq_{\mathbf{m}}$ -degrees are recognizable classes: constant functions, continuous functions, upper semi-continuous functions, and lower semi-continuous functions. See Figure 2.

¹We define these terms in Section 3.

²We will define j_{α} later, but for example $j_1 : 2^{\omega} \rightarrow \mathbb{R}$ is $j_1(X) := \sum_{i \in X} 2^{-(i+1)}$.

FIGURE 1. The \leq_m degrees of functions f with $|f|_\alpha$ a successorFIGURE 2. The smallest \leq_m degrees are recognizable classes

1.3. History and subsequent related work. Several authors have previously considered various notions of reducibility to compare discontinuous functions. The most commonly considered are continuous Weihrauch reducibility, strong continuous Weihrauch reducibility, and a rigid form of Wadge reducibility defined by $f \leq_w g$ if and only if $f = g \circ h$ for some continuous h . These reducibilities are applied in various combinations to the problem of discontinuous functions, for example in [Her96b, Bra05, Myl06, Pau10, Car13]. What all these reducibilities have in common is that the outputs of the functions f and g are considered as indivisible packets of information. By contrast, in this paper, we make rigorous the notion of one “bit” of information about the output of such a function, and this choice is what makes possible the correspondence with the α rank.

As there has been some gap between the time this work was done and the present time, we include a short note about how it developed over time and some work which followed it. Almost all results of this paper were obtained by the authors in 2015 and 2016 while the third author was a postdoctoral fellow at the University of Victoria Wellington. The first major presentation of these results was made by the third author at the February 2017 Dagstuhl Workshop on Computability Theory. At that time, Takayuki Kihara and Arno Pauly each suggested some equivalent definitions; we have included them, with credit as appropriate, and included proofs of their equivalence.

As communicated in multiple correspondences beginning in the summer of 2017, Kihara took some of the main results of the paper and generalized them in various ways. The generalizations, under the additional assumption of AD, extended Theorem 1.2 (about Baire 1 functions) and the folklore Proposition 4.10 (about Baire

functions) to apply to all real-valued functions on Polish spaces. The generalization goes in two directions, extending both the complexity of the functions considered and the class of domains considered. (Following [KL90], the present paper only considered compact Polish spaces as the domains of the functions.) In order to do the more substantial generalization to make these results apply to all functions regardless of complexity, Kihara incorporated diverse techniques including his recent analysis with Montalbán of the Wadge degrees of bco-valued functions [KM19], as well as older theory surrounding the uniform Martin’s conjecture. We will note these extensions as appropriate, and refer the reader to Kihara’s forthcoming paper [Kih] for details.

In the same paper, Kihara also clarifies the relationship between the results of this paper and the work of Elekes, Kiss and Vidnyánszky [EKV16], which was not known to the authors at the time this work was done. They defined a generalization of the α , β and γ ranks of [KL90] into the higher Baire classes. Interestingly, they were able to apply their extension of the β rank to solve an open problem in cardinal characteristics.

The original proof of Theorem 1.1 was done by a relativization of Montalbán’s theory of α -true stages [Mon14] which produced continuous functions of the type required by our $\leq_{\mathbf{m}}$ reductions. In early 2019, the first author presented a sketch of the argument at UCLA, prompting a collaboration with Andrew Marks. In summer 2019 they announced a resolution, conditional on some mild determinacy, to the longstanding Decomposability Conjecture (which was conjectured by several authors—see [Kih15] for a list). They used the relativized α -true stages technique as one of several key ingredients [DM19].

Around the same time, the authors realized that the α -true stages could be replaced with an appeal to a result made famous by Louveau and Saint-Raymond [LSR87, LSR88], which is Proposition 6.4 in this paper. Proposition 6.4 itself is a straightforward consequence of Borel determinacy, but the work of Louveau and Saint-Raymond revealed that this result actually holds in second-order arithmetic. An important consequence is that Borel Wadge determinacy also holds in second-order arithmetic. The arguments of [LSR87, LSR88] were notorious for being impenetrable, but something like Proposition 6.4 had essentially been achieved in the original version of this paper via the relativized α -true stages technique, a technique which can be carried out in second-order arithmetic. The question then naturally arose whether this technique could give a more understandable second-order arithmetic proof of Proposition 6.4. Indeed, in 2021 the first author, with collaborators Greenberg, Harrison-Trainor, and Turetsky, announced success in this endeavor [DGHTT21].

Because the method of the original proof played a role in these developments, we have included it in Appendix A. To cut down on space and improve readability, only the finite case is included.

2. MOTIVATIONS IN DEFINING $\leq_{\mathbf{T}}$

2.1. Weihrauch/computable reducibility. Suppose we want to define $f \leq g$ for functions f and g , with the meaning that g can compute f . The search for a natural notion of $f \leq g$ leads directly to Weihrauch reducibility. For $A, B \in 2^\omega$, it is clear what it means to “know” A . An algorithm or oracle knows A if, given input n , it outputs $A(n)$. Accordingly, a computation of A from B is an algorithm

which can answer these questions about A when given query access to an oracle for B . So, what kinds of questions should we be able to answer if we claim to “know” $f : [0, 1] \rightarrow \mathbb{R}$? At a minimum, an oracle for f ought to be able to produce $f(x)$ when given input x . We take this ability as the defining feature of an oracle for f .

Now, what should it mean for an algorithm to have query access to an oracle for g ? Clearly, given input x , the algorithm should be able to pass it through and query $g(x)$. If $g(x)$ were the only permitted query, the algorithm could not really be said to have access to an oracle for all of g , so we should allow some other queries as well. For example, one would hope for a theory in which the functions $x \mapsto f(x)$ and $x \mapsto f(x + c)$ always compute each other, where c is a computable real. Generalizing this idea, an algorithm with query access to g should be able to ask about $g(y)$ for any $y \leq_T x$. Therefore, the notion of Weihrauch reducibility is a natural starting candidate for a notion of $f \leq g$. Roughly speaking, f is Weihrauch reducible to g ($f \leq_W g$) if $f(x)$ can be computed by a machine with oracle access to $g(\Delta(x))$, where Δ is some fixed computable operation. We refer the reader to Section 3.3 for the full definition.

The name “Weihrauch reducibility” was coined by Brattka and Gherardi [BG11], whereas earlier Weihrauch had called it computable reducibility. Brattka [Bra05] proved effective versions of classical theorems linking the Borel and Baire hierarchies using this reducibility.

2.2. Parallelized Weihrauch reducibility. The above account seems to miss the feature of ordinary \leq_T computation in which the algorithm may use the oracle repeatedly and interactively. We would not like to limit the reduction algorithm to a single use of the g oracle.

However, if the algorithm had access to all of $g(x)$ based on its first query, it would be able to feed this back into the g oracle, obtaining $g(g(x))$ and in general the sequence of $g^{(n)}(x)$. And if we accept some algorithm is uniformly producing the sequence $g^{(n)}(x)$, it could be simultaneously engaged in writing down a summarizing output $g^{(\omega)}(x)$, where $g^{(\omega)}(x)$ is for example defined as $\bigoplus_n g^{(n)}(x)$.³ So we are led to accept $g^{(\omega)} \leq g$. If we accept this, and also wish our notion to be transitive, we must accept $g^{(\omega+1)} \leq g$, otherwise transitivity will be violated in the sequence $g^{(\omega+1)} \leq (g^{(\omega)} \oplus g) \leq g$. In the end, we are forced to say g computes all its iterates up to ω_1^{ck} . The notion just described, complete with all the transfinite iteration, was studied by Kleene [Kle59]. However, this reducibility is coarser than we want (for example, we would not want the jump operator on 2^ω to be able to compute the double-jump operator) and so we choose to go by another route.

Suppose instead we make the following seemingly minor adjustment to our concept of what an oracle for g should do. Instead of querying g with an input $\Delta(x)$, we query with a pair $(\Delta(x), \varepsilon)$, where $\varepsilon \in \mathbb{Q}^+$. Instead of returning the entire $g(\Delta(x))$, the oracle returns some $p \in \mathbb{Q}$ with $|g(\Delta(x)) - p| < \varepsilon$. Now an algorithm which on input x has made finitely many queries to g has only acquired a finite amount of new information, so its future queries are still restricted to those y with $y \leq_T x$. This breaks the cycle above. In order to get more and more precision on $f(x)$, such an algorithm may query $g(y)$ for many different values of y . But there are at most countably many queries to g associated to the computation of a single

³Imagine for the purposes of this hypothetical that g is an operator on 2^ω , so that a joining operation \oplus is available to us; a similar situation could be concocted for operators on the unit interval.

$f(x)$. Therefore, we can naturally express the kind of reducibility described above in the Weihrauch framework: $f \leq g$ could mean $f \leq_W \hat{g}$, where $\hat{g} : X^\omega \rightarrow Y^\omega$ is the *parallelization* of g , defined by applying g componentwise.

2.3. What is a single bit of information about f ? Accepting parallelized Weihrauch reducibility as a higher-type notion of \leq_T , what should \leq_{tt} and \leq_m be? It is particularly informative to consider \leq_m . In classical computability theory, $A \leq_m B$ means that there is an algorithm which, on input n , outputs m such that $A(n) = B(m)$. For us the important features are:

- (1) The oracle's response is accepted unchanged as the output, and
- (2) The question is a yes/no question.

Allowing a more demanding question (such as “approximate $f(x)$ to within ε ”) seems unfair, ruling out m -computations between functions of disjoint ranges that are otherwise computationally identical. (Notions of m -reducibility without point (2) have been considered however, for example by Hertling [Her96b], Pauly [Pau10] and Carroy [Car13].)

Our previous decision on how to finitize the oracle was, upon reflection, rather arbitrary. We could restrict ourselves to yes/no questions with the following convention about oracles, and still end up with a \leq_T notion equivalent to parallelized Weihrauch reducibility. An oracle for f accepts as input a triple (x, p, ε) , with $p \in \mathbb{Q}$ and $\varepsilon \in \mathbb{Q}^+$, and ε -approximately answers the question “is $f(x) < p$ ”? The exact version of this question would be too precise for a computable procedure, so we accept any answer as correct if $|f(x) - p| < \varepsilon$. Now that each query to the oracle yields exactly one bit of information, we can define \leq_m and \leq_{tt} for the higher type objects by placing corresponding restrictions on the oracle use. We give the formal definitions in Section 5.

2.4. Parameters. Another natural question we might ask ourselves is “what parameters would be reasonable for such reductions”? For reductions between objects of type $A : \omega \rightarrow \omega$, we usually allow integer parameters in computation procedures. Therefore, for reductions between objects of type $f : [0, 1] \rightarrow \mathbb{R}$, perhaps we should allow real parameters. We take this approach, which has a substantial simplifying effect. Every continuous function is computable relative to a real parameter, so Weihrauch computability relative to a real parameter is the same as continuous Weihrauch reducibility, the formal definition of which can be found in Section 3. Therefore, our reducibilities have a topological rather than computational character. In particular, we shall define $f \leq_{\mathbf{T}} g$ to mean $f \leq_W^c \hat{g}$, and make similar topological definitions for \leq_{tt} and \leq_m in Section 5. We plan to address the question of the lightface theory in future work.

3. PRELIMINARIES

3.1. Notation. We use standard computability-theoretic notation. Brackets $\langle m, n \rangle$ denote a canonical pairing function identifying $\omega \times \omega$ with ω . The expression 0^ω refers to an ω -length string of 0's. Concatenation of finite or infinite strings σ and τ is denoted by $\sigma \hat{\ } \tau$, which may be shortened to $\sigma\tau$ in cases where it would cause no confusion. If τ is a string with a single entry n , we also denote concatenation by $n \hat{\ } \sigma$ or $\sigma \hat{\ } n$. The n column of an element $X \in \omega^\omega$ is denoted $X^{[n]}$.

The composition of two functions f and g is denoted fg . If multiplication is intended, the notation $f \cdot g$ is used. We usually use X and Y to denote compact

separable metric spaces, A, B, Z, W to denote elements of 2^ω or ω^ω , C, D, P, Q to denote subsets of 2^ω , and \mathcal{C}, \mathcal{D} to denote subsets of $\mathcal{P}(X)$. Usually f, g , and j are arbitrary functions from X to \mathbb{R} (the ones whose complexity we seek to categorize), while h, k, u, v, H and K are typically continuous functions from ω^ω to ω^ω .

3.2. Computability and descriptive set theory. We assume the reader is familiar with Kleene’s \mathcal{O} (but without it, one could still understand the results at the finite levels of each of the hierarchies). The standard reference on this subject is Sacks [Sac90]. The n th jump of a set $A \in \omega^\omega$ is denoted $A^{(n)}$. For any $a \in \mathcal{O}^A$, if $|a| = n$ then let $A_{(a)}$ denote $A^{(n)}$, and if $|a|$ is infinite then let $A_{(a)}$ denote $H_{2^a}^A$. If $a \in \mathcal{O}$ with $|a| = \alpha$, we will often simply write α instead of a . Thus an expression like $A_{(\alpha)}$ is technically ambiguous, but since all the sets which it could refer to are one-equivalent, no problems will arise.

The reason for numbering the jumps in this lower-subscript way is to make them align correctly with the Borel hierarchy. Recall that a set is Σ_1^0 if it is open, Π_α^0 if it is the complement of a Σ_α^0 set, and Σ_α^0 if it is of the form $\cup_{n \in \omega} C_n$ where each C_n is $\Pi_{\beta_n}^0$ for some $\beta_n < \alpha$. Then a set $C \subseteq \omega^\omega$ is Σ_α^0 if and only if there is a parameter $Z \in \omega^\omega$ and an index i such that for all $A \in \omega^\omega$,

$$A \in C \iff i \in (A \oplus Z)_{(\alpha)}$$

(and if no parameter is needed, we say C is Σ_α^0).

Still, at least once we will want to refer to the sets H_a^A , where $|a|$ is a limit ordinal. In this case, we write $A^{(a)}$ to denote H_a^A .

3.3. Representations. Although our results were motivated by considering $f : [0, 1] \rightarrow \mathbb{R}$, they are also applicable in a wider context. If the domain of f is any compact separable metric space,⁴ then computations using this domain can be carried out through the theory of *represented spaces*. Hence, for completeness, we will briefly give an account of such spaces. A standard reference is Weihrauch [Wei00], and a more up-to-date survey is [BGP17].

In order for a machine to interact with a mathematical object, the object must be coded in a format a machine can read, such an element of 2^ω or ω^ω . For example, an element of \mathbb{R} could be coded by a rapidly Cauchy sequence of rational numbers (which is itself coded by an element of ω^ω using some fixed computable bijection $\omega \leftrightarrow \mathbb{Q}$). It is not too hard to see that a similar method will also work for any computable metric space, where the role of the rationals is taken by (codes for) a computable dense subset.

A *representation* of a space X is a partial function $\delta : \subseteq \omega^\omega \rightarrow X$, so that elements $x \in X$ have δ -names A_x (strictly a set $\{A_x \mid \delta(A_x) = x\}$). Note that x can have many names A_x , and not every element of ω^ω is a name. Then if X and Y are represented spaces and $f : X \rightarrow Y$, we say f is *computable* if there is a computable function $F : \omega^\omega \rightarrow \omega^\omega$ such that whenever A_x is a name for x , then $F(A_x)$ is a name for $f(x)$. We say that F *realizes* f . Because x and $f(x)$ each have many names, in general realizers are not unique.

⁴The setting of compact separable metric spaces is surely not the most general domain which could be considered. We thank the anonymous referee for pointing out that one could consider Polish spaces, Quasi-Polish spaces, or even drop the condition of separability. Although we do not consider those generalizations here, we refer the reader to [Kih] for the generalizations of many of the results of this paper to Baire space and general Polish spaces.

Not all representations are created equal. For example, the base 10 representation for reals is a valid representation according to the above definition, but the function $f(x) = 3x$ is not computable with respect the base 10 representation on both sides (what digit should the algorithm output first when seeing input .33333...?). However, it is computable with respect to the Cauchy name representation on both sides. This difference is captured in the following definition: a representation $\delta : \subseteq \omega^\omega \rightarrow X$ is *admissible* if δ is continuous and for every other continuous $\delta' : \subseteq \omega^\omega \rightarrow X$, there is a continuous function $G : \subseteq \omega^\omega \rightarrow \omega^\omega$ such that for all $A \in \text{dom } \delta'$, we have $\delta(A) = \delta'(G(A))$. That is, G transforms δ' -names to δ -names. Observe that it is possible to continuously transform a base 10 name for x into a Cauchy name for x , but not vice versa. Some definition chasing shows that the Cauchy name representation for \mathbb{R} is admissible. Restricting attention to admissible representations allows continuity properties of f to be reflected in its realizers.

Theorem 3.1 (Kreitz and Weihrauch [KW85], Schröder [Sch02]). *If X and Y are admissibly represented separable T_0 spaces, then a partial function $f : \subseteq X \rightarrow Y$ has a continuous realizer if and only if f is continuous.*

All of the pain and suffering involving representations is rewarded when we want to compare functions f and g in topologically incompatible areas, like Cantor space and \mathbb{R} . When comparing $f : X \rightarrow Y$ and $g : U \rightarrow V$, we can do so *via their representations in ω^ω* . Given two represented spaces X and Y , a *Weihrauch problem* is a multivalued partial function $f : X \rightrightarrows Y$. The \rightrightarrows indicated that this definition concerns multivalued partial functions. We will restrict attention to single-valued functions $f : X \rightarrow \mathbb{R}$. Note, however, that each real number has many names.

We conclude this section with the precise definition of Weihrauch reducibility on represented spaces. First, if $f, g \subseteq \omega^\omega \rightrightarrows \omega^\omega$, then we say that f is *Weihrauch reducible* to g , written $f \leq_W g$, if and only there are computable functions $\Delta, \Psi : \subseteq \omega^\omega \rightarrow \omega^\omega$ such that for all $A \in \text{dom}(f)$, if $B \in g(\Delta(A))$, then $\Psi(A, B) \in f(A)$. We say f is *strongly Weihrauch reducible* to g , written $f \leq_{sW} g$, if $\Psi(A, B)$ can be replaced by $\Psi(B)$ above. The notions of *continuous Weihrauch reducibility* and *continuous strong Weihrauch reducibility*, denoted \leq_W^c and \leq_{sW}^c respectively, are obtained by allowing Δ and Ψ to be merely continuous rather than computable.

Let X, Y be represented spaces with representations δ_X and δ_Y , and suppose $f : \subseteq X \rightrightarrows Y$. One can then compare f with other functions using Weihrauch reducibility by composing it with the given representations. So in particular f can be identified with the multivalued function $F : \subseteq \omega^\omega \rightrightarrows \omega^\omega$ defined so that $B \in F(A)$ if and only if $\delta_X(A) \in \text{dom}(f)$ and $\delta_Y(B) \in f(\delta_X(A))$. Of course, different choice of representations may in general result in different Weihrauch complexity of the function F , so some care is needed in the most general case. However, if δ_X and δ_Y are admissible representations, which representation is chosen does not matter, as the following well-known proposition shows. (We show it only for the continuous Weihrauch reducibility, as that is what is used in this paper, but similar statements could be made for Weihrauch reducibility and computably admissible representations.)

Proposition 3.2. *Suppose X and Y are represented spaces and $f : \subseteq X \rightrightarrows Y$. If δ_X, δ'_X are admissible representations for X and δ_Y, δ'_Y are admissible representations for Y , then $F \equiv_{sW}^c F'$, where $B \in F(A)$ if and only if $\delta_X(A) \in \text{dom}(f)$ and $\delta_Y(B) \in f(\delta_X(A))$, and similarly for F' but using δ'_X, δ'_Y .*

Proof. By symmetry it suffices to show that $F \leq_{sW}^c F'$. By admissibility, let Δ be such that $\delta'_X \circ \Delta = \delta_X$, and let Ψ be such that $\delta_Y \circ \Psi = \delta'_Y$. □

The parallelization $\hat{g} : Z^\omega \rightarrow W^\omega$ is the function that applies g countably many times simultaneously: $\hat{g}((z_i)_{i \in \omega}) = (g(z_i))_{i \in \omega}$.

In this paper, we will be dealing for the most part with situations where the coding is clear, and hence suppress the δ_X notation whenever possible. In particular, unless otherwise specified, if $X \subseteq \mathbb{R}$, then δ_X is the Cauchy representation discussed in the previous subsection.

3.4. Baire functions. Baire functions are the most tractable functions we might consider after continuous ones. Baire 1 functions are those which are defined as pointwise limits of a countable collection of continuous functions; $f(x) = \lim_s f_s(x)$ with each f_s continuous. More generally, let X be a compact separable metric space. By $\mathcal{C}(X)$, we mean the continuous functions $f : X \rightarrow \mathbb{R}$. The Baire hierarchy of functions on X is defined as follows. Let $\mathcal{B}_0(X) = \mathcal{C}(X)$. For each $\alpha > 0$, let $\mathcal{B}_\alpha(X)$ be the set of functions which are pointwise limits of sequences of functions from $\cup_{\beta < \alpha} \mathcal{B}_\beta(X)$. The functions in $\mathcal{B}_\alpha(X)$ are also referred to as the Baire α functions when X is clear.

It is well-known that a function f on a separable metrizable space X is Baire α if and only if the inverse image of each open set under f is $\Sigma_{\alpha+1}^0$ [Kec95, Theorems 24.10 & 24.3]. When $X = 2^\omega$, the Baire α functions can also be characterized via the jump.

Proposition 3.3 (Folklore). *For each ordinal α and $f : 2^\omega \rightarrow \mathbb{R}$, $f \in \mathcal{B}_\alpha(2^\omega)$ if and only if there is a Turing functional Γ and $B \in 2^\omega$ such that*

$$f(A) = \delta_{\mathbb{R}}\Gamma((A \oplus B)_{(\alpha)}).$$

Proof. When such Γ and B exist, one can readily check that the inverse images of open sets are $\Sigma_{\alpha+1}^0$. Conversely, if f is Baire α , then the sets $f^{-1}((p, q))$ for $p, q \in \mathbb{Q}$ can each be written as

$$f^{-1}((p, q)) = \{A : i_{p,q} \in (A \oplus B_{p,q})_{(\alpha+1)}\}.$$

Therefore, if B is an oracle containing each $B_{p,q}$ and $i_{p,q}$ in a uniformly accessible manner, one can use an $(A \oplus B)_{(\alpha)}$ oracle to enumerate the rational intervals (p, q) containing $f(A)$, which is enough to make a Cauchy name for $f(A)$. □

3.5. Ranks on Baire 1 functions. In [KL90], Kechris and Louveau defined three ranks α , β and γ on the Baire 1 functions. These ranks had been used either explicitly or implicitly in the literature analyzing this class of functions. Given a Baire 1 function $f : X \rightarrow \mathbb{R}$, the following derivation process is used to define the α rank. Given rational numbers p, q with $p < q$ and a closed set $P \subseteq X$, let

$$P'_{p,q} = P \setminus \cup\{U \subseteq X : U \text{ is open and } f(U) \subseteq (p, \infty) \text{ or } f(U) \subseteq (-\infty, q)\}.$$

For a fixed pair p, q , define an ω_1 -length sequence $\{P_\nu\}_{\nu < \omega_1}$ as follows. Let $P_0 = X$, $P_{\nu+1} = (P_\nu)'_{p,q}$, and $P_\nu = \cap_{\mu < \nu} P_\mu$ if ν is a limit ordinal. Since X is separable, it has a countable basis, so the sequence must stabilize below ω_1 . Let $\alpha(f, p, q)$ be

the least ν such that $P_\nu = \emptyset$; one can show that such ν exists if and only if f is Baire 1.

Finally, the α rank is defined by $\alpha(f) = \sup_{p < q} \alpha(f, p, q)$. The β and γ ranks are also defined by different transfinite derivation processes. Kechris and Louveau show that the levels of the form ω^ν are especially robust in the following sense.

Theorem 3.4 ([KL90]). *For any countable ξ and any bounded Baire 1 function f ,*

$$\alpha(f) \leq \omega^\xi \iff \beta(f) \leq \omega^\xi \iff \gamma(f) \leq \omega^\xi.$$

4. TOPOLOGICAL TURING REDUCIBILITY ON 2^ω

First we define the topological Turing reducibility as mentioned in Section 1. First we give the definition for the special case where $X = 2^\omega$.

Definition 4.1. For $f, g : 2^\omega \rightarrow \mathbb{R}$, let $f \leq_{\mathbf{T}} g$ if $f \leq_W^c \hat{g}$.

Equivalently, $f \leq_{\mathbf{T}} g$ if and only if there is a countable sequence of continuous functions $k_i : 2^\omega \rightarrow 2^\omega$ and a continuous function $h : 2^\omega \rightarrow 2^\omega$ such that whenever $\{B_i\}_{i < \omega}$ are Cauchy names for $\{g(k_i(A))\}_{i < \omega}$, $h(A \oplus \bigoplus_{i < \omega} B_i)$ is a Cauchy name for $f(A)$. Observe that all continuous functions are equivalent under $\leq_{\mathbf{T}}$.

The restriction of the domain to 2^ω is not essential, but helps keep the notation manageable. If X is a compact separable metrizable space and $f : X \rightarrow \mathbb{R}$, then in order to compare f with other functions, we may replace f with $f\delta_X : 2^\omega \rightarrow \mathbb{R}$, where $\delta_X : 2^\omega \rightarrow X$ is any total admissible representation. It is well-known that every compact metric space X has a total admissible representation. We give one standard and simple example. This example will also come in handy later.

We recursively define a function L , whose domain is a subset of 2^ω , and which maps an input string σ to an open ball $L(\sigma) = B(x_\sigma, \varepsilon_\sigma) \subseteq X$, as follows. Let $L(\langle \rangle) = X$. Given that $L(\sigma)$ has been defined, let r be a large enough number and let $\langle x_{\sigma\tau} \rangle_{\tau \in 2^r}$ be a finite sequence of points of $L(\sigma)$ such that the balls $B(x_{\sigma\tau}, 2^{-|\sigma|})$ cover the topological closure of $L(\sigma)$. Then define, for each $\tau \in 2^r$, $L(\sigma\tau) = B(x_{\sigma\tau}, 2^{-|\sigma|})$. Now, given $A \in 2^\omega$, it is clear that if $\sigma_1 \prec \sigma_2 \prec A$ and $\sigma_1, \sigma_2 \in \text{dom}(L)$, then x_{σ_1} and x_{σ_2} are within distance $2^{-|\sigma_1|+1}$ of each other. Thus each $A \in 2^\omega$ can be identified with a Cauchy sequence in X .

Definition 4.2. Given a compact separable metric space X , let the function L be defined as above, and define $\delta_L : 2^\omega \rightarrow X$ by

$$\delta_L(A) = \lim_{\substack{\sigma \prec A \\ \sigma \in \text{dom}(L)}} x_\sigma.$$

It is easy to check that δ_L is a total admissible representation for X . Therefore, there is a well-defined extension of the notion of $\leq_{\mathbf{T}}$ to compact separable metric spaces because, as Proposition 4.3 makes explicit, it does not matter which representation we choose.

Proposition 4.3. *Let X, Y be compact separable metrizable spaces and let $\delta_X, \delta'_X : 2^\omega \rightarrow X$ and $\delta_Y, \delta'_Y : 2^\omega \rightarrow Y$ be any total admissible representations for X and Y respectively. Let $f : X \rightarrow \mathbb{R}$ and $g : Y \rightarrow \mathbb{R}$. Then*

$$f\delta_X \leq_{\mathbf{T}} g\delta_Y \iff f\delta'_X \leq_{\mathbf{T}} g\delta'_Y.$$

Proof. Suppose that $(k_i)_{i < \omega}$ and h witness that $f\delta_X \leq_{\mathbf{T}} g\delta_Y$. By admissibility, let $\phi, \psi : 2^\omega \rightarrow 2^\omega$ be continuous functions such that $\delta'_X = \delta_X\phi$ and $\delta_Y = \delta'_Y\psi$ (note the asymmetry). Then $(\psi k_i \phi)_{i < \omega}$ and h witness that $f\delta'_X \leq_{\mathbf{T}} g\delta'_Y$. The reverse implication follows by symmetry. \square

So we have Definition 4.4.

Definition 4.4. Let X, Y be compact metric spaces and $f : X \rightarrow \mathbb{R}, g : Y \rightarrow \mathbb{R}$. Then we say that $f \leq_{\mathbf{T}} g$ if and only if $f\delta_X \leq_{\mathbf{T}} g\delta_Y$, where $\delta_X : 2^\omega \rightarrow X$ and $\delta_Y : 2^\omega \rightarrow Y$ are any total admissible representations of X and Y respectively.

In most cases we will restrict our attention to functions $f, g : 2^\omega \rightarrow \mathbb{R}$, and obtain more general results as corollaries. A result of Saint-Raymond shows that the Baire class of a function is unchanged by composing it with a representation.

Proposition 4.5. *For any compact separable metric space X , any $f : X \rightarrow \mathbb{R}$, and any continuous onto function $\delta_X : 2^\omega \rightarrow X$, $f \in \mathcal{B}_\alpha(X)$ if and only if $f\delta_X \in \mathcal{B}_\alpha(2^\omega)$.*

Proof. It is a result of Saint-Raymond ([SR76], see also [Kec95, Exercise 24.20]) that if X and Y are any compact metric spaces, Z a separable metric space, and $\delta : Y \rightarrow X$ is continuous and onto, then $f : X \rightarrow Z$ is of Borel class α if and only if $f\delta : Y \rightarrow Z$ is of Borel class α . (A function has Borel class α if the inverse image of each open set is $\Sigma^0_{\alpha+1}$.) In our application $Z = \mathbb{R}$ and X and Y are both separable, so the Borel class and Baire class of f and $f\delta$ coincide [Kec95, Theorems 24.10 & 24.3]. \square

Note that one could also consider the notion defined by $f \leq^c_{sW} \hat{g}$. However, this is almost the same notion as the one defined. If $f \leq_{\mathbf{T}} g$ via $\{k_i\}$ and h , and if g is a non-constant function, then letting B_0 and B_1 be such that $g(B_0) \neq g(B_1)$, one could additionally consider the continuous functions $\{k'_i\}$ which map A to B_0 if $A(i) = 0$ and map A to B_1 otherwise. Then A itself is continuously recoverable from $\bigoplus_i k'_i(A)$, so by adding these to the original $\{k_i\}$, a small modification to the original h will do the job in the strong Weihrauch setting.

Therefore, if g is non-constant, then $f \leq^c_{sW} \hat{g}$ if and only if $f \leq_{\mathbf{T}} g$. On the other hand, if g is constant, then $\{f : f \leq^c_{sW} \hat{g}\}$ is just the set of constant functions. So there is no need to consider the strong variant separately.

Now let us define some jump functions to characterize the $\leq_{\mathbf{T}}$ degrees of the Baire functions. The jump functions we consider are real-valued, because of our original motivation to study functions from $[0, 1]$ to \mathbb{R} . But the jump operator can be represented as a real-valued function in a standard way.

Definition 4.6. For $a \in \mathcal{O}$, let $j_a : 2^\omega \rightarrow \mathbb{R}$ be defined by

$$j_a(A) = \sum_{i \in A_{(a)}} 2^{-(i+1)}.$$

Because each $j_a(A)$ is irrational, its binary expansion can be continuously recovered from any Cauchy name for it. Therefore, by Proposition 3.3, if f is Baire α , then $f \leq_{\mathbf{T}} j_a$ for all a with $|a| = \alpha$.

Observe that the definition just given does not give any name to a function which maps A to (a real number version of) $A^{(\lambda)}$, where λ is any limit ordinal. That was done so that j_a would always be of Baire class $|a|$. Since we do occasionally want to refer to a jump operator which takes a limit number of jumps, we also define a notation for this.

Definition 4.7. If $a \in \mathcal{O}$ is a limit notation, let $j^a : 2^\omega \rightarrow \mathbb{R}$ be defined by

$$j^a(A) = \sum_{i \in A^{(a)}} 2^{-(i+1)}.$$

The following properties are clear.

Proposition 4.8. For any notations $a, b \in \mathcal{O}$,

- (1) $j_a \leq_{\mathbf{T}} j_b$ if and only if $|a| \leq |b|$.
- (2) If a and b are limits with $|a| = |b|$, then $j^a \equiv_{\mathbf{T}} j^b$.
- (3) If a is a limit, $j^a <_{\mathbf{T}} j_a$.

Proof. All parts of the proposition which claim that a reduction exists follow from the fact that $|a| \leq |b|$ implies $H_a^A \leq_T H_b^A$, uniformly in A . For the non-reductions, suppose for the sake of contradiction that $j_a \leq_{\mathbf{T}} j_b$ with $|a| > |b|$ or $j_a \leq_{\mathbf{T}} j^a$. Let $Z \in 2^\omega$ be an oracle strong enough to compute the continuous functions $\langle k_i \rangle$ and h used in the reduction. Then $H_{2^a}^Z \leq_T H_{2^b}^Z$ or $H_{2^a}^Z \leq_T H_a^Z$, which are not possible. \square

Proposition 4.8 justifies the use of notation j_α to refer to j_a for some unspecified $a \in \mathcal{O}$ with $|a| = \alpha$. By relativization, we can go further up the ordinals.

Definition 4.9. For any $Z \in 2^\omega$ and any $a \in \mathcal{O}^Z$, define

$$j_a^Z(A) = \sum_{i \in (A \oplus Z)_{(a)}} 2^{-(i+1)},$$

and similarly for $j^{a,Z}$.

Proposition 4.8 can then be generalized to replace j_a and j_b with j_a^Z and j_b^W , under the assumption that $a, b \in \mathcal{O}^Z \cap \mathcal{O}^W$. We leave both the statement and proof of this generalization to the reader, but for example, part (1) follows from the fact that $H_a^{(A \oplus Z)} \leq_T H_b^{(A \oplus Z) \oplus W}$ uniformly in A ; in the generalization the forward reduction is the continuous map $A \mapsto A \oplus Z$, rather than the identity map as it was in the original. Therefore, for any $\alpha < \omega_1$, we may use j_α to refer to j_a^Z for some pair Z, a with $Z \in 2^\omega$ and $a \in \mathcal{O}^Z$ with $|a|_{\mathcal{O}}^Z = \alpha$, and it does not matter which such Z, a we use because they are all in the same $\leq_{\mathbf{T}}$ equivalence class.⁵ Similar remarks apply to the expression j^α .

We conclude by showing that every Baire function in $\mathcal{F}(2^\omega, \mathbb{R})$ is topologically Turing equivalent to one of the j_α or j^α . To reduce the notational clutter, we prove the version where α is constructive, and leave the relativization to the reader.

Proposition 4.10 (Folklore). *Let α be a constructive ordinal and $f \in \mathcal{B}(2^\omega)$. If $f \notin \mathcal{B}_\alpha(2^\omega)$, then $j_{\alpha+1} \leq_{\mathbf{T}} f$. If α is a limit and $f \notin \mathcal{B}_\beta(2^\omega)$ for any $\beta < \alpha$, then either $f \equiv_{\mathbf{T}} j^\alpha$ or $j_\alpha \leq_{\mathbf{T}} f$.*

Proof. Since f is not Baire α , there is an open set $U \subseteq \mathbb{R}$ such that $f^{-1}(U)$ is not $\Sigma_{\alpha+1}^0$. Since f is Baire, $f^{-1}(U)$ is Borel, so by Wadge determinacy $C_{\alpha+1} \leq_w f^{-1}(U)$, where $C_{\alpha+1}$ is a canonical complete $\Pi_{\alpha+1}^0$ subset of 2^ω , and \leq_w is Wadge reducibility. Let v be a continuous function such that for all Z ,

$$Z \in C_{\alpha+1} \iff v(Z) \in f^{-1}(U).$$

⁵Given a, Z and b, W with $|a|_{\mathcal{O}}^Z = |b|_{\mathcal{O}}^W$ but $a \notin \mathcal{O}^W$ or $b \notin \mathcal{O}^Z$, first fix $a' \in \mathcal{O}^Z \cap \mathcal{O}^W$ with $|a'|_{\mathcal{O}}^Z = |a'|_{\mathcal{O}}^W = |b|_{\mathcal{O}}^W$, then observe $j_a^Z \leq_{\mathbf{T}} j_{a'}^Z \leq_{\mathbf{T}} j_{a'}^W \leq_{\mathbf{T}} j_b^W$.

We now show how to reduce $j_{\alpha+1}$ to f . It suffices to be able to compute each bit of $A_{(\alpha+1)}$ on input A . Given A and i , uniformly compute Z such that $i \notin A_{(\alpha+1)}$ if and only if $Z \in C_{\alpha+1}$. Expressing $i \in A_{(\alpha+1)}$ as the statement $\exists k[u(i, k) \in A_{(\alpha)}]$ for some computable u , compute also a sequence Z_k such that $u(i, k) \in A_{(\alpha)}$ if and only if $Z_k \in C_{\alpha+1}$. Then asking for more and more precision on the values of $f(v(Z))$ and $f(v(Z_k))$, wait until you see one of these enter U . This proves the first part.

Now suppose that α is a limit, $\alpha = \lim_n \alpha_n$. If f is not Baire β for any $\beta < \alpha$, then $f \geq_{\mathbf{T}} j_{\alpha_n}$ for each n . From this it is clear that $f \geq_{\mathbf{T}} j^\alpha$. Suppose that there is an open set U such that $f^{-1}(U)$ is not Σ_α^0 . Then by the same argument as above, $f \geq_{\mathbf{T}} j_\alpha$. On the other hand, if $f^{-1}(U)$ is Σ_α^0 for each open U , then $j^\alpha \geq_{\mathbf{T}} f$ as follows. Let W be an oracle such that $\{(A, p, q) : f(A) \in (p, q)\}$ is $\Sigma_\alpha^0(W)$. Given access to the oracle $j^\alpha(A \oplus W)$, we can enumerate $\{(p, q) : f(A) \in (p, q)\}$. This suffices to compute a Cauchy name for $f(A)$. \square

Corollary 4.11. *Let α be a constructive ordinal, X a compact separable metric space, and $f \in \mathcal{B}(X)$. If $f \notin \mathcal{B}_\alpha(X)$, then $j_{\alpha+1} \leq_{\mathbf{T}} f$. If α is a limit and $f \notin \mathcal{B}_\beta(X)$ for any $\beta < \alpha$, then either $f \equiv_{\mathbf{T}} j^\alpha$ or $j_\alpha \leq_{\mathbf{T}} f$.*

Proof. Let δ_X be any total admissible representation for X . By definition, $f \equiv_{\mathbf{T}} f\delta_X$, and by Proposition 4.5, f and $f\delta_X$ have the same Baire class. \square

So that is the complete picture for $\leq_{\mathbf{T}}$. The particularly strong way in which each Baire α function is reducible to j_α is in fact a continuous Weihrauch reduction. However, the reduction of Proposition 4.10 is not a continuous Weihrauch reduction since we query different values of f for each bit of $A_{(\alpha+1)}$. So the parallelization is certainly used.

After hearing of these results, and under the assumption of AD, in [Kih] Kihara has fully characterized the $\equiv_{\mathbf{T}}$ degrees of functions $f : \omega^\omega \rightarrow \mathbb{R}$. A function $f : 2^\omega \rightarrow 2^\omega$ is called *uniformly order preserving (UOP)* if there is a function $u : \omega \rightarrow \omega$ such that for all $A, B \in 2^\omega$, $A = \phi_e(B)$ implies that $f(A) = \phi_{u(e)}(f(B))$, where ϕ_e denotes the e th Turing functional. Given $f, g : 2^\omega \rightarrow 2^\omega$, f is *Martin reducible to g* , written $f \leq_{\mathbf{T}}^\nabla g$, if $f(X) \leq_{\mathbf{T}} g(X)$ on a cone. Let *UOP* denote those UOP functions $f : 2^\omega \rightarrow 2^\omega$ which are not constant on a cone, and let \mathcal{F} denote the non-constant functions $f : \omega^\omega \rightarrow \mathbb{R}$.

Theorem 4.12 ([Kih]). *The identity map induces an isomorphism between quotients of $(\text{UOP}, \leq_{\mathbf{T}}^\nabla)$ and $(\mathcal{F}, \leq_{\mathbf{T}})$.*

As it is known that the *UOP* classes are well-ordered with the successor operation given by the jump, this result significantly generalizes the structural content of Proposition 4.10.

5. DEFINITION OF TOPOLOGICAL *tt*- AND *m*-REDUCIBILITIES

The classical notions of *tt*- and *m*-reducibility on infinite binary sequences operate by restricting the number of bits of the oracle used and the manner in which they are used. In the case of a *tt*-reduction, in order to get the n th bit of the output, one specifies in advance, using only the number n , finitely many bits of the oracle that will be queried. For each possible way the oracle could respond, one commits to an output for the n th bit. Only then is the oracle queried and the commitment carried out. The *m*-reducibility is even more restrictive. In order to get the n th

bit of the output, one specifies in advance a single bit of the oracle to query, and commits to copy whatever the oracle has there as the n th bit.

As explained in Section 1, we have adopted the convention that one bit of information about f is an ε -approximate answer to the question “Is $f(A)$ greater or less than p ?” where $A \in 2^\omega$ and $p \in \mathbb{Q}$.

Given $A \in 2^\omega$, $p \in \mathbb{Q}$, and $\varepsilon \in \mathbb{Q}^+$, we define the question

$$“f(A) \lesssim_\varepsilon p”?$$

so that “yes” or “1” is a correct answer if $f(A) < p + \varepsilon$ and “no” or “0” is a correct answer if $f(A) > p - \varepsilon$. Observe that either answer is considered correct if $f(A)$ is within ε of p .

We then define a representation of \mathbb{R} whose domain is a subset of 2^ω , where each bit of a name for $y \in \mathbb{R}$ corresponds to a correct answer to a question of the form $y \lesssim_\varepsilon p$.

Definition 5.1. We say $A \in 2^\omega$ is a *separation name* for $y \in \mathbb{R}$ if for every $p \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+$, we have $A(\langle p, \varepsilon \rangle)$ correctly answers $y \lesssim_\varepsilon p$.

One can verify that the function $\delta_{\text{sep}} : \subseteq 2^\omega \rightarrow \mathbb{R}$ mapping separation names to reals is an admissible representation. Now if we take the definition of $\leq_{\mathbf{T}}$ from the previous section, use the δ_{sep} representation for real numbers, and further specify that h be either an m -reduction or a tt -reduction respectively, we obtain the following topological definitions of $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$.

Definition 5.2. For $f, g : 2^\omega \rightarrow \mathbb{R}$, we say $f \leq_{\mathbf{m}} g$ if and only if for every pair of rationals p, ε , there are rationals q, δ and a continuous function $k : 2^\omega \rightarrow 2^\omega$ such that whenever b is a correct answer to $g(k(A)) \lesssim_\delta q$, b is also a correct answer to $f(A) \lesssim_\varepsilon p$.

Definition 5.3. For $f, g : 2^\omega \rightarrow \mathbb{R}$, we say $f \leq_{\mathbf{tt}} g$ if and only if for every pair of rationals p, ε , there are

- finitely many rationals $(q_i, \delta_i)_{i < r}$
- continuous functions $k_i : 2^\omega \rightarrow 2^\omega$, and
- a truth table function $h : 2^r \rightarrow \{0, 1\}$

such that whenever $\sigma \in 2^r$ is a string where each $\sigma(i)$ correctly answers

$$g(k_i(A)) \lesssim_{\delta_i} q_i,$$

then $h(\sigma)$ correctly answers $f(A) \lesssim_\varepsilon p$.

It is clear that the reducibilities $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ are reflexive and transitive, and that

$$f \leq_{\mathbf{m}} g \implies f \leq_{\mathbf{tt}} g \implies f \leq_{\mathbf{T}} g.$$

Exactly as in Proposition 4.3, these reductions may be more generally applied to functions whose domain is any compact separable metrizable space, using admissible representations.

Proposition 5.4. Let X, Y be compact separable metrizable spaces and let $\delta_X, \delta'_X : 2^\omega \rightarrow X$ and $\delta_Y, \delta'_Y : 2^\omega \rightarrow Y$ be any admissible representations for X and Y respectively. Let $f : X \rightarrow \mathbb{R}$ and $g : Y \rightarrow \mathbb{R}$. Then

$$f\delta_X \leq_{\mathbf{tt}} g\delta_Y \iff f\delta'_X \leq_{\mathbf{tt}} g\delta'_Y$$

and

$$f\delta_X \leq_{\mathbf{m}} g\delta_Y \iff f\delta'_X \leq_{\mathbf{m}} g\delta'_Y.$$

Proof. If $k : 2^\omega \rightarrow 2^\omega$ is any function used to translate an $f\delta_X$ question into a $g\delta_Y$ question as a part of the reduction $f\delta_X \leq_{\mathbf{tt}} g\delta_Y$ (resp. $f\delta_X \leq_{\mathbf{m}} g\delta_Y$), then replacing each such k with $\psi k \phi$ witnesses that $f\delta'_X \leq_{\mathbf{tt}} g\delta'_Y$ (resp. $f\delta'_X \leq_{\mathbf{m}} g\delta'_Y$), where ψ and ϕ are as in Proposition 4.3. \square

Therefore, the following extensions are well-defined.

Definition 5.5. Let X, Y be compact metric spaces and $f : X \rightarrow \mathbb{R}, g : Y \rightarrow \mathbb{R}$. Then we say that

- $f \leq_{\mathbf{tt}} g$ if and only if $f\delta_X \leq_{\mathbf{tt}} g\delta_Y$, and
- $f \leq_{\mathbf{m}} g$ if and only if $f\delta_X \leq_{\mathbf{m}} g\delta_Y$,

where $\delta_X : 2^\omega \rightarrow X$ and $\delta_Y : 2^\omega \rightarrow Y$ are any total admissible representations of X and Y respectively.

Finally, all these reductions are primarily suitable for comparing discontinuous functions.

Proposition 5.6. *If f is continuous and g is non-constant, then $f \leq_{\mathbf{m}} g$.*

Proof. Since g is non-constant, let $B_0, B_1 \in 2^\omega$ be such that $g(B_0) < g(B_1)$. Given p, ε , let k be a continuous function which is equal to B_0 on $f^{-1}((-\infty, p - \varepsilon])$ and equal to B_1 on $f^{-1}([p + \varepsilon, \infty))$. Since f is continuous, these sets are closed, so such a k exists. Let q, δ be such that $g(B_0) < q - \delta < q + \delta < g(B_1)$. Then q, δ, k satisfies the part of the \mathbf{m} -reduction associated to p, ε . \square

5.1. Equivalent definitions. After hearing these results, the following equivalent definitions for $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$ reducibilities were observed by Arno Pauly and Takayuki Kihara, respectively.

First some standard notation. If $g : \subseteq X \rightrightarrows Y$ is a Weihrauch problem, g^n is defined as the problem $g \times g : X^n \rightrightarrows Y^n$ where $(y_0, \dots, y_{n-1}) \in g^n(x_0, \dots, x_{n-1})$ if and only if $g(x_i) = g(y_i)$ for all $i < n$. Then g^* is defined as $g^* : \subseteq \cup_n X^n \rightrightarrows \cup_n Y^n$ where $\bar{y} \in g^*(\bar{x})$ if \bar{x} and \bar{y} are the same length n and $\bar{y} \in g^n(\bar{x})$.

For any function $f : 2^\omega \rightarrow \mathbb{R}$, let $S_f : \omega^\omega \rightrightarrows \{0, 1\}$ be defined by

$$b \in S_f((p, \varepsilon) \wedge A) \iff b \text{ correctly answers } f(A) \lesssim_\varepsilon p.$$

Proposition 5.7 (Pauly). *For $f, g : 2^\omega \rightarrow \mathbb{R}$, $f \leq_{\mathbf{tt}} g$ if and only if $S_f \leq_{sW}^c S_g^*$.*

Proof. If g is constant, then each reducibility holds if and only if f is constant as well. So assume that $B_0, B_1 \in 2^\omega$ and $q, \delta \in \mathbb{Q}$ are inputs for which $g(B_0) < q - \varepsilon < q + \varepsilon < g(B_1)$.

If $f \leq_{\mathbf{tt}} g$, then for each p, ε , let $(q_i, \delta_i, k_i)_{i < r}$ and h be witness to this. For each p, ε , let r' be the number of bits sufficient to describe h according to some canonical self-delimiting coding. Then define a strong Weihrauch reduction from S_f to S_g^* as follows:

- Given $(p, \varepsilon) \wedge A$, determine r, r' from (p, ε) and set up a query to $S_g^{r'+r}$.
- Use r' -many queries to ask about $(q, \delta) \wedge B_0$ and $(q, \delta) \wedge B_1$ in a sequence which encodes h .
- Ask about $(q_i, \delta_i) \wedge k_i(A)$ for each $i < r$.
- Given the sequence of answers to these $r' + r$ -many questions, read off h from the first r' bits and apply it to the remaining r bits.

The other direction uses the compactness of 2^ω . Suppose that $S_f \leq_{sW}^c S_g^*$ via K and H . Fix p and ε . By compactness, there are finitely many strings $(\sigma_i)_{i < \ell}$ and for each i there are finitely many rationals $(q_{ij}, \delta_{ij})_{j < r_i}$ such that the cylinders $[\sigma_i]$ cover 2^ω , and for each $A \in 2^\omega$, if $\sigma_i \prec A$, then $K((p, \varepsilon) \frown A)$ has length r_i , and its j th coordinate begins with (q_{ij}, δ_{ij}) .

Let K_j be the function which computes the Cantor space part of the j th coordinate of K , when that coordinate exists. That is, K_j is defined by

$$K((p, \varepsilon) \frown \sigma_i C)(j) = (q_{ij}, \delta_{ij}) \frown K_j(\sigma_i C).$$

Let $(k_{ij})_{j < r_i}$ be functions that do the following:

$$k_{ij}(A) = \begin{cases} B_0 & \text{if } \sigma_i \not\prec A, \\ K_j((p, \varepsilon) \frown A) & \text{if } \sigma_i \prec A. \end{cases}$$

Define also $k'_i(A) = B_j$ where $j = 1$ if $\sigma_i \prec A$ and 0 otherwise, and let (q'_i, δ'_i) be all equal to (q, δ) . Let r be the total number of k_{ij} and k'_i functions defined above. Let $h : 2^r \rightarrow \{0, 1\}$ be the truth table which uses the k', q, δ answers to determine which $\sigma_i \prec A$, then uses the $k_{ij}, q_{ij}, \delta_{ij}$ answers to simulate the reverse reduction H . □

Kihara has also observed an equivalent definition of $\leq_{\mathbf{m}}$ related to partial order valued Wadge reducibility. We refer the reader to [Kih] for details.

6. PROPERTIES OF $\leq_{\mathbf{m}}$

In this section we prove our first main result concerning the $\leq_{\mathbf{m}}$ degrees of the jump functions j_α within the Baire α functions. We start with some easier facts about the structure of the $\leq_{\mathbf{m}}$ degrees. The proof of Proposition 6.1 is due to Kihara.

Proposition 6.1. *For all $f, g \in \mathcal{B}(2^\omega)$, we have either $f \leq_{\mathbf{m}} g$ or $g \leq_{\mathbf{m}} -f$.*

Proof. We can understand the statement $f \leq_{\mathbf{m}} g$ as saying that Player II has a winning strategy in the following game. Player I plays a target bit (p, ε) . Player II plays its intended oracle bit (q, δ) . Player I then starts playing bits of the input A ; Player II also plays bits of a sequence B in response, but Player II can pass (however they must ultimately produce an infinite sequence in order to win.) Player II wins if any correct answer to $g(B) \lesssim_\delta q$ is also a correct answer to $f(A) \lesssim_\varepsilon p$. If Player II has a winning strategy, then q, δ and the continuous function k defined by $k(A) = B$ are as in the definition of $\leq_{\mathbf{m}}$. But if Player I has a winning strategy, then for any q, δ , there are p, ε (in fact, the same p and ε each time, chosen according to the winning strategy of Player I) and a continuous function k' which, following the winning strategy of Player I against Player II playing an arbitrary B , outputs $A = k'(B)$ such that either $g(B) < q + \delta$ and $f(A) \geq p + \varepsilon$ or $g(B) > q - \delta$ and $f(A) \leq p - \varepsilon$. Therefore, if $-f(A) < -p + \varepsilon$, we must be in the first case and thus $g(B) < q + \delta$. Similarly, if $-f(A) > -p - \varepsilon$ then we must be in the second case, so $g(B) > q - \delta$. This shows that $g \leq_{\mathbf{m}} -f$ via k' (observe that $(-p, \varepsilon)$ is the bit of $f(A)$ actually queried). □

Corollary 6.2. *If $f \in \mathcal{B}_\alpha 2^\omega$, then $f \leq_{\mathbf{m}} j_{\alpha+1}$.*

Proof. If not, then by Proposition 6.1 we would have $j_{\alpha+1} \leq_{\mathbf{m}} -f \leq_{\mathbf{T}} j_\alpha$, impossible as $j_{\alpha+1}$ is not $\mathcal{B}_\alpha(2^\omega)$. □

Our first theorem shows that the jump functions are the weakest functions in each Baire class.

Theorem 6.3. *If $f \in \mathcal{B}(2^\omega)$ and $f \notin \mathcal{B}_\alpha 2^\omega$, then either $j_{\alpha+1} \leq_m f$ or $-j_{\alpha+1} \leq_m f$.*

It is easy to see Theorem 6.3 is true when $\alpha = 0$. If f is not continuous, let $(z_n)_{n \in \omega} \rightarrow z$ be a convergent sequence of inputs for which $f(z) \neq \lim_n f(z_n)$. Without loss of generality, there is some $\delta > 0$ such that for all n , $f(z_n) > f(z) + \delta$, or for all n , $f(z_n) < f(z) - \delta$. In the first case, we have that $j_1 \leq_m f$ via the following algorithm. On input (p, ε) , choose (q, δ') so that $[q - \delta', q + \delta'] \subseteq (f(z), f(z) + \delta)$. Then let h be the function which, on input x , outputs bits of z while computing approximations to $j_1(x)$. If it ever sees that $j_1(x) > p - \varepsilon$, it switches to outputting bits of z_n for some n large enough that z and z_n agree on all bits which were already committed to. The case where $f(z_n) < f(z) - \delta$ for all n is similar, only in that case we find that $-j_1 \leq_m f$.

To prove Theorem 6.3 in the general case we will make use of the following generalization of Borel Wadge determinacy. We provide a simple proof of this generalization using Borel determinacy, but it is interesting to note that Louveau and Saint-Raymond [LSR87, LSR88] showed that this generalization is provable in second order arithmetic via a much more intricate argument. Therefore, the use of Borel determinacy here can be avoided.

Proposition 6.4. *Let $D, E_0, E_1 \subseteq \omega^\omega$ be Borel. Then one of the following holds:*

- (1) *There is a continuous function $\varphi : \omega^\omega \rightarrow \omega^\omega$ such that $\varphi(D) \subseteq E_0$ and $\varphi(\omega^\omega \setminus D) \subseteq E_1$.*
- (2) *There is a continuous function $\psi : \omega^\omega \rightarrow \omega^\omega$ such that $\psi(E_0) \subseteq \omega^\omega \setminus D$ and $\psi(E_1) \subseteq D$.*

Proof. Define a two player game, where at turn n player I (who plays first) plays $x(n)$ and player II plays $y(n)$. At the end of the game, II wins if

$$(x \in D \wedge y \in E_0) \vee (x \notin D \wedge y \in E_1).$$

By Borel determinacy, one of the two players has a winning strategy. A winning strategy for II gives a continuous function meeting outcome (1).

If on the other hand I has a winning strategy, then for every play of the game according to I's winning strategy we have that

$$(x \notin D \vee y \notin E_0) \wedge (x \in D \vee y \notin E_1).$$

This gives a continuous function meeting outcome (2). □

We give a new corollary to this theorem.

Corollary 6.5. *Let $V \subseteq \omega^\omega$ be Π_α^0 . Let $W \subseteq \omega^\omega$ be Π_α^0 -hard and let $\{W_i\}_{i \in \mathbb{N}}$ be a partition of W into Borel sets. Then there is a continuous function $\varphi : \omega^\omega \rightarrow \omega^\omega$ and $i \in \mathbb{N}$ such that:*

- (1) $\varphi(V) \subseteq W_i$.
- (2) $\varphi(\omega^\omega \setminus V) \subseteq \omega^\omega \setminus W$.

Proof. For each i , we can apply Proposition 6.4 with $D = V$, $E_0 = W_i$ and $E_1 = \omega^\omega \setminus W$. Assume that for each i , the second option of the theorem holds, i.e. there is a continuous function ψ_i such that

$$\psi_i(W_i) \subseteq \omega^\omega \setminus V \text{ and } \psi_i(\omega^\omega \setminus W) \subseteq V.$$

Now take $K_i = \psi_i^{-1}(\omega^\omega \setminus V)$. Note that K_i is Σ_α^0 and we have that $W = \bigcup_i W_i \subseteq \bigcup_i K_i$. Further, for all i we know that $K_i \cap (\omega^\omega \setminus W) = \emptyset$. Hence $W = \bigcup_i K_i$ and so W is Σ_α^0 . This is a contradiction as we are given that W is Π_α^0 -hard.

Hence for some i we have that the first option of Theorem 6.4 holds. That is, there is a continuous φ such that $\varphi(V) \subseteq W_i$ and $\varphi(\omega^\omega \setminus V) \subseteq \omega^\omega \setminus W$. \square

Proof of Theorem 6.3. Since f is not Baire α , let $U \subseteq \mathbb{R}$ be an open set such that $f^{-1}(U)$ is not $\Sigma_{\alpha+1}^0$. Without loss of generality, U is of the form $(u, +\infty)$ or $(-\infty, u)$. If U is of the form $(-\infty, u)$, then we will have $j_{\alpha+1} \leq_{\mathbf{m}} f$, and in the other case $j_{\alpha+1} \leq_{\mathbf{m}} -f$ (or equivalently, $-j_{\alpha+1} \leq_{\mathbf{m}} f$). Replacing f with $-f$ if necessary let us assume $U = (-\infty, u)$.

Denote $f^{-1}(U)$ by W . Since f is Borel, W is Wadge determined, so it is $\Pi_{\alpha+1}^0$ -hard. We can partition W into the following sets $W_0 = f^{-1}((-\infty, u - 1])$ and for all $i \geq 1$,

$$W_i = f^{-1} \left(\left(u - \frac{1}{i}, u - \frac{1}{i+1} \right] \right).$$

Take any $p, \epsilon \in \mathbb{Q}$ with $\epsilon > 0$. Let $V = j_{\alpha+1}^{-1}((-\infty, p - \epsilon])$. The set V is $\Pi_{\alpha+1}^0$. (We have $A \in V$ if and only if for all finite $F \subseteq \omega$ such that $\sum_{i \in F} 2^{-(i+1)} > p - \epsilon$, there is some $i \in F$ such that $i \notin A_{(\alpha+1)}$. Recall from Section 1 that $\{A : i \in A_{(\eta)}\}$ is a Σ_η^0 set.)

Thus by Corollary 6.5 there is a continuous map φ and an $i \in \mathbb{N}$ such that $\varphi(V) \subseteq W_i$ and $\varphi(2^\omega \setminus V) \subseteq f^{-1}([u, +\infty))$. Hence taking $\delta = \frac{1}{2^{(i+1)}}$ and $q = u - \delta$ we have that for any $A \in 2^\omega$, there is only one correct answer to $f(\varphi(A)) \lesssim_\delta q$. Further, this is also a correct answer to $j_{\alpha+1}(A) \lesssim_\epsilon p$. \square

Corollary 6.6. *Let X be a compact separable metric space. If $f \in \mathcal{B}(X)$ and $f \notin \mathcal{B}_\alpha(X)$, then either $j_{\alpha+1} \leq_{\mathbf{m}} f$ or $-j_{\alpha+1} \leq_{\mathbf{m}} f$.*

Proof. Identical to the proof of Corollary 4.11. \square

Corollary 6.7. *Let X, Y be a compact separable metric space. If $g \in \mathcal{B}(X)$, $g \notin \mathcal{B}_\alpha(X)$ and $f \in \mathcal{B}_\alpha(Y)$, then $f \leq_{\mathbf{m}} g$.*

Proof. By Proposition 6.1, if $f \not\leq_{\mathbf{m}} g$, then $g \leq_{\mathbf{m}} -f$. But this is impossible by the combination of the following three facts. First, either $j_{\alpha+1} \leq_{\mathbf{m}} g$ or $-j_{\alpha+1} \leq_{\mathbf{m}} g$ by Corollary 6.6. Second, if $h \leq_{\mathbf{m}} k$ and k is Baire α , then so is h . Finally, neither $j_{\alpha+1}$ nor $-j_{\alpha+1}$ is Baire α . \square

As discussed in Section 1, Theorem 6.3 was first proved without reference to Proposition 6.4. We include the finite case of the original proof in Appendix A.

7. THE BOURGAIN RANK ON \mathcal{B}_1

The structure of the $\leq_{\mathbf{m}}$ -degrees and $\leq_{\mathbf{tt}}$ -degrees within the Baire 1 functions is related to the α rank, also known as the Bourgain rank [Bou80], which was studied by Kechris and Louveau [KL90]. Here we place that rank in a slightly more general setting that will be suitable for describing both the $\leq_{\mathbf{m}}$ and $\leq_{\mathbf{tt}}$ degrees, and establish some notation that will be used throughout. We begin by considering an arbitrary compact separable metric space X , so that these definitions directly coincide with those given in [KL90].

Definition 7.1. For any collection $\mathcal{C} \subseteq \mathcal{P}(X)$, a *derivation sequence* for \mathcal{C} is defined for $\nu < \omega_1$ by

- $P^0 = X$.
- $P^{\nu+1} \supseteq P^\nu \setminus \cup\{U \text{ open} : \text{for some } C \in \mathcal{C}, P^\nu \cap U \subseteq C\}$
- $P^\lambda \supseteq \cap_{\nu < \lambda} P^\nu$.

By replacing \supseteq with $=$ in two places, we obtain the definition for the *optimal derivation sequence* for \mathcal{C} .

Here are some properties of derivation sequences which will be useful and which follow directly from the definitions.

Proposition 7.2. *Let Q^ν be a derivation sequence for $\mathcal{C} \subseteq \mathcal{P}(X)$.*

- (1) *If P^ν is the optimal derivation sequence for \mathcal{C} , then $P^\nu \subseteq Q^\nu$ for all ν .*
- (2) *If $k : X \rightarrow X$ is continuous, then $R^\nu := k^{-1}(Q^\nu)$ is a derivation sequence for $\{k^{-1}(C) : C \in \mathcal{C}\}$.*
- (3) *If $\mathcal{D} \subseteq \mathcal{P}(X)$ is such that for every $C \in \mathcal{C}$, there is a $D \in \mathcal{D}$ such that $C \subseteq D$, then Q^ν is a derivation sequence for \mathcal{D} .*

Definition 7.3 (Bourgain rank, also known as α rank). For $f \in \mathcal{B}_1(X)$ and rationals p, ε , let $P_{f,p,\varepsilon}^\nu$ be the optimal derivation sequence for $\{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}$. Let $\alpha(f, p, \varepsilon)$ be least ordinal ν such that $P_{f,p,\varepsilon}^\nu = \emptyset$. Let the Bourgain rank of f be

$$|f|_\alpha = \sup_{p,\varepsilon \in \mathbb{Q}} \alpha(f, p, \varepsilon).$$

If f, p, ε are clear from context, we may write P^ν or P_f^ν instead of $P_{f,p,\varepsilon}^\nu$. Observe that the compactness of X implies that $\alpha(f, p, \varepsilon)$ is always a successor, but in general $|f|_\alpha$ may be either a limit or a successor.

In the course of the optimal derivation process, individual points leave at various stages, and we would like to keep track of this.

Definition 7.4. Let $x \in X$. If P^ν is the optimal derivation sequence for sets \mathcal{C} and P^ν is eventually empty, let $|x|_\mathcal{C}$ denote the least ν such that $x \notin P^\nu$. Given $f \in \mathcal{B}_1(X)$, and p, ε , let $|x|_{f,p,\varepsilon}$ be the least ν such that $x \notin P_{f,p,\varepsilon}^\nu$.

If f, p, ε and/or \mathcal{C} are clear from context, we may just write $|x|_f$ or $|x|$. Observe that $|x|$ is always a successor ordinal.

We will need to consider the case when $|f|_\alpha$ is a successor with special care. Supposing we have such an f , let ν, p, ε be defined so that $\nu + 1 = \alpha(f, p, \varepsilon) = |f|_\alpha$. Of course, we may also have $\nu + 1 = \alpha(f, p', \varepsilon')$ for some other rationals p', ε' .

Definition 7.5. Given $f \in \mathcal{B}_1(X)$ with $|f|_\alpha = \nu + 1$, and $p, \varepsilon \in \mathbb{Q}$, say (p, ε) is *maximal* if $f(P_{f,p,\varepsilon}^\nu) \setminus (p - \varepsilon, p + \varepsilon) \neq \emptyset$ and $\alpha(f, p, \varepsilon) = \nu + 1$.

Observe that maximal (p, ε) always exist. If $P_{f,p,\varepsilon}^\nu \neq \emptyset$, but $f(P_{f,p,\varepsilon}^\nu) \setminus (p - \varepsilon, p + \varepsilon) = \emptyset$, then by decreasing ε , one may shrink $(p - \varepsilon, p + \varepsilon)$ to include an element of $f(P_{f,p,\varepsilon}^\nu)$ (which grows in size).

Definition 7.6. Let $f \in \mathcal{B}_1(X)$ with $|f|_\alpha = \nu + 1$. We say f is

- *two-sided* if there is a maximal (p, ε) such that $f(P^\nu) \not\subseteq (p - \varepsilon, \infty)$ and $f(P^\nu) \not\subseteq (-\infty, p + \varepsilon)$;
- *one-sided* otherwise;
- *left-sided* if for every maximal (p, ε) , $f(P^\nu) \subseteq (-\infty, p + \varepsilon)$;

- *right-sided* if for every maximal (p, ε) , $f(P^\nu) \subseteq (p - \varepsilon, \infty)$.

For example j_1 is left-sided, as is any discontinuous lower semi-continuous function. If f is left-sided, then $-f$ is right-sided, and vice versa. However, there are one-sided $f : 2^\omega \rightarrow \mathbb{R}$ which are neither right-sided nor left-sided. For example, consider

$$f(A) = \begin{cases} 1 & \text{if } A \in [0] \setminus \{01^\omega\}, \\ -1 & \text{if } A \in [1] \setminus \{10^\omega\}, \\ 0 & \text{otherwise.} \end{cases}$$

We now restrict attention to the case where $X = 2^\omega$. More general results for compact separable metric spaces will then follow from Proposition 7.7, which verifies that all the important properties of a function $f : X \rightarrow \mathbb{R}$ are shared by $f\delta_X : 2^\omega \rightarrow \mathbb{R}$.

Proposition 7.7. *Let X be any compact separable metric space, let $\delta_X : 2^\omega \rightarrow X$ be any total admissible representation, and let $f \in \mathcal{B}_1(X)$. Then*

- (1) *For all ν, p, ε , we have $\delta_X(P_{f\delta_X, p, \varepsilon}^\nu) = P_{f, p, \varepsilon}^\nu$,*
- (2) *For all p, ε , we have $\alpha(f\delta_X, p, \varepsilon) = \alpha(f, p, \varepsilon)$,*
- (3) *$|f\delta_X|_\alpha = |f|_\alpha$,*
- (4) *For all p, ε , for all $x \in X$, $|x|_{f, p, \varepsilon} = \max\{|A|_{f\delta_X, p, \varepsilon} : \delta_X(A) = x\}$.*
- (5) *If $|f|_\alpha$ is a successor, $f\delta_X$ is two-, one-, left-, or right-sided if and only if f has the same property.*

Proof. First observe that statement (4) immediately implies all the others. Second, we claim it suffices to consider the special case when δ_X is the representation δ_L from Definition 4.2. To see this, observe that if $h : 2^\omega \rightarrow 2^\omega$ is any continuous function, then $h^{-1}(P_{f, p, \varepsilon}^\nu)$ is a derivation sequence for the composition fh , and therefore (fixing and dropping now the p and ε), $P_{fh}^\nu \subseteq h^{-1}(P_f^\nu)$. So by admissibility of δ_X , if we now let $\phi : 2^\omega \rightarrow 2^\omega$ be a continuous function such that $\delta_L = \delta_X\phi$, we may apply this observation to f and δ_X , and to $f\delta_X$ and ϕ . This yields $\phi(P_{f\delta_X}^\nu) \subseteq P_{f\delta_X}^\nu$ and

$$\delta_L(P_{f\delta_L}^\nu) \subseteq \delta_X(P_{f\delta_X}^\nu) \subseteq P_f^\nu.$$

From this we can see that for all $A \in 2^\omega$ we have

$$|A|_{f\delta_L} \leq |\phi(A)|_{f\delta_X} \leq |\delta_L(A)|_f.$$

Therefore, showing item (4) for δ_L suffices to show it in general.

Let us say that $A \in 2^\omega$ is *interior* if $\delta_L(A) \in L(\sigma)$ for every $\sigma \in \text{dom}(L)$ with $\sigma \prec A$. Since each point of X has an interior δ_L -name, it now suffices to show that if A is interior, then $|A|_{f\delta_L} \geq |\delta_L(A)|_f$. So, proceeding by induction, we shall show that for all interior A , if $|\delta_L(A)|_f > \nu$, then $|A|_{f\delta_L} > \nu$. Suppose that A is interior, $A \in P_{f\delta_L}^\nu$, and $\delta_L(A) \in P_f^{\nu+1}$. We must show that $A \in P_{f\delta_L}^{\nu+1}$. Since $\delta_L(A) \in P_f^{\nu+1}$, there must be sequences of points $b_n, c_n \in P_f^\nu$ such that $\lim_n b_n = \lim_n c_n = \delta_L(A)$, and such that for each n , $f(b_n) \leq p - \varepsilon$ and $f(c_n) \geq p + \varepsilon$.

Observe that if A is interior, and if $y \in L(\sigma)$ with $\sigma \prec A$, then there is an interior $B \in L(\sigma)$ such that $\sigma \prec B$ and $\delta_L(B) = y$. Therefore, there are sequences of interior points B_n, C_n such that $\delta_L(B_n) = b_n$, $\delta_L(C_n) = c_n$, and $\lim_{n \rightarrow \infty} B_n = \lim_{n \rightarrow \infty} C_n = A$. By induction, $B_n, C_n \in P_{f\delta_L}^{\nu+1}$. Therefore, $A \in P_{f\delta_L}^{\nu+1}$, as needed. \square

7.1. Some intuition and tools. The Bourgain hierarchy can be understood as a higher type version of the Ershov hierarchy. Recall the Ershov hierarchy stratifies the Δ_2^0 subsets of ω according to the amount of mind-changes needed in an optimal limit approximation to that set. In general, ordinal-many mind-changes can be needed. For $a \in \mathcal{O}$, a function $A : \omega \rightarrow \omega$ is a -computably approximable if there is a partial computable $\varphi(n, b)$ such that $A(n) = \varphi(n, b_n)$, where b_n is the $\leq_{\mathcal{O}}$ -least ordinal $b_n \leq_{\mathcal{O}} a$ for which the computation converges. We picture this process dynamically – a computable procedure makes a guess about $A(n)$ associated to a certain ordinal. If it changes its guess, it must decrease the ordinal. This limits the number of mind-changes.

We can understand each open set removed as a part of the Bourgain derivation process as a guess about the answer to the question $f(x) \lesssim_{\varepsilon} p$. The open sets removed later in the derivation process have a high associated ordinal rank and correspond to early guesses; the open sets removed at the beginning of the derivation process correspond to the latest guesses. The following object, a mind-change sequence, is nothing more than a derivation sequence annotated with the guesses that justified the derivation. It can also be viewed as a higher-type analog of φ as above. To simplify the notation, we assume $\mathcal{C} = \{C_i : i < k\}$, where k could be finite or ω . Recall in this section we have fixed $X = 2^\omega$. Let Ord denote the ordinals.

Definition 7.8. Given $\mathcal{C} = \{C_i : i < k\} \subseteq \mathcal{P}(2^\omega)$, a *mind-change sequence* for \mathcal{C} is a countable subset of $M \subseteq \text{Ord} \times 2^{<\omega} \times k$ for which

- (1) The sequence Q^ν defined by

$$Q^\nu = 2^\omega \setminus \left(\bigcup_{\substack{(\mu, \tau, j) \in M \\ \mu < \nu}} [\tau] \right)$$

is a derivation sequence for \mathcal{C} , and

- (2) For all $(\nu, \sigma, i) \in M$, $[\sigma] \cap Q^\nu \subseteq C_i$.

An *optimal mind-change sequence* for \mathcal{C} is one in which Q^ν is the optimal derivation sequence for \mathcal{C} .

Observe that an optimal mind-change sequence always exists, since it just keeps track of the open sets $[\sigma]$ which are removed at stage ν of the construction of the optimal derivation sets sequence, and keeps track of which set $C \in \mathcal{C}$ caused $[\sigma]$ to be removed at stage ν .

Two “mind-change” based encodings of the Baire 1 functions are suggested by this idea. One encoding of $f \in \mathcal{B}_1(2^\omega)$, following the α rank, would consist of a countable collection of mind-change sequences $M_{p,\varepsilon}$, one for each

$$\mathcal{C}_{p,\varepsilon} = \{f^{-1}((p - \varepsilon, \infty)), f^{-1}((-\infty, p + \varepsilon))\}$$

for $p, \varepsilon \in \mathbb{Q}$. Another encoding, following the β rank, would consist of a different countable collection of mind-change sequences M_ε , one for each

$$\mathcal{C}_\varepsilon = \{f^{-1}((q - \varepsilon, q + \varepsilon)) : q \in \mathbb{Q}\}$$

for each $\varepsilon \in \mathbb{Q}^+$. We will not need to use such encodings explicitly, so we avoid further technical definitions, but this way of thinking about a Baire 1 function motivates all the arguments which follow.

A mind-change sequence can serve as evidence of an upper bound on the length of an optimal derivation sequence for a collection \mathcal{C} . The next notion provides evidence of a lower bound. The idea is that if $[\sigma]$ was not removed at stage ν of the derivation process, then for each $C \in \mathcal{C}$, there was some element $A_{\nu, \sigma, C}$ which witnesses that $P^\nu \cap [\sigma] \not\subseteq C$. If \mathcal{C} is countable and the derivation process lasts only countably many stages, then only countably many A are needed to witness the necessity of an optimal derivation sequence being as long as it is. Below, we define a *scaffolding sequence* to be any countable collection of A 's which can supply all necessary witnesses, together with a record of where in the process these A are slowing things down.

Definition 7.9. Given $\mathcal{C} = \{C_i : i < k\} \subseteq \mathcal{P}(2^\omega)$, let P^ν be its optimal derivation sequence. A *scaffolding sequence* for \mathcal{C} is any enumeration of a countable subset $S \subseteq 2^\omega \times \text{Ord} \times 2^{<\omega} \times k$ such that

- (1) If $(A, \nu, \sigma, i) \in S$, then $A \in P^\nu \cap [\sigma] \setminus C_i$, and
- (2) If $P^\nu \cap [\sigma] \not\subseteq C_i$, there is $A \in 2^\omega$ with $(A, \nu, \sigma, i) \in S$.

Letting S' be the projection of S onto its first coordinate, observe that for all $\mu < \nu$ and σ , if $P^\nu \cap [\sigma] \neq \emptyset$, then $P^\mu \cap [\sigma] \cap S' \neq \emptyset$.

We conclude this subsection with a remark about another notion of “mind-change” that has been considered in the literature in the context of functions. Hertling’s notion of *level* [Her96a, Her96b] is defined as follows. A function $f : X \rightarrow \mathbb{R}$ has level α if there is a sequence of continuous functions $\langle f_\beta \rangle_{\beta < \alpha}$ which have open domains such that $f(x)$ is equal to $f_\beta(x)$ for the least β such that $f_\beta(x)$ is defined. By the Hausdorff-Kuratowski theorem, the functions which have a well-defined level in this sense are precisely the Δ_2^0 -piecewise continuous functions. (A function f is Δ_2^0 -piecewise continuous if there is a partition of the domain of f into countably many Δ_2^0 pieces such that the restriction of f to each piece is continuous.) These are a proper subset of the Baire 1 functions, so this notion of mind-change sequence is different than the one considered here. The difference lies precisely in the fact that for Hertling’s level, one can only change one’s mind finitely often about what continuous operation is going to be applied to the input x to produce the output $f(x)$. In contrast, in the mind-change process defined here, for each bit of $f(x)$, one can change one’s mind only finitely often about the value of that bit. However, $f(x)$ has infinitely many bits and the amount of mind-changing is permitted to vary from bit to bit.

8. CHARACTERIZATION OF THE \leq_m EQUIVALENCE CLASSES IN \mathcal{B}_1

In this section we prove parts (2)-(4) of Theorem 1.2, characterizing the structure of the \leq_m degrees within the Baire 1 functions. We begin by proving Theorem 8.1, which establishes (2)-(4) in case $X = 2^\omega$.

Theorem 8.1. *For $f, g \in B_1 2^\omega$, $|f|_\alpha < |g|_\alpha$ implies $f \leq_m g$. If $|f|_\alpha = |g|_\alpha$, then $f \leq_m g$ if and only if at least one of the following holds:*

- (1) $|f|_\alpha$ is a limit ordinal.
- (2) g is two-sided.
- (3) f is one-sided and g is neither right-sided nor left-sided.
- (4) f and g are either both right-sided or both left-sided.

Proof. We begin with a general observation. Suppose that $p, \varepsilon, q, \delta \in \mathbb{Q}$ and $k : 2^\omega \rightarrow 2^\omega$ is a continuous function such that any correct answer to $g(k(A)) \lesssim_\delta q$ is also a correct answer to $f(A) \lesssim_\varepsilon p$. Then for any A , $g(k(A)) < q + \delta$ implies $f(A) < p + \varepsilon$, so $k^{-1}(g^{-1}((-\infty, q + \delta))) \subseteq f^{-1}((-\infty, p + \varepsilon))$. Similarly, $k^{-1}(g^{-1}((q - \delta, \infty))) \subseteq f^{-1}((p - \varepsilon, \infty))$. Therefore, the sets Q^μ defined by

$$Q^\mu = k^{-1}(P_{g,q,\delta}^\mu)$$

are a derivation sequence for $\{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}$. Therefore $P_{f,p,\varepsilon}^\mu \subseteq Q^\mu$ for each μ , so $\alpha(f, p, \varepsilon) \leq \alpha(g, q, \delta)$. Furthermore, for all $A \in 2^\omega$, we have $|A|_{f,p,\varepsilon} \leq |k(A)|_{g,q,\delta}$.

Now suppose $f \leq_{\mathbf{m}} g$. Then for any p, ε , there are q, δ and k as above, so $|f|_\alpha \leq |g|_\alpha$. The first statement of the theorem now follows by Proposition 6.1 and the observation that $|g|_\alpha = |-g|_\alpha$ for all g .

From now on we consider the case where $|f|_\alpha = |g|_\alpha$.

Suppose that $f \leq_{\mathbf{m}} g$. We claim that if $|f|_\alpha = \nu + 1$ (a successor) then one of (2)-(4) in the statement of the theorem holds.

Let (p, ε) be maximal for f . Since $f \leq_{\mathbf{m}} g$, let q, δ and k be as in the first paragraph. By the choice of p and ε , there is an $A \in 2^\omega$ with $|A|_{f,p,\varepsilon} = |k(A)|_{g,q,\delta} = \nu + 1$ and $f(A) < p - \varepsilon$, or there is a $B \in 2^\omega$ with $|B|_{f,p,\varepsilon} = |k(B)|_{g,q,\delta} = \nu + 1$ and $f(B) > p + \varepsilon$, or perhaps both occur. If such A exists, then $g(k(A)) < q - \delta$ and if such B exists, then $g(k(B)) > q + \delta$.

Therefore, if g is not two-sided, then f is not two-sided; in that case, if g is right-sided or left-sided, then f must match. This completes the proof that $f \leq_{\mathbf{m}} g$ implies the disjunction of (1)-(4).

Assuming now the disjunction of (1)-(4), let p, ε be given. First we choose a pair q, δ which gives us enough room to work. If $|f|_\alpha = \nu + 1$, choose (q, δ) to be maximal for g . Additionally, if g is two-sided, make sure q and δ witness the two-sidedness of g . Or, if f is one-sided and g is neither left-sided nor right-sided, then if $f(P_{f,p,\varepsilon}^\nu) \subseteq (p - \varepsilon, \infty)$ (respectively $f(P_{f,p,\varepsilon}^\nu) \subseteq (-\infty, p + \varepsilon)$) make sure $g(P_{g,q,\delta}^\nu) \subseteq (q - \delta, \infty)$ (respectively $g(P_{g,q,\delta}^\nu) \subseteq (-\infty, q + \delta)$). If f and g are both right- or both left-sided, a maximal choice of q and δ suffices without further restrictions. If $|f|_\alpha$ is a limit, choose q, δ so that $\alpha(f, p, \varepsilon) < \alpha(g, q, \delta)$ and $g(P_{g,q,\delta}^\nu) \setminus (q - \delta, q + \delta) \neq \emptyset$ (decreasing δ if necessary to achieve the latter). In this case, define ν so that $\alpha(g, q, \delta) = \nu + 1$.

We now define a continuous function k such that any correct answer to $g(k(A)) \lesssim_\delta q$ also correctly answers $f(A) \lesssim_\varepsilon p$. Given A , its image $k(A)$ will be defined in stages according to an algorithm which uses oracle information about a mind-change sequence related to f and a scaffolding sequence related to g . By defining $k(A)$ in stages, we guarantee k is continuous.

Let $\mathcal{C} = \{C_0, C_1\}$, where $C_0 = f^{-1}((-\infty, p + \varepsilon))$ and $C_1 = f^{-1}((p - \varepsilon, \infty))$. Let $\mathcal{D} = \{D_0, D_1\}$, where $D_0 = g^{-1}((-\infty, q + \delta))$ and $D_1 = g^{-1}((q - \delta, \infty))$. Let Z be an oracle which contains the following information:

- A well-order W long enough that ν has a code in \mathcal{O}^W (a technical point which allows us to use \mathcal{O}^W in place of Ord in the mind-change and scaffolding sequences).
- An optimal mind-change sequence M for \mathcal{C} .
- A scaffolding sequence S for \mathcal{D} .

Letting A denote the input, at each stage s , we will have defined an initial segment τ_s of $k(A)$. We will be keeping track of an ordinal μ_s , an index $i_s \in \{0, 1\}$, and an element $B_s \in 2^\omega$, where $\tau_s \prec B_s$. We will always maintain the following:

- (i) that $|A|_C \leq \mu_s + 1 \leq |B_s|_{\mathcal{D}}$,
- (ii) that i_s is the only correct answer to $g(B_s) \lesssim_\delta q$, and
- (iii) if $|A|_C = \mu_s + 1$, then i_s correctly answers $f(A) \lesssim_\varepsilon p$.

The idea always is that as long as it seems like $|A|_C = \mu_s + 1$, we are working towards making $k(A) = B_s$. If we later see the bound on $|A|_C$ drop, and i_s no longer looks like a suitable answer, then because $|B_s|_{\mathcal{D}}$ is large, no matter how much of B_s has been copied, we can switch to a nearby B_t for which $i_t = 1 - i_s$ is the only correct answer to $g(B_t) \lesssim_\delta q$, and $|B_t|_{\mathcal{D}}$ is still large.

Let λ denote the empty string. Let $\tau_0 = \lambda$. We begin differently depending on whether g is two-sided. In both of the following cases, the reader can verify that conditions (i)-(iii) are satisfied at stage $s = 0$.

If g is two-sided, we first wait until we see A leave $P_{f,p,\varepsilon}^\mu$ for some $\mu \leq \nu$. That is, we see (μ, σ, i) in M with $\sigma \prec A$. Let $\mu_0 = \mu$ and $i_0 = i$. Now, since g is two-sided, regardless of i , $P_{g,q,\delta}^\nu \setminus D_{1-i}$ is non-empty, and we can find an element B in this set (by looking in S for something of the form $(B, \nu, \lambda, 1 - i)$). Let $B_0 = B$.

If g is not two-sided, then for some j , $P_{g,q,\delta}^\nu \setminus D_j$ is non-empty, so first we wait until we see an element B and a j to witness this (by looking in S for something of the form (B, ν, λ, j)). Let $\mu_0 = \nu$, $i_0 = 1 - j$, and $B_0 = B$. By the choice of q and δ , if $|A|_C = \nu + 1$, then i_0 correctly answers $f(A) \lesssim_\varepsilon p$.⁶

At stage $s + 1$, set μ_{s+1} to be the least μ for which we have seen A leave $P_{f,p,\varepsilon}^\mu$. If $\mu_{s+1} < \mu_s$, that is because (μ_{s+1}, σ, i) just entered M for some $\sigma \prec A$. If $i = i_s$, let $B_{s+1} = B_s$ and $i_{s+1} = i_s$. But if $i \neq i_s$, then set $i_{s+1} = i$, and look through S to find a B so that

$$B \in P_{g,q,\delta}^{\mu_{s+1}} \cap [\tau_s] \setminus D_{i_s}.$$

Such a B must exist because B_s witnesses that $P_{g,q,\delta}^{\mu_s} \cap [\tau_s]$ is non-empty. Let $B_{s+1} = B$. Finally, let $\tau_{s+1} = B_{s+1} \upharpoonright |\tau_s| + 1$. That completes the construction.

At each stage the properties (i)-(iii) are maintained. Now if $|A|_C = \mu + 1$, there is a stage s at which it is seen that A leaves $P_{f,p,\varepsilon}^\mu$. The μ_s, i_s and B_s defined at that stage never change again. Then $k(A) = B_s$, and the only correct answer to $g(k(A)) \lesssim_\delta q$ is i_s , which also correctly answers $f(A) \lesssim_\varepsilon p$, as desired. \square

Corollary 8.2. *Theorem 8.1 holds also if $f \in \mathcal{B}_1(X)$ and $g \in \mathcal{B}_1(Y)$, where X and Y are compact separable metric spaces.*

Proof. Let δ_X and δ_Y be any total admissible representations for X and Y respectively. By Propositions 5.4 and 7.7, replacing f and g with $f\delta_X$ and $g\delta_Y$ does not result in any change to any of the properties of f and g mentioned in Theorem 8.1. \square

The initial segment of the $\leq_{\mathbf{m}}$ -degrees contains some naturally recognizable classes which are blurred together by the α rank. The lowest $\leq_{\mathbf{m}}$ degree consists of the constant functions; right above that is the degree of the continuous

⁶In case (1), by the choice of ν , $P_{f,p,\varepsilon}^\nu = \emptyset$, so $|A|_C < \nu + 1$. In case (3), f is one-sided, so $P_{f,p,\varepsilon}^\nu \subseteq C_i$ for some i . Note that in this case, we have chosen q, δ specifically to make sure that $j = 1 - i$. In case (4), we also have $P_{f,p,\varepsilon}^\nu \subseteq C_{1-j}$ (note that $j = 1$ if f and g are both left-sided and $j = 0$ if f and g are both right-sided).

non-constant functions. Next above that are two incomparable $\leq_{\mathbf{m}}$ -degrees: the upper semi-continuous functions and the lower semi-continuous functions.

Proposition 8.3. *Let g be a lower semi-continuous, discontinuous function (for example, $g = j_1$). The following are equivalent for $f \in \mathcal{B}_1 2^\omega$:*

- (1) $f \leq_{\mathbf{m}} g$
- (2) f is lower semi-continuous.
- (3) For some e and some parameter Z , $f(A) = \delta_{\text{sep}}(W_e^{A \oplus Z})$

where δ_{sep} is the separation name representation from Definition 5.1.

Proof. (1 \implies 2) Given $a \in \mathbb{R}$, we wish to show that $f^{-1}((a, \infty))$ is open. Let $(p_i, \varepsilon_i)_{i < \omega}$ be an infinite sequence of rationals such that $a < p_i - \varepsilon_i$ and $\lim p_i = a$. Let q_i, δ_i and k_i witness the defining property of $f \leq_{\mathbf{m}} j_1$ for each i . Now suppose that $f(A) > a$. For some i , $f(A) > p_i + \varepsilon_i$. Then the only correct answer to $f(A) \lesssim_{\varepsilon_i} p_i$ is 1, so it must be that $g(k_i(A)) > q_i - \delta_i$. The set $C := \{B : g(k_i(B)) > q_i - \delta_i\}$ is open by the lower semi-continuity of g , and since 1 is a correct answer to $f(B) \lesssim_{\varepsilon_i} p_i$ for every $B \in C$, we have $C \subseteq f^{-1}((p_i - \varepsilon_i, \infty)) \subseteq f^{-1}((a, \infty))$.

(2 \implies 3) Assume Z is an oracle which lists, for each p , the collection of rational balls contained in $f^{-1}((p, \infty))$. To define $W_e^{A \oplus Z}(\langle p, \varepsilon \rangle)$, wait to see if A enters $f^{-1}((p - \varepsilon, \infty))$. If it does, enumerate the bit. The result is a separation name of $f(A)$ which has the additional property that it always answers 1 when 1 is a permissible answer.

(3 \implies 2) If $f(A) = \delta_{\text{sep}}(W_e^{A \oplus Z})$, then $f(A) > a$ if and only if for some p, ε , $a < p - \varepsilon$ and $\langle p, \varepsilon \rangle \in W_e^{A \oplus Z}$, which is an open condition.

(2 \implies 1) This follows from Theorem 8.1 because g has rank 2 and is left-sided, and f is either discontinuous and shares these properties or f is continuous, in which case $f \leq_{\mathbf{m}} g$ by Proposition 5.6. □

The authors observed to Kihara that if the lattice structure of the Baire 1 $\leq_{\mathbf{m}}$ -degrees would continue to higher Baire classes in the same pattern described in Theorem 8.1, the $\leq_{\mathbf{m}}$ reducibility could be used to extend the definition of the α rank into higher Baire classes. After seeing these results, Kihara used the theory of Wadge degrees of BQO-valued functions to fully describe the structure of the $\leq_{\mathbf{m}}$ -degrees beyond the Baire 1 functions [Kih], and confirmed that the pattern does continue, even beyond the Baire functions if AD is assumed. He also established that Corollary 8.2 remains true even if X and Y are arbitrary Polish spaces.

Separately and independently of this, Elekes, Kiss and Vidnyánszky defined a generalization of the α, β and γ ranks into the higher Baire classes [EKV16]. Interestingly, they were able to apply their extension of the β rank to solve a problem in cardinal characteristics, but an extension of the α rank was not suitable for that problem. It does not seem easy to modify our work to get a generalization of the β rank. For a discussion of the relationship between the various generalizations, see [Kih].

9. A REDUCIBILITY BETWEEN $\leq_{\mathbf{m}}$ AND $\leq_{\mathbf{tt}}$

There is a reducibility notion which captures the α rank precisely. Consider a truth table reduction $f \leq_{\mathbf{tt}} g$ which looks at only one bit of g , but may use finitely many bits of A .

Definition 9.1. For $f, g : 2^\omega \rightarrow \mathbb{R}$, we say $f \leq_{\mathbf{tt1}} g$ if for all rationals p, ε , there is a continuous $k : 2^\omega \rightarrow 2^\omega$, rationals q, δ , a number r , and a truth table $h : 2^{r+1} \rightarrow \{0, 1\}$ such that for every $A \in 2^\omega$, if b is a correct answer to $g(k(A)) \lesssim_\delta q$, then $h(A \upharpoonright r, b)$ is a correct answer to $f(A) \lesssim_\varepsilon p$.

Proposition 9.2. *The relation $f \leq_{\mathbf{tt1}} g$ is transitive.*

Proof. Suppose $f_1 \leq_{\mathbf{tt1}} f_2$ and $f_2 \leq_{\mathbf{tt1}} f_3$. Given p, ε , let δ, q, k, r and h be as guaranteed by the fact that $f_1 \leq_{\mathbf{tt1}} f_2$. Given $p' = q, \varepsilon' = \delta$, let k', q', δ', r' and h' be as guaranteed by the fact that $f_2 \leq_{\mathbf{tt1}} f_3$. Let $r'' > r'$ be also large enough that r'' bits of any input A are enough to compute r' bits of $k(A)$ (using compactness). Define

$$h''(\tau, b) = h(k(\tau) \upharpoonright r', h'(\tau \upharpoonright r'', b)).$$

Then the reader can verify that $k' \circ k, q', \delta', r''$ and h'' witness $f_1 \leq_{\mathbf{tt1}} f_3$. □

As before, we may extend this notion to any compact separable metrizable space.

Proposition 9.3. *Let X, Y be compact separable metrizable spaces and let $\delta_X, \delta'_X : 2^\omega \rightarrow X$ and $\delta_Y, \delta'_Y : 2^\omega \rightarrow Y$ be any admissible representations for X and Y respectively. Let $f : X \rightarrow \mathbb{R}$ and $g : Y \rightarrow \mathbb{R}$. Then*

$$f\delta_X \leq_{\mathbf{tt1}} g\delta_Y \iff f\delta'_X \leq_{\mathbf{tt1}} g\delta'_Y.$$

Proof. The proof is identical to the proof of Proposition 5.4. □

Therefore, the following extensions are well-defined.

Definition 9.4. Let X, Y be compact metric spaces and $f : X \rightarrow \mathbb{R}, g : Y \rightarrow \mathbb{R}$. Then we say that $f \leq_{\mathbf{tt1}} g$ if and only if $f\delta_X \leq_{\mathbf{tt1}} g\delta_Y$, where $\delta_X : 2^\omega \rightarrow X$ and $\delta_Y : 2^\omega \rightarrow Y$ are any total admissible representations of X and Y respectively.

Theorem 9.5. *If $f, g \in \mathcal{B}_1 2^\omega$, then $f \leq_{\mathbf{tt1}} g$ if and only if $|f|_\alpha \leq |g|_\alpha$.*

Proof. Suppose that $f \leq_{\mathbf{tt1}} g$. Given p, ε , let k, q, δ, r and h witness $f \leq_{\mathbf{tt1}} g$. We claim that $\alpha(f, p, \varepsilon) \leq \alpha(g, q, \delta)$. The proof is very similar to the $\leq_{\mathbf{m}}$ case. Let $Q^\nu = k^{-1}(P_{g,q,\delta}^\nu)$, we claim that Q^ν is a derivation sequence for $\{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}$. If $A \in Q^\nu \setminus Q^{\nu+1}$, then $k(A) \in P^\nu \setminus P^{\nu+1}$, so for some $\tau \prec k(A)$, either $g(P^\nu \cap [\tau]) \subseteq (-\infty, q + \delta)$ or it is a subset of $(q - \delta, \infty)$. Without loss of generality, assume the former. Let $\sigma \prec A$ be long enough that $k([\sigma]) \subseteq [\tau]$ and $|\sigma| \geq r$. Then for all $A' \in [\sigma] \cap Q^\nu$, we have 0 correctly answers $g(k(A')) \lesssim_\delta q$, and $h(\sigma \upharpoonright r, 0)$ correctly answers $f(A) \lesssim_\varepsilon p$. So $f(Q^\nu \cap [\sigma]) \subseteq (-\infty, p + \varepsilon)$ or $(p - \varepsilon, \infty)$.

In the other direction, suppose $|f|_\alpha \leq |g|_\alpha$. Since an $\leq_{\mathbf{m}}$ reduction is a $\leq_{\mathbf{tt1}}$ reduction, Theorem 8.1 implies that it suffices to consider the successor case. Let ν be such that $|f|_\alpha = |g|_\alpha = \nu + 1$. It suffices to show that $f \leq_{\mathbf{tt1}} g$ while assuming that g is left-sided. (The case where g is right-sided is similar.)

Given p, ε , let q, δ be maximal for g . Let $\mathcal{C} = \{C_0, C_1\}$ and $\mathcal{D} = \{D_0, D_1\}$ be as in the proof of Theorem 8.1. Exactly as there, let Z be an oracle which contains a well-order long enough to code ν , an optimal mind-change sequence M for \mathcal{C} , and a scaffolding sequence S for \mathcal{D} .

Let r be long enough that r bits of any input A are enough to see when A first leaves some $P_{f,p,\varepsilon}^\mu$ for some $\mu \leq \nu$. This uses compactness.

Equivalently, r is long enough that for some finite initial segment $(\eta_j, \sigma_j, b_j)_{j < \ell}$ from M , $\cup_j [\sigma_j] = 2^\omega$, and each $|\sigma_j| \leq r$. Without loss of generality, we can assume that the σ_j partition the space.

Define k as follows. At stage 0, on input A , let j be the index for which $\sigma_j \prec A$. Let $\mu_0 = \eta_j$ and $i_0 = b_j$ and $\tau_0 = \lambda$. Now if $b_j = 0$ (matching the natural left-sidedness of g), search through S to find $B \in P_{g,q,\delta}^\nu \setminus D_1$, let $B_0 = B$, and proceed exactly as in the proof of Theorem 8.1. But if $b_j = 1$, then unfortunately $P_{g,q,\delta}^\nu \setminus D_0$ is empty. So in this case also let $B_0 = B$ (the same one found above), but this means i_0 is an incorrect answer to $g(B_0) \lesssim_\delta q$. We will correct this later using h . So if $b_j = 1$, proceed almost exactly as in the proof of Theorem 8.1, except instead of maintaining that i_s is the only correct answer to $g(B_s) \lesssim_\delta q$, now maintain that i_s is incorrect for that question.

The same arguments as in Theorem 8.1 now guarantee that when μ_s, i_s and B_s stabilize, then $|A|_C = \mu_\infty + 1$, $k(A) = B_\infty$, and i_∞ correctly answers $f(A) \lesssim_\varepsilon p$. If $b_j = 0$, i_∞ is the only correct answer to $g(B_\infty) \lesssim_\delta q$. If $b_j = 1$, then $1 - i_\infty$ is the only correct answer to $g(B_\infty) \lesssim_\delta q$.

Define $h(\sigma, b)$ as follows. Let j be the unique index such that $\sigma_j \prec \sigma$. If $b_j = 0$, let $h(\sigma, b) = b$ (letting the doubly correct answer through). If $b_j = 1$, let $h(\sigma, b) = 1 - b$ (changing the only correct answer for $g(k(A)) \lesssim_\delta q$ into a correct answer for $f(A) \lesssim_\varepsilon p$). □

Corollary 9.6. *Theorem 9.5 also holds if $f \in \mathcal{B}_1(X)$, $g \in \mathcal{B}_1(Y)$, where X and Y are any compact separable metrizable spaces.*

Proof. By Propositions 9.3 and 7.7. □

Pauly has alerted us that this notion is also quite natural in the Weihrauch framework. Using the notation of Section 5.1, he asked us whether $f \leq_{\text{tt1}} g$ if and only if $S_f \leq_W^c S_g$. One direction is immediate; below we prove the other using Theorem 9.5. At a first glance, the problem with going directly from a Weihrauch reduction to a \leq_{tt1} reduction is that a Weihrauch reduction, when restricted to inputs starting with p, ε , might use several different choices of q, δ for different parts of the domain. A more subtle point is that in a Weihrauch reduction, the reverse function H does not need to be defined on all of $2^\omega \times \{0, 1\}$, just on the collection of values that it could receive as input. Therefore, we cannot use compactness to automatically transform H into a truth table of the kind used in a \leq_{tt1} reduction.

Proposition 9.7. *For all $f, g \in \mathcal{B}_1 2^\omega$, we have $f \leq_{\text{tt1}} g$ if and only if $S_f \leq_W^c S_g$.*

Proof. A \leq_{tt1} reduction is also a Weihrauch reduction, so one direction is immediate. Suppose that $S_f \leq_W^c S_g$. We claim that then $|f|_\alpha \leq |g|_\alpha$. Let K and H be the continuous functions witnessing the Weihrauch reduction. Note that H takes two arguments, the original input $(p, \varepsilon) \wedge A$, and one bit of output representing a correct answer to $S_g(K((p, \varepsilon) \wedge A))$. Given p, ε , by compactness there are finitely many strings $(\sigma_i)_{i < \ell}$, and for each i rational (q_i, δ_i) such that $\cup_i [\sigma_i] = 2^\omega$, and $\sigma_i \prec A$ implies that $K((p, \varepsilon) \wedge A)$ starts with (q_i, δ_i) . Let K_1 be defined so that

$$K((p, \varepsilon) \wedge \sigma_i C) = (q_i, \delta_i) \wedge K_1(\sigma_i C).$$

For each i , let $P_i^\nu = P_{g,q_i,\delta_i}^\nu$, the optimal derivation sequence for g, q_i, δ_i . Define

$$Q_i^\nu = [\sigma_i] \cap K_1^{-1}(P_i^\nu),$$

and $Q^\nu = \cup_{i < \ell} Q_i^\nu$. We claim that Q^ν is a derivation sequence for $\{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}$. It suffices to check this on the restriction to each $[\sigma_i]$ separately, as these are clopen sets.

Fix one $i < \ell$. Suppose that $A \in Q_i^\nu \setminus Q_i^{\nu+1}$. Then $\sigma_i \prec A$ and $K_1(A) \in P_i^\nu \setminus P_i^{\nu+1}$. So for some $\tau \prec K_1(A)$, either $g(P_i^\nu \cap [\tau]) \subseteq (-\infty, q_i + \delta_i)$ or it is a subset of $(q_i - \delta_i, \infty)$. Without loss of generality, assume the former. Then $(A, 0)$ must be in the domain of H . Let $b = H(A, 0)$. Let $\sigma \prec A$ be long enough that $H(A', 0) = b$ whenever $\sigma \prec A'$, and long enough that $K_1([\sigma]) \subseteq [\tau]$. It is a matter of definition chasing to verify that $f(Q_i^\nu \cap [\sigma]) \subseteq C_b$, where $C_0 = f^{-1}((-\infty, p + \varepsilon))$ and $C_1 = f^{-1}((p - \varepsilon, \infty))$. This shows that Q_i^ν is a derivation sequence on $[\sigma_i]$, and thus Q^ν is a derivation sequence.

It follows that $\alpha(f, p, \varepsilon) \leq \max_{i < \ell} \alpha(g, q_i, \delta_i)$, and therefore $|f|_\alpha \leq |g|_\alpha$. □

10. PROPERTIES OF \leq_{tt}

In this section we characterize the \leq_{tt} degrees inside \mathcal{B}_1 in terms of the Bourgain rank, proving part (1) of Theorem 1.2. Define a coarsening of the order on the ordinals as follows:

Definition 10.1. Let $\alpha \lesssim \beta$ if for every $\gamma < \alpha$, there is $\delta < \beta$ and $n \in \omega$ such that $\gamma < \delta \cdot n$.

This coarsening is quite robust. Recall Cantor normal form for ordinals: every ordinal α can be written uniquely as a sum of the form $\alpha = \omega^{\eta_1} \cdot k_1 + \dots + \omega^{\eta_n} \cdot k_n$, where $\eta_1 > \dots > \eta_n$ and $k_i \in \mathbb{N}^+$. Considering the existence of Cantor normal form, one can see that $\alpha \lesssim \beta$ if for all η , $\beta \leq \omega^\eta$ implies $\alpha \leq \omega^\eta$.

The natural sum $\alpha \# \beta$ is defined by $\alpha \# \beta = \omega^{\xi_1} \cdot k_1 + \dots + \omega^{\xi_r} \cdot k_r$, where $\xi_1 > \dots > \xi_r$ are exactly the exponents in the Cantor normal forms of α and β , and k_i is the sum of the coefficients of ω^{ξ_i} in α and β . One sees also that $\alpha \lesssim \beta$ if for every $\gamma < \alpha$, there is $\delta < \beta$ and $n \in \omega$ such that

$$\gamma < \underbrace{\delta \# \delta \# \dots \# \delta}_n.$$

We will show that the \leq_{tt} degrees inside \mathcal{B}_1 correspond to functions whose ranks are equivalent according to this relation. Lemma 10.2 describes the length of combined derivation sequences.

Lemma 10.2. *Let X be a compact metric space and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(X)$. Let $P_\mathcal{C}^\nu$ and $P_\mathcal{D}^\nu$ be the optimal derivation sequences for \mathcal{C} and \mathcal{D} . Let Q^ν be the optimal derivation sequence for*

$$\{C \cap D : C \in \mathcal{C} \text{ and } D \in \mathcal{D}\}.$$

Then for all ν and μ ,

$$Q^{\nu \# \mu} \subseteq P_\mathcal{C}^\nu \cup P_\mathcal{D}^\mu.$$

Proof. By induction on $\nu \# \mu$. If $\nu \# \mu = 0$, the statement is immediate. Suppose the statement holds for all pairs of ordinals with natural sum less than $\nu \# \mu$. Let $A \notin P_\mathcal{C}^\nu \cup P_\mathcal{D}^\mu$. Then there are ordinals $\eta < \nu$ and $\xi < \mu$, a neighborhood U of A , and sets $C \in \mathcal{C}$ and $D \in \mathcal{D}$ such that $P_\mathcal{C}^\eta \cap U \subseteq C$ and $P_\mathcal{D}^\xi \cap U \subseteq D$.

Let $\zeta = \max(\eta \# \mu, \nu \# \xi)$. Then since $\eta < \nu$ and $\xi < \mu$, we have $\zeta < \nu \# \mu$. So by induction,

$$Q^\zeta \subseteq Q^{\eta \# \mu} \cap Q^{\nu \# \xi} \subseteq (P_\mathcal{C}^\eta \cup P_\mathcal{D}^\mu) \cap (P_\mathcal{C}^\nu \cup P_\mathcal{D}^\xi).$$

Rearranging the right hand side, we have

$$Q^\zeta \subseteq P_C^\nu \cup P_D^\mu \cup (P_C^\eta \cap P_D^\xi).$$

Because $P_C^\nu \cap U = P_D^\mu \cap U = \emptyset$ and $P_C^\eta \cap P_D^\xi \cap U \subseteq C \cap D$, we have $Q^{\zeta+1} \cap U = \emptyset$. So $A \notin Q^{\nu\#\mu}$, because $Q^{\nu\#\mu} \subseteq Q^{\zeta+1}$. \square

The following is then immediate by induction.

Lemma 10.3. *Let X be a compact metric space and let $C_i \subseteq \mathcal{P}(X)$ for all $i < r$. Let P_i^ν be the optimal derivation sequences for C_i , and let Q^ν be the optimal derivation sequence for*

$$\{\cap_{i < r} C_i : C_i \in \mathcal{C}_i\}.$$

Then for all $(\nu_i)_{i < r}$,

$$Q^{\#_{i < r} \nu_i} \subseteq \cup_{i < r} P_i^{\nu_i}.$$

Theorem 10.4. *If $f, g \in \mathcal{B}_1 2^\omega \setminus \mathcal{B}_0 2^\omega$, then $f \leq_{\text{tt}} g$ if and only if $|f|_\alpha \lesssim |g|_\alpha$.*

Proof. Suppose $f \leq_{\text{tt}} g$. Given p, ε , let $(k_i, q_i, \delta_i)_{i < r}$ and h be as in the definition of \leq_{tt} . For each i , define

$$C_i = \{k_i^{-1}(g^{-1}((-\infty, q_i + \delta_i))), k_i^{-1}(g^{-1}((q_i - \delta_i, \infty)))\}.$$

Let

$$\mathcal{C} = \{\cap_{i < r} C_i : C_i \in \mathcal{C}_i\}.$$

We claim that any derivation sequence for \mathcal{C} is also a derivation sequence for

$$\mathcal{D} := \{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}.$$

This follows because for every $\cap_{i < r} C_i \in \mathcal{C}$, there is a $\sigma \in 2^r$ such that $\sigma(i)$ correctly answers $g(k(A)) \lesssim_{\delta_i} q_i$, for every $i < r$ and $A \in \cap_{i < r} C_i$. Therefore, for each $A \in \cap_{i < r} C_i$, $h(\sigma)$ is a correct answer to $f(A) \lesssim_\varepsilon p$. Therefore, for some $D \in \mathcal{D}$, we have $\cap_{i < r} C_i \subseteq D$, and the claim follows by Proposition 7.2.

Define $Q_i^\nu = k_i^{-1}(P_{g, q_i, \delta_i}^\nu)$. By Proposition 7.2, Q_i^ν is a derivation sequence for C_i . Let $\nu_i = \alpha(g, q_i, \delta_i)$, so that $Q_i^{\nu_i} = \emptyset$. Let Q^ν be the optimal derivation sequence for \mathcal{C} . By Lemma 10.3,

$$Q^{\#_{i < r} \nu_i} \subseteq \cup_{i < r} Q_i^{\nu_i}.$$

Therefore, as Q^ν is also a derivation sequence for \mathcal{D} , we have

$$\alpha(f, p, \varepsilon) \leq \#_{i < r} \nu_i \leq \underbrace{\nu \# \dots \# \nu}_r,$$

where $\nu = \max_i \alpha(g, q_i, \delta_i)$. Therefore, $|f|_\alpha \lesssim |g|_\alpha$.

Now suppose that $|f|_\alpha \lesssim |g|_\alpha$. We run a daisy-chain of the kind of argument used in the \leq_{tt1} case. Given p, ε , let q, δ and n be such that $\alpha(f, p, \varepsilon) < \alpha(g, q, \delta) \cdot n$, and $\alpha(g, q, \delta) \geq 2$. Letting $\nu = \alpha(g, q, \delta)$, we may also guarantee that $P_{g, q, \delta}^{\nu-1} \not\subseteq (q - \delta, q + \delta)$, by decreasing δ if necessary.

We will define $3n$ functions k_i , all of them associated to this same pair q, δ . The functions are defined computably relative to an oracle which contains enough information to compute notations up to ν (and thus up to $\nu \cdot n$), a mind-change sequence M for $\{f^{-1}((-\infty, p + \varepsilon)), f^{-1}((p - \varepsilon, \infty))\}$, and a scaffolding sequence S for $\{g^{-1}((-\infty, q + \delta)), g^{-1}((q - \delta, \infty))\}$.

Fix $B_0 \in P_{g, q, \delta}^{\nu-1}$ with $g(B_0) \notin (q - \delta, q + \delta)$, and let b_0 be the unique correct answer to $g(B_0) \lesssim_\delta q$. Since $\nu \geq 2$, $|B_0|_{g, q, \delta} \geq 2$.

Given input A , the first n functions $\{k_i\}_{i < n}$ are used to figure out in which interval

$$I_i = [\nu \cdot i + 1, \nu \cdot (i + 1)]_{i < n}$$

$|A|_{f,p,\varepsilon}$ lies. Define $k_i(A)$ as follows. Copy B_0 until such a time as you see $A \notin P_{f,p,\varepsilon}^{\nu \cdot (i+1)}$. If this occurs, switch to copying a nearby input B_1 with $|B_1|_{g,q,\delta} < |W|_{g,q,\delta}$ and where the unique correct answer to $g(B_1) \lesssim_\delta q$ is $1 - b_0$. That completes the description of the first n functions k_i . By observing the answers for $g(k_i(A)) \lesssim_\delta q$ for $i < n$, one can determine the unique $i < n$ such that $A \in I_i$.

The next n functions $\{k_{n+i}\}_{i < n}$ track the mind-changes of $f(A) \lesssim_\varepsilon p$ under the assumption that $A \in I_i$. Given input A , and letting B_0 and b_0 be as above, first copy B_0 into the output until such a time as you see $A \notin P_{f,p,\varepsilon}^{\nu \cdot (i+1)}$. If this occurs, then we also know a rank $\mu_0 < \nu$ and bit i_0 such that if $|A|_{f,p,\varepsilon} = (\nu \cdot i) + \mu_0 + 1$, then i_0 correctly answers $f(A) \lesssim_\varepsilon p$. Let τ_0 be whatever amount of B_0 has been copied so far. Now proceed similarly as in Theorem 8.1, but maintain the following at each stage:

- (i) that $|A|_C \leq (\nu \cdot i) + \mu_s + 1 \leq (\nu \cdot i) + |B_s|_D$,
- (ii) that the only correct answer to $g(B_s) \lesssim_\delta q$ is i_s if $i_0 = b_0$, and the only correct answer is $1 - i_s$ if $i_0 \neq b_0$.
- (iii) if $|A|_C = (\nu \cdot i) + \mu_s + 1$, then i_s correctly answers $f(A) \lesssim_\varepsilon p$.

Proceeding now just as in Theorem 8.1, the above can be maintained unless A leaves $P_{f,p,\varepsilon}^{\nu \cdot i}$. In that case, the output of this computation will not be used, so one can continue to copy whatever B_s is active at the moment this is discovered. But if μ_s, i_s and B_s stabilize to values μ_∞, i_∞ and B_∞ , then if $A \in I_i$, we have $|A|_{f,p,\varepsilon} = (\nu \cdot i) + \mu_\infty + 1, k_{n+i}(A) = B_\infty, i_\infty$ is a correct answer to $f(A) \lesssim_\varepsilon p$, and the only correct answer to $g(B_\infty) \lesssim_\delta q$ is either i_s or $1 - i_s$ depending on whether $i_0 = b_0$ or not.

The last n functions $\{k_{2n+i}\}_{i < n}$ are simple indicator functions, with k_{2n+1} copying B_0 and silently carrying out the same computation as k_{n+i} until that computation finds an i_0 and a b_0 . If k_{n+i} finds $i_0 \neq b_0$, switch to a nearby B_1 with $|B_1|_{g,q,\delta} < |B_0|_{g,q,\delta}$ and where the unique correct answer to $g(B_1) \lesssim_\delta q$ is $1 - b_0$. Otherwise (including if i_0 is never defined), continue copying B_0 .

Putting this all together, given A , a truth table which has access to separating bits for each $g(k_i(A))$ can correctly answer $f(A) \lesssim_\varepsilon p$ as follows. First use the separating bits of $g(k_i(A))$ for $i < n$ to find the unique i such that $|A|_{f,p,\varepsilon} \in I_i$. Then query $g(h_{2n+i}(A))$ to learn whether $i_0 = b_0$ in the computation of $k_{n+i}(A)$. Finally, query $g(k_{n+i}(A))$ to obtain a bit b which correctly answers $f(A) \lesssim_\varepsilon p$ if $i_0 = b_0$. If $i_0 \neq b_0$, then $1 - b$ will do for a correct answer. \square

Corollary 10.5. *Theorem 10.4 also holds for any discontinuous, Baire 1 functions f and g on any compact separable metrizable spaces.*

Proof. By Propositions 5.4 and 7.7. \square

As a corollary we can give a short algorithmic proof of the following result of Kechris and Louveau, which is a consequence of their Lemma 5 and Theorem 8, and which allows them to conclude that their “small Baire classes” \mathcal{B}_1^ξ are Banach algebras.

Corollary 10.6 ([KL90]). *Let X be a compact separable metrizable space. If $f, g \in \mathcal{B}_1(X)$ are bounded, then*

$$|f + g|_\alpha, |f \cdot g|_\alpha \lesssim \max(|f|_\alpha, |g|_\alpha).$$

Proof. Let δ_X be any total admissible representation for X . Observing that $f\delta_X + g\delta_X = (f+g)\delta_X$ and $f\delta_X \cdot g\delta_X = (f \cdot g)\delta_X$, we may, by replacing f and g everywhere by $f\delta_X$ and $g\delta_X$, assume that $X = 2^\omega$.

Without loss of generality we may also assume that $|f|_\alpha \leq |g|_\alpha$, so $f \leq_{\text{tt}} g$. Also, let $M \in \mathbb{R}$ be chosen so that all outputs of f and g lie in $[-M, M]$.

Then $f+g \leq_{\text{tt}} g$ via the following algorithm. Given A, p, ε , first ask finitely many questions of f and g to determine both $f(A)$ and $g(A)$ to within precision $\varepsilon/2$ (by asking each function $2M/(\varepsilon/2)$ questions of the form $f(A) \lesssim_{\varepsilon/2} q_i$, where the q_i are evenly spaced at intervals of $\varepsilon/2$ in $[-M, M]$). Adding the two approximations gives an approximation to $(f+g)(A)$ which is correct to within ε . Use this approximation to answer $f(A) \lesssim_\varepsilon p$.

Similarly, $fg \leq_{\text{tt}} g$ as follows. Given A, p, ε , first use finitely many questions to approximate $f(A)$ and $g(A)$ to within precision $\varepsilon/(2M)$. Multiplying the results gives an approximation to $(fg)(A)$ that is correct to within ε . □

11. FURTHER DIRECTIONS AND OPEN QUESTIONS

11.1. A road not taken. Recall that we used admissible representations to allow our results about functions on 2^ω to extend to arbitrary compact separable metrizable spaces. Another option for extending these reducibilities would be to transfer the definitions literally to the new spaces, without using representations. For example, one could define $f \leq'_m g$ to mean that for every p, ε , there is a continuous function k and rationals q, δ such that for all x , we have any correct answer to $g(k(x)) \lesssim_\delta q$ is a correct answer to $f(x) \lesssim_\varepsilon p$.

This option behaves very differently from the one we chose, for if X is very connected, then there are not enough continuous functions $k : X \rightarrow Y$ to get the same results. For example, we can define two left-sided, rank 3 functions in $\mathcal{B}_1([0, 1])$ are not \leq'_m -equivalent under this alternate definition. Let $f_1 = \chi_{\{1/n : n \in \omega\}}$. And let $f_2 = \chi_S$ where

$$S = \{x_I^* : I \text{ is a middle third}\},$$

where I is a middle third means that I belongs to the sequence $(1/3, 2/3), (1/9, 2/9), (7/9, 8/9), \dots$ of intervals removed to create the Cantor set in $[0, 1]$, and x_I^* denotes the midpoint of I .

To see that $f_2 \not\leq'_m f_1$ under this less robust definition of \leq'_m , fix $p = 1/2$ and $\varepsilon = 1/3$; q and δ will have to be similarly assigned since we are working with characteristic sets. Then any continuous k that would work for the reduction would have to send the Cantor subset of $[0, 1]$ to 0. For if any z from the Cantor subset of I satisfied $k(z) \in (1/(n+1), 1/n)$, then by pulling back $(1/(n+1), 1/n)$ via k , we'd find a whole neighborhood of z mapped to $(1/(n+1), 1/n)$, impossible since every neighborhood of z includes an element of S . So $h(1/3) = h(2/3) = 0$. Now, what is $k(1/2)$. It must be equal to $1/n$ for some n or the reduction fails. So $k([1/3, 1/2])$ includes both 0 and some $1/n$. Since k is continuous and $[1/3, 1/2]$ is connected, its image is connected so also includes $1/m$ for all $m > n$. But who are getting mapped to $1/m$? The purported reduction is wrong on $k^{-1}(1/m)$ for such m .

In fact f_2 is not even \leq'_m the characteristic function of the rationals, for a similar reason: if $k(1/3)$ is irrational and $k(1/2)$ is rational, then $k([1/3, 1/2])$ contains many rationals.

Since the characteristic function of the rationals is Baire 2, this alternate generalization produces a very different theory, which we did not pursue further.

11.2. Computable reducibilities for discontinuous functions. The original motivation for this work was to devise a notion of *computable* reducibility between arbitrary (especially discontinuous) functions. There is a well-established notion of computable reducibility between continuous functions due to Miller [Mil04], based on the notion of computable function due to Grzegorzczuk [Grz55, Grz57] and Lacombe [Lac55a, Lac55b]. A truly satisfying notion of computable reducibility for arbitrary functions would have its restriction to continuous functions agree with this established notion. Unfortunately, the computable/lightface versions of our reducibilities do not have this property. The reason for this, roughly speaking, is that the Weihrauch-based reductions operate pointwise, whereas the established computable reducibility on continuous functions makes essential use of global information in the form of the modulus of continuity. Therefore, Question 11.1 remains of interest, where of course satisfaction lies in the eye of the beholder.

Question 11.1. Is there a satisfying notion of computable reducibility for arbitrary functions, whose restriction to the continuous functions is exactly continuous reducibility in the sense of Miller?

And of course, it would still be interesting to know more about the structure of arbitrary functions under the computable versions of these reducibilities.

Question 11.2. What can be said about the degree structure of $\mathcal{F}(X, \mathbb{R})$ under the computable versions of $\leq_{\mathbf{T}}$, $\leq_{\mathbf{tt}}$ and $\leq_{\mathbf{m}}$?

We will address further details and progress on these questions in a forthcoming paper.

APPENDIX A. ORIGINAL PROOF OF THEOREM 1.1

In this appendix, we give the original proof of the finite case of Theorem 1.1 (or technically, Theorem 6.3 where $X = 2^\omega$), using a relativized version of Montalbán's theory of α -true stages [Mon14]. The true stages are a modern iteration of a technique whose previous iterations include Ash's η -systems [Ash86], Harrington's worker arguments, and Lempp and Lerman's trees of strategies [LL95]. The most readable presentation is in [GT].

From the development of the k -true stages we use the following metatheorem, whose unrelativized version is given in [Mon14]. By the proof given there, it is easy to see that the following relativized version of the metatheorem holds.

Theorem A.1 ([Mon14]). *Relative to any oracle X , there is a uniformly X -computable sequence of partial orders \leq_k on ω such that*

- \leq_0 is the usual ordering.
- If $a \leq_{k+1} b$, then $a \leq_k b$.
- For each k , the relation \leq_k defines a tree with a single infinite path, called the \leq_k -true path.
- $X^{(k)}$ is uniformly X -computable from the \leq_k -true path.

- If $a \leq_k b \leq_k c$ and $a \leq_{k+1} c$, then $a \leq_{k+1} b$.

Now we state and prove the finite case of Theorem 6.3.

Theorem A.2. *Assume $f : 2^\omega \rightarrow \mathbb{R}$ is Baire, or assume Wadge Determinacy. If f is not Baire class n , then either $j_{n+1} \leq_m f$ or $-j_{n+1} \leq_m f$.*

Proof. Since f is not Baire n , let U be an open set such that $f^{-1}(U)$ is not Σ_{n+1}^0 . Without loss of generality, U is of the form (u, ∞) or $(-\infty, u)$ for some rational u . If U is of the form $(-\infty, u)$, then we will have $j_{n+1} \leq_m f$, and in the other case $j_{n+1} \leq_m -f$. Replacing f with $-f$ if necessary let us assume $U = (-\infty, u)$.

By Wadge determinacy, $f^{-1}(U)$ is Π_{n+1}^0 -hard. Define some canonical Π_n^0 -complete sets C_n by $C_1 = \{1^\omega\}$ and $C_{n+1} = \{X : \forall i(X^{[i]} \notin C_n)\}$, where $X^{[i]}$ denotes the i th column of X , and fix a continuous function h which reduces C_{n+1} to $f^{-1}(U)$.

Now for each m define the following forcing F_m . A condition is a finite tree $T \subseteq \omega^{<\omega}$ of rank $m + 1$ with labeled nodes which satisfies the following properties.

- If $\sigma \frown (i + 1) \in T$ then $\sigma \frown i \in T$.
- Each node is labeled 1 (meaning \exists) or 0 (meaning \forall).
- If σ is labeled 1, at least one of its children must be labeled 0.
- If σ is labeled 0, all of its children must be labeled 1.

Condition T_2 extends T_1 if $T_2 \supseteq T_1$. Given a filter in this forcing, a real X is extracted by reading all the bits off the leaves, by letting the label given to $\langle a_1, \dots, a_m \rangle$ determine the a_m th bit of $X^{[a_1, \dots, a_{m-1}]}$, where we compactly express $(X^{[a]})^{[b]}$ as $X^{[a,b]}$, and similarly for deeper addressing.

By induction, for each forcing F_i , the condition $(\langle \rangle, 0)$ forces $X \in C_i$, while the condition $(\langle \rangle, 1)$ forces $X \notin C_i$. This is without any genericity requirements on the filter. The base case when $i = 1$ is easily seen. For the induction, observe that $(\langle \rangle, 1)$ in F_{i+1} forces $(\langle a \rangle, 0)$ to be included for some a , which forces $X^{[a]} \in C_i$ since the filter below $\langle a \rangle$ is a filter of F_i . Similarly $(\langle \rangle, 0)$ forces each $X^{[a]} \notin C_i$, and thus $X \in C_{i+1}$.

Coming back now to f and considering the forcing F_{n+1} , we see that $(\langle \rangle, 0)$ forces $f(h(X)) \in U$. Fix ε and $T_0 \in F_{n+1}$ such that for any sufficiently F_{n+1} -generic $G \prec T_0$, $f(h(G)) \in (-\infty, u - \varepsilon)$.

Now, let $p < q$ be given, such that we wish to separate $j_{n+1}(X)$ for (p, q) . There is some initial segment of the binary expansion of $j_{n+1}(X)$ which, if we knew it, would suffice to separate $j_{n+1}(X)$ for (p, q) . In fact, it is enough to know only positive information about that initial segment, since if σ is such that $\sum_{i:\sigma(i)=1} 2^{-(i+1)} > p$, then any τ whose positive information is a superset of σ 's also corresponds to the case when $j_{n+1}(X) > p$. Therefore, a correct answer to

$$P(X) \equiv \bigvee_{\sigma: \sum_{i:\sigma(i)=1} 2^{-(i+1)} > p} [\forall i \in \sigma(i \in X^{(n+1)})]$$

would suffice as a separating bit for $j_{n+1}(X)$ for (p, q) . The separating bit we will seek is 1 if $P(X)$ holds and 0 otherwise, and $P(X)$ is $\Sigma_{n+1}(X)$.

Now we show how to continuously transform X into a sufficiently F_{n+1} -generic G such that $P(X)$ implies $G \notin C_{n+1}$ and $\neg P(X)$ implies $G \prec T_0$. The process is an infinite injury priority argument, organized by the notion of k -true stages. Let $\langle \leq_k \rangle_{k \leq n}$ be the uniformly X -computable orderings guaranteed by Theorem A.1.

Define a sequence of F_{n+1} -conditions T_0, T_1, \dots . They are not a descending sequence, but at the end we will identify a descending subsequence. The first condition T_0 is the same T_0 identified above. We will maintain that for each $s < r$ and each $k \leq n$, if $s \leq_k r$ then for each $(\sigma, b) \in T_s$ with $|\sigma| > n - k$, $(\sigma, b) \in T_r$. If T_s and T_r satisfy this labeling agreement condition, we write $T_s \subseteq_k T_r$.

Given T_s for $s \leq r$ already satisfying this condition, we construct T_{r+1} . For each $k = 0, 1, \dots, n$, let $s_k \leq r$ be greatest such that $s_k \leq_k r + 1$. Observe that $s_0 = r$, and for each k , $s_{k+1} \leq_k s_k \leq_k r + 1$ so $s_{k+1} \leq_{k+1} s_k$.

Start with $T = T_r$ without the labels. Because we have $s_n \leq_n s_{n-1} \leq_{n-1} \dots \leq_1 s_0$, we also have $T_{s_n} \subseteq_n T_{s_{n-1}} \subseteq_{n-1} \dots \subseteq_1 T_{s_0}$. All the labels which are referenced in the satisfaction of the latter sweep agree in all the trees T_{s_k} . Populate T with those labels. The result satisfies $T_{s_k} \subseteq_k T$ for each k . The result is consistent in the sense that it never contains the forbidden combination of a node and its child both labeled 0. (Consider any $(\sigma \smallfrown j, 0) \in T_{s_k}$ which was copied into T as a part of satisfying $T_{s_k} \subseteq_k T$. Then if (σ, b) was copied into T , it either was copied from T_{s_k} or it was copied from $T_{s_{k+1}}$, in which case it still occurs in T_{s_k} ; since it occurs with a child labeled 0, it must be that $b = 1$.) The consistency of T can be maintained while adding 1s in the place of every empty label except the root.

To determine the root label for T , use $\{s : s \leq_n r + 1\}$ to attempt to compute an initial segment of $X^{(n)}$ (which is a correct initial segment only if $r + 1$ is actually on the \leq_n -true path). If, based on this initial segment, we would believe $P(X)$, put 1 as the root label. Otherwise, put 0 as the root label. It will always be consistent to use 0 if we want to, because if T_{s_n} has a root labeled 1, then $\{s : s \leq_n s_n\}$ computes enough of its pseudo- $X^{(n)}$ to believe $P(X)$, this is a $\Sigma_1(X^{(n)})$ event and our computation coincides, so we will want to use a 1 at the root also. That means if we want to use a 0, it must be that T_{s_n} has root label 0, so all $(\langle a \rangle, b) \in T_{s_n}$ satisfy $b = 1$ (and these are copied into T) and all remaining $(\langle a \rangle, b) \in T$ also have $b = 1$ because that was how non-copied labels were filled in.

We now have a T which satisfies all properties of an F_{n+1} -condition except perhaps that any node labeled 1 must have a child labeled 0. This can be fixed by added new children labeled 0 as appropriate, resulting in our final $T \in F_{n+1}$. Now let T_{r+1} be the result of extending T in F_{n+1} to meet the first $r + 1$ generic requirements. Meeting the requirements does not alter the fact that $T_s \subseteq_k T_{r+1}$ whenever $s \leq_k r + 1$. This completes the definition of the sequence T_s . Note that leaves are always copied to the next condition, so this sequence uniquely and continuously determines a real G . Now we check that G has the required properties.

Let R be the \leq_n -true path. Then for each $s \in R$, the pseudo- $X^{(n)}$ segment computed by it is correct. Suppose that $P(X)$ holds and that $s \in R$ computes enough of $X^{(n)}$ to witness this. Then $\{T_r : r \in R, s \leq r\}$ defines a filter in F_{n+1} , because each such $r \leq t \in R$ satisfies $T_r \subseteq_n T_t$, which, combined with the fact that their roots both have label 1, gives that $T_r \subseteq T_t$. Since the root is labeled 1, we have that $G \notin C_{n+1}$. On the other hand, if $\neg P(X)$, then every $s \in P$ puts 0 at the root of T_s , including $s = 0$. Then again $\{T_s : s \in P\}$ defines a filter in F_{n+1} , and it induces a G with $G \prec T_0$. Furthermore, G is sufficiently generic, because each T_s has also met the first s -many genericity requirements.

Therefore, given X , we can continuously produce this G , with the intended result that $f(h(G)) \in (-\infty, u - \varepsilon)$ if $\neg P(X)$, and $f(h(G)) \in [u, \infty)$ if $P(X)$. Recalling

that u and ε do not depend on anything but f , we can now separate $j_{n+1}(X)$ for (p, q) with the same bit that separates $f(h(G))$ for $(u - \varepsilon, u - \varepsilon/2)$. \square

The original infinite case required a more delicate technical argument. The limit case in Montalbán's α -true stages is rather complicated and the fact that it relativizes, while true, is not as obvious as with the finite case. Also, the “weak extendability condition” of [Mon14] was not quite weak enough and needed to be weakened further. Meanwhile, [GT] have simplified the limit case of the α -true stages. So the original proof of the limit case of Theorem 6.3 is now deprecated.

ACKNOWLEDGMENTS

The authors would like to thank Vasco Brattka, Takayuki Kihara, Antonio Montalbán, Arno Pauly and Dan Turetsky for many useful discussions on these and related topics. They would also like to thank the anonymous referee for pointing out several possibly related works, and for an increase in the rigor of the results on represented spaces.

REFERENCES

- [Ash86] C. J. Ash, *Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees*, Trans. Amer. Math. Soc. **298** (1986), no. 2, 497–514, DOI 10.2307/2000633. MR860377
- [BG11] Vasco Brattka and Guido Gherardi, *Weihrauch degrees, omniscience principles and weak computability*, J. Symbolic Logic **76** (2011), no. 1, 143–176, DOI 10.2178/jsl/1294170993. MR2791341
- [BGP17] Vasco Brattka, Guido Gherardi, and Arno Pauly, *Weihrauch complexity in computable analysis*, Handbook of computability and complexity in analysis, Theory Appl. Comput., Springer, Cham, 2021, pp. 367–417, DOI 10.1007/978-3-030-59234-9_11. MR4300761
- [Bou80] J. Bourgain, *On convergent sequences of continuous functions*, Bull. Soc. Math. Belg. Sér. B **32** (1980), no. 2, 235–249. MR682645
- [Bra05] Vasco Brattka, *Effective Borel measurability and reducibility of functions*, MLQ Math. Log. Q. **51** (2005), no. 1, 19–44, DOI 10.1002/malq.200310125. MR2099383
- [Car13] Raphaël Carroy, *A quasi-order on continuous functions*, J. Symbolic Logic **78** (2013), no. 2, 633–648. MR3145199
- [DGHTT21] Adam Day, Noam Greenberg, Matthew Harrison-Trainor, and Dan Turetsky, *The strength of Borel Wadge comparability*, 2021, Presentation delivered by Noam Greenberg at the Midwest Computability Seminar joint with the Computability Theory and Applications Online Seminar, April 2021, Slides available at http://www.math.uchicago.edu/~drh/mcs/greenberg_slides2.pdf, accessed July 2021.
- [DM08] Rod Downey and Antonio Montalbán, *The isomorphism problem for torsion-free abelian groups is analytic complete*, J. Algebra **320** (2008), no. 6, 2291–2300, DOI 10.1016/j.jalgebra.2008.06.007. MR2437501
- [DM19] Adam Day and Andrew Marks, *A completeness criterion for Borel sets*, 2019, Presentation delivered by Adam Day at the National University of Singapore Institute for Mathematical Sciences Workshop: Higher Recursion Theory and Set Theory, May–June 2019, Slides available at <https://imsarchives.nus.edu.sg/oldwww2/events/2019/recur/files/adam.pdf>, accessed July 2021.
- [EKV16] Márton Elekes, Viktor Kiss, and Zoltán Vidnyánszky, *Ranks on the Baire class ξ functions*, Trans. Amer. Math. Soc. **368** (2016), no. 11, 8111–8143, DOI 10.1090/tran/6764. MR3546795
- [Grz55] A. Grzegorzczak, *Computable functionals*, Fund. Math. **42** (1955), 168–202, DOI 10.4064/fm-42-1-168-202. MR86756
- [Grz57] A. Grzegorzczak, *On the definitions of computable real continuous functions*, Fund. Math. **44** (1957), 61–71, DOI 10.4064/fm-44-1-61-71. MR89809

- [GT] Noam Greenberg and Daniel Turetsky, *Completeness of the hyperarithmetical isomorphism equivalence relation*, Adv. Math. **403** (2022), Paper No. 108356, 56, DOI 10.1016/j.aim.2022.108356. MR4402976
- [Her96a] Peter Hertling, *Topological complexity with continuous operations*, J. Complexity **12** (1996), no. 4, 315–338, DOI 10.1006/jcom.1996.0021. Special issue for the Foundations of Computational Mathematics Conference (Rio de Janeiro, 1997). MR1422714
- [Her96b] Peter Hertling, *Unstetigkeitsgrade von Funktionen in der effektiven Analysis*, Ph.D. Thesis, FernUniversität Hagen, Informatik-Berichte 208, 1996.
- [HS88] Graham Higman and Elizabeth Scott, *Existentially closed groups*, London Mathematical Society Monographs. New Series, vol. 3, The Clarendon Press, Oxford University Press, New York, 1988. Oxford Science Publications. MR960689
- [Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995, DOI 10.1007/978-1-4612-4190-4. MR1321597
- [Kih] Takayuki Kihara, *Topological reducibilities for discontinuous functions and their structures*, Preprint, arXiv:1906.10573, 2019.
- [Kih15] Takayuki Kihara, *Decomposing Borel functions using the Shore-Slaman join theorem*, Fund. Math. **230** (2015), no. 1, 1–13, DOI 10.4064/fm230-1-1. MR3332281
- [KL90] A. S. Kechris and A. Louveau, *A classification of Baire class 1 functions*, Trans. Amer. Math. Soc. **318** (1990), no. 1, 209–236, DOI 10.2307/2001236. MR946424
- [Kle59] S. C. Kleene, *Recursive functionals and quantifiers of finite types. I*, Trans. Amer. Math. Soc. **91** (1959), 1–52, DOI 10.2307/1993145. MR102480
- [KM19] Takayuki Kihara and Antonio Montalbán, *On the structure of the Wadge degrees of $b\omega$ -valued Borel functions*, Trans. Amer. Math. Soc. **371** (2019), no. 11, 7885–7923, DOI 10.1090/tran/7621. MR3955538
- [KW85] Christoph Kreitz and Klaus Weihrauch, *Theory of representations*, Theoret. Comput. Sci. **38** (1985), no. 1, 35–53, DOI 10.1016/0304-3975(85)90208-7. MR805132
- [Lac55a] Daniel Lacombe, *Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles. I* (French), C. R. Acad. Sci. Paris **240** (1955), 2478–2480. MR72079
- [Lac55b] Daniel Lacombe, *Extension de la notion de fonction récursive aux fonctions d'une ou plusieurs variables réelles. II, III* (French), C. R. Acad. Sci. Paris **241** (1955), 13–14, 151–153. MR72080
- [LL95] Steffen Lempp and Manuel Lerman, *A general framework for priority arguments*, Bull. Symbolic Logic **1** (1995), no. 2, 189–201, DOI 10.2307/421040. MR1335987
- [LSR87] A. Louveau and J. Saint-Raymond, *Borel classes and closed games: Wadge-type and Hurewicz-type results*, Trans. Amer. Math. Soc. **304** (1987), no. 2, 431–467, DOI 10.2307/2000725. MR911079
- [LSR88] Alain Louveau and Jean Saint-Raymond, *The strength of Borel Wadge determinacy*, Cabal Seminar 81–85, Lecture Notes in Math., vol. 1333, Springer, Berlin, 1988, pp. 1–30, DOI 10.1007/BFb0084967. MR960893
- [Mil04] Joseph S. Miller, *Degrees of unsolvability of continuous functions*, J. Symbolic Logic **69** (2004), no. 2, 555–584, DOI 10.2178/jsl/1082418543. MR2058189
- [Mon14] Antonio Montalbán, *Priority arguments via true stages*, J. Symb. Log. **79** (2014), no. 4, 1315–1335, DOI 10.1017/jsl.2014.11. MR3343540
- [My106] U. Mylatz, *Vergleich unstetiger Funktionen: "Principle of Omniscience" und Vollstaendigkeit in der C-Hierarchie*, 2006. Thesis (Ph.D.)–Fernuniversitaet, Gesamthochschule in Hagen.
- [Pau10] Arno Pauly, *On the (semi)lattices induced by continuous reducibilities*, MLQ Math. Log. Q. **56** (2010), no. 5, 488–502, DOI 10.1002/malq.200910104. MR2742884
- [RS] Jan Reimann and Theodore A. Slaman, *Effective randomness for continuous measures*, J. Amer. Math. Soc. **35** (2021), no. 2, 467–512, DOI 10.1090/jams/980. MR4374955
- [Sac90] Gerald E. Sacks, *Higher recursion theory*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990, DOI 10.1007/BFb0086109. MR1080970
- [Sch02] Matthias Schröder, *Extended admissibility*, Theoret. Comput. Sci. **284** (2002), no. 2, 519–538, DOI 10.1016/S0304-3975(01)00109-8. Computability and complexity in analysis (Castle Dagstuhl, 1999). MR1923914

- [SR76] Jean Saint-Raymond, *Fonctions boréliennes sur un quotient* (French), Bull. Sci. Math. (2) **100** (1976), no. 2, 141–147. MR460578
- [Tur39] A. M. Turing, *Systems of logic based on ordinals*, Proc. London Math. Soc. (2) **45** (1939), no. 3, 161–228, DOI 10.1112/plms/s2-45.1.161. MR1576807
- [Wei00] Klaus Weihrauch, *Computable analysis*, Texts in Theoretical Computer Science. An EATCS Series, Springer-Verlag, Berlin, 2000. An introduction, DOI 10.1007/978-3-642-56999-9. MR1795407

SCHOOL OF MATHEMATICS AND STATISTICS, VICTORIA UNIVERSITY OF WELLINGTON, WELLINGTON, NEW ZEALAND

Email address: `aday@mailbox.org`

SCHOOL OF MATHEMATICS AND STATISTICS, VICTORIA UNIVERSITY OF WELLINGTON, WELLINGTON, NEW ZEALAND

Email address: `rod.downey@vuw.ac.nz`

DEPARTMENT OF MATHEMATICS, PENN STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA, 16802

Email address: `westrick@psu.edu`