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NORMAL SUBGROUPS OF BIG MAPPING CLASS GROUPS

DANNY CALEGARI AND LVZHOU CHEN

Abstract. Let S be a surface and let $\mathrm{Mod}(S,K)$ be the mapping class group of S permuting a Cantor subset $K \subset S$. We prove two structure theorems for normal subgroups of $\mathrm{Mod}(S,K)$.

(Purity:) if S has finite type, every normal subgroup of $\mathrm{Mod}(S,K)$ either contains the kernel of the forgetful map to the mapping class group of S, or it is 'pure' — i.e. it fixes the Cantor set pointwise.

(Inertia:) for any n element subset Q of the Cantor set, there is a forgetful map from the pure subgroup $\mathrm{PMod}(S,K)$ of $\mathrm{Mod}(S,K)$ to the mapping class group of (S,Q) fixing Q pointwise. If N is a normal subgroup of $\mathrm{Mod}(S,K)$ contained in $\mathrm{PMod}(S,K)$, its image N_Q is likewise normal. We characterize exactly which finite-type normal subgroups N_Q arise this way.

Several applications and numerous examples are also given.

Contents

1. Introduction

In recent years there has been an surge of interest in the theory of so-called *big mapping class groups*, i.e. mapping class groups of surfaces of infinite type. There are many motivations for studying such objects (see e.g. [4] for an excellent recent survey) but our motivation comes from (2 real dimensional or 1 complex dimensional) dynamics. Thus we are especially interested in big mapping class groups of a certain kind, which we now explain.

If S is a surface of finite type, a hyperbolic dynamical system Γ acting on S will often determine a dynamically distinguished compact subset Λ. The terminology for Λ varies depending on the context: it is an *attractor* if Γ is an Iterated Function System; a *limit set* if Γ is a Kleinian group; a *Julia set* if Γ is the set of iterates of a rational map; or a *hyperbolic set* if Γ is an Axiom A diffeomorphism. Often, Λ will be a Cantor set, and the topology of the moduli spaces of pairs (Γ, Λ) is related to the mapping class group of S minus a Cantor set and its subgroups. Because there is no natural choice of coordinates on Λ, it is the *normal* subgroups

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of such mapping class groups that will arise, and therefore this paper is devoted to exploring the structure and classification of such subgroups.

1.1. **Statement of results.** Let S be a surface of finite type (i.e. S has finite genus and finitely many punctures). Let $\mathrm{Mod}(S)$ denote the mapping class group of S and for any compact totally disconnected subset Q of S let $\mathrm{Mod}(S,Q)$ denote the mapping class group of S rel. Q (i.e. the group of homotopy classes of orientation-preserving self-homeomorphisms of S permuting Q as a set) and $\mathrm{PMod}(S,Q)$ the *pure* mapping class group of S rel. Q (i.e. the group of homotopy classes of orientation-preserving self-homeomorphisms of S fixing Q pointwise). Pure subgroups of big mapping class groups and their properties are studied e.g. in [3] and [15]. Note that $\mathrm{PMod}(S,Q)$ is a normal subgroup of $\mathrm{Mod}(S,Q)$. When Q is finite, we write the cardinality $n := |Q|$. For finite Q the groups $\mathrm{Mod}(S,Q)$ and $\mathrm{PMod}(S,Q)$ depend only on the cardinality of Q, up to conjugation by elements of $\mathrm{Mod}(S,Q)$. By abuse of notation, we denote these equivalence classes of groups by $\mathrm{Mod}(S,n)$ and $\mathrm{PMod}(S,n)$ respectively.

There are forgetful maps $\mathrm{Mod}(S,Q) \to \mathrm{Mod}(S)$ for all Q, and $\mathrm{PMod}(S,R) \to \mathrm{PMod}(S,Q)$ for all pairs (Q,R) with $Q \subset R$. When $K \subset S$ is a Cantor set and $N \subset \mathrm{PMod}(S,K)$ is a normal subgroup of $\mathrm{Mod}(S,K)$, the image $N_n \subset \mathrm{PMod}(S,n)$ is a well-defined normal subgroup of $\mathrm{Mod}(S,n)$.

Our first main theorem is the following:

Purity Theorem 2.1. *Let S be a connected orientable surface of finite type, and let K be a Cantor set in S. Then any normal subgroup of* $\mathrm{Mod}(S,K)$ *either contains the kernel of the forgetful map to* $\mathrm{Mod}(S)$*, or it is contained in* $\mathrm{PMod}(S,K)$.

The two cases in the theorem correspond to normal subgroups of $\mathrm{Mod}(S,K)$ of countable and uncountable index respectively. It implies that all normal subgroups of countable index are pulled back from normal subgroups of $\mathrm{Mod}(S)$.

When S is the plane or the 2-sphere, $\mathrm{Mod}(S)$ is trivial. It follows that every proper normal subgroup of $\mathrm{Mod}(S,K)$ is contained in $\mathrm{PMod}(S,K)$; in particular, its index in $\mathrm{Mod}(S,K)$ is *uncountable.* Thus the mapping class group of the plane or the sphere minus a Cantor set admits no nontrivial homomorphism to a countable group. This fact was proved independently by Nicholas Vlamis [16].

We also obtain the abelianization and generating sets of $\mathrm{Mod}(S,K)$; See Section 2.1.

Our second main theorem concerns the subgroups $N_n \subset \mathrm{PMod}(S,n)$ that arise as the image of subgroups $N \subset \mathrm{PMod}(S,K)$ normal in $\mathrm{Mod}(S,K)$. To state this theorem we must give the definition of an *inert* subgroup of $\mathrm{PMod}(S,n)$.

Let S be any surface and let $Q \subset S$ be a finite subset with cardinality n. Let $D_Q \subset S$ be a collection of n disjoint closed disks, each centered at some point of Q. Let $\mathrm{PMod}(S,D_Q)$ be the mapping class group of S fixing D_Q pointwise (by both homeomorphisms and homotopies). There is a forgetful map $\mathrm{PMod}(S,D_Q) \to \mathrm{PMod}(S,Q)$ and as is well-known, this is a $\mathbb{Z}^n$ central extension (see e.g. [8] Prop 3.19).

Let $\alpha \in \mathrm{PMod}(S,Q)$ be any element, and let $\hat{\alpha}$ be some lift to $\mathrm{PMod}(S,D_Q)$. Let $f: Q \to Q$ be any map (possibly not injective). The *insertion* $f^*\hat{\alpha} \in \mathrm{PMod}(S,n)$ is the $(\mathrm{Mod}(S,n))$-conjugacy class of element defined as follows. Let $\pi: D_Q \to Q$ be the map that takes each component of D_Q to its center, and let $\tilde{f}: Q \to D_Q$ be any injective map for which the composition $\pi\tilde{f} = f$. Then $f^*\hat{\alpha}$ is the image of

$\hat{\alpha}$ under the forgetful map to $\mathrm{PMod}(S, \tilde{f}(Q))$ which may be canonically identified (up to conjugacy in $\mathrm{Mod}(S,n)$) with $\mathrm{PMod}(S,n)$. Note that the class of $f^*\hat{\alpha}$ is invariant under pre-composition of f with a permutation of Q, and it depends only on the cardinalities of the preimages (under f) of the elements of Q.

A subgroup $N \subset \mathrm{PMod}(S,n)$ normal in $\mathrm{Mod}(S,n)$ is said to be *inert* if, after fixing some identification of $\mathrm{PMod}(S,n)$ with $\mathrm{PMod}(S,Q)$, for every $\alpha \in N$ there is a lift $\hat{\alpha}$ in $\mathrm{PMod}(S, D_Q)$ so that for every $f : Q \to Q$ the insertion $f^*\hat{\alpha}$ is in N. For more on this definition see Section 3.1.

Our second main theorem is the following:

Inertia Theorem 3.10. *Let S be any connected, orientable surface, and let K be a Cantor set in S. A subgroup N_n of $\mathrm{PMod}(S,n)$ is equal to the image of some $\mathrm{Mod}(S,K)$-normal pure subgroup $N \subset \mathrm{PMod}(S,K)$ under the forgetful map if and only if it is inert.*

2. The Purity Theorem

The purpose of this section is to prove Theorem 2.1:

Theorem 2.1 (Purity Theorem). *Let S be a connected orientable surface of finite type, and let K be a Cantor set in S. Then any normal subgroup of $\mathrm{Mod}(S,K)$ either contains the kernel of the forgetful map to $\mathrm{Mod}(S)$, or it is contained in $\mathrm{PMod}(S,K)$.*

An immediate corollary is:

Corollary 2.2. *Any homomorphism from $\mathrm{Mod}(S,K)$ to a countable group factors through $\mathrm{Mod}(S,K) \to \mathrm{Mod}(S)$. Thus if $\mathrm{Mod}(S)$ is trivial (for instance if S is the plane or the 2-sphere) then $\mathrm{Mod}(S,K)$ has no nontrivial homomorphisms to a countable group.*

If $\mathrm{Mod}(S)$ is trivial then $\mathrm{Mod}(S,K)/\mathrm{PMod}(S,K)$ is isomorphic to $\mathrm{Aut}(K)$, the group of self-homeomorphisms of a Cantor set. So Corollary 2.2 gives a new proof that $\mathrm{Aut}(K)$ is simple, which is a theorem of Richard Anderson [2].

2.1. Applications.

In this section we give some applications of the Purity Theorem 2.1.

The first application is to give a computation of $H_1(\mathrm{Mod}(S,K))$, the abelianization of $\mathrm{Mod}(S,K)$.

Theorem 2.3. *Let S be a surface of finite type, and let K be a Cantor set in S. Then the forgetful map $\mathrm{Mod}(S,K) \to \mathrm{Mod}(S)$ induces an isomorphism on H_1; i.e.*

$$H_1(\mathrm{Mod}(S,K)) \cong H_1(\mathrm{Mod}(S)).$$

Proof. Let N be the commutator subgroup of $\mathrm{Mod}(S,K)$. Its image in $\mathrm{Aut}(K)$ is nontrivial (actually, its image is all of $\mathrm{Aut}(K)$ by Anderson's theorem, but we do not use this fact). Thus N is not contained in $\mathrm{PMod}(S,K)$, so by Theorem 2.1 it contains the kernel of the forgetful map $\pi : \mathrm{Mod}(S,K) \to \mathrm{Mod}(S)$.

Since π is surjective, $\pi(N)$ is equal to the commutator subgroup of $\mathrm{Mod}(S)$. Thus

$$H_1(\mathrm{Mod}(S,K)) = \mathrm{Mod}(S,K)/N \cong \mathrm{Mod}(S)/\pi(N) = H_1(\mathrm{Mod}(S)).$$

□

The next application lets us determine the normal closure of certain subsets of elements. If T is a subset of a group G, the *normal closure* of T is the subgroup of G (algebraically) generated by conjugates of T. A subset T is said to *normally generate* G if its normal closure is G.

Lemma 2.4. *Let* $T \subset \mathrm{Mod}(S, K)$ *be a subset that is not contained in* $\mathrm{PMod}(S, K)$*. If the image* $\pi(T)$ *of* T *in* $\mathrm{Mod}(S)$ *normally generates* $\mathrm{Mod}(S)$ *then* T *normally generates* $\mathrm{Mod}(S, K)$*.*

Proof. Let N denote the normal closure of T. Then N is not contained in $\mathrm{PMod}(S, K)$ by hypothesis, so by Theorem 2.1, it contains the kernel of π. It follows that N is equal to π^{-1} of the normal closure of $\pi(T)$. □

One interesting case is to take T to be the set of torsion elements.

Lemma 2.5. $\mathrm{PMod}(S, K)$ *is torsion-free.*

Proof. We may identify $\mathrm{Mod}(S, K)$ with $\mathrm{Mod}(S - K)$ in the obvious way. Every torsion element g of $\mathrm{Mod}(S - K)$ is realized by a periodic homeomorphism f of $S - K$ by [1, Thm. 2] which extends to a periodic homeomorphism of S permuting K.

But any nontrivial periodic (orientation-preserving) homeomorphism f of a finite type surface fixes only finitely many points. This follows from a theorem of Béla von Kerékjártó [10], which says that any finite order orientation-preserving homeomorphism of the plane is conjugate to a rotation (and therefore fixes at most two points). This is well-known to experts, but we give an argument to reduce to Kerékjártó's theorem. First extend the homeomorphism f over the punctures (if any) to reduce to the case of a homeomorphism of a closed surface which we also denote S. If there are infinitely many fixed points, there is a fixed point p which is a limit of fixed points p_i. If γ_i is a short path from p to p_i then evidently $f(\gamma_i)$ is homotopic to γ_i rel. endpoints for all sufficiently big i. Let $\tilde{f}$ be a lift of f to the universal cover $\tilde{S}$ of S fixing some point $\tilde{p}$ that covers p. Then $\tilde{f}$ has a finite power that covers the identity on S, and since it fixes $\tilde{p}$ it *is* the identity; i.e. $\tilde{f}$ is a torsion element acting on $\tilde{S}$. Furthermore, $\tilde{f}$ fixes lifts $\tilde{p}_i$ of p_i joined to $\tilde{p}$ by lifts $\tilde{\gamma}_i$ of γ_i. But this contradicts Kerékjártó's theorem. □

Theorem 2.6. *Let* S *be a surface of finite type, and let* K *be a Cantor set in* S*. Then* $\mathrm{Mod}(S, K)$ *is generated by torsion unless* S *has genus two and* $5k+4$ *punctures for some* $k \geq 0$*. Moreover, a torsion element* g *in* $\mathrm{Mod}(S, K)$ *normally generates* $\mathrm{Mod}(S, K)$ *if and only if its image in* $\mathrm{Mod}(S)$ *normally generates* $\mathrm{Mod}(S)$*.*

Remark 2.7. By [12, Thm. 1.1 and Prop. 3.3], when S is closed and has genus at least 3, a torsion element of $\mathrm{Mod}(S)$ normally generates $\mathrm{Mod}(S)$ if and only if it is not a hyperelliptic involution. In this sense, most torsion elements normally generate the entire mapping class group.

Proof of Theorem 2.6*.* The second assertion directly follows from Lemmas 2.4 and 2.5.

For the first assertion, let T be the set of all torsion elements in $\mathrm{Mod}(S, K)$. In view of Lemmas 2.4 and 2.5, it suffices to check that under the forgetful map, the image of T generates $\mathrm{Mod}(S)$. We claim that $\pi(T)$ is exactly the set of torsion elements in $\mathrm{Mod}(S)$. For, each torsion element g in $\mathrm{Mod}(S)$ is represented (by Nielsen realization) by a periodic homeomorphism f of S. If $K' \subset S$ is any Cantor set, so is the union K of its orbits under the (finite) set of powers of f. Thus f permutes some Cantor set K in S and represents a finite order element of $\mathrm{Mod}(S, K)$ mapping to g.

For any finite type surface S, the group $\mathrm{Mod}(S)$ is generated by torsion elements unless S has genus two and $5k+4$ punctures for some $k \geq 0$; see [13] and [11]. This completes the proof of the first assertion. □

Similar results have been obtained independently by Justin Malestein and Jing Tao for surfaces of infinite genus when the space of ends has certain self-similar structure [14].

2.2. **Proof of the Purity Theorem when S has at most one puncture.** We shall prove the Purity Theorem in the case where S has exactly one puncture, denoted ∞. The case of closed S may be deduced from this case by Lemma 2.8. Let's fix notation $\Sigma = S - K$, and define $\widehat{S} := S \cup \{\infty\}$ and $\widehat{\Sigma} := \Sigma \cup \{\infty\} = \widehat{S} - K$. There is a canonical isomorphism of $\mathrm{Mod}(S, K)$ with $\mathrm{Mod}(\Sigma)$; throughout the remainder of this section we usually discuss $\mathrm{Mod}(\Sigma)$.

Lemma 2.8. *Suppose every normal subgroup N of* $\mathrm{Mod}(\Sigma)$ *either lies in* $\mathrm{PMod}(\Sigma)$ *or contains the kernel of* $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$. *Then every normal subgroup $\widehat{N}$ of* $\mathrm{Mod}(\widehat{\Sigma})$ *either lies in* $\mathrm{PMod}(\widehat{\Sigma})$ *or contains the kernel of* $\mathrm{Mod}(\widehat{\Sigma}) \to \mathrm{Mod}(\widehat{S})$.

Proof. Let's denote the kernel of the forgetful map $\pi_S : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ by $\mathrm{Mod}(\Sigma)_0$, and likewise the kernel of the forgetful map $\pi_{\widehat{S}} : \mathrm{Mod}(\widehat{\Sigma}) \to \mathrm{Mod}(\widehat{S})$ by $\mathrm{Mod}(\widehat{\Sigma})_0$.

We have the following commutative diagram of short exact sequences, where each row is the Birman exact sequence obtained by point-pushing ∞:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\widehat{\Sigma}) & \longrightarrow & \mathrm{Mod}(\Sigma) & \xrightarrow{p} & \mathrm{Mod}(\widehat{\Sigma}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \pi_{\widehat{S}}} & & \\ 1 & \longrightarrow & \pi_1(\widehat{S}) & \longrightarrow & \mathrm{Mod}(S) & \longrightarrow & \mathrm{Mod}(\widehat{S}) & \longrightarrow & 1, \end{array}$$

and where the vertical maps π_S and $\pi_{\widehat{S}}$ are the forgetful homomorphisms.

We first show that $p(\mathrm{Mod}(\Sigma)_0) = \mathrm{Mod}(\widehat{\Sigma})_0$. For any $\hat{g} \in \mathrm{Mod}(\widehat{\Sigma})_0$, let $g \in \mathrm{Mod}(\Sigma)$ be any element with $p(g) = \hat{g}$. Then $\pi_S(g)$ must come from point-pushing infinity around a loop $\gamma \in \pi_1(\widehat{S})$. We can lift the isotopy class of γ to some $\tilde{\gamma} \in \pi_1(\widehat{\Sigma})$. Then as a mapping class in $\mathrm{Mod}(\Sigma)$, $\tilde{\gamma}$ and g have the same image under π_S. Hence $\tilde{\gamma}^{-1} \cdot g \in \mathrm{Mod}(\Sigma)_0$ and $p(\tilde{\gamma}^{-1} \cdot g) = \hat{g}$, Thus $p(\mathrm{Mod}(\Sigma)_0) = \mathrm{Mod}(\widehat{\Sigma})_0$ as claimed.

Now for any normal subgroup $\widehat{N}$ of $\mathrm{Mod}(\widehat{\Sigma})$ not contained in $\mathrm{PMod}(\widehat{\Sigma})$, let $N := p^{-1}(\widehat{N})$. Then N is not contained in $\mathrm{PMod}(\Sigma)$, and thus by our assumption it contains $\mathrm{Mod}(\Sigma)_0$. Therefore, by what we showed above, we have $\mathrm{Mod}(\widehat{\Sigma})_0 = p(\mathrm{Mod}(\Sigma)_0) \subset p(N) = \widehat{N}$. □

The remainder of this section is devoted to the proof of Theorem 2.1 under the hypothesis that S is a finite genus surface with exactly one puncture. The case when S has multiple punctures can be proved similarly, which we explain in Section 2.3.

Definition 2.9. We say a disk D in S is a *dividing disk* if both the interior and exterior of D intersect K while its boundary does not. Note that ∞ must lie in the exterior of a dividing disk D.

We say a mapping class g is supported in a dividing disk D if it can be realized as a homeomorphism that is the identity outside D. Denote by N_{div} the subgroup of $\mathrm{Mod}(\Sigma)$ generated by all mapping classes supported in dividing disks. Then N_{div} is normal in $\mathrm{Mod}(\Sigma)$. For brevity of notation, throughout the next section we write Γ for $\mathrm{Mod}(\Sigma)$ and Γ^0 $\mathrm{Mod}(\Sigma)_0$ (i.e. the kernel of $\mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$).

The proof of Theorem 2.1 has two steps. First (Lemma 2.10) we will show the normal closure of any $g \in \Gamma - \mathrm{PMod}(\Sigma)$ contains N_{div}. Second (Proposition 2.17) we will show that $N_{\mathrm{div}} = \Gamma^0$. This will complete the proof.

Lemma 2.10. *The normal closure of any* $g \in \Gamma - \mathrm{PMod}(\Sigma)$ *contains* N_{div}.

Proof. Let $x \in K$ be a point such that $x \neq g(x)$. Then there is a small closed dividing disk D containing x such that $D \cap g(D) = \emptyset$. Since any embedding of a Cantor set in a disk is standard, there exists a sequence of dividing disks $\{D_n\}_{n\geq 0}$ inside D converging to x and a homeomorphism h supported in D preserving $K \cap D$ and fixing x such that $h(D_n) = D_{n+1}$ for all $n \geq 0$; see Figure 1.

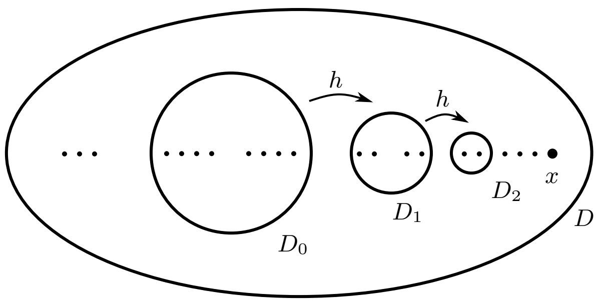

FIGURE 1. A sequence of dividing disks D_n in D converging to x and a mapping class h supported in D with $h(D_n) = D_{n+1}$

Denote $g(D)$ by E and $g(D_n)$ by E_n. Let $b \in \Gamma$ be a mapping class such that $b|_D = g|_D$ and $b|_E = h \circ (g|_D)^{-1} = h \circ (g^{-1})|_E$. Such a mapping class exists since any given orientation-preserving homeomorphism of the boundary of $\Sigma - \mathrm{int}(D \sqcup E)$ extends to an orientation-preserving homeomorphism on all of Σ.

Now let f be any mapping class in Γ supported in a dividing disk. Up to a conjugation, we may assume f is supported in D_0. Then the infinite product $x_f := \prod_{n\geq 0} h^n f h^{-n}$ is a well-defined mapping class in Γ.

Furthermore $a_f := [x_f, g] = (x_f g x_f^{-1}) g^{-1}$ lies in $\langle\langle g \rangle\rangle$, and therefore so does $a_f b a_f b^{-1}$. We claim $a_f b a_f b^{-1} = f$. Indeed, a_f and $b a_f b^{-1}$ are supported in $(\sqcup_{n\geq 0} D_n) \sqcup (\sqcup_{n\geq 0} E_n) \sqcup \{x, g(x)\}$ and $(\sqcup_{n\geq 1} D_n) \sqcup (\sqcup_{n\geq 0} E_n) \sqcup \{x, g(x)\}$ respectively. Note that for all $n \geq 0$, we have

$$(b a_f b^{-1})|_{D_{n+1}} = h g^{-1} \cdot (a_f)|_{E_n} \cdot g h^{-1} = h \cdot (x_f^{-1})|_{D_n} \cdot h^{-1} = (a_f^{-1})|_{D_{n+1}},$$

and

$$(b a_f b^{-1})|_{E_n} = g \cdot (a_f)|_{D_n} \cdot g^{-1} = g \cdot (x_f)|_{D_n} \cdot g^{-1} = (a_f^{-1})|_{E_n}.$$

Thus a_f and ba_fb^{-1} cancel out on all D_{n+1} and E_n for $n \geq 0$. Hence $a_fba_fb^{-1}$ is supported in D_0, on which it agrees with a_f and thus with f. Thus $a_fba_fb^{-1} = f$. □

The next step is to show that $N_{\text{div}} = \Gamma^0$. This will be accomplished in a series of lemmas. First note that $N_{\text{div}} \subset \Gamma^0$ since each mapping class supported in a disk becomes trivial under the forgetful map to $\text{Mod}(S)$. To prove the converse, we shall use notation and terminology consistent with [6]. Recall that a *short ray* in S is an isotopy class of proper embedded ray from ∞ to some point in K, and a *lasso* is a homotopically essential properly embedded copy of $\mathbb{R}$ in S from ∞ to ∞. For any short ray r, let Γ_r be the stabilizer of r and let $\Gamma_{(r)}$ be the subgroup of mapping classes that are the identity in a neighborhood of r.

When $S = \mathbb{R}^2$, Lemmas 6.6 and 6.8 from [6] show that the normal closure of $\Gamma_{(r)}$ is equal to Γ. This proves $N_{\text{div}} = \Gamma = \Gamma^0$ in the special case of $S = \mathbb{R}^2$ since each mapping class in $\Gamma_{(r)}$ is supported in a dividing disk in this situation.

For general S, for any (finite) collection L of disjoint lassos on Σ that are disjoint from r and cut S into a single disk, let $\Gamma_{L,(r)}$ denote the subgroup of $\Gamma_{(r)}$ consisting of mapping classes that fix the isotopy class of each lasso in L. We say that such a collection L is *filling* (with respect to r).

Lemma 2.11. *For each filling collection L, the group $\Gamma_{L,(r)}$ is a subgroup of N_{div}.*

Proof. By definition, the lassos in L cut S into a disk D_1 which contains the ray r. Each mapping class in $\Gamma_{L,(r)}$ is represented by a homeomorphism h that is supported in D_1 and is the identity in an open disk neighborhood $D_2 \subset D_1$ of r. Thus h is supported in $D_1 - D_2$, which is a dividing disk if we choose D_2 suitably; see Figure 2. Hence each mapping class of $\Gamma_{L,(r)}$ lies in N_{div}. □

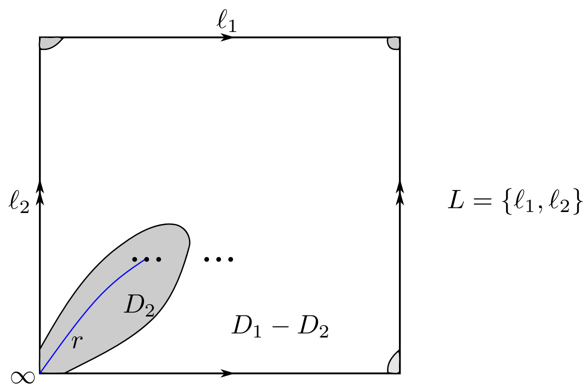

FIGURE 2. The dividing disk $D_1 - D_2$ for a suitable choice of D_2 in the case of $S = T^2 - \infty$

Our strategy to prove $\Gamma^0 \subset N_{\text{div}}$ is to show that these subgroups $\Gamma_{L,(r)}$ of N_{div} together generate Γ^0 as we vary L and r. This is accomplished in two steps: First we show that as we vary L fixing r, these subgroups generate $\Gamma_{(r)} \cap \Gamma^0$ in Lemma 2.12; Second we show that as we vary r, the subgroups $\Gamma_{(r)} \cap \Gamma^0$ generate Γ^0 using Lemmas 2.15 and 2.16.

Lemma 2.12. *For any simple ray r, the group generated by all $\Gamma_{L,(r)}$ over all collections of lassos L filling with respect to r, is equal to $\Gamma_{(r)} \cap \Gamma^0$.*

Proof. Every element of $\Gamma_{L,(r)}$ is supported in a disk and thus is trivial as a mapping class on the surface S. Hence each $\Gamma_{L,(r)}$ is a subgroup of $\Gamma_{(r)} \cap \Gamma^0$.

Conversely, it suffices to factorize an arbitrary mapping class g of $\Gamma_{(r)} \cap \Gamma^0$ as the product of finitely many mapping classes in $\Gamma_{(r)}$, each of which fixes a filling collection of lassos with respect to r.

Pick an arbitrary filling collection $L_0 = \{\ell_1, \cdots, \ell_n\}$ with respect to r. Then $g\ell_1$ is isotopic to ℓ_1 on S minus a neighborhood of r since $g \in \Gamma_{(r)} \cap \Gamma^0$.

Claim 2.13. There is a factorization $g = h_1 g_1$, where $g_1 \in \Gamma_{(r)} \cap \Gamma^0$ fixes ℓ_1 and h_1 is the product of finitely many mapping classes in $\Gamma_{(r)}$ each of which fixes a filling collection of lassos with respect to r.

Proof. First consider the case where $g\ell_1$ and ℓ_1 are disjoint. Then they cobound some open bigon B in $S - r$ since they are isotopic. We may assume that $K \cap B$ is nonempty and thus is also a Cantor set, since otherwise $g\ell_1$ and ℓ_1 are isotopic in Σ and g itself can be represented by a homeomorphism fixing ℓ_1. Also note that $K \cap B$ is a proper subset of K because the endpoint of r (which lies in K) is outside B. We can choose disjoint lassos $\gamma_a, \gamma_b, \gamma_c$ in B that cut both B and $K \cap B$ into four parts, where γ_b is in the middle, γ_a is closer to $g\ell_1$ and γ_c is closer to ℓ_1; see the left of Figure 3. We can choose lassos $\ell'_2, \cdots, \ell'_n$ on Σ such that each ℓ'_i is disjoint from $\ell_1, g\ell_1$ and homotopic to ℓ_i in $S - r$. Then $L = \{\ell_1, \ell'_2, \cdots, \ell'_n\}$ is a filling collection, which cuts Σ into a disk D minus a Cantor set. There is a homeomorphism h_1 preserving K and fixing each lasso in L (in particular ℓ_1) such that

- h_1 is the identity in a neighborhood of r; and
- $h_1 g\ell_1 = \gamma_b$ and $h_1\gamma_b = \gamma_c$.

Refer to Figure 3 to see the effect of h_1 in an example, where the disks D_1 and D_2 are used to track how the Cantor subset outside B changes under h_1. Then $h_1 \in \Gamma_{L,(r)}$. Similarly, $L_g = \{g\ell_1, \ell'_2, \cdots, \ell'_n\}$ is a filling collection, and there is a homeomorphism $h_2 \in \Gamma_{L_g,(r)}$ such that $h_2\ell_1 = \gamma_b$ and $h_2\gamma_b = \gamma_a$. Now $g_1 := h_2^{-1}h_1 g$ preserves ℓ_1 and $g_1 \in \Gamma_{(r)} \cap \Gamma^0$, hence $g = (h_1^{-1}h_2) \cdot g_1$ is the desired factorization.

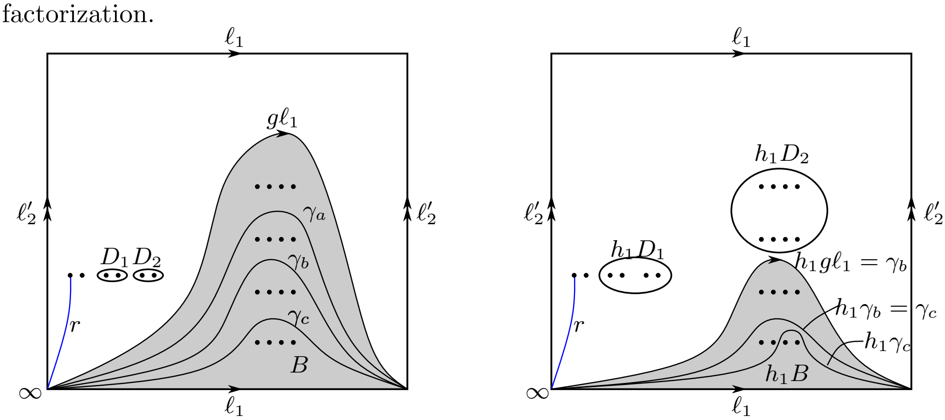

FIGURE 3. A choice of lassos γ_a, γ_b, γ_c in the case $S = T^2 - \infty$ and the effect of the homeomorphism h_1

Suppose $g\ell_1$ and ℓ_1 intersect. Put them in minimal position as lassos on Σ. Then they intersect finitely many times, and there is an innermost disk B in S which is a bigon bounded by arcs $\alpha \subset \ell_1$ and $\alpha_g \subset g\ell_1$. Then $B \cap K$ is nonempty. Note that if we replace α_g by α to modify $g\ell_1$ into a lasso ℓ_1', then ℓ_1' has a smaller intersection number with ℓ_1.

We can decompose B into four parts as in the previous case, and obtain a factorization $g = h'g'$, where $g' \in \Gamma_{(r)} \cap \Gamma^0$ has the property that $g'\ell_1 = \ell_1'$, and h' is a product of two mapping classes in $\Gamma_{(r)}$ that each fix a filling collection with respect to r. Since $g'\ell_1 = \ell_1'$ has a smaller intersection number with ℓ_1, by induction we can reduce to the disjoint case treated earlier and complete the proof of Claim 2.13. □

We continue the proof of Lemma 2.12. Now, g_1 fixes ℓ_1. Furthermore, both $g_1\ell_2$ and ℓ_2 are disjoint from $\ell_1 = g\ell_1$, and arguing as in the proof of Claim 2.13 we obtain a factorization $g_1 = h_2 g_2$, where $g_2 \in \Gamma_{(r)} \cap \Gamma^0$ fixes both ℓ_1 and ℓ_2, and h_2 is the product of finitely many mapping classes in $\Gamma_{(r)}$, each fixing a filling collection (containing ℓ_1) with respect to r. Continuing inductively, we obtain a factorization $g = h_1 h_2 \cdots h_n g_n$ where $g_n \in \Gamma_{(r)} \cap \Gamma^0$ fixes each ℓ_i, and each h_i is a product of finitely many mapping classes in $\Gamma_{(r)}$ that each fixes a filling collection with respect to r, where $1 \leq i \leq n$. Hence $g_n \in \Gamma_{L_0,(r)}$, which completes the proof. □

In what follows, whenever we say two short rays are disjoint we require them to have distinct endpoints in K.

Lemma 2.14. *For any short ray r, the group $\Gamma_r \cap \Gamma^0$ acts transitively on the set of short rays disjoint from r.*

Proof. Let s and s' be short rays disjoint from r. If s and s' are also disjoint, then there is a closed disk D containing ∞ that is the union of disk neighborhoods of r, s and s', so that ∂D does not intersect the K and $D \cap K$ is a Cantor set. Then there is a homeomorphism h supported in D fixing r and taking s to s'. Thus h represents a mapping class $g \in \Gamma_r \cap \Gamma^0$ with $gs = s'$.

For the general case, we can choose a short ray r' very close to r such that r' is disjoint from r, s, s'. Then by the above argument, there are $g, g' \in \Gamma_r \cap \Gamma^0$ such that $gs = r'$ and $g's' = r'$. Hence $g'^{-1}g \in \Gamma_r \cap \Gamma^0$ takes s to s'. □

Lemma 2.15. *For any disjoint short rays r and s, the group $\Gamma_r \cap \Gamma^0$ is generated by $\Gamma_{(r)} \cap \Gamma^0$ and $\Gamma_{(s)} \cap \Gamma_r \cap \Gamma^0$.*

Proof. Let g be a homeomorphism fixing r representing a mapping class in $\Gamma_r \cap \Gamma^0$. Then $g = g_1 g_2$ where g_2 is supported in an arbitrarily small disk neighborhood D of the endpoint of r and g_1 is the identity on $D \cup r$. Hence g_1 represents a mapping class in $\Gamma_{(s)} \cap \Gamma_r \cap \Gamma^0$ by choosing D small enough and g_1 represents a mapping class in $\Gamma_{(r)} \cap \Gamma^0$. □

Lemma 2.16. *Let r and s be disjoint short rays. Then $\Gamma_r \cap \Gamma^0$ and $\Gamma_s \cap \Gamma^0$ generate Γ^0.*

Proof. Let $\Gamma^0(r, s)$ be the group generated by $\Gamma_r \cap \Gamma^0$ and $\Gamma_s \cap \Gamma^0$. It suffices to show that $\Gamma^0(r, s)$ acts transitively on the set of short rays. Here is why. For any $\gamma \in \Gamma^0$, transitivity implies that there is $g \in \Gamma^0(r, s)$ such that $gr = \gamma r$ and thus $\gamma = gh$ where $h := g^{-1}\gamma \in \Gamma_r \cap \Gamma^0 \subset \Gamma^0(r, s)$.

Now we prove transitivity. We show for any short ray x there is some $g \in \Gamma^0(r, s)$ with $gx = s$ by induction on the distance between x and s in the *ray graph* $\mathcal{R}$. The vertex set of $\mathcal{R}$ is the set of short rays, and two vertices are connected by an edge if they are represented by disjoint short rays. The ray graph is connected: any two short rays with distinct endpoints have a finite intersection number, and the result follows from a classical argument proving connectivity of curve graphs of surfaces of finite type; see the proof of [8, Theorem 4.3] for an example.

Hence there is a geodesic path in the ray graph $\mathcal{R}$ connecting x and s, i.e. there is a minimal integer n and short rays $x_0 = x, x_1, \cdots, x_n = s$ such that adjacent short rays are disjoint. For the base case $n = 1$ where x and s are disjoint, since $\Gamma_s \cap \Gamma^0$ acts transitively on rays disjoint from s by Lemma 2.14, there is some $h \in \Gamma_s \cap \Gamma^0$ that takes x to a short ray disjoint from both r and s, which can be further taken to s by an element of $\Gamma_r \cap \Gamma^0$.

Suppose $n \geq 2$ and there is some $g \in \Gamma^0(r, s)$ taking x_1 to $x_n = s$. Then gx is disjoint from $gx_1 = s$ and by the base case there is some $g' \in \Gamma^0(r, s)$ taking gx to s. Hence $g'gx = s$ as desired. This completes the proof. □

Proposition 2.17. *The group $\Gamma^0 = \mathrm{Mod}(\Sigma)_0$ is generated by mapping classes supported in the dividing disks. In other words, $N_{\mathrm{div}} = \Gamma^0$.*

Proof. Evidently $N_{\mathrm{div}} \subset \Gamma^0$ since each mapping class supported in a disk has trivial image in $\mathrm{Mod}(S)$. So we just need to show $\Gamma^0 \subset N_{\mathrm{div}}$. For any short ray r and any collection L of lassos that are disjoint from r and cut S into a disk, the subgroup $\Gamma_{L,(r)}$ is a subgroup of N_{div} by Lemma 2.11. Thus N_{div} contains $\Gamma_r \cap \Gamma^0$ by Lemmas 2.12 and 2.15 for any short ray r. Then it follows from Lemma 2.16 that N_{div} contains Γ^0. □

From this we can deduce the Purity Theorem 2.1 when S has at most one puncture.

Proof of Theorem 2.1 *when S has at most one puncture.* By Lemma 2.8, it suffices to consider the case where S is a closed surface minus one puncture. By Lemma 2.10 and Proposition 2.17, the normal closure of any $g \in \Gamma \setminus \mathrm{PMod}(\Sigma)$ contains $N_{\mathrm{div}} = \Gamma^0$. Hence any normal subgroup not contained in $\mathrm{PMod}(\Sigma)$ must contain Γ^0. □

2.3. **Proof of the Purity Theorem in general.** In this section we prove the Purity Theorem 2.1 in general. Let S be a surface of finite genus with finitely many punctures P. Let K be a Cantor subset of S. Denote $\Sigma = S \setminus K$.

Note that any mapping class in $\mathrm{Mod}(\Sigma) = \mathrm{Mod}(S, K)$ preserves the sets P and K respectively. The subgroup $\mathrm{PMod}(\Sigma, P)$ fixing P pointwise is a proper subgroup if and only if P has more than one puncture.

The proof in Section 2.2 without many changes proves the following version of the Purity Theorem for $\mathrm{PMod}(\Sigma, P)$, from which we will deduce the Purity Theorem for $\mathrm{Mod}(S, K)$.

Theorem 2.18. *Let S be a surface of finite genus with finitely many punctures P. Let K be a Cantor subset of S. Then any normal subgroup N of $\mathrm{PMod}(S \setminus K, P)$ either contains the kernel of the forgetful map to $\mathrm{PMod}(S, P)$, or it is contained in $\mathrm{PMod}(S, K \cup P)$.*

Proof. Let $\Sigma = S \setminus K$. When P has at most one element, we have $\mathrm{PMod}(\Sigma, P) = \mathrm{Mod}(\Sigma) = \mathrm{Mod}(S, K)$, $\mathrm{PMod}(S, P) = \mathrm{Mod}(S)$, and $\mathrm{PMod}(S, K \cup P) = \mathrm{PMod}(S, K)$. Thus this is the Purity Theorem we proved in Section 2.2.

The proof strategy still works when P has more than one element, which we explain as follows. As in Section 2.2, let $\Gamma = \mathrm{PMod}(\Sigma, P)$, let Γ^0 be the kernel of the forgetful map to $\mathrm{PMod}(S, P)$ and let N_{div} denote the subgroup of Γ generated by elements supported in dividing disks. Then Lemma 2.10 using the same proof holds for any $g \in \Gamma \setminus \mathrm{PMod}(S, K \cup P)$. So it remains to show that $N_{\mathrm{div}} = \Gamma^0$.

Fix a puncture $\infty \in P$. Define short rays and lassos as before using the chosen puncture ∞. The only difference in the proof is the definition of a filling collection L: Here for a given short ray r, we say a collection L is *filling* (with respect to r) if it consists of disjoint lassos and $|P| - 1$ arcs in Σ each connecting ∞ to a distinct element of $P \setminus \{\infty\}$ so that they are all disjoint from r and cut S into a single disk.

With this modified definition, the analogs of Lemmas 2.11, 2.12, 2.14, 2.15, and 2.16 can be proved in the same way, and they imply $N_{\mathrm{div}} = \Gamma^0$ as desired. □

Now we can deduce Purity Theorem 2.1 in full generality.

Proof of Purity Theorem 2.1. Let $\Sigma = S \setminus K$ as above. Note that the forgetful map $\pi : \mathrm{Mod}(\Sigma) \to \mathrm{Mod}(S)$ restricts to the forgetful map from $\mathrm{PMod}(\Sigma, P)$ to $\mathrm{PMod}(S, P)$, and that the two maps have the same kernel since any mapping class in $\mathrm{Mod}(\Sigma)$ that acts trivially on S must preserve P pointwise.

First consider the case where the normal subgroup N of $\mathrm{Mod}(S, K)$ contains an element g that acts nontrivially on K and preserves P pointwise. Then the normal subgroup $N \cap \mathrm{PMod}(\Sigma, P)$ of $\mathrm{PMod}(\Sigma, P)$ is not contained in $\mathrm{PMod}(S, K \cup P)$. Thus by Theorem 2.18, $N \cap \mathrm{PMod}(\Sigma, P)$ contains the kernel $\ker \pi$, and so does N.

In general, if a normal subgroup N of $\mathrm{Mod}(S, K)$ contains an element g that acts nontrivially on K, then the image of N in $\mathrm{Aut}(K)$ is a nontrivial normal subgroup since $\mathrm{Mod}(S, K) \to \mathrm{Aut}(K)$ is surjective. As $\mathrm{Aut}(K)$ is simple by Richard Anderson's theorem [2] (which also follows from our proof of Purity Theorem for $S = \mathbb{R}^2$), we may choose g so that its image in $\mathrm{Aut}(K)$ has infinite order. Then a power of g fixes P pointwise and it acts nontrivially on K. Thus N must contain $\ker \pi$ by the previous paragraph. This completes the proof. □

3. The Inertia Theorem

In this section we let S be *any* connected oriented surface (finite type or not). Recall that for any compact totally disconnected subset Q of S the mapping class group of S rel. Q is denoted $\mathrm{Mod}(S, Q)$. This is a subgroup of $\mathrm{Mod}(\Sigma)$ where $\Sigma = S - Q$ but in general it might be smaller, since a typical element of $\mathrm{Mod}(\Sigma)$ might permute ends of S with points of Q. Similarly, $\mathrm{PMod}(S, Q)$ denotes the *pure* subgroup of $\mathrm{Mod}(S, Q)$; i.e. the subgroup of mapping classes fixing Q pointwise. Throughout this section, we say a (normal) subgroup of $\mathrm{Mod}(S, Q)$ is pure (normal) if it is contained in $\mathrm{PMod}(S, Q)$.

For Q finite, the groups $\mathrm{Mod}(S, Q)$ and $\mathrm{PMod}(S, Q)$ depend up to isomorphism only on the cardinality of Q, and by abuse of notation we fix representatives of these isomorphism classes which we denote $\mathrm{Mod}(S, n)$ and $\mathrm{PMod}(S, n)$. Each $\mathrm{Mod}(S, Q)$ is isomorphic to $\mathrm{Mod}(S, n)$ and each $\mathrm{PMod}(S, Q)$ is isomorphic to $\mathrm{PMod}(S, n)$ by an isomorphism (in either case) unique up to an *inner automorphism* (in $\mathrm{Mod}(S, n)$). Thus there is a *canonical* bijection between normal subgroups of any $\mathrm{Mod}(S, Q)$ contained in $\mathrm{PMod}(S, Q)$ and normal subgroups of $\mathrm{Mod}(S, n)$ contained in $\mathrm{PMod}(S, n)$.

Definition 3.1 (Spectrum)**.** Let $K \subset S$ be a Cantor set, and let N be a normal subgroup of $\mathrm{Mod}(S, K)$ contained in $\mathrm{PMod}(S, K)$. For any natural number

n, let $Q \subset K$ be an n-element set. The image of N under the forgetful map $\mathrm{PMod}(S, K) \to \mathrm{PMod}(S, Q)$ determines a normal subgroup N_n of $\mathrm{Mod}(S, n)$ contained in $\mathrm{PMod}(S, n)$ depending only on n (and not on Q). The *spectrum* of N is the sequence $\{N_n\}$ of pure normal subgroups $N_n \subset \mathrm{Mod}(S, n)$.

The spectrum does not determine N; we shall see some examples of different groups with the same spectrum in Example 3.14.

Each group N_n determines N_m for any $m < n$, since N_m is just the image of N_n under any forgetful map $\mathrm{PMod}(S, n) \to \mathrm{PMod}(S, m)$ that forgets $n - m$ marked points. This already imposes nontrivial conditions on the N_n that we formalize as follows:

Definition 3.2 (Algebraically Inert)**.** A family (indexed by $n \in \mathbb{N}$) of normal subgroups $\{N_n\}$ of $\mathrm{Mod}(S, n)$, each N_n contained in $\mathrm{PMod}(S, n)$, is called an *algebraically inert family* if for each $n > m$ the image of N_n in $\mathrm{PMod}(S, m)$ under each of the n choose m forgetful maps $\mathrm{PMod}(S, n) \to \mathrm{PMod}(S, m)$ is equal to N_m.

A normal subgroup N_n is called *algebraically inert* if it is a member of some algebraically inert family.

If $\{N_n\}$ is the spectrum of N then necessarily $\{N_n\}$ is an algebraically inert family and every N_n is algebraically inert. We do not know if every algebraically inert family arises as the spectrum of some N, or even if each individual algebraically inert N_n arises in the spectrum of some N. Nevertheless we *are* able to give a complete characterization of which subgroups of $\mathrm{PMod}(S, n)$ arise as N_n for some normal subgroup N of $\mathrm{PMod}(S, K)$. This is the main result of the section, the *Inertia Theorem* 3.10 and such subgroups N_n are said to be *inert* (see Definition 3.4).

3.1. **Inert subgroups.** Let S be any surface and let $Q \subset S$ be a finite subset with cardinality n. We shall describe an operation on mapping classes in $\mathrm{PMod}(S, Q)$ called *insertion*. Informally, this operation takes as input a pure mapping class α, chooses a representative homeomorphism $\tilde{\alpha}$ which is equal to the identity on a neighborhood D_Q of Q, and taking the conjugacy class of $\tilde{\alpha}$ rel. Q' where Q' is contained in D_Q and Q' is finite with $|Q'| = |Q|$.

Now let's make this more precise. Let $D_Q \subset S$ be a collection of n disjoint closed disks, each centered at some point of Q and let $\pi : D_Q \to Q$ be the retraction that takes each disk to its center. By abuse of notation, let $\mathrm{PMod}(S, D_Q)$ denote the mapping class group of S fixing D_Q *pointwise*. There is a surjective forgetful map $\mathrm{PMod}(S, D_Q) \to \mathrm{PMod}(S, Q)$ and, as is well-known, this is a $\mathbb{Z}^n$ central extension generated by Dehn twists around the boundary components of D_Q (see e.g. Farb–Margalit [8, Prop 3.19]; they call this the *capping homomorphism*).

Definition 3.3 (Insertion)**.** Let $\alpha \in \mathrm{PMod}(S, Q)$ be any element, and let $\hat{\alpha}$ be some preimage in $\mathrm{PMod}(S, D_Q)$. Let $f : Q \to Q$ be any map (possibly not injective) and $n = |f(Q)|$. The *insertion* $f^*\hat{\alpha} \subset \mathrm{PMod}(S, n)$ is the $(\mathrm{Mod}(S, n))$-conjugacy class of the element defined as follows. Let $\tilde{f} : Q \to D_Q$ be any injective map for which the composition $\pi\tilde{f} = f$. Then $f^*\hat{\alpha}$ is represented by the image of $\hat{\alpha}$ under the forgetful map to $\mathrm{PMod}(S, f(Q))$ which may be canonically identified (up to conjugacy in $\mathrm{Mod}(S, n)$) with $\mathrm{PMod}(S, n)$.

Note that the class of $f^*\hat{\alpha}$ is invariant under right composition of f with a permutation of Q, and it depends only on the cardinalities of the preimages (under f) of the elements of Q.

Definition 3.4 (Inert)**.** A subgroup $N \subset \mathrm{PMod}(S,n)$ normal in $\mathrm{Mod}(S,n)$ is said to be *inert* if, after fixing some identification of $\mathrm{PMod}(S,n)$ with $\mathrm{PMod}(S,Q)$, for every $\alpha \in N$ there is a lift $\hat{\alpha}$ in $\mathrm{PMod}(S,D_Q)$ so that for every $f : Q \to Q$ the insertion $f^*\hat{\alpha}$ is in N.

Given N we can define $\hat{N} \subset \mathrm{PMod}(S,D_Q)$ to be the collection of all $\hat{\alpha}$ lifting some $\alpha \in N$ for which every insertion $f^*\hat{\alpha}$ is in N. We call $\hat{N}$ the *corona* of N. By the definition of inert, the corona $\hat{N}$ surjects to N.

Lemma 3.5. *If* $N \subset \mathrm{PMod}(S,n)$ *is inert, the corona* $\hat{N}$ *is a subgroup of* $\mathrm{PMod}(S,D_Q)$.

Proof. For any f and any α, β with lifts $\hat{\alpha}, \hat{\beta}$ there are identities $f^*\hat{\alpha}^{-1} = (f^*\hat{\alpha})^{-1}$ and $f^*\hat{\alpha} f^*\hat{\beta} = f^*(\hat{\alpha}\hat{\beta})$. The proof follows. □

Lemma 3.6 allows one to construct many inert subgroups:

Lemma 3.6. *Let* $X \subset \mathrm{PMod}(S,Q)$ *be a* subset *closed under insertion. In other words, for every* $\alpha \in X$*, there is* $\hat{\alpha} \in \mathrm{PMod}(S,D_Q)$ *so that for any* $f : Q \to Q$ *the insertion* $f^*\hat{\alpha}$ *lies in* X*. Then the subgroup of* $\mathrm{PMod}(S,Q)$ *generated by* X *is inert.*

Proof. Note that for any f and any $\alpha \in \mathrm{PMod}(S,Q)$ there is an identity $f^*\hat{\alpha}^{-1} = (f^*\hat{\alpha})^{-1}$. So without loss of generality we may assume X is closed under taking inverses.

Now, let G be the subgroup of $\mathrm{PMod}(S,Q)$ generated by X, and let $x \in G$ be arbitrary, so that $x = x_1 \cdots x_n$ where every $x_i \in X$. For each i let $\hat{x}_i \in \mathrm{PMod}(S,D_Q)$ be the lift promised by the definition of insertion. If we define $\hat{x} := \hat{x}_1 \cdots \hat{x}_n$ then evidently $\hat{x}$ is a lift of x to $\mathrm{PMod}(S,D_Q)$, and

$$f^*\hat{x} = f^*\hat{x}_1 \cdots \hat{x}_n = (f^*\hat{x}_1) \cdots (f^*\hat{x}_n) \in G.$$

□

Example 3.7 (nth powers of Dehn twists)**.** For any n the subgroup of $\mathrm{PMod}(S,Q)$ generated by nth powers of Dehn twists along embedded loops is inert. For, the *set* of nth powers of Dehn twists along embedded loops is closed under insertion. Now apply Lemma 3.6.

Example 3.8 (Inert-Brunnian subgroup)**.** For any n define the *Inert-Brunnian subgroup* of $\mathrm{PMod}(S,Q)$ to be the group of pure mapping classes $\alpha \in \mathrm{PMod}(S,Q)$ that have lifts $\hat{\alpha} \in \mathrm{PMod}(S,D_Q)$ for which every nontrivial insertion (i.e. every insertion other than a permutation) is the identity. Note that every such mapping class is Brunnian in the usual sense.

Example 3.9 ($\mathrm{PSL}(2,\mathbb{Z})$)**.** Let $S = \mathbb{R}^2$. Then $\mathrm{Mod}(\mathbb{R}^2,3) = \mathrm{PSL}(2,\mathbb{Z})$ and $\mathrm{PMod}(\mathbb{R}^2,3)$ is a free rank 2 subgroup of index 6. Since $\mathrm{PMod}(\mathbb{R}^2,2)$ is trivial it follows that *every* normal subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ contained in $\mathrm{PMod}(\mathbb{R}^2,3)$ is inert.

The main theorem of this section is the *Inertia Theorem*:

Theorem 3.10 (Inertia Theorem)**.** *Let* S *be any connected, orientable surface, and let* K *be a Cantor set in* S*. A subgroup* N_n *of* $\mathrm{PMod}(S,n)$ *is equal to the image of some* $\mathrm{Mod}(S,K)$*-normal pure subgroup* $N \subset \mathrm{PMod}(S,K)$ *under the forgetful map if and only if it is inert.*

Remark 3.11. Insertion bears a family resemblance to Philip Boyland's *forcing* order on braids [5]. It is possible to restate the definitions of inertia and algebraic inertia in the language of operads and FI-modules; see e.g. [7]. One wonders whether such a reformulation could lead to sharper insights into the structure of such groups.

3.2. **Proof of the Inertia Theorem.** The goal of this section is to prove the Inertia Theorem 3.10. One direction is easy: given an inert group $N_n \subset \mathrm{PMod}(S, n)$ we construct a pure normal N with N_n in the spectrum.

Lemma 3.12. *For every inert subgroup $H \subset \mathrm{PMod}(S, n)$ there is a pure normal subgroup N of $\mathrm{Mod}(S, K)$ whose restriction to every n-element subset of K is conjugate to H.*

Proof. Let D_n be a family of n disjoint disks in S and let $\hat{H} \subset \mathrm{PMod}(S, D_n)$ be the corona of H. For every family D_Q of n disjoint disks in S whose interior contains K and each of whose components intersects K, we can fix a homeomorphism of S taking D_n to D_Q and therefore an isomorphism $\mathrm{PMod}(S, D_n) \to \mathrm{PMod}(S, D_Q)$ taking $\hat{H}$ to $\hat{H}_Q$. Under the forgetful map $\mathrm{PMod}(S, D_Q) \to \mathrm{PMod}(S, K)$ the image of $\hat{H}_Q$ is realized by homeomorphisms fixed pointwise on D_Q. Let N be the subgroup of $\mathrm{PMod}(S, K)$ generated by all the images of $\hat{H}_Q$'s. Then by the definition of inert, the restriction of N to every n-element subset of K is conjugate to H. □

We shall have cause to refer to the group N constructed from H in Lemma 3.12. We call it the *flattening* of H, and denote it $N^\flat(H)$. We can use it to give a simple example of distinct normal subgroups in $\mathrm{PMod}(S, K)$ with the same spectrum.

Lemma 3.13. *Let N be equal to $N^\flat(H)$ for some H. Then for every $\alpha \in N^\flat(H)$ there is a neighborhood V of K in S so that α is represented by a homeomorphism fixed pointwise on V.*

Proof. This property holds by definition for the generators (by taking $V = D_Q$), and is preserved under finite products. □

Example 3.14. Let N_{Dehn} be the subgroup of $\mathrm{PMod}(S, K)$ generated by all Dehn twists. Then $N_{\mathrm{Dehn}} := N^\flat(\mathrm{PMod}(S, n))$ for every n and therefore its spectrum is equal to precisely the sequence $\{\mathrm{PMod}(S, n)\}$. On the other hand, the entire group $\mathrm{PMod}(S, K)$ has the same spectrum as N_{Dehn} but fails to have the property of flattened subgroups promised by Lemma 3.13. When S is closed, N_{Dehn} is the subgroup of compactly supported mapping classes and its closure under the compact-open topology is exactly $\mathrm{PMod}(S, K)$ by [15, Theorem 4], which also explains why N_{Dehn} and $\mathrm{PMod}(S, K)$ have the same spectrum in view of Proposition 3.22.

Here is another example. For any p let N_{Dehn^p} be the group generated by pth powers of all Dehn twists; evidently this group is a flattening of any of the inert subgroups from Example 3.7. On the other hand, let N'_p be the group generated by simultaneous $\pm p$th powers of Dehn twists in arbitrary (possibly infinite) families of disjoint curves. Then N'_p and N_{Dehn^p} have the same spectrum but N'_p fails to have the property promised by Lemma 3.13.

The harder direction in the proof of the Inertia Theorem is to show that for every pure normal N, every N_n in the spectrum is inert. In other words:

Lemma 3.15. *Let N be a pure normal subgroup of $\mathrm{Mod}(S, K)$. Then every N_n in the spectrum of N is inert.*

Lemma 3.15 will follow from Proposition 3.16 which might be of independent interest.

Proposition 3.16. *For any* $\mathrm{Mod}(S,K)$*-normal subgroup* N *of* $\mathrm{PMod}(S,K)$*, given any element* $\gamma \in N$*, any positive integer* M *and any* $Q := \{q_1, \cdots, q_n\}$ *an* n*-element subset of* K *for some* n*, there is another element* $\gamma' \in N$*, a neighborhood* $D_Q := \sqcup_{i=1}^n D_i$ *of* Q *consisting of disjoint dividing disks* D_i *and a finite subset* $X = \sqcup_{i=1}^n X_i$ *where each* $X_i \subset K \cap D_i$ *has size at least* M *with* $q_i \in X_i$*, such that*

1. *there exists* $\gamma_D \in \mathrm{PMod}(S, D_Q)$ *(which will play the role of* $\hat{\alpha}$ *in the definition of inert subgroups) whose image under the forgetful map* $\mathrm{PMod}(S, D_Q) \to \mathrm{PMod}(S, X)$ *is the same as the image of* γ' *under the forgetful map* $\mathrm{PMod}(S,K) \to \mathrm{PMod}(S,X)$*; and*
2. γ' *and* γ *have the same restriction to* $\mathrm{PMod}(S,Q)$*.*

We prove Lemma 3.15 assuming Proposition 3.16.

Proof of Lemma 3.15. For any $\alpha \in N_n$, we choose an n-element set $Q := \{q_1, \cdots, q_n\} \subset K$ and by abuse of notation we identify α with a conjugacy class α_Q in $\mathrm{PMod}(S,Q)$. Let $\gamma \in N$ restrict to α_Q in $\mathrm{PMod}(S,Q)$. Choose $M \geq n$. Let the following objects be as promised in Proposition 3.16: $\gamma' \in N$, $\gamma_D \in \mathrm{PMod}(S, D_Q)$, a neighborhood $D_Q = \sqcup_{i=1}^n D_i$ consisting of disjoint dividing disks and subsets $X_i \subset D_i \cap K$ with each $|X_i| \geq n$. If we think of $\mathrm{PMod}(S, D_Q)$ as a central extension of $\mathrm{PMod}(S,Q)$, then γ_D is a lift of α_Q. Furthermore for every map $f : Q \to Q$ we can realize f by an injective map $\tilde{f} : Q \to X$ projecting to f under the obvious projection $X \to Q$ that maps X_i to q_i, and then the restriction of γ' to $\tilde{f}(Q)$ is equal to the restriction of γ_D to $\tilde{f}(Q)$ which is (by definition) equal to the insertion $f^*(\gamma_D)$.

Since $\gamma' \in N$ it follows that the conjugacy class of the insertion $f^*(\gamma_D)$ is in N_n for every f. Thus N_n is inert, as desired. □

It remains to prove Proposition 3.16, for which we need Lemma 3.17.

Lemma 3.17. *Let* D *be a closed disk and* K *be a Cantor set in the interior of* D*. Fix a pure mapping class* $\gamma \in \mathrm{PMod}(D,K)$ *and* $x \in K$*. Then there is an infinite sequence of distinct points* $X = (x_0, x_1, \cdots)$ *in* K *with* $x_0 = x$ *and a sequence of nesting dividing disks* $D_1 \supset D_2 \supset \cdots$ *converging to some* $x_\infty \in K$ *such that*

1. $x_j \in (D_j \setminus D_{j+1}) \cap K$ *for all* $j \geq 0$*, where* $D_0 = D$*, and*
2. *the image of* γ *in* $\mathrm{PMod}(D,X)$ *is represented by a homeomorphism* $h = \prod_{j\geq 0} t_j^{m_j}$*, where* $m_j \in \mathbb{Z}$ *and* t_j *is a (counter-clockwise) Dehn twist around* ∂D_j *supported in a small tubular neighborhood* T_j *of* ∂D_j *in* D_j *away from* x_j *and* D_{j+1}*.*

Proof. Let $x_1 \neq x_0 \in K$ be an arbitrary point. Then $\mathrm{PMod}(D, \{x_0, x_1\}) \cong \mathbb{Z}$ is generated by a (counter-clockwise) Dehn twist, which we represent by a homeomorphism t_0 supported in a tubular neighborhood T_0 of ∂D_0 in $D_0 = D$ so that $K \cap T_0 = \emptyset$. This determines an integer m_0 such that $t_0^{m_0}$ represents the image of γ in $\mathrm{PMod}(D, \{x_0, x_1\})$.

Represent γ by a homeomorphism ϕ in D. Choose any dividing disk D_1 with $x_1 \in D_1$ and $x_0 \notin D_1$, clearly $\phi(D_1)$ is isotopic to D_1 in $(D, \{x_1\} \sqcup (K \setminus D_1))$ as they both shrink to x_1. By continuity, for any Cantor set $K_1 \subset K \cap D_1$ sufficiently close to x_1 so that ∂D_1 stays disjoint from it under the isotopy, we have $\phi(D_1)$ isotopic to D_1 in

$(D, K_1 \sqcup (K \backslash D_1))$. In particular, the image γ_1 of γ in $\mathrm{PMod}(D, K_1 \sqcup \{x_0\})$ preserves the disk D_1 (up to isotopy). Thus we can represent γ_1 as a homeomorphism on D which is the Dehn twist $t_0^{m_0}$ together with a homeomorphism ϕ_1 supported in D_1 fixing its boundary and K_1 pointwise.

Now choose an arbitrary $x_2 \neq x_1$ in K_1 and represent the (counter-clockwise) Dehn twist generating $\mathrm{PMod}(D_1, \{x_1, x_2\}) \cong \mathbb{Z}$ by a homeomorphism t_1 supported in a tubular neighborhood T_1 of ∂D_1 in D_1 so that $K_1 \cap T_1 = \emptyset$. Then ϕ_1 as a mapping class in $\mathrm{PMod}(D_1, \{x_1, x_2\})$ agrees with $t_1^{m_1}$ for some $m_1 \in \mathbb{Z}$.

Repeat the process above inductively. Since D_j and x_j are quite arbitrarily chosen (as long as x_j is sufficiently close to x_{j-1} so that it lies in K_j), we can make sure that the sequence $\{x_j\}$ converges to some $x_\infty \in K$ and $\cap_{j=0}^{\infty} D_j = \{x_\infty\}$. Then for $X = (x_0, x_1, \cdots)$, the image of γ in $\mathrm{PMod}(D, X)$ has the desired property. □

Proof of Proposition 3.16. Let $D_Q := \sqcup_{i=1}^n D_i$ be a neighborhood of Q consisting of disjoint dividing disks D_i with $q_i \in D_i$. As in the proof of Lemma 3.17, by continuity, there are Cantor sets $K_i' \subset K \cap D_i$ for all i such that the image $\bar{\gamma}$ of γ in $\mathrm{PMod}(S, K')$ preserves each D_i up to isotopy, where $K' := \sqcup K_i'$. Denote the image of N in $\mathrm{PMod}(S, K')$ as $\overline{N}$, which is normal in $\mathrm{Mod}(S, K')$.

We show below that there is an element $\bar{\gamma}' \in \overline{N}$ and finite sets $X_i \subset K_i'$ with $q_i \in X_i$ (and $|X_i| \geq M$) such that the image of $\bar{\gamma}'$ under the forgetful map $\mathrm{PMod}(S, K') \to \mathrm{PMod}(S, \sqcup X_i)$ has the desired property. The original assertion follows from this by taking an arbitrary $\gamma' \in N$ which maps to $\bar{\gamma}'$ under the forgetful map $\mathrm{PMod}(S, K) \to \mathrm{PMod}(S, K')$.

Now it remains to construct some element $\bar{\gamma}' \in \overline{N}$ and finite sets X_i's such that

(1) $\bar{\gamma}'|_Q = \bar{\gamma}|_Q (= \gamma|_Q) \in \mathrm{PMod}(S, Q)$,
(2) $\bar{\gamma}'$ preserves each disk D_i up to isotopy in $(S, \sqcup X_i)$, and
(3) $\bar{\gamma}'$ restricts to the identity in each $\mathrm{Mod}(\mathrm{int}(D_i), X_i)$.

We will construct $\bar{\gamma}'$ as a product of conjugates of $\bar{\gamma}$ by mapping classes in $\mathrm{Mod}(S, K')$, where each mapping class is represented by a product of homeomorphisms g_i, $i = 1, \cdots, n$ such that g_i is supported in D_i and preserves K_i'. In particular, bullet (2) will follow automatically from the nature of this construction.

By our choice of K', there is a homeomorphism ϕ on (S, K') representing $\bar{\gamma}$ such that ϕ genuinely preserves each D_i and fixes its boundary pointwise. Thus the restriction $\phi|_{D_i}$ represents a mapping class γ_i in $\mathrm{PMod}(D_i, K_i')$. Applying Lemma 3.17 to $\gamma_i \in \mathrm{PMod}(D_i, K_i')$ gives rise to an infinite sequence of points $Z(i) = (z_0(i), z_1(i), \cdots)$ in K_i and a nesting sequence of dividing disks $D_i = E_0(i) \supset E_1(i) \supset \cdots$, such that $\phi|_{D_i}$ as a mapping class in $\mathrm{PMod}(D_i, Z(i))$ is represented by an infinite product $\prod_{j \geq 0} t_j(i)^{m_j(i)}$, where $t_j(i)$ is a (clockwise) Dehn twist around $\partial E_j(i)$.

Let $X_i = \{z_0(i), \cdots, z_M(i)\}$. Then the image of $\phi|_{D_i}$ in $\mathrm{PMod}(D_i, X_i)$ is $\prod_{j=0}^{M-1} t_j(i)^{m_j(i)}$. This mapping class is encoded by the sequence of integers $(m_0(i), m_1(i), \cdots, m_M(i))$ that records the power of the Dehn twists around the boundaries of the nested dividing disks. If we could somehow arrange for $m_1(i) = m_2(i) = \cdots = m_{M-1}(i) = 0$ then the image of $\phi|_{D_i}$ in $\mathrm{Mod}(\mathrm{int}(D_i), X_i)$ would be the identity, as desired in bullet (3). Our goal is to modify the sequence $(m_0(i), m_1(i), \cdots,)$ to arrive at this case by passing to subsequences of $Z(i)$ and taking (products of) conjugates of $\phi|_{D_i}$ by mapping classes g_i in $\mathrm{Mod}(D_i, K_i')$. Since the modifications can be done simultaneously for different i's without interfering with each other, we fix some i below and omit (in our notation) the dependence of $Z(i)$, $z_j(i)$, $m_j(i)$, $t_j(i)$ and $E_j(i)$ on i for simplicity.

The most important operation is to replace the sequence of points Z by an infinite subsequence $Z' \subset Z$. This has a predictable effect on the associated integer sequence that we now describe. If we take an infinite subsequence $Z' = (z_{j_0}, z_{j_1}, z_{j_2} \cdots)$ of Z, then the image of $\phi|_{D_i}$ in $\mathrm{PMod}(D_i, Z')$ is also isotopic to an infinite product of Dehn twists $\prod_{k=0}^{\infty} t_{j_k}^{m'_k}$, where $m'_k = \sum_{j_{k-1}<j\leq j_k} m_j$ and $j_{-1} := -1$. We refer to such an operation on associated integer sequences $(m_0, m_1, m_2, \cdots) \mapsto (m'_0, m'_1, m'_2, \cdots)$ as an *amalgamation*. In this situation, there is a homeomorphism g_i supported on D_i preserving the Cantor set K'_i such that $g_i(z_{j_k}) = z_k$ and $g_i(E_{j_k}) = E_k$ for all $k \geq 1$, $g_i(z_0) = z_0$, and $g_i(E_{j_0})$ is isotopic to $E_0 = D_i$ in (D_i, Z) (i.e. $g_i(E_{j_0})$ contains all points in Z). Then the conjugate $g_i \phi g_i^{-1}$ represents a mapping class $\phi(Z')$ in $\mathrm{PMod}(D_i, Z)$ and is isotopic to $\prod_{k=0}^{\infty} t_k^{m'_k}$.

Consider the specific subsequences $Z'_0 = (z_1, z_2, z_3, \cdots)$ and $Z'_1 = (z_0, z_2, z_3, z_4, \cdots,)$ obtained by omitting z_0 and z_1 respectively. Then the procedure above gives two corresponding conjugates $g_{i,0} \phi g_{i,1}^{-1}$ and $g_{i,1} \phi g_{i,1}^{-1}$ of ϕ (preserving the family D_Q), whose images in $\mathrm{PMod}(D_i, Z)$ are mapping classes $\phi(Z'_0)$ and $\phi(Z'_1)$ whose associated sequences (that appear as the exponents of t_k's) are $(m_0 + m_1, m_2, m_3, \cdots)$ and $(m_0, m_1 + m_2, m_3, \cdots)$ respectively. Therefore, the homeomorphism $(g_{i,0} \phi g_{i,0}^{-1}) \cdot (g_{i,1} \phi g_{i,1}^{-1})^{-1}$ has the following properties:

(1) it is the identity in $\mathrm{PMod}(S, Q)$ since the restrictions of both $g_{i,0} \phi g_{i,0}^{-1}$ and $g_{i,1} \phi g_{i,1}^{-1}$ to Q are equal to $\bar{\gamma}|_Q$,
(2) it preserves all disks D_j in D_Q,
(3) it is isotopic to $t_0^{m_1} \cdot t_1^{-m_1}$ as a mapping class in $\mathrm{Mod}(D_i, Z(i))$, and is the identity in all D_j for $j \neq i$, and
(4) it represents an element in $\overline{N}$ as a mapping class in $\mathrm{PMod}(S, K') \leq \mathrm{Mod}(S, K')$ since $\overline{N}$ is $\mathrm{Mod}(S, K')$-normal.

Multiplying this homeomorphism with ϕ, we obtain a mapping class in $\overline{N}$ with the following properties:

(1) its image in $\mathrm{PMod}(S, Q)$ is $\bar{\gamma}|_Q$,
(2) it preserves all disks in D_Q, and
(3) it is isotopic to the infinite product $t_0^{m_0+m_1} \cdot \prod_{k\geq 2} t_k^{m_k}$ in $(D_i, Z(i))$ and is the same as ϕ in any $(D_j, Z(j))$ for $j \neq i$.

Thus the net effect of this entire procedure is to change m_1 to 0 and m_0 to $m_0 + m_1$; i.e. its effect is represented on sequences as $(m_0, m_1, m_2, \cdots) \mapsto (m_0 + m_1, 0, m_2, \cdots)$.

Replacing the role of the indices 0 and 1 above by some j and $j + 1$, we can similarly change m_{j+1} to 0 and m_j to $m_j + m_{j+1}$. By applying such operations inductively, we can modify the sequence until it is of the form

$$(\sum_{k=0}^{M-1} m_k, 0, 0, \cdots, 0, m_M, m_{M+1}, \cdots)$$

i.e. all integers are zeros at the j-th location for all $1 \leq j \leq M - 1$. As discussed above, the mapping class associated to this sequence restricts to the identity in each $\mathrm{Mod}(\mathrm{int}(D_i), X_i)$ where $X_i = \{z_0(i), \cdots, z_M(i)\}$.

Performing this modification simultaneously for all D_i's we obtain an element $\bar{\gamma}'$ in $\overline{N}$ with the desired properties. □

This completes the proof of Proposition 3.16 and therefore also Lemma 3.15. Together with Lemma 3.12 this completes the proof of the Inertia Theorem 3.10.

Corollary 3.18. *Every inert subgroup of* $\mathrm{PMod}(S,n)$ *is algebraically inert.*

Proof. Every inert subgroup is equal to N_n for some normal N. But then for any $m > n$, if $N_m \subset \mathrm{PMod}(S,m)$ is in the spectrum of N, then the image of N_m in $\mathrm{PMod}(S,n)$ is equal to N_n for every forgetful homomorphism from $\mathrm{PMod}(S,m) \to \mathrm{PMod}(S,n)$; thus N_n is algebraically inert. □

It seems unlikely that the converse is true—i.e. that every algebraically inert subgroup is inert—but we do not know a counterexample.

3.3. **Operations on normal subgroups.** We give two examples of operations on normal subgroups that preserve the spectrum.

3.3.1. *Inflation.* The self-similarity of a Cantor set gives rise to a natural operation on pure normal subgroups of $\mathrm{Mod}(S,K)$ that we call *inflation*.

Definition 3.19. Let $K \subset S$ be a Cantor set and let $K' \subset K$ be a proper Cantor subset. For N a pure normal subgroup of $\mathrm{Mod}(S,K)$, the *inflation* of N, denoted N^+, is the pure normal subgroup of $\mathrm{Mod}(S,K)$ obtained by choosing a homeomorphism of pairs $h : (S,K') \to (S,K)$ and an associated isomorphism $h_* : \mathrm{Mod}(S,K') \to \mathrm{Mod}(S,K)$ and then defining N^+ to be the image of N under the composition of the restriction $\mathrm{PMod}(S,K) \to \mathrm{PMod}(S,K')$ with h_*.

Because N is normal in $\mathrm{Mod}(S,K)$ it follows that the definition of N^+ does not depend on the choice of $K' \subset K$ or the choice of homeomorphism h.

One nice property of inflation is that it preserves spectrum:

Proposition 3.20. *For any pure normal N the groups N and N^+ have the same spectrum.*

Proof. For any n and any n-element subset $Q \subset K$ there is some homeomorphism of S taking K to K and Q into K'. The proposition follows. □

Example 3.21. Let N be the normal subgroup generated by p-th powers of Dehn twists in the boundary of a dividing disk. Then N^+ is the normal subgroup generated by p-th powers of Dehn twists in all embedded loops of $S \setminus K$ that are homotopically trivial in S. It differs from N by including in the generating set twists around those homotopically trivial loops that enclose all of K. For sufficiently large p these groups are distinct as can be seen e.g. by the method of Louis Funar [9].

In the examples of normal subgroups N we have encountered, N is a subgroup of its inflation N^+. We believe there could be examples where N^+ does not contain N but we do not know of any.

3.3.2. *Closure.* The group $\mathrm{Mod}(S, K)$ has a natural topology in which a sequence $\gamma_i \in \mathrm{Mod}(S, K)$ converges to the identity if there are representative homeomorphisms h_i of (S, K) that converge to the identity in the compact–open topology. This happens for example if each γ_i has a representative supported in some compact subsurface $R_i \subset S$ where $R_{i+1} \subset R_i$ and $\cap_i R_i$ is totally disconnected.

If N is a pure normal subgroup then so is its closure $\bar{N}$. Furthermore, if $x \subset K$ is any n-element set, then for any convergent sequence $\gamma_i \to \gamma$ in $\bar{N}$ the restrictions of γ_i to $\mathrm{PMod}(S, x)$ are eventually constant. Thus:

Proposition 3.22. *For any pure normal N the groups N and $\bar{N}$ have the same spectrum.*

It seems very plausible that there could be infinitely many (and maybe even uncountably many) normal subgroups with the same spectrum. One would like to develop new tools to construct and distinguish them.

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Department of Mathematics, University of Chicago, Chicago, Illinois, 60637
Email address: `dannyc@math.uchicago.edu`

Department of Mathematics, University of Texas at Austin, Austin, Texas, 78712; and Department of Mathematics, Purdue University, West Lafayette, Indiana, 47907
Email address: `lvzhou.chen@math.utexas.edu`