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SHIFT MODULES, STRONGLY STABLE IDEALS, AND THEIR DUALITIES

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Abstract. We enrich the setting of strongly stable ideals (SSI): We introduce shift modules, a module category encompassing SSIs. The recently introduced duality on SSIs is given an effective conceptual and computational setting. We study SSIs in infinite dimensional polynomial rings, where the duality is most natural. Finally a new type of resolution for SSIs is introduced. This is the projective resolution in the category of shift modules.

Contents

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1. Introduction

Strongly stable ideals (SSI) are somewhat hard to place in the landscape of mathematics, but let us venture a brief tour. In algebraic geometry the ideal of every projective variety (char. 0) degenerates to such an ideal, [21] or see [14, Sec.15.9]. Also called Borel-fixed ideals, they are a way to understand and classify components of the Hilbert scheme, [7, 8, 12, 19, 29, 41, 42, 45]. They are the most degenerate of homogeneous ideals in polynomial rings $k[x_1, \ldots, x_n]$: The closed orbits for the action of $GL(n)$ on the Hilbert scheme are precisely the orbits of the strongly stable ideals, see Appendix C. However they being so degenerate, there is hardly any geometry left in them.

In commutative algebra they have a distinguished resolution, the Eliahou-Kervaire resolution [15]. They occur in the study of Hilbert functions and Betti numbers of graded ideals [32], and in particular in the proofs of Macaulay's theorems [22]. Algorithms to generate Borel-fixed ideals with given invariants are given in [31] and [34].

In combinatorics one finds them in shifting theory [28], [10], minimal growth of Hilbert functions [24], and relations to posets [18, Section 6]. One might even consider them so skeletal and degenerate that they are more or less numerical objects. In any case they retain significant invariants of ideals that degenerate to them, for instance regularity [5].

They occur in a number of places but always on the fringe. Their natural position and effective use is however clear. Standard references [24, Ch.4] and [33, Ch.2] have early on specific chapters on them with basic and significant theory. [38] has much the same more distributed, see also [14, Ch.15]. The most comprehensive treatment may be [22] with many examples and relations to algebraic geometry.

In this article we enrich the setting of strongly stable ideals. The following new features are studied:

(1) Shift modules: extending strongly stable ideals to a category of modules,
(2) Dualities: recently discovered in [16] and [44],
(3) Ambient polynomial ring with infinitely many variables: natural setting for the dualities,
(4) Resolutions: new type of projective resolution with new homological invariants.

1.1. **Shift modules.** Over a polynomial ring $k[x_1, \ldots, x_n]$ where k is a field, we introduce a category of modules with shift operations. This class subsumes strongly stable ideals.

Recall that a monomial ideal I in $k[x_1, \ldots, x_n]$ is strongly stable if whenever a monomial $x_j u \in I$ and $i < j$, then $x_i u \in I$. We may write $x_i u = s_{i,j}(x_j u)$. So an ideal I is strongly stable when it is invariant under such shift operations.

This inspires defining the category of multigraded shift modules over polynomial rings. Such a module M comes with shift operations between graded pieces

$$s_{ij} : M_{\mathbf{d}+\mathbf{e}_j} \to M_{\mathbf{d}+\mathbf{e}_i}.$$

A typical example of a shift module comes from an inclusion of two strongly stable ideals $I \subseteq J$: The quotient J/I is a shift module. More generally quotients of maps between sums of such ideals are shift modules.

We define shift modules in three steps. First we define *finite* shift modules, Section 5. Such a module is graded by a finite set of degrees $\mathbf{d} \in \mathbb{N}^{m+1}$ with some fixed total degree. Then we define shift modules over finite dimensional polynomial rings $k[x_1, \ldots, x_m]$. We show that finitely generated such modules come from finite shift modules, Theorem 6.12. Lastly we define shift modules over the infinite dimensional polynomial ring.

Given a monomial m let $\langle m \rangle$ be the smallest strongly stable ideal containing m. For instance $\langle x_1x_2x_3 \rangle$ will be the ideal generated by $x_1^3, x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_2x_3$.

1.2. **Dualities.** Recently a duality on strongly stable ideals was discovered, [16] and [44]. We develop the conceptual framework for this duality, enabling effective arguments and concrete computations of duals. Moreover we extend this duality to shift modules. This is an analog of extending Alexander duality for squarefree monomial ideals to squarefree modules [47].

We get nice formulas such as the dual of $\langle y_m^n \rangle$ being $\langle x_n^m \rangle$, Corollary 3.12, and the following:

Proposition 3.11. *The dual of the strongly stable ideal with one generator* $\langle y_{a_1} y_{a_2} \cdots y_{a_n} \rangle$ *is the strongly stable ideal with generators* $\langle x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n} \rangle$.

Furthermore the duality takes sums of ideals to intersections of their duals, Corollary 3.16.

A complicating aspect with these dualities is that they do not take place in a fixed finite dimensional polynomial ring, as seen by the above. Fixing m the dual of $\langle y_m^n \rangle$ is $\langle x_n^m \rangle$, and the latter requires larger and larger polynomial rings as n gets larger. In order to have a full natural setting we must be in an infinite dimensional polynomial ring.

A basic tool and inspiration for our work here is that strongly stable ideals generated in degree $\leq n$ in $k[x_1, \ldots, x_m]$ are in one-to-one correspondence with poset ideals in $\mathrm{Hom}([m], [n+1])$ [18, Section 6]. To extend this, let $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. The poset of order preserving maps $f : \mathbb{N} \to \hat{\mathbb{N}}$, denoted $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ is a central object for us, and in [17, Section 5] we introduced a *topology* on this poset.

1.3. **Infinite dimensional polynomial rings.** Let $k[x_{\mathbb{N}}]$ be the infinite dimensional polynomial ring in the variables $x_1, x_2, \cdots$ indexed by natural numbers. A basic tool we use is the commutative diagram of bijections:

$$\begin{tikzcd}
\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}}) \arrow[dd, "D"'] \arrow[dr, "\Lambda"] & \\
& \mathrm{Mon}(x_{\mathbb{N}}). \\
\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}}) \arrow[ur, "\Gamma"] &
\end{tikzcd} \tag{1}$$

Here Hom_S are the bounded maps in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$, and Hom^L are the maps which eventually take value ∞. The maps are given by

$$\Gamma(f) = \prod_{f(i)<\infty} x_{f(i)}, \quad \Lambda(f) = \prod_{i\geq 1} x_i^{f(i)-f(i-1)},$$

and D is the duality. We show that strongly stable ideals in $k[x_{\mathbb{N}}]$ are in one-to-one correspondence with *open poset ideals* in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$. The duality for strongly stable ideals in $k[x_{\mathbb{N}}]$ comes about because $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ is a *self-dual poset*, Section 2.1. Not all strongly stable ideals in $k[x_{\mathbb{N}}]$ have duals. We identify precisely which ideals have. In particular all *finitely generated* strongly stable ideals have duals.

We remark that a recent trend in commutative algebra is to consider infinite dimensional polynomial rings [2, 26, 30, 37]. The increasing monoid $\mathrm{Hom}_{inj}(\mathbb{N}, \mathbb{N})$ of injective order-preserving maps $f : \mathbb{N} \to \mathbb{N}$ is in one-one correspondence with $\mathrm{Hom}(\mathbb{N}, \mathbb{N})$ by mapping f to $f - \mathrm{id}_{\mathbb{N}} + 1$. The increasing monoid has been used to study $k[x_{\mathbb{N}}]$ in [37], [30], see also [23]. The use differs however sharply from ours, as $\mathrm{Hom}_{inj}(\mathbb{N}, \mathbb{N})$ is used there to act on $k[x_{\mathbb{N}}]$, while we use $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ in a distinct and intrinsic way by diagram (1).

1.4. **Resolutions.** The most well-known class of explicit resolutions of ideals of polynomial rings is the Eliahou-Kervaire resolutions of stable ideals [15] (a somewhat more general class than strongly stable ideals), see also [39].

The shift module category enables us to define a completely new type of projective resolution of strongly stable ideals, with graded Betti numbers quite distinct from those in the Eliahou-Kervaire resolution.

The indecomposable projectives in the shift module category are precisely the strongly stable ideals with a single strongly stable generator. For instance $x_1x_2^3x_3$ in a polynomial ring S generates a projective $P = \langle x_1x_2^3x_3\rangle$. The ordinary S-module minimal free resolution of P is

$$S(-5)^9 \leftarrow S(-6)^{12} \leftarrow S(-7)^4.$$

On the other hand the resolution in the shift module category is simply P, since it is projective. The invariants of these two resolutions give quite distinct information. The Betti numbers of the shift module resolution of a strongly stable ideal reflect more the combinatorics of the strongly stable generators of the ideal. As an example class, the shift module resolution of universal lex segment ideals has Betti numbers like a Koszul resolution, Section 13.

We also generalize the Eliahou-Kervaire resolution for strongly stable ideals, to resolutions for a subclass of shift modules, the *rear torsion-free* modules, Section 14.

The organization of this article is as follows.

Part I Section 2 recalls basic facts on posets, order preserving maps, dualities, and the topology on $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$.

Part II Section 3 gives the correspondence between strongly stable ideals and open poset ideals in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$. We define the duality on strongly stable ideals, and give basic tools and examples for computing this. Section 4 shows that the duals of universal lex segment ideals are also universal lex segment.

Part III Section 5 defines finite shift modules. Section 6 defines shift modules over finite dimensional polynomial rings. We show that any finitely generated shift module is derived from a finite shift module by the process of *expansion*. In Section

7 we define shift modules over infinite dimensional polynomial rings. At the end, in Section 8 we give examples of shift modules.

Part IV Section 9 defines duals of finite shift modules, and Section 10 defines duals of shift modules over polynomial rings. Section 11 gives examples of duals.

Part V Section 12 gives examples of how the ordinary free resolution of a strongly stable ideal and the shift module resolution differ. In Section 13 we give a shift resolution for strongly stable ideals which is an analog of the Taylor complex. We give conditions ensuring it is minimal. In Section 14 we establish the Eliahou-Kervaire resolution for shift modules.

Appendices. Appendix A recalls incidence algebras, and Appendix B gives the equivalence of categories between shift modules and modules over the incidence algebras of certain posets. Appendix C states and proves folklore knowledge that the strongly stable ideals are the most degenerate ideals.

Part I. Isotone maps between natural numbers

2. Isotone maps between natural numbers

We recall basic notions for partially ordered sets (posets) and distributive lattices. In particular we consider isotone maps $f : \mathbb{N} \to \hat{\mathbb{N}}$, where $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. These form themselves a partially ordered set $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$, which is self-dual. Moreover there is a natural topology on this poset. We see in Part II that it is intimately related to strongly stable ideals.

2.1. **Posets, isotone maps, and distributive lattices.** Given a poset P. A poset ideal I of P is a subset closed under taking smaller elements, and a poset filter F is a subset closed under taking larger elements. The distributive lattice $\hat{P}$ associated to P is the lattice of all *cuts* (I, F) where I is a poset ideal and F the complement filter of I. It is ordered by $(I, F) \leq (J, G)$ if $I \subseteq J$ (or equivalently $F \supseteq G$). The top element $(P, \emptyset)$ in $\hat{P}$ is denoted ∞.

We are particularly interested in this when P is totally ordered:

$$\mathbb{N} = \{1 < 2 < \cdots\}, \quad [n] = \{1 < 2 < \cdots < n\}.$$

Then

$$\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}, \quad \widehat{[n]} = [n+1] = [n] \cup \{\infty\}.$$

For a poset P denote by P^{op} the opposite poset with order relation reversed, so $p^{\mathrm{op}} \leq q^{\mathrm{op}}$ in P^{op} if $p \geq q$ in P. For P, Q two posets, a map $f : P \to Q$ is isotone (order preserving) if $p_1 \leq p_2$ implies $f(p_1) \leq f(p_2)$. The set of such maps is denoted $\mathrm{Hom}(P, Q)$. It is itself a poset by $f \leq g$ if $f(p) \leq g(p)$ for all p. We note that $\hat{P}$ naturally identifies with $\mathrm{Hom}(P^{\mathrm{op}}, \{0 < 1\})$: The cut (I, F) corresponds to the morphism

$$p : P^{\mathrm{op}} \to \{0 < 1\}, \quad p^{-1}(0) = F^{\mathrm{op}}, p^{-1}(1) = I^{\mathrm{op}}.$$

For $p \in P$ let $\uparrow p$ be the filter $\{q \,|\, q \geq p\}$. There is a natural isotone map $P \to \hat{P}$ sending $p \mapsto ((\uparrow p)^c, \uparrow p)$. For the totally ordered sets above, this is the natural inclusion.

By [17, Section 2] there is a natural duality

$$\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \xrightarrow{D} \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$$

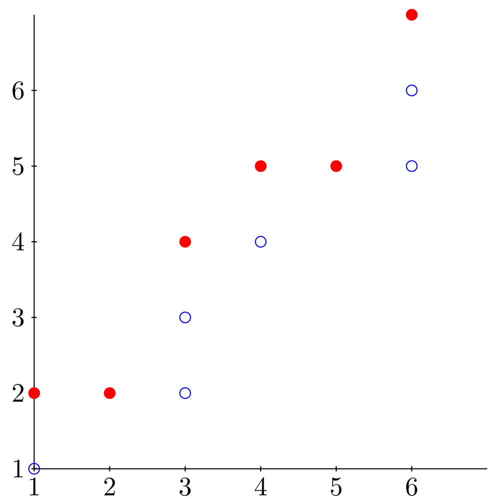

FIGURE 1. Graph (red discs) and dual graph (blue circles)

such that $f \leq g$ iff $Df \geq Dg$. It makes $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ into a self-dual poset, i.e. we have an isomorphism:

$$D : \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})^{\mathrm{op}}. \tag{2}$$

This duality is easy to explain by an example.

Example 2.1. In Figure 1 the red discs form horizontal segments making the graph of f. The values of f are

$$2, 2, 4, 5, 5, 7, \cdots .$$

The blue circles are filled in along vertical segments to make a "connected snake", and the graph of Df is obtained by considering the vertical axis as the argument for Df and its graph given by the blue circles. The values of Df are

$$1, 3, 3, 4, 6, 6, \cdots .$$

Remark 2.2. In general isotone maps $P \to \hat{Q}$ identify as *profunctors* $P \mathrel{\longrightarrow\!\!\!\!\!+} Q$, and are studied in [17]. The duality above is a special case of a duality

$$D : \mathrm{Prof}(P, Q) \xrightarrow{\cong} \mathrm{Prof}(Q, P)^{\mathrm{op}},$$

see [17, Section 2]. In the sequel several results from that article are used.

2.2. **Large and small maps.** This poset of isotone $\mathbb{N} \to \hat{\mathbb{N}}$ decomposes into a disjoint union:

$$\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) = \mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}}) \cup \mathrm{Hom}^u(\mathbb{N}, \hat{\mathbb{N}}) \cup \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}}),$$

where an isotone map $f : \mathbb{N} \to \hat{\mathbb{N}}$ is in

- $\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ if its values are bounded by a finite number in $\mathbb{N}$. These maps are called *small*.
- $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ if $f(n) = \infty$ for some $n \in \mathbb{N}$. These maps are called *large*.
- $\mathrm{Hom}^u(\mathbb{N}, \hat{\mathbb{N}})$ if the image $f(\mathbb{N})$ is in $\mathbb{N} = \hat{\mathbb{N}} \backslash \{\infty\}$ and is unbounded.

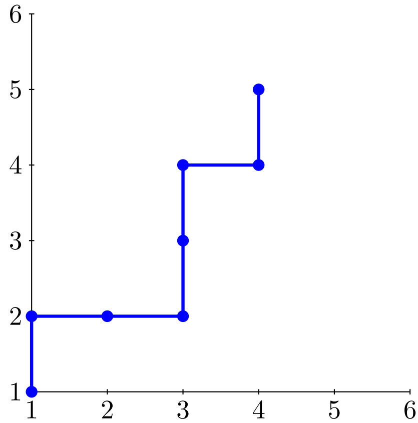

FIGURE 2. A NE-path from $(1,1)$ to $(4,5)$

The duality (2) swaps $\mathrm{Hom}_S(\mathbb{N},\hat{\mathbb{N}})$ and $\mathrm{Hom}^L(\mathbb{N},\hat{\mathbb{N}})$ and maps $\mathrm{Hom}^u(\mathbb{N},\hat{\mathbb{N}})$ to itself.

There is a natural inclusion $[n] \hookrightarrow \mathbb{N}$. It gives $[n]^{\mathrm{op}} \hookrightarrow \mathbb{N}^{\mathrm{op}}$. Applying $\mathrm{Hom}(-,\{0 < 1\})$ we get $\hat{\mathbb{N}} \to \widehat{[n]}$ where all $l > n$ are sent to $\infty \in \widehat{[n]}$.

We get a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(\mathbb{N},\hat{\mathbb{N}}) \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & \mathrm{Hom}(\mathbb{N},\widehat{[n]}) \arrow[d, twoheadrightarrow] \\ \mathrm{Hom}([m],\hat{\mathbb{N}}) \arrow[r, twoheadrightarrow] & \mathrm{Hom}([m],\widehat{[n]}). \end{tikzcd} \tag{3}$$

It restricts to diagrams

$$\begin{tikzcd} \mathrm{Hom}_S(\mathbb{N},\hat{\mathbb{N}}) \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & \mathrm{Hom}(\mathbb{N},\widehat{[n]}) \arrow[d, twoheadrightarrow] \\ \mathrm{Hom}_S([m],\hat{\mathbb{N}}) \arrow[r, twoheadrightarrow] & \mathrm{Hom}([m],\widehat{[n]}) \end{tikzcd} \quad , \quad \begin{tikzcd} \mathrm{Hom}^L(\mathbb{N},\hat{\mathbb{N}}) \arrow[r, twoheadrightarrow] \arrow[d, twoheadrightarrow] & \mathrm{Hom}^L(\mathbb{N},\widehat{[n]}) \arrow[d, twoheadrightarrow] \\ \mathrm{Hom}([m],\hat{\mathbb{N}}) \arrow[r, twoheadrightarrow] & \mathrm{Hom}([m],\widehat{[n]}) \end{tikzcd} . \tag{4}$$

All maps in these diagrams respect duality appropriately, as is readily seen by Example 2.1.

2.2.1. *Large and small maps identify with finite paths.* A finite *North-East path* is a path starting from $(1,1)$ going steps of length one in the north and east directions, see Figure 2. If it ends at (m,n) we get a function $f : [m] \to \mathbb{N}$ with $f(m) = n$, by letting $f(i)$ be the largest value j such that (i,j) is on the path. So we have a one-one correspondence

$$\text{NE-path from } (1,1) \text{ to } (m,n) \xleftrightarrow{1-1} \text{isotone maps } f : [m] \to \mathbb{N} \text{ with } f(m) = n.$$

Let a partial map $f : \mathbb{N} \dashrightarrow \mathbb{N}$ be an isotone map $[m] \to \mathbb{N}$, defined for some initial interval $[m]$ where m is finite. Denote by $\mathrm{Hom}^P(\mathbb{N},\mathbb{N})$ the set of partial maps. Note that we have a fibration

$$\mathrm{Hom}^P(\mathbb{N},\mathbb{N}) \to \mathbb{N}^2$$

by sending f to (m,n) if $f(m)=n$. The fibers identify as the NE-paths from $(1,1)$ to (m,n). The cardinality of this fiber is $\binom{m+n-2}{n-1}$.

Given a partial map f define two new maps on $\mathbb{N}$ by

$$f_S(i)=\begin{cases} f(i), & i<m \\ n+1, & i\geq m \end{cases}, \quad f^L(i)=\begin{cases} f(i), & i\leq m, \\ \infty, & i>m. \end{cases} \tag{5}$$

There are then one-to-one correspondences

$$\mathrm{Hom}_S(\mathbb{N},\hat{\mathbb{N}}) \xleftrightarrow{1-1} \mathrm{Hom}^P(\mathbb{N},\mathbb{N}) \xleftrightarrow{1-1} \mathrm{Hom}^L(\mathbb{N},\hat{\mathbb{N}}).$$

2.3. **The topology on Hom**$(\mathbb{N},\hat{\mathbb{N}})$**.** The isotone maps $\mathrm{Hom}(\mathbb{N},\hat{\mathbb{N}})$ may be given a topology. Let $\underline{f},\overline{f}:\mathbb{N}\to\hat{\mathbb{N}}$ be respectively a small and large isotone map and

$$U(\underline{f},\overline{f})=\{f:\mathbb{N}\to\hat{\mathbb{N}}\,|\,\underline{f}\leq f\leq\overline{f}\}.$$

These sets $U(\underline{f},\overline{f})$ form a basis for a topology on $\mathrm{Hom}(\mathbb{N},\hat{\mathbb{N}})$. The map D is a homeomorphism of topological spaces. In the diagram (3), we give the other spaces the quotient topology. On $\mathrm{Hom}([m],\widehat{[n]})$ this becomes the discrete topology.

On a topological space X, and Y a subset of X, denote by Y^c its complement $X\backslash Y$, by $\overline{Y}$ its closure, and by Y° its interior (the union of all open subsets contained in Y). For any topological space X we have a distinguished subclass of open subsets, those that are the interiors of their closures. These are called *regular* open sets, and we denote this class as $\mathrm{reg}X$. There is an involution, see [17, Section 4]

$$\mathrm{reg}X \xrightarrow{i} \mathrm{reg}X, \quad U\mapsto(\overline{U})^c=(U^c)^\circ. \tag{6}$$

If $\mathcal{I}$ is a poset ideal in $\mathrm{Hom}(\mathbb{N},\hat{\mathbb{N}})$ both its closure $\overline{\mathcal{I}}$ and its interior $(\mathcal{I})^\circ$ are poset ideals, and similarly concerning poset filters, see [17, Section 4]. We have the following criterion for an open poset ideal to be regular.

Definition 2.3. Let $\mathcal{I}$ be an open poset ideal and $F:\mathbb{N}\to\mathbb{N}$ an isotone map taking finite values. This is a *bounding function* for $\mathcal{I}$ if any $f\in\mathcal{I}$ with $f(p)>F(p)$ is dominated by a $g\in\mathcal{I}$ (i.e. $g\geq f$) with $g(p)=\infty$.

Proposition 2.4. *An open poset ideal* $\mathcal{I}$ *in Hom*$(\mathbb{N},\hat{\mathbb{N}})$ *is regular iff it has a bounding function.*

Proof. By Proposition 6.6b in [17] we have the following criterion: Consider sequences $f_1\leq f_2\leq f_3\leq\cdots$ in $\mathcal{I}$ and let f be $\mathrm{colim} f_i$. Then $\mathcal{I}$ is regular iff f is in $\mathcal{I}$ whenever f is large.

So suppose $\mathcal{I}$ has a bounding function. Suppose f large and let m such that $f(m-1)<\infty$ and $f(m)=\infty$. Since $f_i(j)\leq f(j)$ for $j\in[m-1]$ and $f_i(j)$ eventually becomes $f(j)$, there is an N such that $f_i(j)=f(j)$ for $j\in[m-1]$ and $i\geq N$. We also will have $\lim f_i(m)=\infty$ so $f_i(m)>F(m)$ for i large. Then there is $g_i\in\mathcal{I}$ such that $g_i\geq f_i$ and $g_i(m)=\infty$. But then $g_i\geq f$ and so $f\in\mathcal{I}$.

Conversely assume $\mathcal{I}$ is regular. Suppose we have defined $F(i)$ for $i<m$ such that if $f\in\mathcal{I}$ and $f(p)>F(p)$ for some $p<m$, then there is $g\in\mathcal{I}$ with $g\geq f$ and $g(p)=\infty$. Let

$$T=\{f\in\mathcal{I}\,|\,f(m)\text{ finite, and there is no } g\in\mathcal{I},\, g\geq f \text{ with } g(m)=\infty\}.$$

If $T=\emptyset$, let $F(m)=F(m-1)$. If $T\neq\emptyset$ consider $f\in T$. If $f(p)>F(p)$ for some $p<m$, there would be $g\in\mathcal{I}$ with $g\geq f$ and $g(p)=\infty$. This contradicts $f\in T$, so $f(p)\leq F(p)$ for every $p<m$.

We show the values of $f(m)$ for $f \in T$ are bounded. Suppose they were not. Then there must be an isotone $\phi : [m-1] \to \mathbb{N}$ such that for any N there is an $f \in T$ with $f(m) \geq N$ and the restriction $f_{|[m-1]} = \phi$. Let then

$$f_m(p) = \begin{cases} \phi(p), & p < m \\ f(m), & p \geq m \end{cases} \quad (f_m \text{ depending on } N).$$

Then clearly $f_m \in T$ since if there is a $g \in \mathcal{I}$ dominating f_m with $g(m) = \infty$, such a g would contradict f being in T. So we get an increasing sequence of f_m's with limit ϕ^L (see (5)), which is in $\mathcal{I}$ since $\mathcal{I}$ is regular. But this contradicts f_m being in T.

The upshot is that the elements in T have bounded $f(m)$. Let $F(m)$ be the maximal of these. □

Part II. Strongly stable ideals and their duality

3. Strongly stable ideals and their duals

We recall the notion of strongly stable ideals in a polynomial ring over a field k. For the infinite dimensional polynomial ring $k[x_{\mathbb{N}}]$ these correspond precisely to open poset ideals $\mathcal{I}$ in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$. Using the topology on $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ and that it is a self-dual poset, we define the dual of a strongly stable ideal in $k[x_{\mathbb{N}}]$. We call the ideal dualizable iff its double dual is the ideal itself. The class of dualizable strongly stable ideals are those corresponding to regular open poset ideals $\mathcal{I}$. We provide results on how to compute the duals of strongly stable ideals.

3.1. **Correspondence between Hom$(\mathbb{N}, \hat{\mathbb{N}})$ and monomials.** For a set R denote by $k[x_R]$ the polynomial ring in the variables $\{x_r\}_{r \in R}$ and by $\mathrm{Mon}(x_R)$ the monomials in this ring. Let $\mathrm{Mon}_{\leq d}(x_R)$ be the monomials of degrees $\leq d$. There is a bijection [17, Section 8]

$$\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}}) \xrightarrow{\Lambda} \mathrm{Mon}(x_{\mathbb{N}}), \quad f \mapsto \prod_{i \geq 1} x_i^{f(i) - f(i-1)}.$$

Here by convention $f(0) = 1$. Similarly there is a bijection

$$\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}}) \xrightarrow{\Gamma} \mathrm{Mon}(x_{\mathbb{N}}), \quad f \mapsto \prod_{f(i) < \infty} x_{f(i)}.$$

We get a commutative diagram of bijections, by [17, Section 8]

$$\begin{tikzcd} \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}}) \arrow[dd, "D"] \arrow[dr, "\Lambda"] & \\ & \mathrm{Mon}(x_{\mathbb{N}}). \\ \mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}}) \arrow[ur, "\Gamma"'] & \end{tikzcd} \tag{7}$$

There are also commutative diagrams of bijections

(8)

$$
\begin{tikzcd}
\mathrm{Hom}(\mathbb{N}, \widehat{[n]}) \arrow[dr, "\Lambda"] \arrow[dd, "D"] & \\
& \mathrm{Mon}_{\leq n}(x_{\mathbb{N}}) \\
\mathrm{Hom}([n], \hat{\mathbb{N}}) \arrow[ur, "\Gamma"] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathrm{Hom}_S([m], \hat{\mathbb{N}}) \arrow[dr, "\Lambda"] \arrow[dd, "D"] & \\
& \mathrm{Mon}(x_{[m]}) \\
\mathrm{Hom}^L(\mathbb{N}, \widehat{[m]}) \arrow[ur, "\Gamma"] &
\end{tikzcd}
\quad
\begin{tikzcd}
\mathrm{Hom}([m], [\hat{n}]) \arrow[dr, "\Lambda"] \arrow[dd, "D"] & \\
& \mathrm{Mon}_{\leq n}(x_{[m]}). \\
\mathrm{Hom}([n], \widehat{[m]}) \arrow[ur, "\Gamma"] &
\end{tikzcd}
$$

Definition 3.1. If R is a totally ordered set, a monomial ideal I of the polynomial ring $k[x_R]$ is *strongly stable* (sst) if a monomial $x_j u \in I$ implies $x_i u \in I$ for $i < j$. The transitive closure of the relations (i) $x_i u \geq x_j u$ if $i \leq j$ and (ii) $x_i u \geq u$ for any i and monomial u gives a partial order $\geq_{st}$ on monomials, the *strongly stable order*. (See also [24, Lemma 4.2.5] where it is called the Borel order.)

If $\{u_i\}_{i \in I}$ is a set of monomials in $k[x_R]$, we write $\langle u_i \rangle_{i \in I}$ for the smallest strongly stable monomial ideal containing all the u_i. It consists of all monomials u for which $u \geq_{st} u_i$ for some i. We say it is the *strongly stable ideal generated* by the u_i's. We denote by $\mathrm{STS}(x_R)$ the strongly stable ideals in $k[x_R]$. (See [20] for more on the perspective of strongly stable generators.)

The following is immediately verified.

Lemma 3.2. *The order relation $\geq_{st}$ corresponds to the order relation on Hom$(\mathbb{N}, \hat{\mathbb{N}})$ in the following way:*

$$
\begin{aligned}
(9) \qquad f \leq g \text{ in } \mathit{Hom}^L(\mathbb{N}, \hat{\mathbb{N}}) &\Leftrightarrow \Gamma f \geq_{st} \Gamma g, \\
f \leq g \text{ in } \mathit{Hom}_S(\mathbb{N}, \hat{\mathbb{N}}) &\Leftrightarrow \Lambda f \leq_{st} \Lambda g.
\end{aligned}
$$

If $\mathcal{I}$ is an open poset ideal in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$, it is fully determined by its intersection

$$\mathcal{I}^L = \mathcal{I} \cap \mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}}).$$

Similarly if $\mathcal{F}$ is an open poset filter it is fully determined by

$$\mathcal{F}_S = \mathcal{F} \cap \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}}).$$

Proposition 3.3. *There is a one-to-one correspondence between open poset ideals $\mathcal{I}$ in Hom$(\mathbb{N}, \hat{\mathbb{N}})$ and strongly stable ideals I in $k[x_{\mathbb{N}}]$. For an open poset ideal $\mathcal{I}$ the associated strongly stable ideal is $I = \Gamma(\mathcal{I}^L)$.*

Proof. By Observation (9), the image by Γ in (7) of $\mathcal{I}^L$ is a strongly stable ideal I in $\mathrm{Mon}(x_{\mathbb{N}})$.

When $\mathcal{I}$ is an open poset ideal, it is fully determined by its elements in $\mathcal{I}^L$, and so distinct $\mathcal{I}$ give distinct strongly stable ideals. Conversely the elements of a strongly stable ideal give elements of $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ which generate an open poset ideal $\mathcal{I}$ in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$. $\square$

Remark 3.4. There are also one-to-one correspondences by Γ between open poset ideals in $\mathrm{Hom}(\mathbb{N}, [\hat{n}]), \mathrm{Hom}([m], \hat{\mathbb{N}})$ and $\mathrm{Hom}([m], [\hat{n}]]$ and strongly stable ideals in respectively $k[x_{[n]}], k[x_{\mathbb{N}}]_{\leq m}$ and $k[x_{[n]}]_{\leq m}$.

3.2. **Dualizable strongly stable ideals.** If $\mathcal{I}$ is a *regular* open poset ideal of $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$, let $\mathcal{F}$ be the poset filter $(\mathcal{I}^c)^\circ$, the interior of the complement of $\mathcal{I}$. Since i of (6) is an involution, it also turns $\mathcal{F}$ into $\mathcal{I}$. We say the pair $[\mathcal{I}, \mathcal{F}]$ is a *Dedekind cut* of $\mathrm{Hom}(\mathbb{N}, \widehat{\mathbb{N}})$. Since D is a homeomorphism, the dual $D\mathcal{F}$ is again a regular open poset ideal and similarly $D\mathcal{I}$ a regular open poset filter, and $[D\mathcal{F}, D\mathcal{I}]$ is the dual Dedekind cut.

Definition 3.5. A strongly stable ideal in $k[x_\mathbb{N}]$ is *dualizable strongly stable* if it corresponds by Γ to a *regular* open poset ideal $\mathcal{I}$ in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$. Its *dual strongly stable ideal* in $k[y_\mathbb{N}]$ is the ideal corresponding to the dual regular open poset ideal $D\mathcal{F}$ by the construction above.

By the commutative diagram (7) the dual of the ideal $\Gamma(\mathcal{I}^L)$ is $\Lambda(\mathcal{F}_S)$.

Example 3.6. The following two strongly stable ideals are not dualizable.

- $I_1 = \langle x_1, x_2, x_3, \ldots \rangle$, the maximal irrelevant ideal in $k[x_\mathbb{N}]$
- $I_2 = \langle x_1, x_2^2, x_3^3, x_4^4, \ldots \rangle$

The corresponding open poset ideals $\mathcal{I}$ are not regular since they do not have bounding functions, Proposition 2.4.

- $I_3 = \langle x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_2x_3x_4^2, \ldots \rangle$.

This ideal is dualizable since the corresponding poset ideal is regular open. The identity map $\mathrm{id}_\mathbb{N}$ is a bounding function. The ideal I_3 is the ideal I_1^2 of Example 3.9. The dual of I_3 is $\langle y_1, y_2^2, y_2y_3^2, y_2y_3y_4^2, \ldots \rangle$.

Remark 3.7. Definition 3.5 is for strongly stable ideals in $k[x_\mathbb{N}]$ but is easily adapted to the more restricted cases. For instance in $\mathrm{Hom}([m], \hat{[n]})$ (which has the discrete topology) let $(\mathcal{I}, \mathcal{F})$ a cut. By Γ the (open) poset ideal $\mathcal{I}$ corresponds to a strongly stable ideal in $k[x_{[n]}]_{\leq m}$. The Alexander dual poset ideal $\mathcal{J} = D\mathcal{F}$ in $\mathrm{Hom}([n], \hat{[m]})$ then gives the dual strongly stable ideal $J = \Gamma(\mathcal{J}) = \Lambda(\mathcal{F})$ in $k[x_{[m]}]_{\leq n}$. Similarly with other cases of Remark 3.4. All maps and correspondences in (4) and (8) interact well. For instance sst-ideals in $k[x_{[m]}]$ correspond to (open) poset ideals in $\mathrm{Hom}(\mathbb{N}, \widehat{[m]})$ (here the topology is the discrete topology). The dual (open) poset ideal in $\mathrm{Hom}([m], \widehat{\mathbb{N}})$ then via Γ corresponds to a strongly stable ideal in $k[x_\mathbb{N}]_{\leq m}$.

For a Dedekind cut $[\mathcal{I}, \mathcal{F}]$ for $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$, its *gap* is:

$$\mathcal{G} = \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \backslash (\mathcal{I} \cup \mathcal{F}).$$

The gap is a subset of $\mathrm{Hom}^u(\mathbb{N}, \hat{\mathbb{N}})$ by [17, Cor.4.9], so it consists of unbounded functions. The following is [17, Theorem 5.10].

Theorem 3.8. *Let $\mathcal{I}$ be an open poset ideal in Hom$(\mathbb{N}, \hat{\mathbb{N}})$. Then $\mathcal{I}$ is also closed iff the strongly stable ideal corresponding to $\mathcal{I}$ is finitely generated.*

So finitely generated strongly stable ideals correspond precisely to clopen $\mathcal{I}$, or alternatively to regular open pairs with empty gap. Such strongly stable ideals are therefore dualizable

Proof. By [17, Theorem 5.10] $\mathcal{I}$ is clopen iff it is finitely generated (and these generators can be chosen large). But this corresponds to the corresponding strongly stable ideal being finitely generated. □

Here are examples where the gap consists of a single element.

Example 3.9. Let $\mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ be the identity function. For $1 \leq a < p$ consider the strongly stable ideal I_a^p generated by

$$\{x_1 x_2 \cdots x_r x_{r+a}^p \,|\, r \geq 0\}. \tag{10}$$

Let J be any finitely generated strongly stable ideal, not containing the monomial $x_1 x_2 \cdots x_{r-1} x_r$ for any $r \geq 1$. Consider the strongly stable ideal $I_a^p + J$ and let $\mathcal{I}$ be the corresponding open poset ideal. It is regular by Proposition 2.4 and Lemma 3.14. The monomials of (10) are the Γh of the large isotones h_r defined by

$$h_r(i) = \begin{cases} i, & i \leq r \\ r+a, & r < i \leq r+p \\ \infty, & i > r+p \end{cases}.$$

These are then in $\mathcal{I}$. The function f_r below is $\leq h_r$

$$f_r(i) = \begin{cases} i, & i \leq r, \\ r+1, & i \geq r+1 \end{cases}$$

and so is also in $\mathcal{I}$. Then $\lim_r f_r = \mathrm{id}_{\mathbb{N}}$ and is in the closure of $\mathcal{I}$ by [17, Proposition 5.10]. But $\mathrm{id}_{\mathbb{N}}$ is not in $\mathcal{I}$, since if it were, for some r the map

$$g_r(i) = \begin{cases} i, & i \leq r, \\ \infty, & i > r \end{cases}$$

would be in $\mathcal{I}$ due to $\mathcal{I}$ being open. But Γg_r is not in J and not in the ideal I_a^p for $p > a$. One may argue that $\mathrm{id}_{\mathbb{N}}$ is the only function in the gap for $\mathcal{I}$.

Now consider when $p \leq a$. Note that when $r = 0$ in (10) the monomial x_a^p is in I_a^p. Since now $p \leq a$ this implies that $x_1 x_2 \cdots x_p$ is in the ideal I_a^p. All generators of I_a^p with $r \geq p$ are a consequence of this, so I_a^p in this case is finitely generated.

Problem 3.10. What subsets of $\mathrm{Hom}^u(\mathbb{N}, \hat{\mathbb{N}})$ can be gaps for Dedekind cuts $[\mathcal{I}, \mathcal{F}]$?

3.3. **Computing duals of strongly stable ideals.** By Theorem 3.8 any finitely generated ideal is dualizable. A package to compute their duals is given in [11]. We describe the duals of principal strongly stable ideals. When $A\colon a_1, a_2, \ldots, a_m$ is a finite non-decreasing sequence in $\mathbb{N}$ we get two strongly stable ideals

$$\langle y_A \rangle = \langle y_{a_1} y_{a_2} \cdots y_{a_m} \rangle, \quad \langle x^A \rangle = \langle x_1^{a_1}, x_2^{a_2}, \cdots, x_m^{a_m} \rangle.$$

Proposition 3.11. *The dual of the strongly stable ideal $\langle y_A \rangle$ with a single sst-generator is the strongly stable ideal $\langle x^A \rangle$.*

Proof. Define the isotone map f by

$$f(i) = \begin{cases} a_i, & i = 1, \ldots, m, \\ \infty, & i > m. \end{cases}$$

Let $\mathcal{I}$ be the poset ideal generated by f, i.e. consisting of all isotone maps g such that $g \leq f$. Then $\Gamma(\mathcal{I}^L)$ is the strongly stable ideal generated by $\Gamma f = y_{a_1} y_{a_2} \cdots y_{a_m}$. The complement filter $\mathcal{F}$ of $\mathcal{I}$ is then generated by the bounded functions $g_1, \ldots, g_m$ where

$$g_p(i) = \begin{cases} 1, & i < p, \\ a_p + 1, & i \geq p. \end{cases}$$

The dual ideal is then $J = \Lambda(\mathcal{F}_S)$, generated by the $\Lambda g_p = x_p^{a_p}$. ∎

Corollary 3.12. *The strongly stable ideals $\langle y_a^m \rangle$ and $\langle x_m^a \rangle$ are duals of each other.*

Proof. The monomial y_a^m corresponds to the sequence $a, a, \cdots, a$ of length m. The dual of $\langle y_a^m \rangle$ is then the ideal $\langle x_1^a, x_2^a, \cdots, x_m^a \rangle$ but this is $\langle x_m^a \rangle$. □

Remark 3.13. Strongly stable ideals with one generator are studied in [20]. In Section 3 they give the minimal primary decomposition. In Section 5 so called Catalan diagrams are introduced associated to principal Borel ideals, giving effective computation of Hilbert series, and in Section 7 they relate Betti numbers of the principal ideal $\langle x_1 x_2 \cdots x_n \rangle$ to pseudo-triangulations. In [15, Example 2] they note that Catalan numbers occur in computing total Betti numbers of $\langle x_1, x_2^2, \cdots, x_n^n \rangle$. In [25] p-Borel principal ideals are studied.

Corollary 3.16 shows that strongly stable duality behaves quite similar to Alexander duality for squarefree monomial ideals, see [24, Cor. 1.5.5] or [33, Def. 5.20].

Lemma 3.14. *Let I_1 and I_2 be dualizable strongly stable ideals, with duals J_1 and J_2. Then $I_1 + I_2$ is dualizable strongly stable with dual $J_1 \cap J_2$.*

In particular, when I_1 is dualizable and I_2 is finitely generated, then $I_1 + I_2$ is dualizable.

Proof. Let I_i correspond to $\mathcal{I}_i$, and J_i to $\mathcal{J}_i$. Then $\mathcal{I}_1 \cup \mathcal{I}_2$ is regular open by Proposition 2.4. In general the closure $\overline{\mathcal{I}_1 \cup \mathcal{I}_2} = \overline{\mathcal{I}_1} \cup \overline{\mathcal{I}_2}$. Thus the complement

$$\overline{\mathcal{I}_1 \cup \mathcal{I}_2}^c = \overline{\mathcal{I}_1}^c \cap \overline{\mathcal{I}_2}^c = \mathcal{J}_1 \cap \mathcal{J}_2.$$

In conclusion: $\mathcal{I}_1 \cup \mathcal{I}_2$ is regular open with dual regular open poset ideal $\mathcal{J}_1 \cap \mathcal{J}_2$. □

Remark 3.15. In a topological space $U_1 \cup U_2$ may not be regular open when U_1 and U_2 are so. For instance if $[\mathcal{I}, \mathcal{F}]$ is a regular open pair, then $\mathcal{I} \cup \mathcal{F}$ will not be regular open if the gap is non-empty.

Corollary 3.16. *Let the $A_1, \ldots, A_r$ each be finite weakly increasing sequences of natural numbers. The following strongly stable ideals are then duals*

$$\langle y_{A_1}, \cdots, y_{A_r} \rangle, \quad \langle x^{A_1} \rangle \cap \langle x^{A_2} \rangle \cap \cdots \cap \langle x^{A_r} \rangle.$$

In addition to the above, the below seems to be an effective tool to compute the dual of a strongly stable ideal. It is [17, Lemma 2.5].

Lemma 3.17. *Let $\mathcal{I}$ be a poset ideal in Hom$(\mathbb{N}, \hat{\mathbb{N}})$ and $\mathcal{J}$ its dual poset ideal. Then g is in $\mathcal{J}$ iff for each $f \in \mathcal{I}$ there is $p \in \mathbb{N}$ (depending on f) such that $g(f(p)) \leq p$.*

Note. We have no assumptions on these ideals being open or regular.

Proof. By Lemma 2.2 a and b of [17], the following holds for $p, q \in \mathbb{N}$ (and is easily checked by Figure 1):

$$q < Dg(p) \text{ iff } g(q) \leq p.$$

Now g is in $\mathcal{J}$ iff Dg is in $\mathcal{F}$, the complement of $\mathcal{I}$. This holds iff for every $f \in \mathcal{I}$ there is a p with $f(p) < Dg(p)$. By the above this is equivalent to $g(f(p)) \leq p$. □

4. Universal lex-segment ideals

If $I \subseteq k[x_1, \ldots, x_n]$ is a lex segment ideal the extended ideal $(I) \subseteq k[x_1, \ldots, x_n, x_{n+1}]$ in a polynomial ring with one more variable is usually not lex segment. However there is a class, universal lex segment ideals, which has this property. They were introduced in [4] for finite dimensional polynomial rings. We show these ideals correspond precisely to either an isotone map $f : \mathbb{N} \to \mathbb{N}$ (infinitely generated ideals) or partial isotones $f : \mathbb{N} \dashrightarrow \mathbb{N}$ (finitely generated ideals). These ideals also have duals, the universal lex segment ideals corresponding to the dual isotones Df.

On $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ we have the lexicographic order: $f \succeq_{\mathrm{lex}} g$ if $f = g$ or $f(r) > g(r)$ where

$$r = \min\{n \in \mathbb{N} \mid f(n) \neq g(n)\}.$$

This is a total order which refines the partial order on $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$.

Lemma 4.1. *$f \succeq_{lex} g$ iff $Df \preceq_{lex} Dg$.*

Proof. Let p be minimal with $f(p) > g(p)$ and let $m = g(p)$. Then $Df(m) = p$ and $Dg(m) > p$ as we readily see from Figure 1, and $Df(l) = Dg(l)$ for $l < m$. □

Let $f : \mathbb{N} \to \mathbb{N}$ be an unbounded function. Define a poset ideal and poset filter by (note the order relation is strict)

$$\begin{aligned} \mathcal{I}(f) =& \{g \in \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \mid g \prec_{\mathrm{lex}} f\}, \\ \mathcal{F}(f) =& \{g \in \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \mid g \succ_{\mathrm{lex}} f\}. \end{aligned}$$

Proposition 4.2. *$\mathcal{I}(f)$ and $\mathcal{F}(f)$ form a Dedekind cut with gap $\{f\}$.*

Proof. The ideal $\mathcal{I}(f)$ is generated by the large maps

$$f_r(i) = \begin{cases} f(i), & i < r \\ f(r) - 1, & i = r \\ \infty, & i > r \end{cases}, \quad \text{for } r \geq 1 \text{ and } f(r-1) < f(r), \tag{11}$$

and so is an open subset. By Proposition 2.4, $\mathcal{I}(f)$ is regular. By Lemma 4.1 the dual by D of $\mathcal{F}(f)$ is $\mathcal{I}(Df)$. Hence also $\mathcal{F}(f)$ is regular. □

Definition 4.3. Recall that $\mathcal{I}^L(f)$ are the large functions in $\mathcal{I}(f)$, and $\mathcal{F}_S(f)$ the small functions in $\mathcal{F}(f)$.

- $\tilde{\Gamma}(f)$ is the strongly stable ideal $\Gamma(\mathcal{I}^L(f))$.
- $\tilde{\Lambda}(f)$ is the strongly stable ideal $\Lambda(\mathcal{F}_S(f))$.

Example 4.4. In the case when $f = \mathrm{id}_{\mathbb{N}} : \mathbb{N} \to \mathbb{N}$ is the identity, $\tilde{\Gamma}(\mathrm{id}_{\mathbb{N}})$ is the strongly stable ideal I_1^2 (where $a = 1$ and $p = 2$) in Example 3.9.

Note also that

$$I_{p-1}^p \supseteq \tilde{\Gamma}(\mathrm{id}_{\mathbb{N}}) = I_1^2 \supseteq I_1^p.$$

Thus the $\mathcal{I}(f)$ are not extremal regular open poset ideals with gap $\{f\}$.

We get the following diagram

$$\begin{tikzcd} & \mathrm{Hom}^u(\mathbb{N},\hat{\mathbb{N}}) \arrow[dl, "\tilde{\Lambda}"'] \arrow[dr, "\tilde{\Gamma}"] \arrow[dd, "D"] & \\ \mathrm{STS}(x_{\mathbb{N}}) & & \mathrm{STS}(x_{\mathbb{N}}) \\ & \mathrm{Hom}^u(\mathbb{N},\hat{\mathbb{N}}). \arrow[ul, "\tilde{\Gamma}"] \arrow[ur, "\tilde{\Lambda}"'] & \end{tikzcd}$$

Proposition 4.5. *The left and right triangles above commute. Furthermore* $\tilde{\Gamma}(f)$ *and* $\tilde{\Lambda}(f)$ *are dual strongly stable ideals.*

Proof. By Lemma 4.1 the dual of $\mathcal{I}(f)$ is $\mathcal{F}(Df)$. Hence by the commutative diagram (7), $\Gamma(\mathcal{I}^L(f)) = \Lambda(\mathcal{F}_S(Df)$, showing that the triangles above commute.

For the strongly stable ideal $\tilde{\Gamma}(f) = \Gamma(\mathcal{I}^L(f))$, by Definition 3.5 the dual ideal is $\Lambda(\mathcal{F}_S(f)) = \tilde{\Lambda}(f)$. $\square$

Let us describe these ideals more in detail. They may be called (infinitely generated) universal lex segment ideals, due to the comment after Proposition 4.7.

Proposition 4.6. *Let the unbounded isotone* f *take values* $f(i) = a_i$, *so* $a_1, a_2, a_3, \cdots$, *is the sequence of values. Recall that we set* $a_0 = 1$.

(a) $\tilde{\Gamma}(f)$ *is the strongly stable ideal sst-generated by*

$$x_{a_1} \cdots x_{a_{r-1}} x_{a_r - 1}, \quad \textit{for } r \geq 1 \textit{ and } a_{r-1} < a_r.$$

(b) *Its dual* $\tilde{\Lambda}(f)$ *is the strongly stable ideal sst-generated by*

$$x_1^{a_1 - a_0} x_2^{a_2 - a_1} \cdots x_{r-1}^{a_{r-1} - a_{r-2}} x_r^{a_r - a_{r-1} + 1}, \quad r \geq 1.$$

Proof. Part (a) is due to the generators of $\mathcal{I}(f)$ being (11). Part (b) follows in a similar way by considering the generators of the filter $\mathcal{F}(f)$. $\square$

The above ideals are not finitely generated since f is unbounded. Now we consider universal lex-segment ideals which are finitely generated. Let $f : [m] \to \mathbb{N}$ be an isotone map. Recall the functions f_S and f^L from (5). Note that $f_S \succ_{\text{lex}} f^L$ and this is a covering relation for the lex order. Define a poset ideal and filter by

$$\mathcal{I}(f) = \{g \in \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \mid g \preceq_{\text{lex}} f^L\},$$
$$\mathcal{F}(f) = \{g \in \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}}) \mid g \succeq_{\text{lex}} f_S\}.$$

In the same way as above, these form a regular open pair. In fact a clopen pair. The associated ideals $\tilde{\Gamma}(f)$ and $\tilde{\Lambda}(f)$ are dual strongly stable ideals.

Proposition 4.7. *Let the partial isotone* f *take values* $a_1, a_2, a_3, \cdots, a_m$ *(recall that we set* $a_0 = 1$*).*

(a) $\tilde{\Gamma}(f)$ *is the strongly stable ideal generated by*

$$x_{a_1} \cdots x_{a_{r-1}} x_{a_r - 1}, \quad 1 \leq r \leq m \textit{ and } a_{r-1} < a_r$$

together with $x_{a_1} \cdots x_{a_m}$.

(b) $\tilde{\Lambda}(f)$ *is the ideal generated by*

$$x_1^{a_1-a_0}x_2^{a_2-a_1}\cdots x_{r-1}^{a_{r-1}-a_{r-2}}x_r^{a_r-a_{r-1}+1}, \quad 1 \leq r \leq m.$$

These ideals are dual finitely generated universal lex-segment ideals.

The description of universal lex segment ideals as above was given in Proposition 1.2 and Corollary 1.3 in [36], as well as other characterizations. The Hilbert functions of these ideals were in [35] characterized as *critical*, i.e. all ideals with this Hilbert functions have the same Betti numbers.

Part III. **Shift modules**

We define the category of shift modules. They are the module-theoretic generalization of strongly stable ideals, much in the same way as squarefree modules [47] are the generalization of squarefree ideals. As it turns out, in the natural setting for the duality, our base ring should be the infinite dimensional polynomial ring. In order to get there, we go through some steps.

5. Finite shift modules

Let $\Delta_{m+1}(n)$ be all sequences $\mathbf{d} = (d_1, \ldots, d_{m+1})$ of non-negative integers such that the sum $|\mathbf{d}| = \sum_{i=1}^{m+1} d_i = n$. We define shift modules graded by $\Delta_{m+1}(n)$.

5.1. **Finite combinatorial shift modules.** Denote the i'th basis vector for $\mathbb{N}_0^{m+1}$ by $\mathbf{e}_i$, so we may write $\mathbf{d}$ above as $\sum_{i=1}^{m+1} d_i\mathbf{e}_i$.

Definition 5.1. Let V be a finite dimensional vector space graded by $\Delta_{m+1}(n)$ so

$$V = \bigoplus_{\mathbf{d}\in\Delta_{m+1}(n)} V_{\mathbf{d}}.$$

This is a *combinatorial shift module* if for each $p = 1, \ldots, m$ and $\mathbf{d} = (d_1, \ldots, d_{m+1})$ in $\Delta_{m+1}(n)$ with $d_{p+1} > 0$ there are linear maps

$$s_p : V_{\mathbf{d}} \to V_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}}$$

such that if $1 \leq p < q \leq m$ and $\mathbf{d}$ has both $d_{p+1}, d_{q+1} > 0$, the maps s_p and s_q commute:

$$s_p \circ s_q = s_q \circ s_p : V_{\mathbf{d}} \to V_{\mathbf{d}+\mathbf{e}_p+\mathbf{e}_q-\mathbf{e}_{p+1}-\mathbf{e}_{q+1}}.$$

If $d_{p+1} = 0$ we define s_p to be zero.

Homomorphisms $f : V \to W$ between combinatorial shift modules on $\Delta_{m+1}(n)$ are then naturally defined by requiring f to commute with the shift maps. These modules form an abelian category.

For each pair $p < q$ in $[m + 1]$ we moreover define shift maps:

$$s_{p,q} : V_{\mathbf{d}} \longrightarrow V_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_q},$$

as the composition

$$s_{p,q} = s_p \circ s_{p+1} \circ \cdots \circ s_{q-1}.$$

Note then:

- For $p < r < q$ we have

$$s_{p,q} = s_{p,r} \circ s_{r,q}.$$

- When d_q and d_ℓ are both > 0, the two maps $s_{p,q}$ and $s_{k,\ell}$ commute. In fact they commute save possibly in the following cases:
  - $d_\ell = 0, d_q > 0$ and $p = \ell$,
  - $d_q = 0, d_\ell > 0$ and $q = k$.

5.2. **Relations to the incidence algebra and projectives.** Appendix B shows that for the poset $\mathrm{Hom}([m], \widehat{[n]})$, the category of finite dimensional modules over this incidence algebra is isomorphic to the category of combinatorial shift modules on $\Delta_{m+1}(n)$. In particular, by Corollary B.1 this module category has projective dimension $\min\{m, n\}$.

The projectives in the category of shift modules over $\Delta_{m+1}(n)$ are given as follows: Consider the partial order $\geq_{st}$ on $\Delta_{m+1}(n)$ given by $\mathbf{d} \geq_{st} \mathbf{e}$ if $\sum_{i=1}^{j} d_i \geq \sum_{i=1}^{j} e_i$ for $j = 1, \ldots, m+1$.

Lemma 5.2. $\mathbf{d} \geq_{st} \mathbf{e}$ *iff there is a sequence of shifts taking an element of degree* $\mathbf{e}$ *to an element of degree* $\mathbf{d}$.

Proof. The if direction is clear since $\mathbf{d} + \mathbf{e}_p - \mathbf{e}_{p+1} \geq_{st} \mathbf{d}$. For the only if direction, let p be minimal such that $d_j > e_j$. Then $\mathbf{d}' = \mathbf{d} - \mathbf{e}_p + \mathbf{e}_{p+1} \geq_{st} \mathbf{e}$, and it follows by induction. $\square$

Then for each $\mathbf{d}$ consider the module which is the one-dimensional vector space k in all degrees $\mathbf{e} \geq_{st} \mathbf{d}$ and zero in all other degrees, and with all shift maps the identity maps. By the equivalence with the incidence algebra, this is a projective $P(\mathbf{d})$ in the category of combinatorial shift modules over $\Delta_{m+1}(n)$, and all indecomposable projectives are of this form.

5.3. **Digression: Algebraic shift modules.** Given a combinatorial shift module V on $\Delta_{m+1}(n)$, its associated *algebraic shift module* is defined by the maps, for $p = 1, \ldots, m$:

$$a_p := d_{p+1} \cdot s_p : V_{\mathbf{d}} \to V_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}}, \tag{12}$$

where d_{p+1} is the $(p+1)$'th coordinate of $\mathbf{d}$. More generally we put $a_{p,q} = d_q \cdot s_{p,q}$.

We then have the commutator relation

$$[a_{p,r}, a_{r,q}] = a_{p,q}. \tag{13}$$

This makes the module V a module over the Lie algebra U_{m+1} of strictly upper triangular $(m+1) \times (m+1)$-matrices.

Note however that for any $v \in V_{\mathbf{d}}$ the images of the maps included in relation (13)

$$a_{p,r} \circ a_{r,q}(v), \quad a_{r,q} \circ a_{p,r}(v), \quad a_{p,q}(v)$$

form at most a one-dimensional space, and not two-dimensional (as one would have expected if one defined algebraic shift maps (12) as only fulfilling the commutator relation (13)). So the algebraic shift modules we get from combinatorial shift modules form a special subclass of the algebraic shift modules.

In this article we are concerned with combinatorial shifting and we explain why in Section 6.3.

6. Shift modules over the polynomial ring $k[x_1, x_2, \cdots, x_m]$

We define shift modules over finite dimensional polynomial rings $k[x_{[m]}]$. When such a module is finitely generated, we show it is induced by a finite shift module over $\Delta_{m+1}(n)$ for some n. Write $\mathbb{N}_0^m$ for all m-tuples $\mathbf{d} = (d_1, \ldots, d_m)$ such that d_i are natural numbers ≥ 0. For uniformity of statements we define $d_{m+1} = \infty$ and sometimes consider an element $\mathbf{d}$ as $\sum_{i=1}^{m+1} d_i \mathbf{e}_i$.

6.1. **Combinatorial shift modules.**

Definition 6.1. An $\mathbb{N}_0^m$-graded vector space M with finite dimensional graded parts $M_{\mathbf{d}}$ is a *combinatorial shift module* if there for $p = 1, \ldots, m$ are linear maps

$$s_p : M_{\mathbf{d}} \to M_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}},$$

whenever $d_{p+1} > 0$, such that s_p and s_q commute if d_{p+1} and d_{q+1} are both > 0. If $d_{p+1} = 0$ we define s_p to be zero.

For $1 \leq p < q \leq m+1$ we define

$$s_{p,q} = s_p \circ s_{p+1} \circ \cdots \circ s_{q-1}.$$

Again $s_{p,q}$ and $s_{k,\ell}$ commute if $d_q > 0$ and $d_\ell > 0$.

We make an $\mathbb{N}_0^m$-graded shift module into a module over the polynomial ring $S = k[x_1, \ldots, x_m]$ as follows. For an element u of $M_{\mathbf{d}}$ define

$$x_i \cdot u = s_{i,m+1}(u), \quad (\text{recall } d_{m+1} = \infty).$$

Note that for $i < j$:

$$x_i \cdot u = s_{i,m+1}(u) = s_{i,j} \circ s_{j,m+1}(u) = s_{i,j}(x_j \cdot u).$$

In particular the polynomial ring S itself becomes a shift module, by defining

$$s_p(\cdots x_p^{d_p} x_{p+1}^{d_{p+1}} \cdots) = \cdots x_p^{d_p+1} x_{p+1}^{d_{p+1}-1} \cdots .$$

Note that the power d_{p+1} of x_{p+1} does not contribute to the coefficient of the monomial on the right side, see Section 6.3.

The maps s_p are almost S-module maps, but not quite.

Lemma 6.2.

(a) *Let $u \in M_{\mathbf{d}}$ have degree $\mathbf{d}$. We have $s_p(x_i u) = x_i s_p(u)$ except when $i = p+1$ and $d_{p+1} = 0$. In this latter case we have $s_{i-1}(x_i u) = x_{i-1} u$ while $x_i s_{i-1}(u) = 0$.*

(b) *Generally we have*

$$s_p(x^{\mathbf{a}} u) = \begin{cases} 0, & \text{if } d_{p+1} = 0 \text{ and } a_{p+1} = 0, \\ s_p(x^{\mathbf{a}})u, & \text{if } a_{p+1} > 0, \\ x^{\mathbf{a}} s_p(u), & \text{if } d_{p+1} > 0. \end{cases} \tag{14}$$

Note that when s_p acts non-trivially on both $x^{\mathbf{a}}$ and u (i.e. both $a_{p+1} > 0$ and $d_{p+1} > 0$), we can chose which of these to act on:

$$s_p(x^{\mathbf{a}} u) = s_p(x^{\mathbf{a}})u = x^{\mathbf{a}} s_p(u).$$

Proof. (a)

$$\begin{aligned} s_p(x_i u) &= s_p \circ s_{i,m+1}(u) \\ &= s_{p,p+1} \circ s_{i,m+1}(u). \end{aligned}$$

If $d_{p+1} > 0$ this is

$$= s_{i,m+1} \circ s_{p,p+1}(u)$$
$$= x_i \cdot s_p(u).$$

If $d_{p+1} = 0$ and $p+1 \neq i$ then

$$s_p(x_i u) = 0 = x_i s_p(u).$$

If $p+1 = i$ and $d_i = 0$ we have

$$\begin{aligned} s_{i-1}(x_i u) &= s_{i-1,i} \circ s_{i,m+1}(u) \\ &= s_{i-1,m+1}(u) \\ &= x_{i-1} u \end{aligned}$$

and $x_i s_{i-1}(u) = 0$.

(b) The last property of (14) follows due to the S-module property in part (a) when $d_{p+1} > 0$. The middle property of (14) follows by using the S-module property in part (a) on $x^{\mathbf{a}} = x_{p+1}^{a_{p+1}} x^{\mathbf{a}'}$ as long as $a_{p+1} \geq 2$, and then in the last instance using that $s_p(x_{p+1} x^{\mathbf{a}'} u) = x_p x^{\mathbf{a}'} u$. □

The shift modules over $\mathbb{N}_0^m$ or equivalently $k[x_1, \ldots, x_m]$ form an abelian category. We denote this category as $\mathtt{shmod}\, k[x_{[m]}]$.

Example 6.3. Any strongly stable ideal I in $k[x_1, \ldots, x_m]$ is a shift module, our primary example of a shift module. Also any quotient ring S/I of a strongly stable ideal I is a shift module. More generally if $\{I_a\}$ is a finite family of strongly stable ideals and $\{J_b\}$ is another finite family, we get maps

$$\oplus_a I_a \to \oplus_b J_b,$$

where the component $I_a \to J_b$ is either zero or a scalar multiple of an inclusion map. The kernel and cokernel of this map are shift modules. We see later that any shift module is a cokernel of such a map.

A basic result on free shift modules is the following.

Lemma 6.4. *The free S-module $Su_{\mathbf{d}}$ with generator $u_{\mathbf{d}}$ of degree $\mathbf{d} \in \mathbb{N}_0^m$ can be a shift module iff $\mathbf{d} = d_1 \mathbf{e}_1$ for some $d_1 \geq 0$. Then $Su_{\mathbf{d}}$ is isomorphic to the strongly stable ideal $\langle x_1^d \rangle$.*

Proof. Suppose for some $1 \leq p \leq m-1$ that $d_{p+1} > 0$. If $Su_{\mathbf{d}}$ is a shift module, then note that $s_p(u_{\mathbf{d}}) = 0$ since $Su_{\mathbf{d}}$ is zero in degree $\mathbf{d} + \mathbf{e}_p - \mathbf{e}_{p+1}$. Thus

$$\begin{aligned} x_p \cdot u_{\mathbf{d}} = s_{p,m+1}(u_{\mathbf{d}}) &= s_{p,p+1} \circ s_{p+1,m+1}(u_{\mathbf{d}}) \\ &= s_{p+1,m+1} \circ s_{p,p+1}(u_{\mathbf{d}}) = s_{p+1,m+1}(0) = 0, \end{aligned}$$

which is wrong. Here we use that $s_{p,p+1}$ and $s_{p+1,m+1}$ commute since $d_{p+1} > 0$ (and $d_{m+1} = \infty > 0$).

Conversely, if $\mathbf{d} = d_1 \mathbf{e}_1$ the map $x^{\mathbf{a}} u_d \mapsto x_1^{d_1} x^{\mathbf{a}}$ defines an isomorphism of shift modules by Lemma 6.2(b). □

More generally we may prove as above:

Lemma 6.5. *$Su_{\mathbf{d}}/(x_1, \ldots, x_{p-1}) \cdot u_{\mathbf{d}}$ is a shift module if $\mathbf{d} = \sum_{i=1}^{p} d_i \mathbf{e}_i$, but not if $d_i > 0$ for some $p+1 \leq i \leq m$.*

This shift module is isomorphic to the cokernel of the inclusion

$$\langle x^{\mathbf{d}} \cdot x_{p-1} \rangle \hookrightarrow \langle x^{\mathbf{d}} \rangle.$$

6.2. **Projectives.** The following is an immediate consequence of the definition of shift modules.

Lemma 6.6. *If $s_q \circ s_p(u)$ is non-zero with $q > p$, then $s_p \circ s_q(u) = s_q \circ s_p(u)$.*

Corollary 6.7. *Given $q_1 \leq q_2 \leq \cdots \leq q_r$ and let $q_1', \ldots q_r'$ be some reordering of these. Then*

$$s_{q_1} \circ \cdots \circ s_{q_r}(u) = s_{q_1'} \circ \cdots \circ s_{q_r'}(u)$$

if the latter is non-zero.

Lemma 6.8. *If $s_{p_1} \circ \cdots \circ s_{p_r}(u)$ and $s_{q_1} \circ \cdots \circ s_{q_t}(u)$ are non-zero of the same degree, with the p's and the q's in weakly increasing order, then $r = t$ and each $p_i = q_i$.*

Proof. It is easy to see that we must have $p_1 = q_1$. Then we continue by induction. $\square$

For a monomial $x^{\mathbf{d}}$ recall that $\langle x^{\mathbf{d}} \rangle$ is the strongly stable ideal generated by $x^{\mathbf{d}}$.

Proposition 6.9. *The ideal $\langle x^{\mathbf{d}} \rangle$ is a projective in the category of shift modules over $k[x_{[m]}]$ and all indecomposable projectives are of this form.*

Proof. Let M be a shift module and $m \in M_{\mathbf{d}}$. By Corollary 6.7 and Lemma 6.8 there is a unique morphism of shift modules $\langle x^{\mathbf{d}} \rangle \to M$ sending $x^{\mathbf{d}} \mapsto m$, and such that if $x^{\mathbf{e}} = s_{p_1} \circ \cdots \circ s_{p_r}(x^{\mathbf{d}})$ then $x^{\mathbf{e}} \mapsto s_{p_1} \circ \cdots \circ s_{p_r}(m)$.

Let $M \to N$ be a surjection, and given $\langle x^{\mathbf{d}} \rangle \to N$ sending $x^{\mathbf{d}} \mapsto n \in N_{\mathbf{d}}$. Let $m \in M_{\mathbf{d}}$ be a lifting of n. We then get map $\langle x^{\mathbf{d}} \rangle \to M$ with $x^{\mathbf{d}} \mapsto m$ lifting the map $\langle x^{\mathbf{d}} \rangle \to N$.

If P is a projective shift module, let $\mathbf{d}$ be minimal for the order $\geq_{st}$ such that $P_{\mathbf{d}} \neq 0$. Consider the short exact sequence

$$0 \to \langle x^{\mathbf{d}} \rangle_{>_{st}} \to \langle x^{\mathbf{d}} \rangle \to k \cdot x^{\mathbf{d}} \to 0.$$

There is a map $P \to k \cdot x^{\mathbf{d}}$ which is a map of shift modules. It lifts to a map of shift modules $P \to \langle x^{\mathbf{d}} \rangle$, which must be a surjection, and hence the latter is a summand of P. $\square$

6.3. **Digression: Algebraic shifting.** For a combinatorial shift module M over $k[x_1, \ldots, x_m]$ its associated *algebraic shift maps*

$$a_p : M_{\mathbf{d}} \to M_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}}$$

are $a_p = d_{p+1} \cdot s_p$ for $p < m$ and $a_m = s_m$. More generally define $a_{p,q} = d_q \cdot s_{p,q}$ for $q \leq m$ and $a_{p,m+1} = s_{p,m+1}$. Again we have the commutator relation

$$[a_{p,r} \circ a_{r,q}] = a_{p,q}.$$

Then M becomes a graded module over the Lie algebra U_{m+1}. We also have the natural shift operations s_p and a_p on the polynomial ring S. By this the a_p act as derivations:

$$a_p(x^{\mathbf{b}} u) = a_p(x^{\mathbf{b}})u + x^{\mathbf{b}} a_p(u).$$

In this article we are only concerned with combinatorial shifting. The reason is that we want to define dual shift modules in Part IV. A shift map $s_p : M_{\mathbf{d}} \to M_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}}$ dualizes to a shift map $t_{D_p} : N_{\mathbf{c}} \to N_{\mathbf{c}+\mathbf{e}_{D_p}-\mathbf{e}_{D_p+1}}$ where $D_p = 1 + \sum_{i=1}^p d_i$. If $d_{p+1} \geq 2$ then s_p and s_{p+1} dualize to t_{D_p} and $t_{D_{p+1}}$ where the difference $|D_{p+1} - D_p| \geq 2$. These latter shift maps then commute and so, in a setting where we have duals, s_p and s_{p+1} should commute when $d_{p+1} \geq 2$. When $d_{p+1} = 1$ we have the possibility of divergence between combinatorial and algebraic shift modules, in the presence of duals. In this case it is consistent for algebraic shift modules that $s_p \circ s_{p+1}$ and $s_{p+1} \circ s_p$ are different maps, as well as their duals $t_{D_p} \circ t_{D_{p+1}}$ and $t_{D_{p+1}} \circ t_{D_p}$ being different maps. We stick to combinatorial shift modules where we require also these two to commute. The category of shift modules is then equivalent to the category of modules over an incidence algebra, Appendix B. Since we only consider combinatorial shift modules, we henceforth call them simply shift modules.

6.4. **Expanded shift modules.** Given a shift module V over $\Delta_{m+1}(n)$ we may extend it to a shift module M over $k[x_1, \ldots, x_m]$. Let $\mathbf{d} = \sum_{i=1}^m d_i \mathbf{e}_i \in \mathbb{N}_0^m$. If the total degree $|\mathbf{d}| \leq n$, let $d_{m+1} = n - |\mathbf{d}|$ and $\hat{\mathbf{d}} = \mathbf{d} + d_{m+1}\mathbf{e}_{m+1}$ which is of degree n. Then let $M_{\mathbf{d}} = V_{\hat{\mathbf{d}}}$. If $|\mathbf{d}| \geq n$ then write $\mathbf{d} = \mathbf{d}^1 + \mathbf{d}^2$ where

$$\mathbf{d}^1 = (d_1^1, d_2^1, \ldots, d_r^1, 0, \cdots, 0) \quad \mathbf{d}^2 = (0, \ldots, 0, d_r^2, d_{r+1}^2, \ldots), \tag{15}$$

where the break r is such that

$$|\mathbf{d}^1| = n, \quad d_r = d_r^1 + d_r^2, \quad d_r^1 > 0.$$

We let $M_{\mathbf{d}} = V_{\mathbf{d}^1}$. Then M becomes a shift module over $\mathbb{N}_0^m$ as follows.

- If $|\mathbf{d}| < n$ let $s_p^M : M_{\mathbf{d}} \to M_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}}$ be $s_p : V_{\hat{\mathbf{d}}} \to V_{\hat{\mathbf{d}}+\mathbf{e}_p-\mathbf{e}_{p+1}}$.
- If $|\mathbf{d}| \geq n$ let:
  - If $p < r$ then

    $$s_p = s_p^M : M_{\mathbf{d}} \to M_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}} \text{ is } s_p^V : V_{\mathbf{d}^1} \to V_{\mathbf{d}^1+\mathbf{e}_p-\mathbf{e}_{p+1}}.$$
  - If $p \geq r$ then s_p^M is the identity map if $d_{p+1} > 0$, otherwise 0. (Recall when $p = m$ the convention that $d_{m+1} = \infty$.)

This gives an exact functor from the category of shift modules over $\Delta_{m+1}(n)$ to the category of shift modules over $k[x_{[m]}]$.

Definition 6.10. A shift module M over $k[x_1, \ldots, x_m]$ is *expanded* if it is isomorphic to a module induced from a shift module over $\Delta_{m+1}(n)$ for some n. More specifically it is called an n-expanded shift module.

We denote the category of n-expanded shift modules over $k[x_{[m]}]$ by $\mathtt{shmod}_{\leq n} k[x_{[m]}]$. So this is a category equivalent to the category of shift modules over $\Delta_{m+1}(n)$. Strongly stable ideals in $k[x_{[m]}]$ generated in degree $\leq n$ are typical examples of n-expanded shift modules.

Let A, B be finite subsets of $\mathbb{N}_0^m$ and let

$$\phi : \oplus_{\mathbf{b} \in B} \langle x^{\mathbf{b}} \rangle \to \oplus_{\mathbf{a} \in A} \langle x^{\mathbf{a}} \rangle$$

be a morphism of shift modules. Note that there is a non-zero morphism of shift modules $\langle x^{\mathbf{b}} \rangle \to \langle x^{\mathbf{a}} \rangle$ iff $\mathbf{b} \geq_{st} \mathbf{a}$ and in this case $x^{\mathbf{b}} \mapsto \alpha x^{\mathbf{b}}$ for some non-zero $\alpha \in k$.

Let

$$\hat{b} = \max\{|\mathbf{b}| \mid \mathbf{b} \in B\}, \quad \hat{a} = \max\{|\mathbf{a}| \mid \mathbf{a} \in A\}.$$

Proposition 6.11. *Given a morphism ϕ which is a minimal presentation of its cokernel. The regularity of the image $\operatorname{im}\phi$ is $\hat{b}$, and the regularity of $\operatorname{coker}\phi$ is $\max(\hat{a}, \hat{b} - 1)$.*

Proof. Let $n = \hat{b}$. Let $A' \subseteq A$ be those $\mathbf{a} \in A$ such that $|\mathbf{a}| \leq n$. The image of ϕ is contained in $\oplus_{\mathbf{a}\in A'}\langle x^{\mathbf{a}}\rangle$. So let ϕ' be the map ϕ restricted to this codomain. Both $\oplus_{\mathbf{a}\in A'}\langle x^{\mathbf{a}}\rangle$ and $\oplus_{\mathbf{b}\in B}\langle x^{\mathbf{b}}\rangle$ are n-expanded, and the map ϕ' comes from a map

$$\overline{\phi'} : \oplus_{\mathbf{b}\in B} P(\mathbf{b}) \to \oplus_{\mathbf{a}\in A'} P(\mathbf{a})$$

of shift modules over $\Delta_{m+1}(n)$. The kernel $\ker \overline{\phi'}$ has finite projective resolution with terms being finite sums of projective shift modules over $\Delta_{m+1}(n)$. Each projective $P(\mathbf{c})$ in this resolution must have $|\mathbf{c}| \leq n$. Since expanding modules is exact, $\operatorname{im}\phi'$ has a projective resolution $(F_\bullet, d_\bullet)$ of shift modules where all terms are projectives $\langle x^{\mathbf{c}}\rangle$ with $|\mathbf{c}| \leq n$. Such a projective has regularity $|\mathbf{c}|$. By taking successive mapping cones of this resolution, we get that each $\operatorname{im} d_i$ has regularity $\leq n$, and in the end $\operatorname{im}\phi' = \operatorname{im}\phi$ has regularity $\leq n$. But since $\operatorname{im}\phi'$ has a generator of degree $\hat{b} = n$, its regularity is n.

Consider the exact sequence

$$0 \to \operatorname{im}\phi \xrightarrow{i} \oplus_{\mathbf{a}\in A}\langle x^{\mathbf{a}}\rangle \to \operatorname{coker}\phi \to 0,$$

and taking the mapping cone of i, the regularity of $\operatorname{coker}\phi$ is $\max(\hat{a}, \hat{b} - 1)$. □

Theorem 6.12. *A shift module M over $k[x_{[m]}]$, finitely generated as an S-module, is expanded. The least n such that M is n-expanded is either the regularity $\operatorname{reg} M$ or $\operatorname{reg} M - 1$.*

Proof. Since M is finitely generated it has a minimal presentation

$$\oplus_{\mathbf{b}\in B}\langle x^{\mathbf{b}}\rangle \xrightarrow{\phi} \oplus_{\mathbf{a}\in A}\langle x^{\mathbf{a}}\rangle \to M$$

with A, B finite subsets of $\mathbb{N}_0^m$. Let $n = \max(\hat{a}, \hat{b})$. The proof of Proposition 6.11 shows that ϕ is induced from a map of shift modules over $\Delta_{m+1}(n)$:

$$\oplus_{\mathbf{b}\in B} P(\mathbf{b}) \xrightarrow{\overline{\phi}} \oplus_{\mathbf{a}\in A} P(\mathbf{a}).$$

Let $\overline{M}$ be the cokernel of $\overline{\phi}$. Expanding up to shift modules over $k[x_{[m]}]$ and using that expansion is exact, we get that M is expanded from $\overline{M}$. The minimal presentation of ϕ is unique up to isomorphism, and so we get that the minimal n is $\max(\hat{b}, \hat{a})$. If this n is $\hat{a}$, it is the regularity of M by Proposition 6.11. If $\hat{a} < \hat{b}$, the regularity of M is $\hat{b} - 1 = n - 1$. □

Example 6.13. Any n-expanded shift module is generated in degrees $\leq n$, but may be generated in much smaller degrees, for instance if I is a strongly stable ideal generated in degree n, then S/I is generated in degree 0. It is n-expanded, but not $(n-1)$-expanded.

7. Shift modules over the infinite dimensional polynomial ring $k[x_{\mathbb{N}}]$

We define shift modules over the infinite dimensional polynomial ring $k[x_{\mathbb{N}}]$. Let $\mathbb{N}_0^\infty = \oplus_{i\geq 1}\mathbb{N}_0$ consist of all infinite sequences $\mathbf{d} = (d_1, d_2, \ldots,)$ where all $d_i \geq 0$ and only a finite number of the d_i may be non-zero. We also put $d_\infty = \infty$.

Definition 7.1. An $\mathbb{N}_0^\infty$ graded module M over the infinite dimensional polynomial ring is a *combinatorial shift module* if for every natural number $p \geq 1$ there are linear maps

$$s_p : M_{\mathbf{d}} \to M_{\mathbf{d}} + \mathbf{e}_p - \mathbf{e}_{p+1},$$

with s_p the zero map if $d_{p+1} = 0$, such that:

- The maps s_p and s_q commute if d_q and d_q are both > 0.
- $s_p(x_{p+1}u) = x_p u$ and if u has degree $\mathbf{d}$ with $d_{p+1} > 0$ this also equals $x_{p+1}s_p(u)$.

For $1 \leq p < q$ we define

$$s_{p,q} = s_p \circ s_{p+1} \circ \cdots \circ s_{q-1},$$

and $s_{p,\infty}$ to be multiplication by x_p. Again $s_{p,q}$ and $s_{k,\ell}$ commute if $d_q > 0$ and $d_\ell > 0$.

We could alternatively in a more uniform way have defined an $\mathbb{N}_0^\infty$-graded shift modules as a graded vector space with maps for every $1 \leq p < q \leq \infty$

$$s_{p,q} : M_{\mathbf{d}} \to M_{\mathbf{d}} + \mathbf{e}_p - \mathbf{e}_q,$$

with $s_{p,q}$ the zero map if $d_q = 0$ such that:

- $s_{p,r} \circ s_{r,q} = s_{p,q}$,
- $s_{p,q}$ and $s_{k,\ell}$ commute when $d_q > 0$ and $d_\ell > 0$.

In this case $s_{p,\infty}$ would define multiplication with x_p.

In the same way as before we may also define the algebraic shift maps $a_{p,q}$ making M an algebraic shift module, a module over the infinite-dimensional Lie algebra U_∞.

7.1. **Extending shift modules.** A shift module M over $k[x_1, \ldots, x_m]$ may be extended to a shift module

$$\tilde{M} = M \otimes_{k[x_1,\ldots,x_m]} k[x_{\mathbb{N}}]$$

over $k[x_{\mathbb{N}}]$. The shift maps are given as:

$$\tilde{s}_{p,\infty}(u \otimes x^{\mathbf{d}}) = \begin{cases} x_p \cdot u \otimes x^{\mathbf{d}}, & p \leq m, \\ u \otimes x_p \cdot x^{\mathbf{d}}, & p > m, \end{cases}$$

and

- For $p < m$
$$\tilde{s}_p : M_{\mathbf{d}} \otimes x^{\mathbf{c}} \xrightarrow{s_p \otimes \mathbf{1}} M_{\mathbf{d}+\mathbf{e}_p-\mathbf{e}_{p+1}} \otimes x^{\mathbf{c}}.$$
- For $p > m$
$$\tilde{s}_p : M_{\mathbf{d}} \otimes x^{\mathbf{c}} \xrightarrow{\mathbf{1} \otimes s_p} M_{\mathbf{d}} \otimes s_p(x^{\mathbf{c}}).$$
- For $p = m$, if $x^{\mathbf{c}}$ contains x_{m+1}
$$\tilde{s}_m : M_{\mathbf{d}} \otimes x^{\mathbf{c}} \xrightarrow{(\cdot x_m) \otimes (: x_{m+1})} M_{\mathbf{d}+\mathbf{e}_m} \otimes x^{\mathbf{c}-\mathbf{e}_{m+1}},$$
and $\tilde{s}_m$ is zero if $x^{\mathbf{c}}$ does not contain x_{m+1}.

8. Examples of shift modules

In these examples S is a finite dimensional polynomial ring $k[x_1, \ldots, x_m]$ or infinite dimensional polynomial ring $k[x_{\mathbb{N}}]$.

8.1. **Strongly stable ideals.** Any strongly stable ideal in a finite or infinite dimensional polynomial ring is shift module.

Here are examples in the infinite dimensional polynomial ring $k[x_{\mathbb{N}}]$ which we discussed in Example 3.6 and will discuss in Section 11.

(1) The maximal irrelevant ideal in $k[x_{\mathbb{N}}]$, $I_1 = \langle x_1, x_2, x_3, \ldots \rangle$.
(2) $I_2 = \langle x_1, x_2^2, x_3^3, x_4^4, \ldots \rangle$.
(3) $I_3 = \langle x_1^2, x_1x_2^2, x_1x_2x_3^2, x_1x_2x_3x_4^2, \ldots \rangle$.

8.1.1. *Quotients of strongly stable ideals.* Any quotient module S/I of a strongly stable ideal is a shift module. More generally if $\{I_a\}$ is a finite family of strongly stable ideals and $\{J_b\}$ is another finite family, the quotient by a map

$$\oplus_a I_a \to \oplus_b J_b$$

is a shift module, where each component $I_a \to J_b$ is either zero or a scalar multiple of an inclusion map.

8.2. **Non-finitely generated shift modules I.** For $a \in \mathbb{N}_0$ let u_a be an element of degree $a \cdot \mathbf{e}_1$.

(1) The module $M = \oplus_{a \in \mathbb{N}_0} S u_a$ is a shift module.
(2) The module

$$M_1 = \oplus_{a \in \mathbb{N}_0} S u_a / (x_1 u_a)$$

is a shift module with the same multigraded dimensions as S, all are one-dimensional. For M_1 the shift map s_1 is zero.

For $a, b \in \mathbb{N}_0$ let $u_{a,b}$ be an element of degree $a \cdot \mathbf{e}_1 + b \cdot \mathbf{e}_2$.

(3) The module

$$M_2 = \oplus_{(a,b) \in \mathbb{N}_0^2} S u_{a,b} / (x_1, x_2) u_{a,b}$$

is again a shift module with the same multigraded dimensions as S.

8.3. **Non-finitely graded shift modules II.**

(1) Let the module N_1 be the graded vector space which is the direct sum of all the one-dimensional spaces generated by $y_{\mathbf{a}} = y_{a_1} y_{a_2} \cdots y_{a_r}$ for weakly increasing sequences $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$, with $r \geq 1$. As a graded vector space it identifies as $S_{\geq 1}$, the subspace of S spanned by monomials in S of degree ≥ 1. We make this a shift module over $k[y_{\mathbb{N}}]$ by

(i) $s_{a_1 - 1}(y_{\mathbf{a}}) = 0$
(ii) For $p \geq 2$ if $a_{p-1} < a_p$ then

$$s_{a_p - 1}(y_{\mathbf{a}}) = y_{a_1} \cdots y_{a_p - 1} y_{a_{p+1}} \cdots y_{a_r}.$$

(2) Let the module N_2 be the graded vector space which is the direct sum of all the one-dimensional spaces generated by $y_{\mathbf{a}} = y_{a_1} y_{a_2} \cdots y_{a_r}$ for weakly increasing sequences $1 \leq a_1 \leq a_2 \leq \cdots \leq a_r$, with $r \geq 2$. As a graded vector space it identifies as $S_{\geq 2}$, the subspace of S spanned by monomials in S of degree ≥ 2. We make this a shift module over $k[y_{\mathbb{N}}]$ by:

(i) $s_{a_1 - 1}(y_{\mathbf{a}}) = 0$ and $s_{a_2 - 1}(y_{\mathbf{a}}) = 0$.
(ii) For $p \geq 3$ if $a_{p-1} < a_p$ then

$$s_{a_p - 1}(y_{\mathbf{a}}) = y_{a_1} \cdots y_{a_p - 1} y_{a_{p+1}} \cdots y_{a_r}.$$

Part IV. Duals of shift modules

For a shift module there is a dual shift module, in much the same way as for a squarefree module there is a dual squarefree module. However shift modules over polynomial rings are more subtle. This is due to there not being a simple stable correspondence between the degrees in the two polynomial rings where the module and its dual module live over. However for finite shift modules this problem is not there and we first do this case.

9. Duals of finite shift modules

9.1. **Degree correspondences.** Recall that $\Delta_{m+1}(n)$ is the set of all $(m+1)$-tuples $\mathbf{d} = (d_1, \ldots, d_{m+1})$ of integers ≥ 0 such that their sum is n. This set is in bijection with monomials in $k[x_{[m]}]$ of degree $\leq n$ by sending $\mathbf{d}$ to the monomial $\prod_{i=1}^m x_i^{d_i}$. Recall the rightmost commutative triangle in (8). Splicing this triangle with its mirror we get a commutative diagram of bijections.

(16)

$$\begin{tikzcd} & \mathrm{Hom}([m], \hat{[n]}) \arrow[dl, "\Gamma"'] \arrow[dd, "D"] \arrow[dr, "\Lambda"] & \\ \Delta_{n+1}(m) \cong \mathrm{Mon}_{\leq m}(y_{[n]}) & & \mathrm{Mon}_{\leq n}(x_{[m]}) \cong \Delta_{m+1}(n) \\ & \mathrm{Hom}([n], \hat{[m]}) \arrow[ul, "\Lambda"] \arrow[ur, "\Gamma"'] & \end{tikzcd}$$

The right Γ-map sends a map f to the monomial

$$\prod_{i=1}^n x_{f(i)}.$$

For a map g in $\mathrm{Hom}([m], \hat{[n]})$ define $g(0) = 1$ and $g(m+1) = n+1$. Then the right Λ-map sends a map g to the monomial

$$\prod_{i=1}^m x_i^{g(i)-g(i-1)}$$

and further to the element in $\Delta_{m+1}(n)$

$$(g(1) - g(0), g(2) - g(1), \cdots, g(m+1) - g(m)).$$

The essential thing for our purpose it that the diagram gives a bijection between $\Delta_{m+1}(n)$ and $\Delta_{n+1}(m)$. The practical way to compute this bijection seems to be using the diagram above. Here is an example.

Example 9.1. Consider $(2,1,0,3,1)$ in $\Delta_{4+1}(7)$ corresponding to the monomial $x_1^2 x_2 x_4^3$ in $k[x_{[4]}]_{\leq 7}$. We want to compute the corresponding element in $\Delta_{7+1}(4)$. The most convenient way is perhaps to use the lower path in (16).

(1) Via Γ this monomial $x_1x_1x_2x_4x_4x_4$ corresponds to the function f in $\mathrm{Hom}([7], \hat{[4]})$ with values $1,1,2,4,4,4,5$. To compute the image by Λ we add 1 and 5 at the ends

$$1-1,1,2,4,4,4,5-5$$

and take the differences $0,0,1,2,0,0,1,0$ giving the element in $\Delta_{7+1}(4)$, corresponding to the monomial $y_3y_4^2y_7$ in $k[y_{[7]}]_{\leq 4}$.

We may also use the upper path in (16):

(2) From $(2,1,0,3,1)$ we take the partial sums $2,3,3,6$, and add 1 to each. Then it corresponds to the function g in $\mathrm{Hom}([4],\hat{[7]})$ with values $3,4,4,7$. By applying the map Γ this gives $y_3y_4^2y_7$, which in turn may be converted to the element $(0,0,1,2,0,0,1,0)$ in $\Delta_{7+1}(4)$.

Remark 9.2. The element 1 in $\mathrm{Mon}_{\leq n}(x_{[m]})$ corresponds to $(0,0,\cdots,0,n)$ in $\Delta_{m+1}(n)$. Via the procedure (1) above this corresponds to the function f in $\mathrm{Hom}([n],\hat{[m]})$ with values $m+1,m+1,\cdots,m+1$. Taking the differences of

$$1-(m+1),(m+1),\cdots,(m+1)-(m+1),$$

we get the sequence $(m,0,0,\cdots,0)$ giving the monomial y_1^m in $k[y_{[n]}]_{\leq m}$. In particular we see that as m increases the element 1 goes to different elements.

9.2. **Duals of finite modules.** For a vector space V denote its dual as $V^* = \mathrm{Hom}(V,k)$. Let $V = \oplus_{\mathbf{d}\in\Delta_{m+1}(n)}V_{\mathbf{d}}$ be a shift module over $\Delta_{m+1}(n)$. We want to define a dual shift module $W = \oplus_{\mathbf{c}\in\Delta_{n+1}(m)}W_{\mathbf{c}}$ over $\Delta_{n+1}(m)$. Take f in $\mathrm{Hom}([n],\hat{[m]})$ and set

$$W_{\Lambda f} = (V_{\Gamma f})^*.$$

Next we define the shift maps. For $p\in[n]$ let i_p be the map with domain $[n]$ such that $i_p(p)=1$ while all other values of i_p are zero. Suppose that $f(p)<f(p+1)$ (where we set $f(n+1)=m+1=\infty$). Note the shift map

$$s_p : V_{\Lambda f} \to V_{\Lambda(f+i_p)}.$$

Let $g = D(f+i_p)$ be the dual. Letting $k=f(p)$ we see by Figure 3 that the dual $Df = g+i_k$.

Now we have

$$W_{\Lambda(g+i_k)} = (V_{\Gamma(g+i_k)})^* = (V_{\Lambda D(g+i_k)})^* = (V_{\Lambda f})^*.$$

Similarly $W_{\Lambda g} = (V_{\Lambda(f+i_p)})^*$. The dual of the map s_p above is then by definition our shift map t_k for W

$$t_k : W_{\Lambda g} \to W_{\Lambda(g+i_k)}.$$

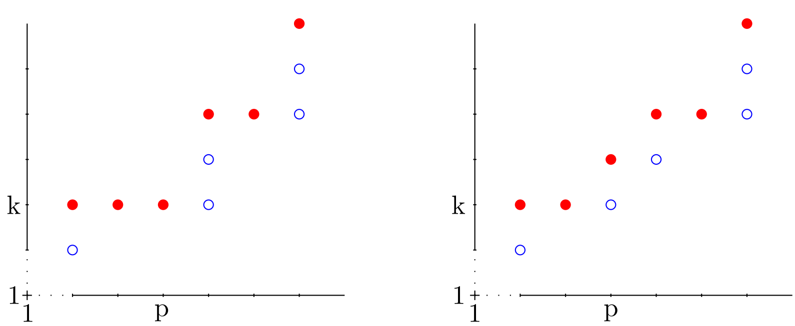

FIGURE 3. f with $\bullet$ and $g+i_k$ with $\circ$ $\quad f+i_p$ with $\bullet$ and g with $\circ$

Lemma 9.3. *When V is a shift module over $\Delta_{m+1}(n)$, the module $W = V^*$ is a shift module over $\Delta_{n+1}(m)$ with shift maps the t_k.*

Proof. Let $f : [n] \to [\hat{m}]$ be an isotone map. Let $1 \leq p < q \leq n$ with $f(p) < f(p+1)$ and $f(q) < f(q+1)$. Since V is a shift module we have a commutative diagram

$$
\begin{tikzcd}
V_{\Lambda f} \arrow[r, "s_p"] \arrow[d, "s_q"] & V_{\Lambda(f+i_p)} \arrow[d, "s_q"] \\
V_{\Lambda(f+i_q)} \arrow[r, "s_p"] & V_{\Lambda(f+i_q+i_p)}.
\end{tikzcd}
\tag{17}
$$

Now let $g = D(f + i_p + i_q)$ and $k = f(p)$ and $\ell = f(q)$. Then as we can infer from Figure 3:

$$g + i_\ell = D(f + i_p), \quad g + i_k = D(f + i_q), \quad g + i_k + i_\ell = Df.$$

Dualizing the diagram (17) we get a commutative diagram

$$
\begin{tikzcd}
W_{\Lambda(g+i_k+i_\ell)} & W_{\Lambda(g+i_\ell)} \arrow[l, "t_k"'] \\
W_{\Lambda(g+i_k)} \arrow[u, "t_\ell"] & W_{\Lambda g}. \arrow[l, "t_k"] \arrow[u, "t_\ell"']
\end{tikzcd}
$$

□

We observe that dualization is an exact functor on the category of shift modules over $\Delta_{m+1}(n)$.

10. Duals of shift modules over polynomial rings

We now want to define duals of shift modules over polynomial rings $k[x_{[m]}]$ and $k[x_\mathbb{N}]$. By the diagram (16) there is a one-one correspondence between monomials in $k[x_{[m]}]$ of degree $\leq n$ and monomials in $k[y_{[n]}]$ of degree $\leq m$. However by Remark 9.2 such a correspondence does not stabilize to a correspondence when n and m go to infinity. The element 1 in the first ring corresponds to the element y_1^m in the second ring. Also the diagram (7) does not extend to a diamond diagram like (16) since the map Λ cannot be defined on $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ and the map Γ not on $\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$. The notion of dual modules cannot then rest on a bijection between the monomials in $k[x_\mathbb{N}]$ and $k[y_\mathbb{N}]$. However we may accomplish our objective using limit considerations.

Definition 10.1.

- For f in $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ let r be the integer such that $f(r) = \infty$ and $f(r-1) < \infty$. For $m \geq f(r-1)$ define $f^{\underline{m}}$ to be the small function in $\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ given by

  $$f^{\underline{m}}(i) = \begin{cases} f(i), & i < r \\ m, & i \geq r \end{cases}.$$

  Note that we have

$$\Lambda f^{\underline{m+1}} = x_r \cdot \Lambda f^{\underline{m}}. \tag{18}$$

- For g in $\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ let p be such that $g(p-1) < g(p)$ and $g(i) = g(p)$ for $i \geq p$. For $n \geq p$ let $g_{|n}$ be the element in $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ given by

$$g_{|n}(i) = \begin{cases} g(i), & i < n \\ \infty, & i \geq n \end{cases}.$$

  Note that we have

$$\Gamma g_{|n+1} = x_{g(p)} \cdot \Gamma g_{|n}. \tag{19}$$

The following is easily verified, since the duality D is essentially reflection about the axis $x = y$.

Lemma 10.2. *Let $f \in Hom^L(\mathbb{N}, \hat{\mathbb{N}})$ and $g \in Hom_S(\mathbb{N}, \hat{\mathbb{N}})$ correspond via D. Then $f^{\underline{m}} \in Hom_S(\mathbb{N}, \hat{\mathbb{N}})$ and $g_{|m} \in Hom^L(\mathbb{N}, \hat{\mathbb{N}})$ correspond via D.*

To define the dual of a module M we need to extend the meaning of Λ to $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ and of Γ to $\mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$.

Definition 10.3. Let M be a shift module over $k[x_{\mathbb{N}}]$.

- For $g \in \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ we have $g_{|n} \in \mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$. By (19) we have multiplication maps

$$M_{\Gamma g_{|n}} \xrightarrow{\cdot x_{g(p)}} M_{\Gamma g_{|n+1}}.$$

  We define

$$M_{\Gamma g} = \mathrm{colim}_n M_{\Gamma g_{|n}}.$$

- For $f \in \mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ we have $f^{\underline{m}} \in \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$. By (18) we have multiplication maps

$$M_{\ \Lambda f^{\underline{m}}} \xrightarrow{\cdot x_r} M_{\Lambda f^{\underline{m+1}}}.$$

  We define

$$M_{\Lambda f} = \mathrm{colim}_m M_{\Lambda f^{\underline{m}}}.$$

Corollary 10.4. *For M a shift module over $k[x_{\mathbb{N}}]$ and dual elements $f \in Hom^L(\mathbb{N}, \hat{\mathbb{N}})$ and $g \in Hom_S(\mathbb{N}, \hat{\mathbb{N}})$, then $M_{\Lambda f} = M_{\Gamma g}$.*

Proof.

$$M_{\Lambda f} = \mathrm{colim}_m M_{\Lambda f^{\underline{m}}} = \mathrm{colim}_m M_{\Gamma g_{|m}} = M_{\Gamma g}.$$

□

Definition 10.5. Let M be a shift module over $k[x_{\mathbb{N}}]$ such that for every $g \in \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ the graded part $M_{\Gamma g}$ is finite dimensional. The *dual module* $N = M^\vee$ over $k[y_{\mathbb{N}}]$ is defined by letting

$$N_{\Lambda g} = (M_{\Gamma g})^*.$$

Note that if $f = Dg$ is the dual large map, the dual module N also by Corollary 10.4 fulfills

$$N_{\Gamma f} = N_{\Lambda g} = (M_{\Gamma g})^* = (M_{\Lambda f})^*.$$

The shift maps are defined as follows. Let $g \in \mathrm{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$. Suppose $g(p) < g(p+1)$. For n large enough, we have shift maps

$$s_{g(p)} : M_{\Gamma (g+i_p)_{|n}} \to M_{\Gamma g_{|n}}.$$

Taking the colimit w.r.t. n we get a shift map and a dual map

$$s_{g(p)} : M_{\Gamma(g+i_p)} \to M_{\Gamma g}, \quad t_p : N_{\Lambda g} \to N_{\Lambda(g+i_p)},$$

where t_p is the shift map for N. That the shift maps t_p and t_q commute when the $(p+1)$'th and $(q+1)$'th degree coordinates of Γf are positive is checked like in Lemma 9.3.

Since colim is an exact functor on vector spaces, we observe that dualization is an exact functor on the category of shift modules over $k[x_{\mathbb{N}}]$.

Definition 10.6. A shift module M over $k[x_{\mathbb{N}}]$ is *dualizable* if it has a dual module N such that the dual module of N is M again.

For a shift module M over $k[x_{\mathbb{N}}]$ and $u \in \mathrm{Mon}(x_{\mathbb{N}})$ with $n \geq \max(u)$ the largest index of a variable in u, there are maps

$$\begin{tikzcd} M_u \arrow[r, "\cdot x_n^m"] \arrow[dr, "x_{n+1}^m"'] & M_{u \cdot x_n^m} \\ & M_{u \cdot x_{n+1}^m}. \arrow[u, "(s_n)^m"'] \end{tikzcd}$$

Here $(s_n)^m$ is the shift maps s_n applied m times. This gives a diagram

$$\begin{tikzcd} M_u \arrow[r] \arrow[dr] & \mathrm{colim}_m M_{u \cdot x_n^m} \\ & \mathrm{colim}_m M_{u \cdot x_{n+1}^m} \arrow[u, "(s_n)^m"'] \end{tikzcd}$$

and then a map

$$M_u \to \lim_n \mathrm{colim}_m M_{u \cdot x_m^n}.$$

Proposition 10.7. *Let M be a shift module over $k[x_{\mathbb{N}}]$ which has a dual module. (In particular this holds if there is a uniform bound on the dimensions of the M_u.) Then M is dualizable if and only if the natural map*

$$M_u \to lim_n\, colim_m M_{u \cdot x_n^m}$$

is an isomorphism for every $u \in \mathbb{N}_0^\infty$.

Proof. Suppose M is dualizable with dual module N. Then for large f

$$\begin{aligned} M_{\Gamma f} &= (\mathrm{colim}_n N_{\Lambda f \underline{n}})^* \\ &= \lim_n\, (N_{\Lambda f \underline{n}})^*. \end{aligned}$$

But

$$(N_{\Lambda f \underline{n}})^* = M_{\Gamma f \underline{n}} = \mathrm{colim}_m M_{\Gamma (f \underline{n})_{|m}}.$$

If $u = \Gamma f$ then $\Gamma(f\underline{n})_{|m} = u \cdot x_n^{m-c_0}$ for suitable c_0. Whence

$$\begin{aligned} M_{\Gamma f} &= \lim_n \mathrm{colim}_m M_{\Gamma (f \underline{n})_{|m}}, \\ M_u &= \lim_n \mathrm{colim}_m M_{u \cdot x_n^m}. \end{aligned}$$

Conversely, suppose the natural map is an isomorphism. Let N be the dual module of M. Then

$$N_{\Lambda f \underline{n}} = (\mathrm{colim}_m M_{\Gamma (f\underline{n})_{|m}})^*.$$

Hence for the dual M' of N:

$$M'_{\Gamma f} = (N_{\Lambda f})^* = \lim_n (N_{\Lambda f^{\underline{n}}})^* = \lim_n \operatorname{colim}_m M_{\Gamma (f^{\underline{n}})_{|m}} = M_{\Gamma f}.$$

□

11. Examples of duals

We give examples of duals of the modules given in Section 8. We first give three examples of ideals where the dual module is the polynomial ring $S = k[x_{\mathbb{N}}]$.

11.1. **Duals of ideals.** $I = S = k[x_{\mathbb{N}}]$ *the polynomial ring.* Then for f in $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ with $f(n-1) < f(n) = \infty$, the multiplication maps for $m \geq n$

$$S_{\Lambda f^{\underline{m}}} \xrightarrow{\cdot x_n} S_{\Lambda f^{\underline{m+1}}}$$

are always isomorphisms of one-dimensional vector spaces. Hence the dual module of S is S itself.

I is the ideal $m^r = (x_1, x_2, x_3, \cdots,)^r$. As above the multiplication maps

$$I_{\Lambda f^{\underline{m}}} \xrightarrow{\cdot x_n} I_{\Lambda f^{\underline{m+1}}}$$

are always isomorphisms of one-dimensional vector spaces for m sufficiently large. Hence the dual module of I is again S.

I is the ideal $\langle x_1, x_2^2, x_3^3, \cdots, x_p^p, \cdots \rangle$. Note that for any monomial u and m large, the maps

$$I_{u \cdot x_n^m} \xrightarrow{\cdot x_n} I_{u \cdot x_n^{m+1}}$$

are isomorphism between one-dimensional spaces for $m \geq n$. Whence the dual of I is the module S.

In the last two examples, the ideals were modules which were not dualizable. When the ideal is dualizable we have the following, which is analogous to what happens for Alexander duality for squarefree ideals [47].

Proposition 11.1. *Let I and J be dual strongly stable ideals.*

(a) *The dual of I is the module S/J.*
(b) *The dual of S/I is the ideal J.*

Proof. (a) Let $[\mathcal{I}, \mathcal{F}]$ be a Dedekind cut in $\mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$, with $I = \Gamma(\mathcal{I}^L)$ and J the dual ideal of I, which is

$$J = \Lambda(\mathcal{F}_S) = \Gamma((D\mathcal{F})^L).$$

Note that for f in $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$:

$$(S/J)_{\Gamma f} = \begin{cases} k, & f \notin (D\mathcal{F})^L, \\ 0, & f \in (D\mathcal{F})^L. \end{cases}$$

Let N be the dual shift module of I. We show that N lives in precisely the same degrees as S/J above. Let f be an element of $\mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ and $g = Df$ the dual small function. Then

$$N_{\Gamma f} = N_{\Lambda g} = (I_{\Gamma g})^*, \quad I_{\Gamma g} = \operatorname{colim}_n I_{\Gamma g_{|n}}.$$

(i) Suppose f is in $(D\mathcal{F})^L$ so g is in $\mathcal{F}_S$. Note that $g_{|n}$ is then also in $\mathcal{F}$. Whence $I_{\Gamma g_{|n}} = 0$. Then $N_{\Gamma f}$ above is zero.

(ii) If f is not in $(D\mathcal{F})^L$ then g is not in $\mathcal{F}_S$. Since g is small, it is not in the gap between $\mathcal{I}$ and $\mathcal{F}$, and hence it is in $\mathcal{I}$. Since $\mathcal{I}$ is open, $g_{|n}$ is in $\mathcal{I}$ for n big, and so

$$\mathrm{colim}_n I_{\Gamma g_{|n}} = k.$$

This shows that N and S/J have precisely the same dimensions in each degree. It is readily verified that the shift maps also correspond, so $N = S/J$.

(b) The exact sequence

$$0 \to I \to S \to S/I \to 0$$

dualizes to the exact sequence

$$0 \to (S/I)^\vee \to S^\vee \to I^\vee \to 0.$$

Here the map between the last shift modules identify as $S \to S/J$ and so the dual of S/I is J. □

I is the principal strongly stable ideal $\langle y_{a_1} y_{a_2} \cdots y_{a_r}\rangle$. By Proposition 3.11 and Proposition 11.1, the dual module is $k[x_{\mathbb{N}}]/\langle x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}\rangle$.

11.2. **Duals of modules.** *The module $M = \oplus_{a\in\mathbb{N}_0} S u_a$* where u_a is a generator with degree $(a, 0, 0, \cdots)$. Then $M_{x_1^m}$ is a vector space of dimension $m+1$ and the maps

$$M_{x_1^m} \to M_{x_1^{m+1}}$$

are injections. Thus

$$\mathrm{colim}_m M_{x_1^{m+1}}$$

is not finite-dimensional, and M has no dual.

Recall the modules N_1 and M_1 in Sections 8.2 and 8.3.

Proposition 11.2. *The modules N_1 and M_1 are dual modules. Similarly the modules N_2 and M_2 are dual modules.*

Proof. We show that N_1 is the dual module of M_1. That M_1 is the dual module of N_1 is an analogous argument. So let N' be the dual module of M_1. Let $f \in \mathrm{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$. Then

$$(N')_{\Gamma f} = (M_{1,\Lambda f})^*,$$

where

$$M_{1,\Lambda f} = \mathrm{colim}_m M_{1,\Lambda f^{\underline{m}}}. \tag{20}$$

If f is the function taking value ∞ at every $i \in \mathbb{N}$, then

$$M_{1,\Lambda f^{\underline{m}}} = M_{1,x_1^m}$$

and the colimit above is 0 (the multiplication by x_1 on M_1 is zero). Thus $(N')_{\Gamma f} = (N')_{\mathbf{0}} = 0$.

If f is not the above constant function, then for m big $\Lambda f^{\underline{m}}$ is $u \cdot x_p^{m-c_0}$ for some fixed monomial u and fixed $p \geq 2$. Then the colimit in (20) is k. Such f correspond to monomials $u = \Gamma f$ of positive degree and so $N'_u = k$ for these u.

As for the shift maps in N' let f have values the finite sequence $a_1, \cdots, a_{r-1}, a_r, a_{r+1}, \cdots, a_p$, with $a_{r-1} < a_r$, and let f' have values the finite sequence $a_1, a_2 \cdots, a_{r-1}, a_r - 1, a_{r+1}, \cdots, a_p$. The map

$$t_{a_r - 1} : (N')_{\Gamma f} \to (N')_{\Gamma f'}$$

is the dual of the colimit of the shift maps

$$s_r : (M_1)_{\Lambda f'\underline{m}} \to (M_1)_{\Lambda f\underline{m}}.$$

If $r \geq 2$ this is an isomorphism of one-dimensional spaces. If $r = 1$ this is the zero map. We find that dual N' of M_1 identifies as N_1.

In a similar way we show that the dual of N_1 is M_1. □

Part V. Resolutions

12. Examples of resolutions

We give simple examples of minimal projective shift module resolutions, and in particular see how they differ from the ordinary minimal free resolutions as S-modules. Recall from Proposition 6.9 that the projective shift modules over the polynomial ring $k[x_{[m]}]$ are the principal strongly stable ideals $\langle x^{\mathbf{d}}\rangle$.

12.1. **Ideals with projective dimension one as shift modules.** Consider the ideal in $k[x_1, x_2, x_3]$ with strongly stable generators

$$x_1^a x_2^b,\ x_1^{a-1+r} x_2^{b-r} x_3,\ x_1^{a-2+s} x_2^{b-s} x_3^2,$$

where $r \geq 1$ and $s \geq r+1$. These generators are illustrated with bullets in Figure 4. The strongly stable ideal generated by these three monomials will have

- One generator, x_1^{a+b}, whose highest index variable is x_1,
- b generators whose highest index variable is x_2,
- $2b+2-r-s$ generators whose highest index variable is x_3.

The minimal free resolution as $S = k[x_1, x_2, x_3]$-modules, by Eliahou-Kervaire [15], has the following form

$$S(-a-b)^{3b+3-r-s} \leftarrow S(-a-b-1)^{5b+4-2r-2s} \leftarrow S(-a-b-2)^{2b+2-r-s}.$$

On the other hand the minimal projective shift resolution of this ideal has the following form when $r \geq 2$ and $s \geq r+2$:

$$\begin{aligned} &\langle x_1^a x_2^b\rangle \oplus \langle x_1^{a-1+r} x_2^{b-r} x_3\rangle \oplus \langle x_1^{a-2+s} x_2^{b-s} x_3^2\rangle \\ \leftarrow &\langle x_1^{a-1+r} x_2^{b+1-r}\rangle \oplus \langle x_1^{a-2+s} x_2^{b+1-s} x_3\rangle. \end{aligned}$$

When $r = 1$ and $s \geq 3$ the minimal resolution becomes

$$\langle x_1^a x_2^{b-1} x_3\rangle \oplus \langle x_1^{a-2+s} x_2^{b-s} x_3^2\rangle \leftarrow \langle x_1^{a-2+s} x_2^{b+1-s} x_3\rangle.$$

When $r \geq 2$ and $s = r+1$ the minimal resolution becomes

$$\langle x_1^a x_2^b\rangle \oplus \langle x_1^{a-1+r} x_2^{b-1-r} x_3^2\rangle \leftarrow \langle x_1^{a-1+r} x_2^{b-r} x_3\rangle.$$

Finally when $r=1$ and $s=2$ the ideal becomes the projective module $\langle x_1^a x_2^{b-2} x_3^2\rangle$. The two resolutions give quite distinct information.

- By Figure 4 the Betti numbers in the shift resolution reflect better the combinatorial nature of the shift generators.
- For the various pairs (a, b) with $a+b$ fixed, the shift resolutions above have the same graded Betti numbers, but those in the EK-resolution vary.
- It is easy to find strongly stable ideals with the same graded Betti numbers in the EK-resolution above (and so in particular have the same Hilbert series), but with distinct Betti numbers in the shift resolution.

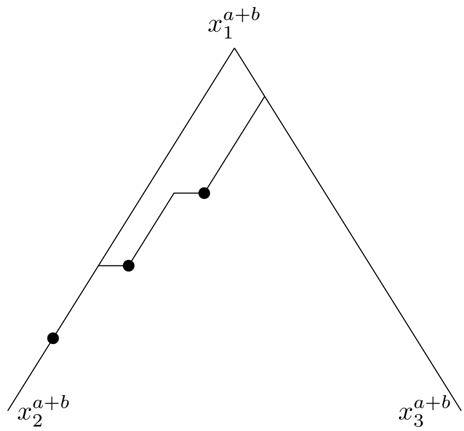

FIGURE 4

In general all strongly stable ideals in three variables with generators of the same degree will have shift projective dimension one or zero.

12.2. **Ideals with projective dimension two as shift modules.** Consider the ideal with strongly stable generators

$$x_1^{a+t}x_2^bx_3^c,\ x_1^ax_2^{b+r}x_3^c,\ x_1^ax_2^bx_3^{c+s}.$$

The minimal projective shift-resolution of this when $s > r > t$ is

$$\begin{aligned}
&\langle x_1^{a+t}x_2^bx_3^c\rangle \oplus \langle x_1^ax_2^{b+r}x_3^c\rangle \oplus \langle x_1^ax_2^bx_3^{c+s}\rangle\\
\leftarrow&\langle x_1^{a+t}x_2^{b+r-t}x_3^c\rangle \oplus \langle x_1^ax_2^{b+r}x_3^{c+s-r}\rangle \oplus \langle x_1^{a+t}x_2^bx_3^{c+s-t}\rangle\\
\leftarrow&\langle x_1^{a+t}x_2^{b+r-t}x_3^{c+s-r}\rangle.
\end{aligned}$$

This is a special case of Proposition 13.2.

13. Koszul-type shift resolutions

If the sst-generators of a strongly stable ideal are sufficiently generic we expect the minimal resolution to be given by a Koszul-type resolution: If there are n generators, the p'th Betti number should be $\binom{n}{p}$. We will see there is always such a resolution for any set of generators, and give conditions so that it is minimal. In particular the minimal shift resolution of universal lex-segment ideals has this form.

Recall the strongly stable partial order on $\mathrm{Mon}(x_{\mathbb{N}})$.

$$x_{a_1}\cdots x_{a_r} \geq_{st} x_{b_1}\cdots x_{b_s}$$

iff $r \geq s$ and $a_i \leq b_i$ for $i = 1, \cdots, s$.

Recall that $\langle x^{\mathbf{d}}\rangle$ and $\langle x^{\mathbf{e}}\rangle$ are indecomposable projectives in $\mathtt{shmod}\, k[x_{\mathbb{N}}]$, where $\mathbf{d}, \mathbf{e}$ are in $\mathbb{N}_0^\infty$. There is an inclusion of shift modules $\langle \mathbf{d}\rangle \hookrightarrow \langle \mathbf{e}\rangle$ iff $x^{\mathbf{d}} \geq_{\mathrm{st}} x^{\mathbf{e}}$. If the latter does not hold the only map from $\langle \mathbf{d}\rangle$ to $\langle \mathbf{e}\rangle$ is the zero map. By Lemma 3.2 for $f, g \in \mathrm{Hom}(\mathbb{N}, \hat{\mathbb{N}})$ there is an inclusion of shift modules $\langle \Gamma f\rangle \hookrightarrow \langle \Gamma g\rangle$ iff $f \leq g$.

Given isotone maps

$$f_i : \mathbb{N} \to \hat{\mathbb{N}} \quad (\text{or } [m] \to \hat{\mathbb{N}}), \quad i \in J. \tag{21}$$

When explicitly mentioned later we might have Condition 13.1 on the f_i.

Condition 13.1. For each i there exists $q_i \in \mathbb{N}$ (or $[m]$) such that $f_i(q_i) < f_j(q_i)$ for every $j \in J\backslash\{i\}$. So each f_i has some q_i where f_i is the unique function having minimal value at q_i.

Thus q gives an injective function $q : J \to \mathbb{N}$ (or $[m]$). For R a finite subset of J let f_R be the isotone map which is the meet of the functions $f_i, i \in R$, so:

$$f_R(p) = \min\{f_i(p) \mid i \in R\}.$$

Let $P(f_R)$ be the projective module $\langle \Gamma(f_R) \rangle$. There is an inclusion map

$$i_{R,S} : P(f_R) \hookrightarrow P(f_S)$$

when $S \subseteq R$. For $r \in R$ denote by $i_{R,r} = i_{R,R\backslash\{r\}}$.

Let I be the ideal $\langle \Gamma f_i \rangle_{i \in J}$. We can now give the terms in the resolution of the quotient ring $k[x_{\mathbb{N}}]/I$. Let the first term be $F_0 = k[x_{\mathbb{N}}]$ and for $p \geq 1$ let

$$F_p = \oplus_{R \subseteq J, |R| = p} P(f_R).$$

For $R = \{r_1 < r_2 < \cdots < r_p\}$, there are natural maps

$$P(f_R) \xrightarrow{[-i_{R,r_1}, i_{R,r_2}, \cdots, (-1)^p i_{R,r_p}]} \bigoplus_{i=1}^{p} P(f_{R\backslash\{r_i\}}) \subseteq F_{p-1}.$$

This gives natural maps

$$F_p \xrightarrow{d_p} F_{p-1}, \quad p \geq 1.$$

Proposition 13.2. *The map d is a differential, and $F_\bullet$ is a projective shift resolution of the quotient ring $k[x_{\mathbb{N}}]/I$.*

If the $\{f_i\}$ fulfills Condition 13.1*, then $F_\bullet$ is a minimal projective resolution.*

Remark 13.3. The above is the analog of the Taylor resolution for square free modules [33, 4.3.2, 6.1], [38, Sec.26] or [24, Ch.7]. The condition for minimality is also similar to that for the Taylor complex [33, Lem.6.4].

Proof. Let $\sum \alpha_R u_R$ be a syzygy in F_p where the α_R are non-zero constants in k and each $\alpha_R u_R$ is in $P(f_R)$. We may assume it is homogeneous so all monomials u_R equal a fixed monomial u. Put a total order on I. Order the p-subsets of I such that $S > R$ if for the maximal ℓ such that s_ℓ and r_ℓ differ, we have $s_\ell < r_\ell$. (So R is dragged down by having a heavy rear.)

Let R_0 be *minimal* among the R where α_R is non-zero in the sum. Write $R_0 = \{r_1 < r_2 < \cdots < r_p\}$. Then

$$P(f_{R_0}) \text{ maps to } \bigoplus_{r_i \in R_0} P(f_{R_0\backslash\{r_i\}}).$$

The image of $\alpha_{R_0} u_{R_0}$ in $P(f_{R_0\backslash\{r_1\}})$ must cancel against a term in the image of $P(f_S)$ for some S occurring in the syzygy $\sum \alpha_R u_R$. So $R_0\backslash\{r_1\} = S\backslash\{s_t\}$ for some t. If $t \geq 2$ then $s_i = r_i$ for $i > t$ while $r_t = s_{t-1} < s_t$. This contradicts $R_0 < S$. Thus $t = 1$ and since $R_0 < S$ we have $r_1 > s_1$. Let $r_0 = s_1$ and $R' = R_0 \cup \{r_0\}$. If we add plus or minus the image of $d(\alpha_R \Gamma(f_{R'}))$ to the syzygy $\sum \alpha_R u_R$, we get a syzygy with larger minimal f_R. We may thus continue and in the end see that all syzygies are in the image of d.

When Condition 13.1 is fulfilled, the maps $f_R < f_{R\backslash\{r\}}$ for every $r \in R$. Hence the resolution is minimal. $\square$

Corollary 13.4. *The above gives the minimal shift resolutions of the following ideals:*

- $\langle x_1, x_2^2, x_3^3, \cdots \rangle$
- *Universal lex-segment ideals.*

14. Generalized Eliahou-Kervaire resolution

The resolution of strongly stable ideals and more generally stable ideals is the celebrated Eliahou-Kervaire resolution [15], a resolution where the terms and differentials are explicitly described. See [39] or [38, Sec.28] for a simple exposition. Here we generalize this to shift modules. Another direction where the Eliahou-Kervaire resolution has recently been generalized is to the resolution of co-letterplace ideals [13].

14.1. **Rear torsion-free modules.** For a degree $\mathbf{d}$ in $\mathbb{N}_0^\infty$ let $\max(\mathbf{d})$ be the largest index i such that d_i is non-zero. Also let $\min(\mathbf{d})$ be the smallest such index.

Definition 14.1. A shift module M over $k[x_\mathbb{N}]$ is *rear torsion-free* if for every $m \in M_\mathbf{d}$ and every monomial $x^\mathbf{a}$ with $\max(\mathbf{d}) \leq \min(\mathbf{a})$, if $x^\mathbf{a} \cdot m = 0$ then $m = 0$.

Note that since M is graded by $\mathbb{N}_0^\infty$ and the $M_\mathbf{d}$ are finite-dimensional, the module M has a minimal homogeneous generating set. Let $\{m_\mathbf{d}^i\}$ be such a minimal generating set for M, with $m_\mathbf{d}^i$ of degree $\mathbf{d}$.

Lemma 14.2. *Let M be a rear-torsion free module and $m \in M$. There is a unique way of writing*

$$m = \sum_{i,\mathbf{d}} \alpha_\mathbf{d}^i x^{\mathbf{a}_\mathbf{d}^i} m_\mathbf{d}^i$$

with $\alpha_\mathbf{d}^i \in k$ and $m_\mathbf{d}^i \in M_\mathbf{d}$, and for each term $\max(\mathbf{d}) \leq \min(\mathbf{a}_\mathbf{d}^i)$.

Proof. First we do existence. We may in some way write $m = \sum \alpha_\mathbf{d}^i x^{\mathbf{a}_\mathbf{d}^i} m_\mathbf{d}^i$. Consider $x_p \cdot m_\mathbf{d}^i$ where $p < \max \mathbf{d} = b$. This is

$$\begin{aligned} s_{p,\infty}(m_\mathbf{d}^i) &= s_{p,b} \circ s_{b,\infty}(m_\mathbf{d}^i) \\ &= s_{b,\infty} \circ s_{p,b}(m_\mathbf{d}^i) = x_b \cdot s_{p,b}(m_\mathbf{d}^i). \end{aligned}$$

This means that whenever $x_p m_\mathbf{d}^i$ occurs in a term above, we may replace it with the term $x_b \cdot s_{p,b}(m_\mathbf{d}^i)$ where $b \geq \max s_{p,b}(m_\mathbf{d}^i)$. Continuing in this way we get the existence of an expression as claimed.

Now consider uniqueness. If we do not have uniqueness, we have a homogeneous expression of degree $\mathbf{e}$

$$0 = \sum_{\mathbf{a}_\mathbf{d}^i + \mathbf{d} = \mathbf{e}} \alpha_\mathbf{d}^i x^{\mathbf{a}_\mathbf{d}^i} m_\mathbf{d}^i,$$

where not all the $\alpha_\mathbf{d}^i$ are zero. Let $p = \max(\mathbf{e})$. If x_p does not divide $x^{\mathbf{a}_\mathbf{d}^i}$ then we would have $x^{\mathbf{a}_\mathbf{d}^i} = 1$, which is not so since the $m_\mathbf{d}^i$ are part of a minimal generating set. Hence x_p divides each $x^{\mathbf{a}^i_{\mathbf{d}}}$. By rear torsion-freeness we may divide out by x_p and get

$$0 = \sum \alpha^i_{\mathbf{d}} x^{\mathbf{a}^i_{\mathbf{d}}}/x_p \cdot m^i_{\mathbf{d}}.$$

In this way we may continue until some $m^i_{\mathbf{d}}$ is a combination of the other ones, contradicting minimality of the generators. □

Remark 14.3. Such a unique way of writing an element in the module is more or less exactly the same as the unique way of writing an element in a quasi-stable submodule in terms of its Pommaret basis, [1, Thm.3.3] or [43, Prop.4.4, 4.6].

A difference to Pommaret bases is that those are submodules of free modules. In contrast the class of rear torsion-free shift modules also includes modules that are not submodules of free modules. For instance let the multidegree $\mathbf{d} = (d_1, d_2, d_3, 0, \ldots, 0)$ have $d_3 > 0$. Then the module $Su_{\mathbf{d}}/(x_1, x_2)u_{\mathbf{d}}$ (which is a shift module by Lemma 6.5) is rear-torsion free if $d_3 > 0$. However it is not rear torsion-free if $d_3 = 0$.

Another difference is the quasi-stable modules are essentially direct sums of ideals, they are generated by terms $x^\alpha e_k$. In contrast for a shift submodule of a free module, this may not be so.

Let $\{u^i_{\mathbf{d}}\}$ be a set of symbols where $u^i_{\mathbf{d}}$ has degree $\mathbf{d}$. Let $T^i_{\mathbf{d}}$ be the subspace of $Su^i_{\mathbf{d}}$ with basis $x^{\mathbf{a}} u^i_{\mathbf{d}}$ where $\max(\mathbf{d}) \leq \min(\mathbf{a})$, and

$$T = \bigoplus_{i,\mathbf{d}} T^i_{\mathbf{d}} \subseteq \bigoplus_{i,\mathbf{d}} S^i_{\mathbf{d}}.$$

Corollary 14.4. *The natural map $T \to M$ sending $u^i_{\mathbf{d}}$ to $m^i_{\mathbf{d}}$ is an isomorphism of vector spaces.*

Proof. This is clear. □

For $p \in \mathbb{N}$ we may then transport the shift map s_p on M to a shift map s_p on T. Since s_p and s_q commute on $M_{\mathbf{d}}$ when d_{p+1} and d_{q+1} are non-zero, the same holds for s_p and s_q on T. Explicitly we have

$$s_p(x^{\mathbf{a}} \cdot u^i_{\mathbf{d}}) = \begin{cases} s_p(x^{\mathbf{a}}) \cdot u^i_{\mathbf{d}}, & p \geq \max(\mathbf{d}), \\ x^{\mathbf{a}} \cdot s_p(u^i_{\mathbf{d}}), & p < \max(\mathbf{d}). \end{cases}$$

14.2. **The complex giving the resolution.** Let F_p be the free S-module generated by all symbols $(i_1, \ldots, i_p \,|\, u^i_{\mathbf{d}})$ where

$$i_1 < i_2 < \cdots < i_p < \max(\mathbf{d}).$$

This symbol has multidegree $\mathbf{d} + \sum_{j=1}^p e_{i_j}$. For a monomial $x^{\mathbf{a}}$, we also let $(\mathbf{i} \,|\, x^{\mathbf{a}} u^i_{\mathbf{d}})$ be $x^{\mathbf{a}} \cdot (\mathbf{i} \,|\, u^i_{\mathbf{d}})$. For each $\mathbf{d}$ choose an arbitrary total order on the $u^i_{\mathbf{d}}$'s. Define a total order on the symbols $(\mathbf{i} \,|\, u^i_{\mathbf{d}})$ by $(\mathbf{j} \,|\, u^j_{\mathbf{d}}) > (\mathbf{i} \,|\, u^i_{\mathbf{e}})$ when we have the following.

- If $\mathbf{d} \neq \mathbf{e}$ let $p = \max\{i \,|\, d_i \neq e_i\}$. Then we have $d_p > e_p$ (and write also $\mathbf{d} > \mathbf{e}$).
- If $\mathbf{d} = \mathbf{e}$ and $\mathbf{i} \neq \mathbf{j}$ let $q = \max\{r \,|\, i_r \neq j_r\}$. Then we have $j_q > i_q$.
- If $\mathbf{d} = \mathbf{e}$ and $\mathbf{i} = \mathbf{j}$ then $u^j_{\mathbf{d}} > u^i_{\mathbf{d}}$.

In the following $b = \max(\mathbf{d})$. Define maps $\delta, \mu : F_p \to F_{p-1}$ by

$$(i_1, i_2, \cdots, i_p \,|\, u^i_{\mathbf{d}}) \overset{\delta}{\mapsto} \sum_q (-1)^q x_{i_q} \cdot (i_1, \cdots, \hat{i_q}, \cdots, i_p \,|\, u^i_{\mathbf{d}}),$$

$$(i_1, i_2, \cdots, i_p \,|\, u^i_{\mathbf{d}}) \overset{\mu}{\mapsto} \sum_q (-1)^q x_b \cdot (i_1, \cdots, \hat{i_q}, \cdots, i_p \,|\, s_{i_q,b}(u^i_{\mathbf{d}})).$$

Note that $s_{i_q,b}(u^i_{\mathbf{d}})$ will typically be rewritten as a linear combination of products of monomials and other $u^j_{\mathbf{e}}$.

Lemma 14.5. *$d = \delta - \mu$ is a differential, i.e. $d^2 = 0$.*

This is a simple check using that the maps $s_{i,j}$ commute when they are non-zero. Note that the free modules F_p are generally not shift modules, as a free S-module only is a shift-module if its generator has a multidegree $\mathbf{d} = d_1 e_1$, Lemma 6.4. The following is the generalized Eliahou-Kervaire resolution.

Theorem 14.6. *Let M be a rear torsion-free shift module. The complex $F_\bullet$ is a free resolution M.*

When M is a strongly stable ideal, the resolution $F_\bullet$ is the Eliahou-Kervaire resolution.

Proof. Given a *homogeneous* (for the $\mathbb{N}_0^\infty$-grading) syzygy in F_p

$$\sum \alpha_{\mathbf{i},u} x^{a_{\mathbf{i},u}} (\mathbf{i} \,|\, u). \tag{22}$$

Let $(\mathbf{i}^0 \,|\, u^0)$ be maximal of the non-zero terms with respect to the order above.

Note that

$$d(\mathbf{i}^0 \,|\, u^0) = x_{i^0_1} \cdot (i^0_2, \cdots, i^0_p \,|\, u^0) + \text{lower terms}.$$

Since (22) is a syzygy and so maps by d to zero, in order for the above to cancel, we must in (22) have a term $x^{a_{\mathbf{i}^{0\prime},u^0}} \cdot (i^{0\prime}_1, i^0_2, \cdots, i^0_p \,|\, u^0)$ where $i^{0\prime}_1 < i^0_1$. Then we must in (22) have terms

$$\alpha n \cdot x_{i^{0\prime}_1} (\mathbf{i}^0 \,|\, u^0) - \alpha n \cdot x_{i^0_1} (\mathbf{i}^{0\prime} \,|\, u^0).$$

Then subtracting the image of $\alpha n \cdot (i^{0\prime}_1, i^0_1, i^0_2, \cdots, i_p \,|\, u^0)$ from (22), we reduce (22) to a syzygy with smaller initial term. We may continue until we get zero, and so the kernel of $F_p \xrightarrow{d} F_{p-1}$ is the image of $F_{p+1} \xrightarrow{d} F_p$. $\square$

Remark 14.7. A similar resolution occurs in W. Seiler [43, Thm.7.2] for the class of quasi-stable modules, and generalizing the Eliahou-Kervaire resolution for stable ideals [15]. Again in [43] the differential decomposes into two parts, which are completely analogous to our δ and μ.

However quasi-stable modules are essentially direct sums of ideals. So the resolution of [43, Thm.7.2] is essentially a resolution of an ideal.

A difference concerning the terms in the resolution is then that our term $S_{i_q,b}(u^i_{\mathbf{d}})$ may be a linear combination of products of monomials and other basis terms $u^i_{\mathbf{e}}$, while in [43, Thm.7.2] the corresponding term is only a product of a monomial and a basis element. In Section 6 of [43] there is a more general form for the differential when taking resolutions of polynomial submodules. This resolution may however not be minimal.

Appendix A. Incidence algebras

Incidence algebras are constructed from partially ordered sets. They can be viewed as quiver algebras with relations.

Let P be a poset. It gives a quiver with an arrow for each pair $p' > p$ in P which is a covering relation. Denote this arrow as $[p', p]$. A path

$$q = p_n > p_{n-1} > \cdots > p_0 = p$$

of covering relations gives a product

$$[p_n, p_{n-1}] \cdots [p_2, p_1] \cdot [p_1, p_0]$$

in the quiver algebra. We form the quiver algebra with relations by setting these products equal for any two paths from p to q. This is the incidence algebra $I(P)$.

A module M over the incidence algebra is a direct sum $M = \oplus_{p \in P} M_p$ such that for each $q > p$ we have a map $M_q \xleftarrow{\cdot [q,p]} M_p$ such that all path relations are respected. The indecomposable projective modules for the incidence algebra are the modules, one for each $y \in P$

$$P(y) = \prod_{x \geq y} k_x,$$

where k_x is a copy of k in degree x. The multiplication with $[q,p]$ on $P(y)$ is the identity map from k_p to k_q and is zero on the k_x where $x \neq p$.

Let $\hat{P} = \mathrm{Hom}(P^{\mathrm{op}}, \omega)$ be the associated distributive lattice to P. Then $\hat{P}$ is a Cohen-Macaulay poset by for instance [9, Cor.4.5, Ex. 4.6]. By [40] or [46] the incidence algebra $I(\hat{P})$ is then Koszul. The elements of $\hat{P}$ are poset ideals in P with the ordering on $\hat{P}$ induced by inclusions of poset ideals. Let $I \subseteq J$ be poset ideals with $J \backslash I = \{x, y\}$ and x, y incomparable. The ideals of relations for $I(\hat{P})$ are generated by the quadratic relations as I and J vary

$$[I \cup \{x,y\}, I \cup \{x\}] \cdot [I \cup \{x\}, I] = [I \cup \{x,y\}, I \cup \{y\}] \cdot [I \cup \{y\}, I].$$

The Koszul dual $E(\hat{P})$ of $I(\hat{P})$ is then generated by the relations

(1) For incomparable x and y:

$$[I \cup \{x,y\}, I \cup \{x\}] \cdot [I \cup \{x\}, I] = -[I \cup \{x,y\}, I \cup \{y\}] \cdot [I \cup \{y\}, I],$$

(2) When $y > x$:

$$[I \cup \{x,y\}, I \cup \{x\}] \cdot [I \cup \{x\}, I] = 0.$$

Lemma A.1. *The largest degree d for which $E(\hat{P})_d$ is non-zero is the largest cardinality of an antichain in P.*

Proof. We claim that if $I \subseteq J$ are poset ideals and

$$I = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = J$$

a sequence of covering relations (meaning each I_{p+1} has cardinality one more than I_p), then the product

$$[I_n, I_{n-1}] \cdots [I_1, I_0] \tag{23}$$

is zero iff $J \backslash I$ contains at least two elements x, y which are comparable by a covering relation, say $y > x$: By using Relation (1) above repeatedly, we only change the product (23) by a sign, and eventually get to a chain where we have successive terms

$$I_{r+1} = I_r \cup \{y\}, \quad I_r = I_{r-1} \cup \{x\}.$$

But then by Relation (2) above this product is zero. If $J \backslash I$ is an antichain, the product (23) only can change sign when we take different paths, and is non-zero. $\square$

Corollary A.2. *The global dimension of the incidence algebra $I(\hat{P})$ is the longest antichain in P.*

Proof. The semi-simple part $\prod_{p \in \hat{P}} k_p$ of the incidence algebra $I(\hat{P})$ has minimal resolution of length the highest degree in which the Koszul dual algebra $E(\hat{P})$ lives. This is due to the resolution of the semi-simple part being given by the Koszul dual algebra [6], and then [3, I.5.1]. This degree is the length of the longest antichain in P. $\square$

Appendix B. Equivalence with modules over incidence algebras

We show that the categories of shift modules are equivalent to module categories for incidence algebras of the partially ordered sets that occur in our setting.

B.1. **The finite case.** Let $\omega = \{0 < 1\}$. The distributive lattice $\hat{P}$ then identifies as $\mathrm{Hom}(P^{\mathrm{op}}, \omega)$. Considering the poset $\mathrm{Hom}([m], \hat{[n]})$ we then have

$$\mathrm{Hom}([m], \hat{[n]}) = \mathrm{Hom}([m], \mathrm{Hom}([n]^{\mathrm{op}}, \omega)) = \mathrm{Hom}([m] \times [n]^{\mathrm{op}}, \omega).$$

So this is the distributive lattice associated to $[m] \times [n]^{\mathrm{op}}$. As a consequence of Corollary A.2 we have:

Corollary B.1. *The global dimension of the incidence algebra of*

(a) *Hom*$([m], \hat{[n]})$ *is* $\min\{m, n\}$.
(b) *Hom*$(\mathbb{N}, \hat{[n]})$ *and Hom*$([n], \hat{\mathbb{N}})$ *is* n.

Proof. (a) The longest antichain in $[m] \times [n]^{\mathrm{op}}$ has length $\min\{m, n\}$. Similarly the longest antichain in $\mathbb{N} \times [n]^{\mathrm{op}}$ has length n. $\square$

Denote the incidence algebra of $\mathrm{Hom}([m], \hat{[n]})$ as $I(m, n)$. Now given a finite dimensional module $M = \oplus_{f \in \mathrm{Hom}([m], \hat{[n]})} M_f$ over this incidence algebra. Recall the map Λ in (8) in Section 3. Let $M_{\Lambda f} = M_f$. We get a vector space ΛM graded by the monomials $\mathrm{Mon}_{\leq n}(x_{[m]})$ (These monomials are in one-one correspondence with $\Delta_{m+1}(n)$.)

$$\Lambda M := \bigoplus_{f \in \mathrm{Hom}([m], \hat{[n]})} M_{\Lambda f}.$$

Proposition B.2. *The correspondence $M \to \Lambda M$ gives an isomorphism of categories of finite dimensional modules:*

$$\textit{modules over } I(m, n) \leftrightarrow \textit{shift modules over } \Delta_{m+1}(n) \cong \textit{\textbf{shmod}}_{\leq n} k[x_{[m]}].$$

Proof. Let α be such that $\alpha(p) < \alpha(p+1)$ and i_p the bump function which takes value 1 at p and zero elsewhere, and $\beta = \alpha + i_p$. The multiplication map

$$M_\beta \xleftarrow{\cdot [\beta, \alpha]} M_\alpha$$

in the incidence algebra corresponds to the shift map

$$s_p : M_{\Lambda\alpha} \to M_{\Lambda\beta}.$$

If $p < q$ such that $\alpha(q) < \alpha(q+1)$, let

$$\gamma = \alpha + i_q, \quad \phi = \alpha + i_p + i_q.$$

The relation

$$[\phi, \beta] \cdot [\beta, \alpha] = [\phi, \gamma] \cdot [\gamma, \alpha]$$

then corresponds to s_p and s_q commuting. □

Each element f of the poset $\mathrm{Hom}([m], \hat{[n]})$ gives an indecomposable projective module $P(f)$ of the incidence algebra $I(m, n)$. The module in $\mathtt{shmod}_{\leq n} k[x_{[m]}]$ corresponding to $P(f)$ is the principal strongly stable ideal $\langle \Lambda f \rangle \subseteq k[x_{[m]}]$. As a consequence of Corollary B.1 we get:

Corollary B.3. *The global dimension of the module category* $\mathtt{shmod}_{\leq n} k[x_{[m]}]$ *is* $\min\{m, n\}$.

In particular if I is a strongly stable ideal in $k[x_{[m]}]$ generated by monomials of degree 2, it has projective dimension one in this category (or zero if it has only a single strongly stable generator). In contrast, in the ordinary category of modules over the polynomial ring $S = k[x_{[m]}]$, it may have any projective dimension up to $m - 1$.

All of the above may be extended to the poset $\mathrm{Hom}([m], \hat{\mathbb{N}})$ giving an incidence algebra $I(m, \mathbb{N})$. Thus the indecomposable projectives in $\mathtt{shmod}\, k[x_{[m]}]$ are precisely the principal strongly stable ideals for this ring.

We may also use the correspondence Γ to get shift modules. Given again a module $M = \oplus_{f \in \mathrm{Hom}([m], \hat{[n]})} M_f$ over the incidence algebra $I(m, n)$ we get a shift module over $\Delta_{n+1}(m)$:

$$\Gamma M = \bigoplus_{f \in \mathrm{Hom}([m], \hat{[n]})} (M_{\Gamma f})^*.$$

As above we get:

Proposition B.4. *The correspondence $M \to \Gamma M$ gives an isomorphism of categories of finite dimensional modules*

$$\textit{modules over } I(m, n) \longrightarrow \textit{shift modules over } \Delta_{n+1}(m) \cong \mathtt{shmod}_{\leq m} k[x_{[n]}].$$

B.2. **Duals.** If $M = \oplus_{p \in P} M_p$ is a module over an incidence algebra $I(P)$, we get a module $M^\vee$ over the incidence algebra $I(P^{\mathrm{op}})$ of the opposite poset (where $*$ denotes dual vector space)

$$M^\vee = \bigoplus_{p^{\mathrm{op}} \in P^{\mathrm{op}}} (M^\vee)_{p^{\mathrm{op}}} := \bigoplus_{p \in P} (M_p)^*.$$

Since $\mathrm{Hom}([m], \hat{[n]})$ and $\mathrm{Hom}([n], \hat{[m]})$ are opposite posets, by the third diagram of (8), we get a commutative diagram (modulo identifying the double dual V^{**} of a finite dimensional vector space with V).

$$\begin{tikzcd}
 & \operatorname{mod} I(m,n) \arrow[dl, "\Gamma"'] \arrow[dd, "()^\vee"] \arrow[dr, "\Lambda"] & \\
\mathtt{shmod}\,\Delta_{n+1}(m) & & \mathtt{shmod}\,\Delta_{m+1}(n) \\
 & \operatorname{mod} I(n,m). \arrow[ul, "\Lambda"] \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

B.3. **The non-finite case.** Consider the poset $\operatorname{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ of small maps, and its incidence algebra $I_S(\mathbb{N}, \hat{\mathbb{N}})$. Again we get a functor

$$\operatorname{mod} I_S(\mathbb{N}, \hat{\mathbb{N}}) \xrightarrow{\Lambda} \mathtt{shmod}\, k[x_{\mathbb{N}}]$$

which is an equivalence of categories.

Moreover for the poset $\operatorname{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ of large maps and its incidence algebra we get an equivalence of categories

$$\operatorname{mod} I^L(\mathbb{N}, \hat{\mathbb{N}}) \xrightarrow{\Gamma} \mathtt{shmod}\, k[x_{\mathbb{N}}].$$

Since $\operatorname{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})$ and $\operatorname{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})$ are opposite posets, a module

$$M = \oplus_{f \in \operatorname{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})} M_f$$

over $I_S(\mathbb{N}, \hat{\mathbb{N}})$ gives a dual module over $I^L(\mathbb{N}, \hat{\mathbb{N}})$

$$M^\vee = \bigoplus_{g \in \operatorname{Hom}^L(\mathbb{N}, \hat{\mathbb{N}})} (M^\vee)_g := \bigoplus_{f \in \operatorname{Hom}_S(\mathbb{N}, \hat{\mathbb{N}})} (M_f)^*,$$

where $g = Df$.

We obtain a commutative diagram

$$\begin{tikzcd}
\operatorname{mod} I_S(\mathbb{N}, \hat{\mathbb{N}}) \arrow[dd, "()^*"] \arrow[dr, "\Lambda"] & \\
 & \mathtt{shmod}\, k[x_{\mathbb{N}}] \\
\operatorname{mod} I^L(\mathbb{N}, \hat{\mathbb{N}}). \arrow[ur, "\Gamma"'] &
\end{tikzcd}$$

Appendix C. The most degenerate ideals

In Section 1 we stated that the strongly stable ideals are the most degenerate ideals in a polynomial ring (characteristic k is 0). We state this in precise form and give the argument as it seems not easy to come by in the literature.

The group $GL(n+1)$ of invertible linear operators on the linear space generated by the variables acts by coordinate change on homogeneous ideals in $I \subseteq k[x_0, \ldots, x_n]$, where in this appendix k may have any characteristic. Let $B = B(n+1)$ be the Borel subgroup of upper triangular matrices of $G = GL(n+1)$, those invertible linear maps sending $x_j \mapsto \sum_{i=1}^{j} \alpha_{ij} x_i$, where the $\alpha_{ij} \in k$. An ideal

I is *Borel-fixed* if $g.I = I$ for every $g \in B$. When char. $k = 0$ this is the same as I being strongly stable [24, Prop. 4.2.4].

Let $\mathrm{Hilb}^{\mathbb{P}^n}$ be the Hilbert scheme of subschemes of the projective space $\mathbb{P}^n$. We get the action

$$GL(n+1) \times \mathrm{Hilb}^{\mathbb{P}^n} \to \mathrm{Hilb}^{\mathbb{P}^n}, \tag{24}$$

where if x corresponds to the ideal I then $g.x$ corresponds to $g.I$.

Theorem C.1.

(a) *The closed orbits of the action* (24) *are precisely the orbits of Borel-fixed ideals.*

(b) *(char. $k = 0$) Any such orbit has exactly one Borel-fixed ideal.*

Note. Part (b) is likely true in arbitrary characteristic but needs a more elaborate proof.

Proof. (a) Given $x \in \mathrm{Hilb}^{\mathbb{P}^n}$ we get a morphism

$$G \longrightarrow \mathrm{Hilb}^{\mathbb{P}^n}, \quad g \mapsto g.x.$$

When x corresponds to a Borel-fixed ideal, it is fixed by B and so by [27, Sec. 12.1] we get a morphism

$$G/B \longrightarrow \mathrm{Hilb}^{\mathbb{P}^n}.$$

But G/B is a projective variety [27, Section 21.3], hence complete and so the image, the orbit of x, is a closed subvariety of $\mathrm{Hilb}^{\mathbb{P}^n}$ [27, Section 21.1].

Conversely suppose an orbit Y of the action (24) is closed, and let Y have the reduced scheme structure. We get a morphism $G \times Y \to \mathrm{Hilb}^{\mathbb{P}^n}$ which factors through Y (since it is reduced) to give $G \times Y \to Y$. The restriction $B \times Y \to Y$ has a fixed point by the Borel fixed point theorem [27, Section 21.2]. This fix point corresponds to an ideal I such that $g.I = I$ for every $g \in B$. So I is a Borel-fixed ideal, and Y is its orbit by $G = GL(n+1)$.

(b) Let I be a strongly stable ideal, and suppose $J = g.I$ is also strongly stable. We show that I and J are equal. Given g, for each $\langle x_1, \ldots, x_i\rangle$ let $\tau(i)$ be minimal such that $g.\langle x_1, \ldots, x_i\rangle \subseteq \langle x_1, \ldots, x_{\tau(i)}\rangle$. Then $\tau(i) \geq i$. Let $S = \{i \,|\, \tau(i) = i\}$. Clearly $n \in S$. If τ' is the associated function to g^{-1}, it is clear that the associated S' must equal S.

Suppose now first $S = \{n\}$. We show that I and J are both the ideal x_n^d for some d. Let $m = \prod_{p=1}^n x_p^{i_p}$ be a minimal strongly stable generator for I. Since $\tau(p) > p$, for each $p < n$ there is a $q = q(p) \leq p$ such that the $g(x_q)$ has a variable with index $> p$ if $p < n$ and index n if $p = n$. Let this index be $r = r(p)$. Then $\prod_{p=1}^n x_{q(p)}^{i_p}$ is in I by it being strongly stable, and $\ell = \prod_{p=1}^n x_{r(p)}^{i_p}$ is in J, since J is monomial. If m is not a power of x_n, we note that $m >_{st} \ell$. By applying the same argument to g^{-1} and ℓ we get an element m' in I with $\ell \geq m'$. But if m is not a power of x_n, this contradicts m being minimal. So $m = x_n^d$ for some d, and so also x_n^d is in J, and these ideals must be equal (they contain all monomials of degree d).

Now consider the general case $S = \{s_1 < s_2 < \cdots < s_r = n\}$. We claim that every minimal strongly stable generator of I has the form $\prod_{u=1}^r x_{s_u}^{j_u}$. Then $J = g.I$ must also have these as generators and so $J = g.I$. Let m be a minimal generator for I and write $m = \prod_{u=1}^r m_u$, where the variables in m_u are x_j with $s_{u-1} < j \leq s_u$. By the same type of argument as in the $S = \{n\}$ case, we will have $m' = \prod_{u=1}^r x_{s_u}^{j_u}$ in I where $d_u = \deg m_u$. $\square$

Corollary C.2. *(char. $k = 0$) If I is Borel-fixed and $>$ any term ordering, then for g in an open subset of $GL(n+1)$ the initial ideal (the generic initial ideal) $in_{>}(g.I) = I$.*

Proof. The initial ideal $\mathrm{in}_{>}(g.I)$ is the limit at $t = 0$ of a family of ideals parametrized by $\mathrm{Spec} k[t]$ [24, Section 3.2], and whose general member is a coordinate change of $g.I$ and so of I. Thus $\mathrm{in}_{>}(g.I)$ is in the closure of the orbit of $g.I$ and so is *in* the orbit of $g.I$ or equivalently of I. But since the generic initial ideal $\mathrm{in}_{>}(g.I)$ is Borel-fixed [14, Chap.15], it is then equal to I. ☐

Remark C.3. Several people have to their surprise observed Corollary C.2. It is stated and shown for g in an open subset of $GL(n+1)$ in [24, Prop. 4.2.6(b)], and they attribute it to A. Conca. Theorem C.1 has been known by M. Stillman since the late 1980's, who learned it from D. Bayer. He also informed that Theorem C.1 and Corollary C.2 essentially follow from Borel's fixed point theorem, as shown above.

Galligo's theorem [21] that any ideal degenerates to a Borel ideal and Theorem C.1 are inspiration for approaches to the classification of Hilbert scheme components using Borel ideals, [8], [12, 19, 29], and recently [41, 45].

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