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RIESZ AND GREEN ENERGY ON PROJECTIVE SPACES

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Abstract. In this paper we study Riesz, Green and logarithmic energy on two-point homogeneous spaces. More precisely we consider the real, the complex, the quaternionic and the Cayley projective spaces. For each of these spaces we provide upper estimates for the mentioned energies using determinantal point processes. Moreover, we determine lower bounds for these energies of the same order of magnitude.

1. Introduction

Motivated by classical potential theory (see, for instance [40]) discrete energies of point sets on manifolds have been studied. More precisely, for a given symmetric and lower semi-continuous kernel $K : \Omega \times \Omega \to \mathbb{R}$ on a metric space Ω the *discrete energy* of a set $\omega_N = \{x_1, \dots, x_N\} \subset \Omega$ is given by

$$E_K(\omega_N) = \sum_{\substack{i,j=1\\ i\neq j}}^{N} K(x_i, x_j).$$

In rather general settings the empirical measures associated to minimizing configurations of $E_K(\omega_N)$ for $N \to \infty$ converge weakly to the minimizing measure of the continuous energy

$$I_K(\mu) = \iint\limits_{\Omega\times\Omega} K(x, y)\, d\mu(x)\, d\mu(y)$$

amongst all Borel probability measures. For more details and a comprehensive introduction to the subject we refer to [16].

For a sufficiently repulsive potential, one expects the minimizing configurations of the discrete energy to be well-distributed, in some sense. Perhaps the best-known example of such potentials are the classical Riesz s-energies ($s > 0$) for infinite compact sets $\Omega \subseteq \mathbb{R}^d$

$$J_s(x, y) := \frac{1}{\|x - y\|^s}.$$

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It has been shown in [32] that, under rather general conditions, if $s \geq \dim(\Omega)$, then the minimizers of E_{J_s} are uniformly distributed. This uniformity of minimizers does not hold in general for $s < \dim(\Omega)$. However, due to the highly symmetric structure of the sphere $\mathbb{S}^{d-1}$, one finds that for the Riesz potentials J_s with $0 < s < d-1$ and the logarithmic potential

$$J_0(x, y) = -\log(\|x - y\|),$$

which is obtained by a limiting process for $s \to 0$, the continuous energies I_{J_s} and I_{J_0} are uniquely minimized by the uniform measure on the sphere σ, and the minimizers of the discrete energies are uniformly distributed.

The minimal energy of N points for the kernel J_s ($s \geq 0$) on a space Ω,

$$\mathcal{E}_{J_s}(\Omega, N) = \min_{\omega_N \subset \Omega} E_{J_s}(\omega_N),$$

has been investigated especially for the sphere $\mathbb{S}^{d-1}$, see for instance [54, 55]. For $0 < s < d-1$ it satisfies

$$-C_1 N^{1+\frac{s}{d-1}} \leq \mathcal{E}_{J_s}(\mathbb{S}^{d-1}, N) - I_{J_s}(\sigma) N^2 \leq -C_2 N^{1+\frac{s}{d-1}},$$

where C_1 and C_2 are positive constants. The term $I_{J_s}(\sigma) N^2$ of highest order reflects the fact that the empirical measures of the discrete minimizers weakly tend to σ. It is conjectured that a more precise asymptotic equation

$$\mathcal{E}_{J_s}(\mathbb{S}^{d-1}, N) = I_{J_s}(\sigma) N^2 - C N^{1+\frac{s}{d-1}} + o(N^{1+\frac{s}{d-1}})$$

holds, where the precise value of the constant C is believed to reflect the local structure of minimizing configuration. For more details we refer to [19] and [32]. The conjectural values of the constant are related to zeta functions of certain lattices, which relates the question to lattice energies on Euclidean spaces.

Motivated by these results, as well as certain other recent works mentioned below, we extend these results known for spheres to the projective spaces $\mathbb{FP}^{d-1}$ over scalar domains $\mathbb{F}$ (the real or complex numbers, the quaternions, or the octonions). These spaces together with the spheres are the only compact connected two-point homogeneous spaces (see [56]). On these projective spaces, we study the energies given by the *chordal* Riesz s-kernels

$$K_s(x, y) = \frac{1}{\rho(x, y)^s} = \frac{1}{\sin(\vartheta(x, y))^s} \quad \text{for } s > 0,$$

and chordal logarithmic kernel

$$K_0(x, y) = -\log(\rho(x, y)) = -\log \sin(\vartheta(x, y)),$$

where ρ and ϑ are the chordal and geodesic metrics, respectively, discussed below. The case of $s < \dim(\Omega)$ is the subject of classical potential theory (see, for instance [40]), the corresponding kernels are called *singular*, whereas the kernels for $s \geq \dim(\Omega)$ are called *hypersingular*. Each of the projective spaces can be embedded in a sufficiently high dimensional unit sphere, in which case the chordal distance becomes the Euclidean distance, making these energies the natural generalization of the classical Riesz and logarithmic energies on the sphere, but without requiring the embedding itself.

The study of Riesz energies on projective spaces, particularly $\mathbb{CP}^{d-1}$, has been a subject of recent interest. In[1, 4, 5], the authors studied various potential theoretic properties of the logarithmic energy on complex projective spaces. The expected Riesz and logarithmic energies of zero sets of independent Gaussian polynomials on

$\mathbb{CP}^{d-1}$ (and more generally on Kähler manifolds) was determined in [26], whereas in [9], the authors computed the expected energies for certain determinantal point processes to find asymptotic upper bounds on the Riesz and logarithmic energies. More qualitative properties of the minimizers of the logarithmic and Riesz energies on the real and complex projective spaces were studied in [22]. The authors found that the minimizers of these energies are uniformly distributed, including the hypersingular case, and that these minimizers approximate tight frames (acting as element on the real and complex spheres). Moreover, as $s \to \infty$, the minimizers of the Riesz energies approximate best packings on these spaces (i.e. frames with low coherence). Both tight frames on real and complex spaces as well as best packings on Grassmannians have applications to signal processing (see, e.g., [21, 38, 45]).

Another natural kernel to study on the projective spaces is the Green function, $G(x, y)$, associated to the Laplace-Beltrami operator. The Green function is a smooth potential, intrinsic to any Riemannian manifold, that behaves similarly to a Riesz energy at short ranges. On the sphere and projective spaces, it is in fact a function of distance only, making computations much more feasible. The minimization of Green energies on compact Riemannian manifolds was first studied in [7], where it was shown that the continuous Green energy is uniquely minimized by the uniform measure and that minimizers of the discrete Green energy are uniformly distributed. The minimizers of the Green energy have more recently been shown to be well-seperated in [25], and to have the optimal asymptotic quadratic Wasserstein distance from the uniform measure in [51]. Upper bounds for the minimal discrete Green energy on complex projective spaces were determined in [9] using determinantal point processes different from the ones used in our paper. This upper bound was of the optimal order, which was determined for general compact Riemannian manifolds in [51].

1.1. **Summary of paper and main results.** In Sections 1.2 and 1.3 we list some necessary notation and properties of Jacobi polynomials, which we make extensive use of in this paper.

In order to make this paper (mostly) self-contained and gather the required material, in Section 2 we cover the necessary background for harmonic analysis on compact connected two-point homogeneous spaces, as well as some of its consequences. In Section 2.5, we obtain explicit formulae for the Green functions on the projective spaces. In Section 2.6, we show that the Riesz and logarithmic kernels are strictly positive definite, and obtain the following result:

Theorem 1.1. *The continuous logarithmic energy I_{K_0}, Green energy I_G, and Riesz s-energies I_{K_s}, for $0 < s < \dim(\mathbb{FP}^{d-1})$, are uniquely minimized by the uniform measure σ.*

Moreover, if $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of minimizers for the discrete energies E_{K_0}, E_G, or E_{K_s}, for $0 < s < \dim(\mathbb{FP}^{d-1})$, then $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of uniformly distributed point configurations.

We finish Section 2 by discussing the properties of the heat kernel we use to find lower bounds on the minimal discrete Green energy.

In Section 3, we define determinantal point processes given by rotation invariant kernels on the projective spaces. The processes are defined by projections to spaces of harmonic functions, thus they are called *harmonic ensembles* following [12].

In Section 4, we provide lower and upper asymptotic bounds on the minimal discrete Riesz, logarithmic, and Green energies on the projective spaces, cumulating in the following three results:

Theorem 1.2. *Let $d > 2$. For each projective space $\mathbb{FP}^{d-1}$ and $0 < s < \dim(\mathbb{FP}^{d-1})$ there exist positive constants C_s, C'_s such that for $N \geq 2$*

$$-C_s N^{1+\frac{s}{\dim(\mathbb{FP}^{d-1})}} \leq \mathcal{E}_{K_s}(\mathbb{FP}^{d-1}, N) - I_{K_s}(\sigma)N^2 \leq -C'_s N^{1+\frac{s}{\dim(\mathbb{FP}^{d-1})}}.$$

Theorem 1.3. *Let $d > 2$. For each projective space $\mathbb{FP}^{d-1}$, there exist positive constants C_0, C'_0 such that for $N \geq 2$*

$$-C_0 N \log(N) \leq \mathcal{E}_{K_0}(\mathbb{FP}^{d-1}, N) - I_{K_0}(\sigma)N^2 \leq -C'_0 N \log(N).$$

Theorem 1.4. *Let $d > 2$. For each projective space $\mathbb{FP}^{d-1}$, there exist positive constants C_G, C'_G such that for $N \geq 2$*

$$-C_G N^{2-\frac{2}{\dim(\mathbb{FP}^{d-1})}} \leq \mathcal{E}_G(\mathbb{FP}^{d-1}, N) \leq -C'_G N^{2-\frac{2}{\dim(\mathbb{FP}^{d-1})}},$$

unless $\mathbb{FP}^{d-1} = \mathbb{RP}^2$, in which case

$$-C_G N \log(N) \leq \mathcal{E}_G(\mathbb{FP}^{d-1}, N) \leq -C'_G N \log(N).$$

The order of the upper bounds for each of these results is proved in Section 4.1 through jittered sampling using equal area partitions with some extra control on the diameters. Such partitions exist on general Ahlfors regular metric measure spaces by [30]. However, without a deeper understanding of the geometry of these spaces, the method of jittered sampling does not give explicit values for the constants C'_s, C'_0, and C'_G in the above theorems. In Section 4.2, we compute the expected Riesz, logarithmic, and Green energies of the harmonic ensemble. This provides a more concrete upper bound on the minimal energies, with an explicit constant for the next-order term and the order of the error term, though only for certain values of N (see (4.5), (4.7), (4.8), and (4.9)). In addition, we compute the expected Riesz s-energy for $s = \dim \mathbb{FP}^{d-1}$ of this ensemble, resulting in an asymptotic upper bound on the minimum of this hypersingular energy. In Section 4.3, we determine the order of the lower bounds for the Riesz and logarithmic energies through linear programming using the complete monotonicity of the corresponding kernels as a function of chordal distance. Finally, in Section 4.4 we give lower bounds for the Green energy, with explicit values of the next-order term. We achieve these lower bounds again via linear programming, this time making use of lower bounds for the Green function obtained from the positivity of the heat kernel.

We collect our explicit upper and lower bounds for the minimal discrete Green energies on projective spaces, with the lower bounds holding for all N and the upper bound holding only for certain values of N, in Table 1.

For the complex projective spaces $\mathbb{CP}^{d-1}$ with $d > 4$, this upper bound is an improvement upon the previously best known upper bound

$$\mathcal{E}_G(N) \leq -\frac{d-1}{4(d-2)}\left(\frac{1}{(d-1)!}\right)^{\frac{1}{d-1}} N^{2-\frac{2}{2d-2}}$$

for $N = \binom{d+n-1}{n}$ [9, Theorem 1.3]. We note that in their paper, Beltrán and Etayo took the volume of the $\mathbb{CP}^{d-1}$ to be $\frac{\pi^{d-1}}{(d-1)!}$, so we have adjusted their result to match our normalization (volume being 1). This upper bound was achieved through a determinantal point process on the complex space $\mathbb{C}^{d-1}$ with a kernel constructed

TABLE 1. Lower and upper bounds for the minimal discrete Green energy on projective spaces in terms of the number of points N. The upper bound only holds for $N = \frac{(\alpha+\beta+2)_n(\alpha+2)_n}{(\beta+2)_n n!}$ (defined in 2.1).

Ω	lower bound	upper bound
$\mathbb{RP}^2$	$-\frac{1}{2}N\log(N)+\mathcal{O}(N)$	$-\frac{1}{2}N\log(N)+\mathcal{O}(N)$
$\mathbb{RP}^3$	$-\frac{3}{4}(\pi)^{\frac{1}{3}}N^{2-\frac{2}{3}}+\mathcal{O}(N\log(N))$	$-\frac{9}{16}\left(\frac{4}{3}\right)^{\frac{2}{3}}N^{2-\frac{2}{3}}+\mathcal{O}(N)$
$\mathbb{RP}^{d-1}$	$-\frac{d-1}{4(d-3)}\left(\frac{\sqrt{\pi}}{\Gamma(d/2)}\right)^{\frac{2}{d-1}}N^{2-\frac{2}{d-1}}$ $+\mathcal{O}(N^{2-\frac{3}{d-1}})$	$-\frac{(d-1)^2}{8(d-2)(d-3)}\left(\frac{\sqrt{\pi}}{\Gamma(\frac{d}{2})\Gamma(\frac{d+1}{2})}\right)^{\frac{2}{d-1}}$ $\times N^{2-\frac{2}{d-1}}+\mathcal{O}(N^{2-\frac{3}{d-1}})$
$\mathbb{CP}^{d-1}$	$-\frac{d-1}{4(d-2)}\left(\frac{1}{\Gamma(d)}\right)^{\frac{1}{d-1}}N^{2-\frac{2}{2d-2}}$ $+\mathcal{O}(N^{2-\frac{3}{2d-2}})$	$-\frac{(d-1)}{4(2d-3)}\left(\frac{1}{\Gamma(d)}\right)^{\frac{2}{d-1}}N^{2-\frac{2}{2d-2}}$ $+\mathcal{O}(N^{2-\frac{3}{2d-2}})$
$\mathbb{HP}^{d-1}$	$-\frac{d-1}{2(2d-3)}\left(\frac{1}{\Gamma(2d)}\right)^{\frac{1}{2d-2}}N^{2-\frac{2}{4d-4}}$ $+\mathcal{O}(N^{2-\frac{3}{4d-4}})$	$-\frac{(d-1)^2}{(2d-3)(4d-5)}\left(\frac{1}{\Gamma(2d)\Gamma(2d-1)}\right)^{\frac{1}{2d-2}}$ $\times N^{2-\frac{2}{4d-4}}+\mathcal{O}(N^{2-\frac{3}{4d-4}})$
$\mathbb{OP}^2$	$-\frac{2}{7}\left(\frac{6}{11!}\right)^{\frac{1}{8}}N^{2-\frac{2}{16}}+\mathcal{O}(N^{2-\frac{3}{16}})$	$-\frac{16}{21}\left(\frac{6}{(11!)(8!)}\right)^{\frac{1}{8}}N^{2-\frac{2}{16}}$ $+\mathcal{O}(N^{2-\frac{3}{16}})$

from functions on this space, and then mapping this process to the complex projective space $\mathbb{CP}^{d-1}$ via the map $z \mapsto (1, z)$, creating the *projective ensemble*. As pointed out, our upper bound also comes from a determinantal point process, but the kernel in this instance is built directly from functions on $\mathbb{CP}^{d-1}$. The rotational invariance that results from this seems to lead to an improvement; however, it also means that our bound holds for different values of N ($N = \frac{((d-1)!)^2}{n+1}\binom{d+n-1}{n}^2$).

Similarly, our estimates for the minimal Riesz and logarithmic energies on complex projective spaces resulting from the harmonic ensemble ((4.5) and (4.7)) generally match or improve upon previously known results. A first instance of applying point processes to obtain estimates for the Riesz and logarithmic energies on $\mathbb{CP}^{d-1}$ is [26]. There, Feng and Zelditch studied the expectation of these energies for the zero set of $d-1$ degree m Gaussian random polynomials. They obtain the correct main term and the correct order of the second asymptotic term for $0 \le s \le \min(4, D)$; nevertheless, their second order term becomes positive for s close to 4.

More recently, Beltrán and Etayo also applied their projective ensemble to find estimates for the minimal Riesz and logarithmic energies on $\mathbb{CP}^{d-1}$. For $0 < s < 2d-2$ and $N = \binom{d+n-1}{n}$, they obtained the upper bound [9, Theorem 3.3])

$$\mathcal{E}_{K_s}(N) \le I_{K_s}(\sigma)N^2 - \frac{(d-1)\Gamma\left(d-1-\frac{s}{2}\right)}{(\Gamma(d))^{1-\frac{s}{2d-2}}}N^{1+\frac{s}{2d-2}} + o(N^{1+\frac{s}{2d-2}}).$$

Numerical evidence strongly suggests that the coefficient of the next-order term in this bound and the one we obtain in Theorem 4.5 equate at some unique value $s = s_{d-1}$ in each dimension, with their bound winning for values below s_{d-1} and ours being better for values above s_{d-1}. These values s_{d-1} appear to be bounded from above by some constant s^*. Assuming such an s^* exists, we believe $s^* \approx 6.0365$, which we obtain by solving

$$e^{-\frac{s^*}{2}} \frac{\Gamma\left(1+s^*\right)}{\Gamma\left(1+\frac{s^*}{2}\right)^2} = 1,$$

which results from letting d to tend to infinity in the ratio of the two coefficients and assuming them to be equal. In particular, our bound appears to represent an improvement for all s exceeding a fixed value independent of dimension.

We also provide the expected logarithmic energy of our harmonic ensemble on $\mathbb{CP}^{d-1}$ in Theorem 4.7, achieving the same next order term, $-\frac{1}{2d-2} N \log(N)$, as achieved by the expected logarithmic energies in [9, Corollary 3.4] and [26, Corollary 1]. The lower bound we determine in Theorem 4.12 shows that this is indeed the best possible coefficient for the second order term.

1.2. **Notation.** Throughout the paper we will use the following notations:

- The *Pochhammer symbol*

$$(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)},$$

- the *digamma function*

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+x}\right),$$

- the *harmonic numbers*

$$H_k = \sum_{\ell=1}^{k} \frac{1}{\ell} = \psi(k+1) + \gamma,$$

- where

$$\gamma = -\Gamma'(1) = \lim_{k\to\infty} H_k - \log(k)$$

  denotes the *Euler-Mascheroni constant.*
- We will also make frequent use of the asymptotic relations

$$\frac{\Gamma(n+x)}{\Gamma(n+y)} = n^{x-y}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty$$

  and

$$\binom{n+x}{n} = \frac{n^x}{\Gamma(x+1)}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right) \text{ as } n \to \infty.$$

- We will denote the set of finite Borel measures on a space Ω as $\mathcal{B}(\Omega)$, the set of Borel probability measures as $\mathbb{P}(\Omega)$, the set of finite signed Borel measures as $\mathcal{M}(\Omega)$, and the set of finite signed Borel measures with total mass zero, i.e. $\nu \in \mathcal{M}(\Omega)$ satisfying $\nu(\Omega) = 0$, as $\mathcal{Z}(\Omega)$.

1.3. **Jacobi polynomials.** The classical Jacobi polynomials will play a prominent role in this paper. Thus we collect some basic facts about them. The *Jacobi polynomials* $P_n^{(\alpha,\beta)}(t)$ are the orthogonal polynomials for the weight function $(1-t)^\alpha(1+t)^\beta$ on the interval $[-1,1]$. Throughout the paper we will use the substitution $t=\cos(2\vartheta)$. The measure is then normalised and transformed to a measure on the interval $[0,\frac{\pi}{2}]$ that we denote by

$$d\nu^{(\alpha,\beta)}(\vartheta)=\frac{1}{\gamma_{\alpha,\beta}}\sin(\vartheta)^{2\alpha+1}\cos(\vartheta)^{2\beta+1}\,d\vartheta, \tag{1.1}$$

where

$$\gamma_{\alpha,\beta}=\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{2\Gamma(\alpha+\beta+2)}.$$

The Jacobi polynomials can be given by Rodrigues' formula (see [43, p. 213])

$$P_n^{(\alpha,\beta)}(t)=\frac{(-1)^n}{2^n n!}\frac{1}{(1-t)^\alpha(1+t)^\beta}\frac{d^n}{dt^n}\left((1-t)^{n+\alpha}(1+t)^{n+\beta}\right).$$

The value

$$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$$

and the relation

$$\int_0^{\frac{\pi}{2}}\left(P_n^{(\alpha,\beta)}(\cos(2\vartheta))\right)^2 d\nu^{(\alpha,\beta)}(\vartheta)=\frac{\alpha+\beta+1}{2n+\alpha+\beta+1}\frac{(\alpha+1)_n(\beta+1)_n}{n!(\alpha+\beta+1)_n}$$

will occur frequently throughout.

We will use the summation formula

$$\sum_{k=0}^{n}\frac{2k+\alpha+\beta+1}{\alpha+\beta+1}\frac{(\alpha+\beta+1)_k}{(\beta+1)_k}P_k^{(\alpha,\beta)}(t)=\frac{(\alpha+\beta+2)_n}{(\beta+1)_n}P_n^{(\alpha+1,\beta)}(t) \tag{1.2}$$

at several occasions; this is a special case of a connection formula for Jacobi polynomials with different parameters given in [3, Theorem 7.1.3].

Furthermore the orthogonality of the Jacobi polynomials allows expanding functions $F(\cos(2\vartheta))\in L^2([0,\frac{\pi}{2}],d\nu^{(\alpha,\beta)})$ in terms of $P_n^{(\alpha,\beta)}$:

$$F(t)=\sum_{n=0}^{\infty}\widehat{F}(n)P_n^{(\alpha,\beta)}(t), \tag{1.3}$$

where

$$\widehat{F}(n)=\frac{m_n}{\left(P_n^{(\alpha,\beta)}(1)\right)^2}\int_0^{\frac{\pi}{2}}F(\cos(2\vartheta))P_n^{(\alpha,\beta)}(\cos(2\vartheta))\,d\nu^{(\alpha,\beta)}(\vartheta) \tag{1.4}$$

and

$$m_n=\frac{2n+\alpha+\beta+1}{\alpha+\beta+1}\frac{(\alpha+\beta+1)_n(\alpha+1)_n}{n!(\beta+1)_n}.$$

The convergence of the series (1.3) is *a priori* in the L^2-sense. In the case that F is continuous on $[-1,1]$ and all the coefficients $\widehat{F}(n)$ are non-negative, Mercer's theorem (see, for instance [27]) ensures absolute and uniform convergence. This will be the content of Lemma 2.16.

2. Harmonic analysis on two-point homogeneous spaces

2.1. **Classification of two-point homogeneous spaces.** We call a connected Riemannian manifold (Ω, p) *homogeneous* if there is a Lie group G acting transitively on Ω. This implies that Ω is homeomorphic to the quotient space G/G_a, where $G_a := \{g \in G : ga = a\}$ is the stabilizer of a point $a \in \Omega$. The choice of $a \in \Omega$ does not matter in this instance, as all stabilizers are conjugate by transitivity. Let ϑ_p be the metric induced by the metric tensor p. The metric induces a volume form, which we normalize to obtain the normalized surface measure σ. If G acts by isometries this equals the measure induced by the Haar-measure on G.

If G is the isometry group of the homogeneous space Ω, we call Ω *two-point homogeneous* if for all $x_1, x_2, y_1, y_2 \in \Omega$ with $\vartheta_p(x_1, x_2) = \vartheta_p(y_1, y_2)$, there is an isometry $g \in G$ such that $gx_i = y_i$, $i = 1, 2$. All two-point homogeneous Riemannian manifolds have been classified (see [36, Chapter I.4]). The noncompact spaces are the Euclidean spaces $\mathbb{R}^d$, the real, complex, and quaternionic hyperbolic spaces, and the hyperbolic analogue of the Cayley plane [53]. The only compact connected two-point homogeneous Riemannian manifolds are the real unit spheres $\mathbb{S}^{d-1}$, the real projective spaces $\mathbb{RP}^{d-1}$, the complex projective spaces $\mathbb{CP}^{d-1}$, the quaternionic projective spaces $\mathbb{HP}^{d-1}$, and the Cayley projective plane $\mathbb{OP}^2$ (see [56]), with the quotient representations given in [57, pp. 28-29]

$$\begin{aligned}
\mathbb{S}^{d-1} &\cong \mathrm{SO}(d)/\mathrm{SO}(d-1)\\
\mathbb{RP}^{d-1} &\cong \mathrm{O}(d)/\Big(\mathrm{O}(d-1)\times \mathrm{O}(1)\Big),\\
\mathbb{CP}^{d-1} &\cong \mathrm{U}(d)/\Big(\mathrm{U}(d-1)\times \mathrm{U}(1)\Big),\\
\mathbb{HP}^{d-1} &\cong \mathrm{Sp}(d)/\Big(\mathrm{Sp}(d-1)\times \mathrm{Sp}(1)\Big),\\
\mathbb{OP}^{2} &\cong F_4/\mathrm{Spin}(9).
\end{aligned}$$

When talking about the projective spaces in general we denote the scalar domain by $\mathbb{F}$.

Note that it suffices to consider $\mathbb{FP}^{d-1}$ for $d > 2$ only, as $\mathbb{FP}^1$ is isomorphic to the sphere $\mathbb{S}^{\dim_{\mathbb{R}}(\mathbb{F})}$ (see [6, p. 170]), so those will not be considered in what follows.

For each two-point homogeneous space with underlying scalar domain $\mathbb{F}$, we associate parameters

$$\alpha = (d-1)\frac{\dim_{\mathbb{R}}(\mathbb{F})}{2} - 1, \quad \beta = \begin{cases} \alpha, & \text{for } \Omega = \mathbb{S}^{d-1};\\ \frac{\dim_{\mathbb{R}}(\mathbb{F})}{2} - 1, & \text{for } \Omega = \mathbb{FP}^{d-1}. \end{cases} \tag{2.1}$$

The dependence of α and β on the space and its dimension will be clarified in Section 2.3. Furthermore, we denote $D = \dim(\Omega) = 2\alpha + 2$ the dimension of the space Ω as a real manifold.

From now on, Ω always refers to a two-point homogeneous space, equipped with metric tensor p and corresponding G-invariant probability measure σ, i.e. the normalized uniform surface measure. We let ϑ denote the geodesic distance, normalized to take values in $[0, \frac{\pi}{2\kappa}]$, where $\kappa = \frac{1}{2}$ or $\kappa = 1$, if Ω is a sphere or projective space, respectively. We can also define a *chordal metric* ρ on each of these spaces by

$$\rho(x, y) = \sin(\kappa\vartheta(x, y)) = \sqrt{\frac{1 - \cos(2\kappa\vartheta(x, y))}{2}}, \quad x, y \in \Omega. \tag{2.2}$$

Note that on the sphere, this is the Euclidean distance $\frac{1}{2}\|x-y\|$ in ambient space, and on the complex projective space this is also known as the *Fubini-Study metric.*

Each projective space $\mathbb{FP}^{d-1}$ can be canonically embedded into the unit sphere $\mathbb{S}^{\tilde{d}-1}$, where $\tilde{d} = d(\alpha+2) = \frac{d(d-1)}{2}\dim_{\mathbb{R}}(\mathbb{F}) + d$, so that the chordal metric is equivalent to the Euclidean metric on this embedding, which we will now show.

Let $\mathcal{H}(\mathbb{F}^d)$ be the set of all Hermitian $d \times d$ matrices with entries in $\mathbb{F}$. We see that $\mathcal{H}(\mathbb{F}^d)$ is a linear space over $\mathbb{R}$ of dimension $\tilde{d}$, equipped with the symmetric real-valued inner product

$$\langle A, B\rangle = \frac{1}{2}\mathrm{tr}(A\overline{B^{\mathsf{T}}} + B\overline{A^{\mathsf{T}}}) = \mathrm{Re}(\mathrm{tr}(A\overline{B^{\mathsf{T}}})) = \mathrm{Re}\sum_{i,j=1}^{d} a_{i,j}\overline{b_{i,j}}$$

and norm

$$\|A\|_{\mathcal{H}(\mathbb{F}^d)} = (\mathrm{tr}(A\overline{A^{\mathsf{T}}}))^{\frac{1}{2}} = \Big(\sum_{i,j=1}^{d} |a_{i,j}|^2\Big)^{\frac{1}{2}}. \tag{2.3}$$

For $\mathbb{F} \neq \mathbb{O}$, the orthogonal projector $\Pi_x \in \mathcal{H}(\mathbb{F}^d)$ ($x \in \mathbb{F}^d$, $\|x\| = 1$) onto a one-dimensional subspace $x\mathbb{F}$ can be given by the matrix $\Pi_x = (x_i\overline{x}_j)_{1\le i,j\le d}$, with $x = (x_1, \ldots, x_d)$. Thus, the projective space can be written as

$$\mathbb{FP}^{d-1} \cong \{\Pi \in \mathcal{H}(\mathbb{F}^d) : \Pi^2 = \Pi, \mathrm{Tr}(\Pi) = 1\}. \tag{2.4}$$

The group of isometries $U(d, \mathbb{F})$ acts on these projectors by $g(\Pi) = g\Pi g^{-1}$.

For the Cayley plane, a similar model (as well as a detailed discussion) is given in [6, 28]. In this model, one defines the Cayley plane by

$$\mathbb{OP}^2 \cong \{\Pi \in \mathcal{H}(\mathbb{O}^3) : \Pi^2 = \Pi, \mathrm{Tr}(\Pi) = 1\}. \tag{2.5}$$

Each matrix can be written as $\Pi_x = (x_i\overline{x_j})_{1\le i,j\le 3}$, for a vector $x = (x_1, x_2, x_3) \in \mathbb{O}^3$ with $\|x\|^2 = |x_1|^2 + |x_2|^2 + |x_3|^2 = 1$ and $(x_1x_2)x_3 = x_1(x_2x_3)$ [33, Lemma 14.90].

Equations (2.3), (2.4), and (2.5) show us that for any $\Pi \in \mathbb{FP}^{d-1}$, as defined by the above models,

$$\|\Pi\|^2_{\mathcal{H}(\mathbb{F}^d)} = \mathrm{Tr}(\Pi^2) = \mathrm{Tr}(\Pi) = 1,$$

so the projective spaces are submanifolds in the unit sphere

$$\mathbb{FP}^{d-1} \subset \{\Pi \in \mathcal{H}(\mathbb{F}^d) : \|\Pi\|_{\mathcal{H}(\mathbb{F}^d)} = 1\} \subset \mathcal{H}(\mathbb{F}^d) \cong \mathbb{R}^{\tilde{d}}.$$

This provides an embedding A of $\mathbb{FP}^{d-1}$ into the sphere $\mathbb{S}^{\tilde{d}-1}$. The chordal metric $\rho(\Pi_1, \Pi_2)$, for $\Pi_1, \Pi_2 \in \mathbb{FP}^{d-1}$, is then defined as the Euclidean distance in the embedding

$$\begin{aligned}\rho(\Pi_1, \Pi_2) = \sqrt{1 - \langle \Pi_1, \Pi_2\rangle} &= \frac{1}{\sqrt{2}}\|\Pi_1 - \Pi_2\|_{\mathcal{H}(\mathbb{F}^d)} \\ &= \frac{1}{\sqrt{2}}\|A(\Pi_1) - A(\Pi_2)\|.\end{aligned}$$

2.2. **The Laplace operator and its eigenfunctions.** Let $\triangle_\Omega$ be the Laplace-Beltrami operator on Ω induced by the Riemannian metric p, and let $0 = \lambda_0 < \lambda_1 < \cdots$ be the eigenvalues of $\triangle_\Omega$ and for each $k \in \mathbb{N}$, let $V_k := V(\triangle_\Omega, \lambda_k)$ be the corresponding eigenspace and $m_k = \dim(V_k)$ be the multiplicity of λ_k. Notice that we follow the convention of geometry choosing the sign of the operator so that the eigenvalues are non-negative.

We then have the following version of the Spectral Theorem [24, Chapter 3, Theorem 1.3, and Remark 1.2]

Theorem 2.1. *The eigenvalues of $\triangle_\Omega$ can be arranged in increasing order $0 = \lambda_0 < \lambda_1 < \cdots$, where $\lim_{k\to\infty} \lambda_k = \infty$, each eigenspace V_k has finite dimension, and*

$$L^2(\Omega, \sigma) = \overline{\bigoplus_{k=0}^{\infty} V_k}.$$

For any Riemannian manifold M, we can use geodesic polar coordinates $(r, \vartheta_1, \ldots, \vartheta_{\dim_{\mathbb{R}}(M)-1})$ to parametrize a sufficiently small neighborhood U of any point a, giving U a positive orientation and having a radial component $r(x)$ which is the distance of $x \in M$ from a, as described in [13, Chapter 2], [48, Chapter 2.4], and [34, Chapters IX.5, X.4, and X.7.4].

In particular, on a two-point homogeneous space Ω, such a polar coordinate parameterization can be defined on $\Omega_a := \Omega \setminus \mathbb{S}_\Omega(a, \frac{\pi}{2\kappa})$, where $\mathbb{S}_\Omega(a, r) = \{x \in \Omega : \vartheta(a, x) = r\}$ for $r \in [0, \frac{\pi}{2\kappa}]$ [34, Chapters IX.5, X.4, and X.7.4]. This allows us to separate $\triangle_\Omega$ into a radial and angular component on this set.

Theorem 2.2 ([34, Chapter X.7.4, Lemma 7.12])**.** *If $f \in C^\infty(\Omega)$ and $a \in \Omega$, then on Ω_a the Laplace operator can be expressed in terms of geodesic polar coordinates by*

$$\triangle_\Omega f = -\frac{1}{A(r)} \frac{\partial}{\partial r} \left(A(r) \frac{\partial f}{\partial r} \right) + \triangle_\vartheta f,$$

where $\triangle_\vartheta$ is the Laplace operator on $\mathbb{S}_\Omega(a, r)$, and $A(r)$ denotes the surface measure of $\mathbb{S}_\Omega(a, r)$.

For each $r \in [0, \frac{\pi}{2\kappa}]$, $\mathbb{S}_\Omega(a, r)$ is a submanifold of Ω with Riemannian structure induced by that of Ω. In this case we have for $0 < r < \frac{\pi}{2\kappa}$ (see [35, Proposition 5.6 and p. 171])

$$A(r) = c\kappa^{-2\alpha-1} \sin^{2\alpha+1}(\kappa r) \cos^{2\beta+1}(\kappa r), \tag{2.6}$$

where c is a constant depending on the structure of Ω, and the values α and β are given by (2.1). Recall that for the spaces $\mathbb{FP}^{d-1}$ we choose $\kappa = 1$, whereas for $\mathbb{S}^{d-1}$ we set $\kappa = \frac{1}{2}$.

For functions f only depending on r the Laplace operator then becomes

$$\triangle_r = -\frac{1}{\sin^{2\alpha+1}(\kappa r) \cos^{2\beta+1}(\kappa r)} \frac{d}{dr} \left(\sin^{2\alpha+1}(\kappa r) \cos^{2\beta+1}(\kappa r) \frac{d}{dr} \right),$$

which, making the substitution $z = \cos(2\kappa r)$, becomes

$$\triangle_z = -\frac{4\kappa^2}{(1-z)^\alpha (1+z)^\beta} \frac{d}{dz} \left((1-z)^{\alpha+1} (1+z)^{\beta+1} \frac{d}{dz} \right),$$

see [29, pp. 177-178]. This is the Jacobi operator, for which the only eigenfunctions continuous on $[-1, 1]$ are the Jacobi polynomials $P_k^{(\alpha,\beta)}(z)$, with corresponding eigenvalues $\lambda_k = 4\kappa^2 k(k + \alpha + \beta + 1)$ (see [52, Theorem 4.2.2]). Summing up this discussion we have shown

Theorem 2.3. *Let Ω be a two-point homogeneous space of diameter $\frac{\pi}{2\kappa}$ and $a \in \Omega$. Then the eigenfunctions of $\triangle_\Omega$ on Ω depending only on $\vartheta(a, x)$ are given by*

$$cP_k^{(\alpha,\beta)}(\cos(2\kappa\vartheta(x, a))) \quad \textit{with } c \in \mathbb{R}$$

and the corresponding eigenvalues are

$$\lambda_k = 4\kappa^2 k(k+\alpha+\beta+1)$$

with the values of α *and* β *given by* (2.1).

The eigenvalues and the dimensions of their corresponding eigenspaces are given in Table 2 (see [14, 20, 31, 49]).

TABLE 2. The eigenvalues and dimensions of the eigenspaces of the Laplace operator for two-point homogeneous spaces

Ω	α	β	λ_k	$m_k = \dim(V_k)$
$\mathbb{S}^{d-1}$	$\frac{d-3}{2}$	$\frac{d-3}{2}$	$k(k+d-2)$	$\frac{2k+d-2}{d-2}\binom{k+d-3}{d-3}$
$\mathbb{RP}^{d-1}$	$\frac{d-3}{2}$	$-\frac{1}{2}$	$2k(2k+d-2)$	$\frac{4k+d-2}{d-2}\binom{2k+d-3}{d-3}$
$\mathbb{CP}^{d-1}$	$d-2$	0	$4k(k+d-1)$	$\frac{2k+d-1}{d-1}\binom{d+k-2}{d-2}^2$
$\mathbb{HP}^{d-1}$	$2d-3$	1	$4k(k+2d-1)$	$\frac{2k+2d-1}{(2d-1)(2d-2)}\binom{k+2d-2}{2d-2}\binom{k+2d-3}{2d-3}$
$\mathbb{OP}^2$	7	3	$4k(k+11)$	$\frac{2k+11}{1320}\binom{k+10}{7}\binom{k+7}{7}$

For all two-point homogeneous spaces, we have the general formula

$$m_k = \frac{2k+\alpha+\beta+1}{\alpha+\beta+1}\frac{(\alpha+\beta+1)_k(\alpha+1)_k}{k!(\beta+1)_k}.$$

2.3. **Irreducibility of representations.** Let Ω be a two-point homogeneous space, with isometry group G and G_a the stabilizer of some $a \in \Omega$, meaning that $\Omega \simeq G/G_a$. We then have an orthogonal representation of G in the pre-Hilbert space $C^\infty(\Omega)$ given by $g \mapsto (f(x) \mapsto f(gx))$. Let V be a finite-dimensional subspace of $C^\infty(\Omega)$, invariant under this representation, and let q_V be the representation induced in V.

Definition 2.4. A function f is called *zonal* with respect to $a \in \Omega$ if $f(a) \neq 0$, and for all $g \in G_a$ and $x \in \Omega$, $f(gx) = f(x)$. The zonal functions of V form a vector subspace of V, denoted by $Z_a(V)$.

Lemma 2.5. *If* $V \neq \{0\}$, *then* $Z_a(V) \neq \{0\}$. *If* $\dim(Z_a(V)) = 1$ *for one (and thus all)* $a \in \Omega$, q_V *is irreducible.*

Proof. Consider the linear map $\phi : V \to \mathbb{R}$ defined by

$$\phi(f) = f(a).$$

Since, by assumption, V contains a non-zero function, G is transitive on Ω, and V is G-invariant there exists some $f \in V$ such that $\phi(f)$ is not zero. Thus $\ker(\phi)$ is a subspace of V, which is invariant under G_a. The orthogonal subspace $(\ker(\phi))^\perp$ is also invariant under G_a. Now, let $f \in (\ker(\phi))^\perp$ and μ be the Haar measure on G_a. Define $f^*(x) = \int_{G_a} f(hx)\, d\mu(h)$ for all $x \in \Omega$. Then f^* is in V, non-zero at a, and is G_a-invariant, meaning it is a zonal function, proving our first statement.

Suppose that q_V is reducible. Then, since it is an orthogonal representation, we have a decomposition

$$V = V' \oplus V'',$$

where V' and V'' are both G-invariant and nontrivial. Then each of the two spaces contains a nontrivial subspace of zonal functions, so $\dim(Z_a(V)) \geq 2$. $\square$

Proposition 2.6. *For each $k \in \mathbb{N}_0$, let V_k be the space of eigenfunctions of $\triangle_\Omega$ associated to the k-th eigenvalue λ_k, and let q_k be the orthogonal representation of G in V_k. The zonal functions of V_k are exactly $cP_k^{(\alpha,\beta)}(\cos(2\kappa\vartheta(x,a)))$ for $c \in \mathbb{R}$, and so q_k is irreducible.*

Proof. By Lemma 2.5, $Z_a(V_k) \neq \{0\}$, and it suffices to show that $\dim(Z_a(V_k)) \leq 1$. Let f be a zonal function. Then two-point homogeneity and Theorem 2.3 tell us that $f(x) = cP_k^{(\alpha,\beta)}(\cos(2\kappa\vartheta(x,a)))$ for some $c \in \mathbb{R}$. Thus the space $Z_a(V_k)$ is one-dimensional, and the proposition is proved. $\square$

As a consequence of Theorem 2.1 and Proposition 2.6 we have

Proposition 2.7. *Let H be a finite-dimensional G-invariant subspace of $L^2(\Omega,\sigma)$. Then there exist $0 \leq k_1 < \cdots < k_m$ such that*

$$H = V_{k_1} \oplus \cdots \oplus V_{k_m},$$

where V_k is the eigenspace of $\triangle_\Omega$ corresponding to the eigenvalue λ_k.

2.4. **The addition formula for eigenfunctions of the Laplace operator.** For each $k \in \mathbb{N}_0$, let $Y_{k,1},\ldots,Y_{k,m_k}$ be real-valued functions that form an orthonormal basis of V_k under the inner product $\langle Y,Z\rangle = \int_\Omega Y(x)Z(x)\,d\sigma(x)$.

Theorem 2.8 (Addition formula)**.** *For each $k \in \mathbb{N}_0$ and $x,y \in \Omega$,*

$$\sum_{m=1}^{m_k} Y_{k,m}(x)Y_{k,m}(y) = \frac{m_k}{P_k^{(\alpha,\beta)}(1)} P_k^{(\alpha,\beta)}(\cos(2\kappa\vartheta(x,y))). \tag{2.7}$$

Proof. Let

$$Y_k(x) := \begin{pmatrix} Y_{k,1}(x) \\ \vdots \\ Y_{k,m_k}(x) \end{pmatrix},$$

and note that $\sum_{m=1}^{m_k} Y_{k,m}(x)Y_{k,m}(y) = Y_k(x)^\mathsf{T} Y_k(y)$. For any $\gamma \in G$, $Y_{k,1}(\gamma x),\ldots,$ $Y_{k,m_k}(\gamma x)$ is also an orthonormal basis of V_k, thus there is a $q_k(\gamma) \in \mathrm{O}(V_k)$, so that

$$\begin{aligned} Y_k(\gamma x)^\mathsf{T} Y_k(\gamma y) &= (q_k(\gamma)Y_k(x))^\mathsf{T} q_k(\gamma)Y_k(y) \\ &= Y_k(x)^\mathsf{T} q_k(\gamma)^\mathsf{T} q_k(\gamma) Y_k(y) = Y_k(x)^\mathsf{T} Y_k(y). \end{aligned}$$

In particular, this means that for all $\gamma \in G_y$ $(y = \gamma y)$,

$$Y_k(\gamma x)^\mathsf{T} Y_k(y) = Y_k(\gamma x)^\mathsf{T} Y_k(\gamma y) = Y_k(x)^\mathsf{T} Y_k(y)$$

making $Y_k(x)^\mathsf{T} Y_k(y)$ a zonal function of x, thus it must be a multiple of $P_k^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))$:

$$Y_k(x)^\mathsf{T} Y_k(y) = c_k P_k^{(\alpha,\beta)}(\cos(2\kappa\vartheta(x,y))).$$

Setting $x = y$ and integrating yields

$$\sum_{m=1}^{m_k} \int_\Omega Y_{k,m}(x)^2 \, d\sigma(x) = m_k = c_k P_k^{(\alpha,\beta)}(1),$$

which gives (2.7). □

We note that one can achieve such an addition formula without making use of properties of the Laplace operator and its eigenfunctions, and rather making use of representation theory (see [23, Chapter 9.2.1], [42, Section 3.2]).

Remark 2.9. From now on we only consider the projective spaces $\mathbb{RP}^{d-1}$, $\mathbb{CP}^{d-1}$, $\mathbb{HP}^{d-1}$, and $\mathbb{OP}^2$; the case of the sphere $\mathbb{S}^{d-1}$ has been treated in a very similar manner in [12]. Thus we set $\kappa = 1$ for the remaining part of the paper.

As a corollary of the area formula (2.6), we can see that the weighted measure $\nu^{(\alpha,\beta)}$ given by (1.1) is related to integration on Ω in the following way: for any $a \in \Omega$, and $F(\cos(2\vartheta)) \in L^1([0, \frac{\pi}{2}], \nu^{\alpha,\beta})$,

$$\int\limits_\Omega F(\cos(2\vartheta(x,a)))\, d\sigma(x) = \int\limits_0^{\frac{\pi}{2}} F(\cos(2\vartheta))\, d\nu^{\alpha,\beta}(\vartheta). \tag{2.8}$$

2.5. **The Green function.** The *Green function* G of the Laplace-Beltrami operator is given by the integral operator solving the equation

$$\triangle_\Omega g = f$$

by

$$g(x) = \int_\Omega G(x,y) f(y)\, d\sigma(y)$$

with the additional condition $\int_\Omega g(x)\, d\sigma(x) = 0$. In the case of two-point homogeneous spaces the bivariate function $G(x, y)$ actually only depends on the distance between x and y (see [7]). In fact, as we will show below, the Green functions on the projective spaces have the following closed form expressions.

Proposition 2.10. *The Green function for a projective space* Ω *is given by*

$$\begin{aligned} G(x,y) =& \frac{1}{4(\alpha+\beta+1)} \Bigg(\sum_{\ell=1}^{\alpha} \frac{\binom{\alpha}{\ell}(\ell-1)!}{(\beta+\alpha+1-\ell)_\ell} \frac{1}{\rho(x,y)^{2\ell}} - 2\log(\rho(x,y)) \\ & - \psi(\alpha+\beta+1) - \psi(\alpha+\beta+2) - \gamma + \psi(\beta+1) \Bigg) \end{aligned}$$

for $\alpha \in \mathbb{N}_0$ *and*

$$\begin{aligned} G(x,y) =& \frac{\cos(\vartheta(x,y))}{2\alpha+1} \left(\frac{\pi}{2} - \vartheta(x,y) \right) \sum_{\ell=0}^{\alpha-\frac{1}{2}} \frac{(\frac{1}{2})_\ell}{\ell!} \frac{1}{\rho(x,y)^{2\ell+1}} \\ & + \frac{1}{4\alpha+2} \sum_{\ell=1}^{\alpha-\frac{1}{2}} \frac{1}{\ell} \left(\frac{(\alpha+1-\ell)_\ell}{(\alpha+\frac{1}{2}-\ell)_\ell} - \frac{(\frac{1}{2})_\ell}{\ell!} \right) \frac{1}{\rho(x,y)^{2\ell}} \\ & - \frac{1}{2\alpha+1} H_{\alpha-\frac{1}{2}} - \frac{1}{(2\alpha+1)^2} \end{aligned}$$

for $\alpha \in \frac{1}{2} + \mathbb{N}_0$.

Proof. Using the addition formula (2.7) we obtain the *formal* expression

$$G(x,y)=\sum_{n=1}^{\infty}\frac{2n+\alpha+\beta+1}{4(\alpha+\beta+1)n(n+\alpha+\beta+1)}\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))).$$

Indeed, this series converges absolutely only for $\alpha<\frac{1}{2}$, converges conditionally for $\frac{1}{2}\le\alpha<\frac{3}{2}$, and does not converge at all for $\alpha\ge\frac{3}{2}$. We will show that the series converges in the sense of Abel's summation method (see [58]) and we will compute the limit.

We start with [3, p. 301, eq. (6.4.7)]

$$\sum_{n=0}^{\infty}\frac{(\alpha+\beta+1)_n}{(\beta+1)_n}P_n^{(\alpha,\beta)}(t)z^n=\frac{1}{(1+z)^{\alpha+\beta+1}}\,{}_2\mathrm{F}_1\left(\begin{matrix}\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta+2}{2}\\ \beta+1\end{matrix}\,\middle|\,\frac{2z(1+t)}{(1+z)^2}\right)$$

and define

$$\begin{aligned}\mathcal{G}_{\alpha,\beta}(\cos(2\vartheta))&=\frac{1}{\eta}\lim_{z\to1}\sum_{n=1}^{\infty}\frac{2n+\eta}{4n(n+\eta)}\frac{(\eta)_n}{(\beta+1)_n}P_n^{(\alpha,\beta)}(\cos(2\vartheta))z^n\\&=\int_0^1\left(\frac{1}{(1+z)^{\eta}}\,{}_2\mathrm{F}_1\left(\begin{matrix}\frac{\eta}{2},\frac{\eta+1}{2}\\ \beta+1\end{matrix}\,\middle|\,\frac{4\cos(\vartheta)^2z}{(1+z)^2}\right)-1\right)\frac{1+z^{\eta}}{4\eta z}\,dz,\end{aligned}$$

where the convergence of the integral shows the existence of the limit for $|\cos(\vartheta)|<1$. Here we have set $\eta=\alpha+\beta+1$ for short, which we will use for this computation in the sequel.

The coefficient of $\cos(\vartheta)^{2n}$ for $n\ge1$ is given by

$$\frac{(\eta)_{2n}}{(\beta+1)_n n!}\int_0^1\frac{z^n}{(1+z)^{2n+\eta}}\frac{1+z^{\eta}}{4\eta z}\,dz,$$

where we have used

$$4^n\left(\frac{\eta}{2}\right)_n\left(\frac{\eta+1}{2}\right)_n=(\eta)_{2n}.$$

The integral is split into two parts; we first use the substitution $z=\frac{1}{t}$ to obtain

$$\int_0^1\frac{z^{n+\eta-1}}{(1+z)^{2n+\eta}}\,dz=\int_1^{\infty}\frac{t^{n-1}}{(1+t)^{2n+\eta}}\,dt.$$

This shows that (see [43, p. 6])

$$\int_0^1\frac{z^n}{(1+z)^{2n+\eta}}\frac{1+z^{\eta}}{4\eta z}\,dz=\frac{1}{4\eta}\int_0^{\infty}\frac{z^{n-1}}{(1+z)^{2n+\eta}}\,dz=\frac{(n-1)!}{4\eta(n+\eta)_n}.$$

For the constant term in the Taylor expansion we obtain

$$C_\eta=-\int_0^1\left(1-\frac{1}{(1+z)^{\eta}}\right)\frac{1+z^{\eta}}{4\eta z}\,dz=-\frac{1}{4\eta}\left(\psi(\eta+1)+\gamma\right).$$

This can be obtained by splitting the integral in a similar way as above with some slight modification to preserve convergence.

Putting everything together yields

$$\begin{aligned}\mathcal{G}_{\alpha,\beta}(\cos(2\vartheta))=\frac{1}{4(\alpha+\beta+1)}\Big(&-(\gamma+\psi(\alpha+\beta+2))\\&+\sum_{n=1}^{\infty}\frac{(\alpha+\beta+1)_n}{n(\beta+1)_n}\cos(\vartheta)^{2n}\Big).\end{aligned}$$

The derivation up to now was valid for all $\alpha,\beta>-1$. For the specific values of α and β occurring in the context of projective spaces it turns out that the functions $\mathcal{G}_{\alpha,\beta}$ are indeed elementary. For an alternative derivation of the Green function on the spaces $\mathbb{FP}^{d-1}$ we refer to [7, Appendix A.1].

For $\alpha=k\in\mathbb{N}_0$ this is immediate from the observation that

$$\frac{(\alpha+\beta+1)_n}{n(\beta+1)_n}=\frac{(n+\beta+1)_k}{n(\beta+1)_k},$$

which is a polynomial in n of degree k divided by n. More precisely, we have

$$(n+\beta+1)_k=\sum_{\ell=0}^{k}\binom{k}{\ell}(\beta+1)_{k-\ell}(n)_\ell,$$

from which we derive

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{(\alpha+\beta+1)_n\cos(\vartheta)^{2n}}{n(\beta+1)_n}\\ =&\frac{1}{(\beta+1)_k}\sum_{\ell=0}^{k}\binom{k}{\ell}(\beta+1)_{k-\ell}\sum_{n=1}^{\infty}\frac{(n)_\ell}{n}\cos(\vartheta)^{2n}\\ =&\sum_{\ell=1}^{k}\frac{\binom{k}{\ell}(\ell-1)!}{(\beta+k+1-\ell)_\ell}\frac{1}{\sin(\vartheta)^{2\ell}}+\log\frac{1}{\sin(\vartheta)^2}\\ &-\left(\psi(\beta+k+1)-\psi(\beta+1)\right).\end{aligned}$$

For $\alpha\in\mathbb{N}_0$ this gives

$$\begin{aligned}\mathcal{G}_{\alpha,\beta}(\cos(2\vartheta))=&\frac{1}{4(\alpha+\beta+1)}\left(\sum_{\ell=1}^{\alpha}\binom{\alpha}{\ell}\frac{(\ell-1)!}{(\beta+\alpha+1-\ell)_\ell}\frac{1}{\sin(\vartheta)^{2\ell}}-\gamma\right.\\ &\left.+\log\frac{1}{\sin(\vartheta)^2}-\psi(\alpha+\beta+1)-\psi(\alpha+\beta+2)-\psi(\beta+1)\right).\end{aligned}$$

The case that $\alpha\in\frac12+\mathbb{N}_0$ only occurs in the case of real projective spaces, when $\beta=-\frac12$. In this case we consider the functions

$$\begin{aligned}F_k(z)&=\sum_{n=1}^{\infty}\frac{(k+1)_n}{n(\frac12)_n}z^n\\ G_k(z)&=2(k+1)\sum_{n=0}^{\infty}\frac{(k+2)_n}{(\frac32)_n}z^n=2(k+1)\,{}_2\mathrm{F}_1\left(\left.\begin{matrix}1,k+2\\ \frac32\end{matrix}\right|z\right);\end{aligned}$$

these satisfy $F_k'=G_k$. From [47, p. 463, eqns. 132, 133] we infer that

$$G_k(z)=\frac{(\frac32)_k}{k!}\frac{\arcsin(\sqrt{z})}{\sqrt{z}}\frac{1}{(1-z)^{k+\frac32}}+\sum_{\ell=0}^{k}\frac{(k+\frac32-\ell)_\ell}{(k+1-\ell)_\ell}\frac{1}{(1-z)^{\ell+1}}.$$

From this we compute

$$F_k(z) = \frac{2\sqrt{z}\arcsin(\sqrt{z})}{\sqrt{1-z}} \sum_{\ell=0}^{k} \frac{(\frac{1}{2})_\ell}{\ell!} \frac{1}{(1-z)^\ell} + \sum_{\ell=1}^{k} \frac{1}{\ell}\left(\frac{(k+\frac{3}{2}-\ell)_\ell}{(k+1-\ell)_\ell} - \frac{(\frac{1}{2})_\ell}{\ell!}\right)\frac{1}{(1-z)^\ell} - H_k,$$

which can be checked by differentiation. We get

$$\begin{aligned}\mathcal{G}_{k+\frac{1}{2},-\frac{1}{2}}(\cos(2\vartheta)) =& \frac{1}{2(k+1)}\cos(\vartheta)\left(\frac{\pi}{2}-\vartheta\right)\sum_{\ell=0}^{k}\frac{(\frac{1}{2})_\ell}{\ell!}\frac{1}{\sin(\vartheta)^{2\ell+1}} \\ &+ \frac{1}{4(k+1)}\sum_{\ell=1}^{k}\frac{1}{\ell}\left(\frac{(k+\frac{3}{2}-\ell)_\ell}{(k+1-\ell)_\ell} - \frac{(\frac{1}{2})_\ell}{\ell!}\right)\frac{1}{\sin(\vartheta)^{2\ell}} \\ &- \frac{1}{2(k+1)}H_k - \frac{1}{4(k+1)^2}.\end{aligned}$$

□

Putting everything together we derive the asymptotic main term of the Green function

$$\mathcal{G}_{\alpha,\beta}(\cos(2\vartheta)) = \frac{\Gamma(\alpha)\Gamma(\beta+1)}{4\Gamma(\alpha+\beta+2)}\frac{1}{\sin(\vartheta)^{2\alpha}} + \mathcal{O}\left(\frac{1}{\sin(\vartheta)^{2\alpha-1}}\right), \tag{2.9}$$

for $\alpha > 0$ and

$$\mathcal{G}_{0,-\frac{1}{2}}(\cos(2\vartheta)) = -\log(\sin(\vartheta)) + \mathcal{O}(1), \tag{2.10}$$

for $\Omega = \mathbb{RP}^2$.

2.6. **Minimizers of the Riesz, Green, and logarithmic energies.** In this section, we prove that the uniform measure on Ω induced by the Haar measure of the group acting on Ω minimizes the Riesz, Green and logarithmic energies.

Theorem 2.11. *The logarithmic energy I_{K_0}, the Green energy I_G, and the Riesz s-energies I_{K_s}, for $0 < s < 2\alpha+2$, are uniquely minimized by the uniform measure σ, with*

$$I_{K_s}(\sigma) = \frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+1-\frac{s}{2})}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2-\frac{s}{2})}, \tag{2.11}$$

$$I_{K_0}(\sigma) = \frac{1}{2}(\psi(\alpha+\beta+2) - \psi(\alpha+1)), \tag{2.12}$$

and

$$I_G(\sigma) = 0. \tag{2.13}$$

Moreover, if $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of minimizers for the discrete energies E_{K_0}, E_{K_s}, or E_G, then the normalized counting measures

$$\nu_{\omega_N} = \frac{1}{N}\sum_{j=1}^{N}\delta_{z_j}$$

converge weakly to σ.

We note that this has already been proven for Green energies in [7, Theorem 1.1], and the uniformity of minimizers for logarithmic and Riesz energies on $\mathbb{RP}^{d-1}$ and $\mathbb{CP}^{d-1}$ was proven in [22, Theorem 4.1].

In general, the minimal discrete and continuous energies of some kernel K are related in the following way

Theorem 2.12 ([16, Theorem 4.2.2])**.** *Let K be a lower semi-continuous kernel on Ω, and μ be a minimizer of I_K. Then for all $N \in \mathbb{N} \setminus \{1\}$,*

$$\mathcal{E}_K(N) \leq N(N-1)I_K(\mu),$$

and

$$\lim_{N\to\infty} \frac{\mathcal{E}_K(N)}{N^2} = I_K(\mu).$$

If $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of optimal K-energy configurations and ν is a weak* limit point of the normalized counting measures ν_{ω_N}, then ν is an equilibrium measure of I_K.

Thus, if the continuous energy is minimized by the uniform measure, we should expect uniformly distributed discrete minimizers. That the uniform measure (uniquely) minimizes the energies follows from the strict positive definiteness of the kernels, which we show below.

Definition 2.13. We call a kernel *conditionally positive definite* if for all $\nu \in \mathcal{Z}(\Omega)$, for which the energy is well defined, $I_K(\nu) \geq 0$. We call a kernel *positive definite* if for all $\nu \in \mathcal{M}(\Omega)$ for which the energy is well defined, $I_K(\nu) \geq 0$. We call a kernel *strictly positive definite* or *conditionally strictly positive definite* if it is positive definite or conditionally positive definite, respectively, and $I_K(\nu) = 0$ only if $\nu = 0$.

Theorem 2.14 (see [16, Theorem 4.2.7])**.** *If K is conditionally strictly positive definite, then I_K has a unique minimizer in $\mathbb{P}(\Omega)$.*

We note that the proofs for Theorems 2.12 and 2.14 given in [16] are for compact subsets of $\mathbb{R}^d$, but they can be generalized to compact metric spaces by straight forward adaptations of the arguments.

Corollary 2.15. *If K is conditionally strictly positive definite and isometry invariant, then σ is the unique minimizer of I_K.*

Proof. Suppose that μ minimizes I_K. Since K is isometry invariant, we have for all $g \in G$,

$$\begin{aligned} I_K(\mu) &= \iint_{\Omega\times\Omega} K(x,y)\, d\mu(x)\, d\mu(y) \\ &= \iint_{\Omega\times\Omega} K(g(x),g(y))\, d\mu(x)\, d\mu(y) \\ &= I_K(g_{\#}(\mu)), \end{aligned}$$

where $g_{\#}(\mu)$ is the pushforward measure of μ under g. Since μ must be unique, it must be isometry invariant, giving us our claim. $\square$

Lemma 2.16 gives more information on the convergence of Jacobi series as in (1.3).

Lemma 2.16. *If $F \in C([-1,1])$ and $K(x,y) = F(\cos(2\vartheta(x,y)))$ is conditionally positive definite, then*

$$F(t) = \sum_{n=0}^{\infty} \widehat{F}(n) P_n^{(\alpha,\beta)}(t),$$

where the series converges absolutely and uniformly on $[-1,1]$.

Proof. We first observe that $K(x,y)+C$ is positive definite for a large enough constant C by [15, Theorem 2]. Thus we can assume without loss of generality that $K(x,y)$ is positive definite. The operator $T_K : f \mapsto \int_\Omega K(x,y)f(y)\,d\sigma(y)$ is then compact and thus has finite dimensional eigenspaces. By the G-invariance of K and σ, these eigenspaces are also G-invariant. Thus by Proposition 2.7 the eigenfunctions of T_K are eigenfunctions of the Laplace operator and the corresponding eigenvalues are positive. Then the claim follows from Mercer's theorem (see, for instance [27]). $\square$

Theorem 2.17. *If $F \in C([-1,1])$, then $K(x,y) = F(\cos(2\vartheta(x,y)))$ is*

(1) *conditionally positive definite if and only if $\widehat{F}(n) \geq 0$ for all $n \in \mathbb{N}$,*
(2) *positive definite if and only if $\widehat{F}(n) \geq 0$ for all $n \in \mathbb{N}_0$,*
(3) *conditionally strictly positive definite if and only if $\widehat{F}(n) > 0$ for all $n \in \mathbb{N}$,*
(4) *strictly positive definite if and only if $\widehat{F}(n) > 0$ for all $n \in \mathbb{N}_0$.*

Proof. The forward direction for each of these follows from [15, Theorem 2 and Section III], with a slight alteration for the strict case.

Part (2) then follows from [15, Lemma 2] and that uniform limits of positive definite functions are positive definite.

Now, suppose that $\widehat{F}(n) > 0$ for all $n \in \mathbb{N}_0$. Then K is positive definite, so by Lemma 2.16, the addition formula (2.7), and the density of $\{Y_{n,k} : n \in \mathbb{N}_0, k \in \{1,\dots,\dim(V_n)\}\}$ in $C(\Omega)$, we have for any $\mu \in \mathcal{M}(\Omega)$, not identically zero,

$$\begin{aligned} I_K(\mu) &= \sum_{n=0}^{\infty} \widehat{F}(n) \iint\limits_{\Omega\times\Omega} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))\,d\mu(x)\,d\mu(y) \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{m_n} \frac{\widehat{F}(n)P_n^{(\alpha,\beta)}(1)}{m_n} \iint\limits_{\Omega\times\Omega} Y_{n,k}(x)Y_{n,k}(y)\,d\mu(x)\,d\mu(y) \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{m_n} \frac{\widehat{F}(n)P_n^{(\alpha,\beta)}(1)}{m_n} \Big(\int_\Omega Y_{n,k}(x)\,d\mu(x)\Big)^2 > 0, \end{aligned}$$

proving (4). Parts (1) and (3) now follow from [15, Theorem 2]. $\square$

Theorem 2.18. *For $0 \leq s < D$, the Riesz kernel K_s is strictly positive definite.*

Proof. For $\varepsilon > 0$ and $s \geq 0$, let

$$K_{s,\varepsilon}(x,y) = \begin{cases} \left(\varepsilon + \frac{1-\cos(2\vartheta(x,y))}{2}\right)^{-\frac{s}{2}} & \text{for } s > 0 \\ -\frac{1}{2}\log\left(\varepsilon + \frac{1-\cos(2\vartheta(x,y))}{2}\right) & \text{for } s = 0 \end{cases}$$

and

$$F_{s,\varepsilon}(t) = \begin{cases} \left(\varepsilon + \frac{1-t}{2}\right)^{-\frac{s}{2}} & \text{for } s > 0 \\ -\frac{1}{2}\log\left(\varepsilon + \frac{1-t}{2}\right) & \text{for } s = 0 \end{cases}.$$

Note that for all $x, y \in \Omega$, $K_{s,\varepsilon}(x,y)$ is strictly decreasing in ε, and $\lim_{\varepsilon\to 0} K_{s,\varepsilon}(x,y) = K_s(x,y)$. Now, suppose that $\mu \in \mathcal{M}(\Omega)$ (not identically zero) such that $I_{K_s}(\mu)$ is well defined. There exists $\mu^+, \mu^- \in \mathcal{B}(\Omega)$ such that $\mu = \mu^+ - \mu^-$, and so, by the Monotone Convergence Theorem, we have

$$\begin{aligned} I_{K_s}(\mu) &= I_{K_s}(\mu^+) - 2\iint\limits_{\Omega\times\Omega} K_s(x,y)d\mu^+(x)d\mu^-(y) + I_{K_s}(\mu^-) \\ &= \lim_{\varepsilon\to 0}\left(I_{K_{s,\varepsilon}}(\mu^+) - 2\iint\limits_{\Omega\times\Omega} K_{s,\varepsilon}(x,y)d\mu^+(x)d\mu^-(y) + I_{K_{s,\varepsilon}}(\mu^-)\right) \\ &= \lim_{\varepsilon\to 0} I_{K_{s,\varepsilon}}(\mu). \end{aligned}$$

We now show that $I_{K_{s,\varepsilon}}(\mu)$ is positive and strictly decreasing as a function of ε, for all $s \geq 0$.

For $\varepsilon > 0$,

$$\left(\varepsilon + \frac{1-t}{2}\right)^{-\frac{s}{2}} = \left(\varepsilon + 1\right)^{-\frac{s}{2}} \sum_{k=0}^{\infty} \binom{k+\frac{s}{2}-1}{k}\left(\frac{t+1}{2(\varepsilon+1)}\right)^k \tag{2.14}$$

and

$$-\log\left(\varepsilon + \frac{1-t}{2}\right) = -\log\left(\varepsilon+1\right) + \sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{t+1}{2(\varepsilon+1)}\right)^k, \tag{2.15}$$

with the series converging uniformly on $[-1,1]$.

The polynomials $(t+1)^k$ can be expressed as linear combinations of Jacobi polynomials; the coefficients are given by

$$\begin{aligned} &\frac{m_n}{P_n^{(\alpha,\beta)}(1)^2}\int_0^{\frac{\pi}{2}} (1+\cos(2\vartheta))^k P_n^{(\alpha,\beta)}(\cos(2\vartheta))\, d\nu^{\alpha,\beta}(\vartheta) \\ &\qquad = \frac{m_n}{P_n^{(\alpha,\beta)}(1)^2}\binom{k}{n}\frac{2^k(\alpha+1)_n(\beta+1)_k}{(\alpha+\beta+2)_{n+k}}. \end{aligned}$$

Together with the series expansions (2.14) and (2.15) this shows that $\widehat{F_{s,\varepsilon}}(n) > 0$ for all $s \geq 0, n \in \mathbb{N}_0$, for ε sufficiently small.

These coefficients are positive, meaning that $K_{s,\varepsilon}$ is strictly positive definite, for ε sufficiently small, by Theorem 2.17. Since $\mu \neq 0$, we know (from the density of $\operatorname{span}(\{Y_{n,k} : n \in \mathbb{N}_0, 1 \leq k \leq \dim(V_n)\})$ in $C(\Omega)$) that for some $m \in \mathbb{N}_0$, we must have

$$\iint\limits_{\Omega\times\Omega} P_m^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))\, d\mu(x)\, d\mu(y) > 0.$$

Since the coefficients $\widehat{F_{s,\varepsilon}}(n)$, for $n \in \mathbb{N}_0$, are strictly decreasing in ε and positive, the positive definiteness of the Jacobi polynomials and Lemma 2.16 gives us, for

$0 \leq s < D$,

$$\begin{aligned}
I_{K_s}(\mu) &= \lim_{\varepsilon \to 0} I_{K_{s,\varepsilon}}(\mu) \\
&= \lim_{\varepsilon \to 0} \sum_{n=0}^{\infty} \widehat{F_{s,\varepsilon}}(n) \iint_{\Omega \times \Omega} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))\, d\mu(x)\, d\mu(y) \\
&\geq \lim_{\varepsilon \to 0} \widehat{F_{s,\varepsilon}}(m) \iint_{\Omega \times \Omega} P_m^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))\, d\mu(x)\, d\mu(y) \\
&> 0.
\end{aligned}$$

This proves conditional strict positive definiteness. ☐

Theorem 2.19 (see [7, Proposition 3.14])**.** *The Green function $G(x, y)$ is conditionally strictly positive definite.*

Proof of Theorem 2.11*.* The first part of our claim follows from Theorems 2.18 and 2.19 and Corollary 2.15. Equations (2.11), (2.12), and (2.13) follow from direct computation or our assumptions on G.

By Theorem 2.12, any weak* limit point of $\{\nu_{\omega_N} : N \geq 2\}$ must be σ. We know, by the Banach-Alaoglu Theorem, that $\mathbb{P}(\Omega)$ is weak* compact. Thus the sequence (ν_{ω_N}) has a limit point, which has to be σ by Theorem 2.12. Thus the sequence (ν_{ω_N}) converges to σ and the second part of our claim now follows. ☐

2.7. **The heat kernel.** Using the theory of the heat kernel for a compact Riemannian manifold (see, for example, [48]) we obtain a lower bound for the Green function $G(x, y)$ on each of the (compact) projective spaces.

The *heat kernel* on Ω is the unique function $H_t(x, y) := H(t, x, y) \in C^\infty(\mathbb{R}^+ \times \Omega \times \Omega)$ satisfying

$$\triangle_x H(t,x,y) + \frac{\partial}{\partial t} H(t,x,y) = 0 \tag{2.16}$$

$$\lim_{t \to 0^+} \int_\Omega H(t,x,y) f(y)\, d\sigma(y) = f(x) \tag{2.17}$$

for each $f \in C^\infty(\Omega)$, where $\triangle_x = \triangle_\Omega$ is the Laplace operator in the variable x.

Similar to the case of the Green function, the spectral theorem and the addition formula (2.7) together imply that $H_t(x, y)$ has a series expansion in terms of the Jacobi polynomials over $L^2(\Omega, \sigma)$:

$$\begin{aligned}
H_t(x,y) = &\sum_{n=0}^{\infty} e^{-4n(n+\alpha+\beta+1)t} \frac{2n+\alpha+\beta+1}{(\alpha+\beta+1)} \\
&\times \frac{(\alpha+\beta+1)_n (\alpha+1)_n}{n!(\beta+1)_n} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))).
\end{aligned}$$

Contrasting with the formal expansion for $G(x, y)$, the series expansion of $H_t(x, y)$ is uniformly convergent in $x, y \in \Omega$ for all $t > 0$.

Integrating $(1 - H_t(x, y))$ with respect to t, we arrive at the kernel

$$\begin{aligned}
G_t(x,y) = &\sum_{n=1}^{\infty} e^{-4n(n+\alpha+\beta+1)t} \frac{2n+\alpha+\beta+1}{4n(n+\alpha+\beta+1)} \\
&\times \frac{(\alpha+\beta+2)_{n-1} (\alpha+1)_n}{(\beta+1)_n n!} P_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y))),
\end{aligned} \tag{2.18}$$

which we may use to provide an explicit lower bound on the Green function.

Lemma 2.20. *For all $t > 0$ and $x \neq y$ we have*

$$G(x,y) \geq G_t(x,y) - t. \tag{2.19}$$

This approximation is originally due to Elkies, and Lang presents a proof in [41, Lemma 5.2]. We shall also provide a proof, which uses the non-negativity of the heat kernel.

Theorem 2.21. *For all $x, y \in \Omega$ and $t > 0$,*

$$H_t(x,y) \geq 0.$$

Non-negativity follows from the fact that $\frac{\partial}{\partial t} + \triangle_x$ is a parabolic differential operator, thus satisfying a *strong maximum principle* (see [46, Chapter 3, Theorem 3], [41, page 152].)

Proof of Lemma 2.20. We observe that

$$G_t(x,y) = \int_\Omega G(x,z) H_t(z,y)\, d\sigma(z).$$

This function is defined for all $(x,y) \in \Omega^2$ for $t > 0$ by the integrability of $G(x,y)$. Furthermore, for all $x \neq y$ we have

$$\begin{aligned} \triangle_x G_t(x,y) + \frac{\partial}{\partial t} G_t(x,y) &= 0 \\ \lim_{t \to 0} G_t(x,y) &= G(x,y) \end{aligned}$$

by (2.16), (2.17), the integrability of G, and the continuity of G for $x \neq y$.

Then, by the defining property of G, we have

$$\triangle_x G_t(x,y) = H_t(x,y) - 1.$$

From this, and Theorem 2.21, we derive, for $t_1 > t_0 > 0$,

$$-(t_1 - t_0) \leq \int_{t_0}^{t_1} \left(H_t(x,y) - 1 \right)\, dt = G_{t_0}(x,y) - G_{t_1}(x,y).$$

Taking the limit $t_0 \to 0$ and using (2.17), we obtain (2.19). ☐

In Section 4.4, we use this result to obtain lower estimates for the Green energy on each of the projective spaces.

3. Rotation invariant determinantal point processes on two-point homogeneous spaces

We denote as $\mathscr{X}$ a (simple) random point process in the space Ω. A point process $\mathscr{X}$ is a random measure taking integer values. The random variable $\mathscr{X}(F)$ is then counting the number of points of the process in F, for all Borel sets $F \subset \Omega$. A process is called *simple*, if for any $p \in \Omega$ we have $\mathscr{X}(\{p\}) \leq 1$, almost surely.

The joint intensities $\rho(x_1, \ldots, x_k)$ are functions defined in Ω such that for any family of mutually disjoint subsets $F_1, \ldots, F_k \subset \Omega$

$$\mathbb{E}[\mathscr{X}(F_1) \cdots \mathscr{X}(F_k)] = \int \cdots \int_{F_1 \times \cdots \times F_k} \rho(x_1, \ldots, x_k)\, d\sigma(x_1) \cdots d\sigma(x_k),$$

and we assume that $\rho(x_1, \ldots, x_k) = 0$, when $x_i = x_j$ for $i \neq j$.

Definition 3.1. A random point process (see, e.g., [39, Chapter 4]) is called *determinantal* with kernel $\mathcal{K} : \Omega \times \Omega \to \mathbb{C}$ if it is simple and the joint intensities with respect to a background measure σ are given by

$$\rho(x_1, \dots, x_k) = \det(\mathcal{K}(x_i, x_j))_{1 \le i,j \le k},$$

for every $k \ge 1$ and $x_1, \dots, x_k \in \Omega$.

In [39], it is shown that a determinantal process samples exactly N points if and only if it is associated to the projection of $L^2(\Omega, \sigma)$ to an N-dimensional subspace H. Let $\phi_1, \dots, \phi_N$ be an orthornormal basis of H, then the projection kernel is given by

$$\mathcal{K}_H(x, y) = \sum_{k=1}^{N} \phi_k(x)\overline{\phi_k(y)}.$$

By the Macchi–Soshnikov theorem (see, e.g., [39, Theorem 4.5.5]) the projection kernel $\mathcal{K}$ defines a determinantal point process.

By Proposition 2.7 the only finite-dimensional G-invariant subspaces of $L^2(\Omega, \sigma)$ are finite orthogonal sums of eigenspaces of $\triangle_\Omega$. Thus it is natural to consider the subspace

$$H = V_0 \oplus \cdots \oplus V_k$$

and the corresponding projection kernel given by

$$\mathcal{K}_n^{(\alpha,\beta)}(x, y) = \sum_{k=0}^{n} \sum_{m=1}^{m_k} Y_{k,m}(x) Y_{k,m}(y), \quad x, y \in \Omega.$$

This defines a G-invariant determinantal point process.

Using the addition formula (2.7) and (1.2) the kernel $\mathcal{K}_n^{(\alpha,\beta)}(x, y)$ can be written in the form

$$\begin{aligned} \mathcal{K}_n^{(\alpha,\beta)}(x, y) &= \sum_{k=0}^{n} \frac{m_k}{P_k^{(\alpha,\beta)}(1)} P_k^{(\alpha,\beta)}(\cos(2\vartheta(x, y)) \\ &= \frac{(\alpha+\beta+2)_n}{(\beta+1)_n} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta(x, y)), \quad x, y \in \Omega. \end{aligned} \tag{3.1}$$

Then, for these kernels we have that

$$\begin{aligned} N = \operatorname{tr}(\mathcal{K}_n^{(\alpha,\beta)}(x, x)) &= \int_\Omega \mathcal{K}_n^{(\alpha,\beta)}(x, x)\, d\sigma(x) = \mathcal{K}_n^{(\alpha,\beta)}(1) \\ &= \sum_{k=0}^{n} m_k = \frac{(\alpha+\beta+2)_n (\alpha+2)_n}{(\beta+1)_n n!} \sim \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{n^{2\alpha+2}}{\Gamma(\alpha+2)}, \end{aligned} \tag{3.2}$$

which by the Macchi–Soshnikov theorem is the number of points sampled by the determinantal process associated to the kernel $\mathcal{K}_n^{(\alpha,\beta)}$, almost surely. We shall call these determinantal point processes *harmonic ensembles*.

Determinantal point processes are very convenient probabilistic models for the study of energy expressions due to the following theorem (see e.g., [39, Equation (1.2.2)]).

Proposition 3.2. *Let $\mathcal{K}(x,y)$ be a projection kernel with trace N in Ω, and let $\omega_N = \{x_1, \dots, x_N\} \subset \Omega$ be N random points generated by the corresponding determinantal point process $\mathscr{X}$. Then, for any measurable $f : \Omega \times \Omega \to [0, \infty)$, we have*

$$\mathbb{E}_{\mathscr{X}_N}\left(\sum_{k \neq j} f(x_k, x_j)\right) = \iint_{\Omega \times \Omega} \left(\mathcal{K}(x,x)\mathcal{K}(y,y) - |\mathcal{K}(x,y)|^2\right) f(x,y)\, d\sigma(x)\, d\sigma(y).$$

4. Bounds for the Green, Riesz, and logarithmic energies on projective spaces

One often studies random configurations of points to find upper estimates for the minimal energy. However, no local repulsion occurs between i.i.d. random points, meaning that sampled points may concentrate near one another, and so the expected discrete energy, $N(N-1)I_K(\mu)$, is too coarse of an estimate for a good upper bound. One can prevent this clumping of sampled points by distributing one point in each part of a partition of Ω (i.e. jittered sampling). Alternatively, one can also generate random point sets with local repulsion built in, using determinantal point processes. Recently, determinantal point processes have been used to find bounds on energies in various symmetric spaces (see, e.g. [2, 8–12, 37, 44]).

Here, we use both jittered sampling and the determinantal point processes introduced in Section 3 to compute the expectations of the discrete Riesz, Green, and logarithmic energies under these models. These, of course, provide upper bounds for the minimal energies. For the lower bounds of these energies, we use linear programming. As in the case of the sphere, the next-order terms in the upper and lower bounds obtained by these ideas have the same orders of magnitude in terms of the number of points N.

We recall that $D = (d-1)\dim_{\mathbb{R}}(\mathbb{F})$ is the dimension of the space $\mathbb{FP}^{d-1}$. In this section it is convenient to use this notation to make the statement of the results transparent.

4.1. **Upper bounds using jittered sampling.** Since each projective space is a connected Ahlfors regular metric measure space with finite measure, we may use the following (formulated for our context):

Proposition 4.1 ([30, Theorem 2])**.** *For each projective space Ω, there exist positive constants c_1 and c_2 such that for all N sufficiently large, there is a partition of Ω into N regions each of measure $\frac{1}{N}$, contained in a geodesic ball of radius $c_1 N^{-\frac{1}{D}}$, and containing a geodesic ball of radius $c_2 N^{-\frac{1}{D}}$.*

Proposition 4.2. *For the projective space Ω and $0 \le s < D$, there is some positive constant $c_{\Omega,s}$ such that for $N \in \mathbb{N}$ sufficiently large,*

$$\mathcal{E}_{K_s}(N) \le \begin{cases} N^2 I_{K_s}(\sigma) - c_{\Omega,s} N^{1+\frac{s}{D}} & \text{for } s > 0 \\ N^2 I_{K_s}(\sigma) - c_{\Omega,s} N \log(N) & \text{for } s = 0 \end{cases}.$$

Proof. Since $\sin(\vartheta) < \vartheta < 2\sin(\vartheta)$ for ϑ sufficiently small, Proposition 4.1 gives us that there exists some positive constant c_3 such that for N sufficiently large, there is a partition of Ω into N regions, $D_1, \ldots, D_N$, each of measure $\frac{1}{N}$ and contained in a chordal ball of radius $c_3 N^{-\frac{1}{2\alpha+2}}$.

Letting $d\sigma_j(x) := N\mathbf{1}_{D_j} d\sigma(x)$, we have for $0 < s < D$,

$$\begin{aligned}
\mathcal{E}_{K_s}(N) &\leq \int_\Omega \cdots \int_\Omega \sum_{i\neq j} K_s(z_i, z_j)\, d\sigma_1(z_1) \cdots d\sigma_N(z_N) \\
&= N^2 \sum_{i\neq j} \iint\limits_{D_i \times D_j} K_s(z_i, z_j)\, d\sigma(z_i)\, d\sigma(z_j) \\
&= N^2 \Bigg(\iint\limits_{\Omega\times\Omega} K_s(x,y)\, d\sigma(x)\, d\sigma(y) - \sum_{j=1}^N \iint\limits_{D_j\times D_j} K_s(x,y)\, d\sigma_j(x)\, d\sigma_j(y) \Bigg) \\
&\leq N^2 I_{K_s}(\sigma) - \sum_{j=1}^N \frac{1}{\operatorname{diam}(D_j)^s} \leq N^2 I_{K_s}(\sigma) - \frac{1}{(2c_3)^s} N^{1+\frac{s}{D}}.
\end{aligned}$$

The logarithmic case works similarly. □

Using (2.9) and (2.10) we immediately have Corollary 4.3.

Corollary 4.3. *For the projective space Ω there is some positive constant $c_{\Omega,G}$ such that for $N \in \mathbb{N}$ sufficiently large,*

$$\mathcal{E}_G(N) \leq \begin{cases} -c_{\Omega,G} N \log(N) & \text{for } \Omega = \mathbb{RP}^2 \\ -c_{\Omega,G} N^{2-\frac{2}{D}} & \text{for } \Omega \neq \mathbb{RP}^2 \end{cases}.$$

4.2. **Upper bounds using determinantal point processes.** In this section we study the expectation of E_{K_s} and E_G under the harmonic ensemble given by the projection kernel (3.1). Taking this kernel restricts the number of points to the subsequence $N = \frac{(\alpha+\beta+2)_n(\alpha+2)_n}{(\beta+2)_n n!}$ for $n \in \mathbb{N}_0$. For the Riesz and logarithmic energies, this amounts to the computation of integrals of the form

$$\mathbb{E}_{\mathscr{X}_N}\left[E_{K_s}\right] = \iint\limits_{\Omega\times\Omega} \frac{\mathcal{K}_n^{(\alpha,\beta)}(1)^2 - \mathcal{K}_n^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))^2}{\sin(\vartheta(x,y))^s}\, d\sigma(x)\, d\sigma(y).$$

Using (2.8) this simplifies to

$$\frac{(\alpha+\beta+2)_n^2}{(\beta+1)_n^2} \int_0^{\frac{\pi}{2}} \frac{1}{\sin(\vartheta)^s} \left(P_n^{(\alpha+1,\beta)}(1)^2 - P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2 \right) d\nu^{\alpha,\beta}(\vartheta).$$

We first observe that the integral can be written as

$$\frac{1}{\gamma^{\alpha,\beta}} \int_0^{\frac{\pi}{2}} \frac{P_n^{(\alpha+1,\beta)}(1)^2 - P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2}{\sin(\vartheta)^2} \sin(\vartheta)^{2\alpha+3-s} \cos(\vartheta)^{2\beta+1}\, d\vartheta.$$

Now the limit of the quotient exists for $\vartheta \to 0$, which shows that the integral converges if $2\alpha + 3 - s > -1$. This gives

Theorem 4.4. *The expected Riesz s-energy $\mathbb{E}_{\mathscr{X}_N}\left[E_{K_s}\right]$ is finite if and only if $s < D + 2$.*

Furthermore, using a similar reasoning the integral

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin(\vartheta)^s} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2\, d\nu^{\alpha,\beta}(\vartheta) \tag{4.1}$$

converges, if and only if $s < D$. In order to derive the asymptotic behaviour of these integrals for $n \to \infty$ we use the classical Hilb approximation for the Jacobi polynomials (see [52, Theorem 8.21.12])

$$\begin{aligned}&\frac{1}{\binom{n+\alpha+1}{n}} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta)) \sin(\vartheta)^{\alpha+\frac{3}{2}} \cos(\vartheta)^{\beta+\frac{1}{2}} \\ &= \Gamma(\alpha+2)\tilde{n}^{-\alpha-1}\sqrt{\vartheta} J_{\alpha+1}(2\tilde{n}\vartheta) + \begin{cases} \mathcal{O}\left(\vartheta^{\alpha+3}\right) & \text{for } 0 \le \vartheta \le \frac{c}{n} \\ \mathcal{O}\left(\vartheta^{\frac{1}{2}} n^{-\alpha-\frac{5}{2}}\right) & \text{for } \frac{c}{n} \le \vartheta \le \frac{\pi}{4} \end{cases},\end{aligned} \tag{4.2}$$

where $J_\alpha(x)$ denotes the Bessel function and $\tilde{n} = n + \frac{1}{2}(\alpha+\beta+1)$. The $\mathcal{O}$-terms are uniform in $\vartheta \in [0, \frac{\pi}{4}]$ (we use a weaker formulation here than what is known).

For studying the asymptotic behaviour of (4.1) we insert (4.2) and obtain

$$\begin{aligned}&\int_0^{\frac{\pi}{4}} \frac{1}{\sin(\vartheta)^s} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2 \, d\nu^{\alpha,\beta}(\vartheta) \\ &= \frac{1}{\gamma^{\alpha,\beta}} \binom{n+\alpha+1}{n}^2 \Gamma(\alpha+2)^2 \tilde{n}^{-2\alpha-2} \int_0^{\frac{\pi}{4}} \frac{\vartheta}{\sin(\vartheta)^{s+2}} J_{\alpha+1}(2\tilde{n}\vartheta)^2 \, d\vartheta \\ &\quad + \mathcal{O}\left(n^{2\alpha+2} \int_0^{\frac{1}{n}} \vartheta^{2\alpha+\frac{5}{2}-s} \, d\vartheta + n^{-2} \int_{\frac{1}{n}}^{\frac{\pi}{4}} \vartheta^{-\frac{3}{2}-s} \, d\vartheta \right),\end{aligned}$$

where we have used the estimates

$$J_{\alpha+1}(x) = \begin{cases} \mathcal{O}(x^{\alpha+1}) & \text{for } x \to 0 \\ \mathcal{O}(x^{-\frac{1}{2}}) & \text{for } x \to \infty \end{cases}$$

for the first and the second integral in the $\mathcal{O}$-term, respectively. We also estimated $\sin(\vartheta)$ trivially with ϑ. The error term then turns into $\mathcal{O}(n^{s-\frac{3}{2}})$.

Thus we are left with the asymptotic evaluation of the integral

$$\int_0^{\frac{\pi}{4}} \frac{\vartheta}{\sin(\vartheta)^{s+2}} J_{\alpha+1}(2\tilde{n}\vartheta)^2 \, d\vartheta.$$

We substitute $2\tilde{n}\vartheta = \tau$ and split the integral to obtain

$$\begin{aligned}(2\tilde{n})^s \Bigg(&\int_0^{\sqrt{n}} \tau^{-s-1} J_{\alpha+1}(\tau)^2 \, d\tau + \mathcal{O}\left(\frac{1}{n}\right) \\ &+ \int_{\sqrt{n}}^{\frac{\tilde{n}\pi}{2}} \frac{\tau}{(2\tilde{n})^{s+2} \sin(\tau/2\tilde{n})^{s+2}} J_{\alpha+1}(\tau)^2 \, d\tau \Bigg);\end{aligned}$$

here we have replaced $\sin(\vartheta)$ with ϑ and controlled the error in the range $\vartheta < 1/\sqrt{n}$. The second integral can be estimated by $\mathcal{O}(n^{-\frac{s+1}{2}})$. For the first integral we use [43, Section 3.8.5]

$$\int_0^\infty \tau^{-s-1} J_{\alpha+1}(\tau)^2 \, d\tau = \frac{1}{2^{s+1}} \frac{\Gamma(s+1)\Gamma(\alpha+1-\frac{s}{2})}{\Gamma(\frac{s+1}{2})^2 \Gamma(\alpha+2+\frac{s}{2})} \tag{4.3}$$

with an error

$$\int_{\sqrt{n}}^\infty \tau^{-s-1} J_{\alpha+1}(\tau)^2 \, d\tau = \mathcal{O}(n^{-\frac{s+1}{2}}).$$

For the remaining integral we estimate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin(\vartheta)^s} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2 \, d\nu^{\alpha,\beta}(\vartheta) \\ = \mathcal{O}\left(\int_0^{\frac{\pi}{2}} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2 \, d\nu^{\alpha+1,\beta}(\vartheta)\right) = \mathcal{O}\left(\frac{1}{n}\right).$$

Putting everything together, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin(\vartheta)^s} P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2 \, d\nu^{\alpha,\beta}(\vartheta) \\ = \frac{\binom{n+\alpha+1}{n}^2}{2\gamma_{\alpha,\beta}} \frac{\Gamma(s+1)\Gamma(\alpha+1-\frac{s}{2})}{\Gamma(\frac{s+2}{2})^2\Gamma(\alpha+2+\frac{s}{2})} \tilde{n}^{s-2\alpha-2} + \mathcal{O}(n^{\frac{s-1}{2}}).$$

This gives

$$\begin{aligned} \mathbb{E}_{\mathscr{X}_N}\left[E_{K_s}\right] &= \mathcal{K}_n^{(\alpha,\beta)}(1)^2\Gamma(\alpha+\beta+2)\Gamma\left(\alpha+1-\frac{s}{2}\right) \\ &\times\left(\frac{1}{\Gamma(\alpha+\beta+2-\frac{s}{2})} - \frac{\Gamma(s+1)\Gamma(\alpha+2)^2}{\Gamma(\frac{s+2}{2})^2\Gamma(\alpha+2+\frac{s}{2})\Gamma(\beta+1)}\tilde{n}^{s-2\alpha-2}\right) \\ &+ \mathcal{O}\left(n^{2\alpha+2+\frac{s-1}{2}}\right). \end{aligned} \tag{4.4}$$

Theorem 4.5. *For $0 < s < D$ the expected value of the Riesz energy satisfies*

$$\mathbb{E}_{\mathscr{X}_N}\left[E_{K_s}\right] = I_{K_s}(\sigma)\mathcal{K}_n^{(\alpha,\beta)}(1)^2 \\ - \frac{\Gamma(s+1)\Gamma(\alpha+1-\frac{s}{2})\Gamma(\beta+1)}{\Gamma(\frac{s}{2}+1)^2\Gamma(\alpha+2+\frac{s}{2})\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)} n^{s+D} + \mathcal{O}(n^{s+D-1}).$$

In terms of the number of points N this gives

$$\begin{aligned} \mathcal{E}_{K_s}(N) \leq{}& \mathbb{E}_{\mathscr{X}_N}\left[E_{K_s}\right] \\ ={}& I_{K_s}(\sigma)N^2 - \frac{\Gamma(s+1)\Gamma(\alpha+1-\frac{s}{2})\Gamma(\beta+1)}{\Gamma(\frac{s}{2}+1)^2\Gamma(\alpha+2+\frac{s}{2})\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)} \\ &\times\left(\frac{\Gamma(\alpha+\beta+2)\Gamma(\alpha+2)}{\Gamma(\beta+1)}\right)^{\frac{2\alpha+s+2}{2\alpha+2}} N^{1+\frac{s}{D}} + \mathcal{O}(N^{1+\frac{s-1}{D}}). \end{aligned} \tag{4.5}$$

For the limiting case $s = D$, we observe that the implicit constant in the error term in (4.4) remains bounded for $s \to D$. Thus we can take the limit $s \to D$ to obtain

Theorem 4.6. *The expected energy in the limiting case $s = D$ satisfies*

$$\mathbb{E}_{\mathscr{X}_N}\left[E_{K_D}\right] = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)}\mathcal{K}_n^{(\alpha,\beta)}(1)^2\Big(2\log(n) \\ + \psi(2\alpha+3) - 2\psi(\alpha+2) - \psi(\beta+1)\Big) + \mathcal{O}(n^{D-1}).$$

In terms of the number of points N this gives

$$
\begin{aligned}
\mathcal{E}_{K_D}(N) \le \mathbb{E}_{\mathscr{X}_N}\left[E_{K_D}\right] &= \frac{\Gamma(\alpha+\beta+2)}{(\alpha+1)\Gamma(\beta+1)} \\
&\times N^2\Big(\log(N) + \log\left(\frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)}\right) \\
&+ 2(\alpha+1)\left(\psi(2\alpha+3) - 2\psi(\alpha+2) - \psi(\beta+1)\right)\Big) + \mathcal{O}(N^{2-\frac{1}{D}}).
\end{aligned}
\tag{4.6}
$$

For the logarithmic energy we follow the same line of reasoning as above. We first compute the integral

$$
\int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\sin(\vartheta)}\right) d\nu^{\alpha,\beta}(\vartheta) = \frac{1}{2}\left(\psi(\alpha+\beta+2) - \psi(\alpha+1)\right),
$$

which is done by computing

$$
\frac{\partial}{\partial s}\int_0^{\frac{\pi}{2}} \sin(\vartheta)^{-s}\, d\nu^{\alpha,\beta}(\vartheta)\bigg|_{s=0}.
$$

Then we derive the asymptotic behaviour of

$$
\int_0^{\frac{\pi}{2}} \log\left(\frac{1}{\sin(\vartheta)}\right) P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2\, d\nu^{\alpha,\beta}(\vartheta)
$$

by using (4.2) again. This gives

$$
\begin{aligned}
&\int_0^{\frac{\pi}{4}} \log\left(\frac{1}{\sin(\vartheta)}\right) P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2\, d\nu^{\alpha,\beta}(\vartheta) \\
=&\frac{1}{\gamma^{\alpha,\beta}}\binom{n+\alpha+1}{n}^2 \Gamma(\alpha+2)^2\tilde{n}^{-2\alpha-2} \\
\times&\int_0^{\frac{\pi}{4}} \log\left(\frac{1}{\sin(\vartheta)}\right) \frac{\vartheta}{\sin(\vartheta)^2} J_{\alpha+1}(2\tilde{n}\vartheta)^2\, d\vartheta \\
+&\mathcal{O}\left(n^{2\alpha+2}\int_0^{\frac{1}{n}} \log(\vartheta)\vartheta^{2\alpha+\frac{5}{2}}\, d\vartheta + n^{-2}\int_{\frac{1}{n}}^{\frac{\pi}{4}} \log(\vartheta)\vartheta^{-\frac{3}{2}}\, d\vartheta\right).
\end{aligned}
$$

The error term becomes $\mathcal{O}(n^{-\frac{3}{2}}\log(n))$. The remaining integral is then treated as above, which gives

$$
\begin{aligned}
&\int_0^{\frac{\pi}{4}} \log\left(\frac{1}{\sin(\vartheta)}\right) \frac{\vartheta}{\sin(\vartheta)^2} J_{\alpha+1}(2\tilde{n}\vartheta)^2\, d\vartheta \\
=&\int_0^{\infty} \log\left(\frac{2\tilde{n}}{\tau}\right) J_{\alpha+1}(\tau)^2\,\frac{d\tau}{\tau} + \mathcal{O}\left(\frac{\log n}{n}\right) \\
=&\frac{1}{2(\alpha+1)}\log(2n) \\
-&\frac{1}{4(\alpha+1)^2}\left(2(\alpha+1)(\psi(\alpha+1)+\log(2))+1\right) + \mathcal{O}\left(\frac{\log n}{n}\right),
\end{aligned}
$$

where we have used

$$\int_0^\infty \frac{1}{t} J_{\alpha+1}(t)^2\,dt = \frac{1}{2(\alpha+1)}$$

$$\int_0^\infty \frac{\log(t)}{t} J_{\alpha+1}(t)^2\,dt = \frac{1}{2(\alpha+1)}\left(\psi(\alpha+1)+\log(2)\right) + \frac{1}{4(\alpha+1)^2},$$

which can be derived from (4.3).

The remaining integral

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \log\left(\frac{1}{\sin(\vartheta)}\right) P_n^{(\alpha+1,\beta)}(\cos(2\vartheta))^2\,d\nu^{(\alpha,\beta)}(\vartheta)$$

can be estimated by $\mathcal{O}(\frac{1}{n})$ as above. Putting everything together, this shows:

Theorem 4.7. *The expected value of the logarithmic energy satisfies*

$$\begin{aligned}\mathbb{E}_{\mathscr{X}_N}\left[E_{K_0}\right] &= I_{K_0}(\sigma)\mathcal{K}_n^{(\alpha,\beta)}(1)^2\\ &\quad - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)} n^D \left(\log(n) - \frac{\psi(\alpha+1)+\psi(\alpha+2)}{2}\right)\\ &\quad + \mathcal{O}(n^{D-1}\log n).\end{aligned}$$

In terms of the number of points this gives

$$\begin{aligned}\mathcal{E}_{K_0}(N) &\le \mathbb{E}_{\mathscr{X}_N}\left[E_{K_0}\right]\\ &= I_{K_0}(\sigma)N^2 - \frac{N}{D}\Bigg(\log(N) + \log\left(\frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\right)\\ &\quad - (\alpha+1)(\psi(\alpha+1)+\psi(\alpha+2))\Bigg) + \mathcal{O}(N^{1-\frac{1}{D}}\log N).\end{aligned} \tag{4.7}$$

Remark 4.8. A different approach to the asymptotic study of integrals of the form (4.1) was used in [50] and [18]. There the integral was rewritten as a sum using connection formulas and the orthogonality relations. The generating functions of these expressions can then be expressed in terms of hypergeometric functions; in some special cases there is a closed form expression.

By (2.9) and (2.10), the Green energy is closely related to the Riesz energy with exponent $s = D-2 = 2\alpha$ (or the logarithmic energy for $D=2$); the main difference between the Green energy and this Riesz energy is that the integral of the Green function vanishes. Inserting (2.9) into Theorem 4.5 we have

Theorem 4.9. *If $\alpha > 0$, the expected value of the Green energy under the harmonic ensemble is given by*

$$\mathbb{E}_{\mathscr{X}_N}\left[E_G\right] = -\frac{\Gamma(\beta+1)^2}{4\alpha(2\alpha+1)\Gamma(\alpha+1)^2\Gamma(\alpha+\beta+2)^2} n^{2D-2} + \mathcal{O}(n^{2D-3}).$$

In terms of the number of points N this gives

$$\begin{aligned}\mathcal{E}_G(N) &\le \mathbb{E}_{\mathscr{X}_N}\left[E_G\right]\\ &= -\frac{(\alpha+1)^2}{4\alpha(2\alpha+1)}\left(\frac{\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)}{\Gamma(\beta+1)}\right)^{\frac{1}{\alpha+1}} N^{2-\frac{2}{D}} + \mathcal{O}(N^{2-\frac{3}{D}}).\end{aligned} \tag{4.8}$$

If $\Omega = \mathbb{RP}^2$*, then*

$$\begin{aligned}\mathcal{E}_G(N) &\leq \mathbb{E}_{\mathscr{X}_N}\left[E_G\right] \\ &= -\frac{N}{2}\left(\log(N) - 1 + 2\gamma\right) + \mathcal{O}(N^{\frac{1}{2}}\log N).\end{aligned} \tag{4.9}$$

4.3. **Lower bounds for Riesz energy.** In this section we develop lower bounds for the Riesz and logarithmic energies using a method which originates from [54] for the case of the sphere. This method has then been refined in [17] to provide matching asymptotic orders for the upper and lower bounds; a general version, again for the sphere, is given in [16, Theorem 6.4.4].

We present a slightly simplified proof for the lower bounds. For this purpose we need some lemmas.

Lemma 4.10. *If* $K(x,y) = g(\cos(2\vartheta(x,y)))$ *is continuous and positive definite, then*

$$\mathcal{E}_K(N) \geq \widehat{g}(0)N^2 - g(1)N.$$

We recall that a function $f : I \to \mathbb{R}$ is called completely monotone on an interval I if for all $n \geq 0$

$$\forall u \in I : (-1)^n f^{(n)}(u) \geq 0.$$

Lemma 4.11. *Let* f *be completely monotone on* $[0,\infty)$*, and* $g(1-2u) = f(u)$ *for* $u \in [0,1]$*. Then the coefficients* $\widehat{g}(n)$ *as given in* (1.4) *are all non-negative for* $n \geq 0$*.*

Taking $u = \sin(\vartheta(x,y))^2$, and using (2.2), we see that g is a function of $\cos(2\vartheta(x,y))$. Moreover $g(t)$ is absolutely monotonic on $[-1,1]$ (i.e. all derivatives of g are non-negative). The proof is then essentially the same as the proof of [16, Theorem 5.2.14], changing Gegenbauer polynomials and weights to Jacobi polynomials and weights.

Let f be completely monotone on $(0,\infty)$. From Taylor's formula with the integral form of the remainder term we obtain, for $u > 0$,

$$f(u) = \sum_{k=0}^{n} \frac{\delta^k}{k!}(-1)^k f^{(k)}(u+\delta) + \frac{(-1)^{n+1}}{n!}\int_0^\delta t^n f^{(n+1)}(u+t)\,dt.$$

This observation was the main ingredient in [17]. All summands are positive and finite for $\delta > 0$ and $u \in [0,\infty)$. Furthermore, all summands are positive definite, taking $u = \sin(\vartheta(x,y))^2$ by Lemma 4.11 and Theorem 2.17.

We apply Lemma 4.10 to the function

$$F_{n,\delta}(u) = \sum_{k=0}^{n} \frac{\delta^k}{k!}(-1)^k f^{(k)}(u+\delta) \leq f(u) - \frac{(-1)^{n+1}}{n!}\int_0^\delta t^n f^{(n+1)}(u+t)\,dt,$$

with the inequality being an equality for $u > 0$, to obtain

$$\begin{aligned}E_{K_f}(\omega_N) \geq & N^2\Bigg(\int_0^{\frac{\pi}{2}} f(\sin(\vartheta)^2)\,d\nu^{(\alpha,\beta)}(\vartheta) \\ & - \frac{(-1)^{n+1}}{n!}\int_0^\delta t^n \int_0^{\frac{\pi}{2}} f^{(n+1)}\left(t + \sin(\vartheta)^2\right)\,d\nu^{(\alpha,\beta)}(\vartheta)\,dt\Bigg) \\ & - NF_{n,\delta}(0).\end{aligned} \tag{4.10}$$

We now apply the above observations to the functions

$$f_s(u)=\begin{cases}u^{-s/2} & \text{for } s>0\\ -\frac{1}{2}\log(u) & \text{for } s=0.\end{cases}$$

Then the derivatives are given by

$$f_s^{(k)}(u)=(-1)^k c_{s,k} f_{s+2k}(u),$$

with

$$c_{s,k}=\begin{cases}1 & \text{for } k=0\\ \left(\frac{s}{2}\right)_k & \text{for } s>0 \quad \text{and } k>0\\ \frac{1}{2}(k-1)! & \text{for } s=0 \quad \text{and } k>0.\end{cases}$$

Then (4.10) becomes

$$\begin{aligned}&E_{K_s}(\omega_N)\\ &\quad\geq N^2\left(\int_0^{\frac{\pi}{2}} f_s(\sin(\vartheta)^2)\,d\nu^{(\alpha,\beta)}(\vartheta)-\frac{c_{s,n+1}}{n!}\int_0^{\delta} t^n\int_0^{\frac{\pi}{2}}\frac{d\nu^{(\alpha,\beta)}(\vartheta)}{(t+\sin(\vartheta)^2)^{\frac{s}{2}+n+1}}\,dt\right)\\ &\qquad - N\left(f_s(\delta)+\delta^{-\frac{s}{2}}\sum_{k=1}^{n}\frac{c_{s,k}}{k!}\right).\end{aligned}$$

The inner integral then computes as

$$\int_0^{\frac{\pi}{2}}\frac{d\nu^{(\alpha,\beta)}(\vartheta)}{(t+\sin(\vartheta)^2)^{\frac{s}{2}+n+1}}=(1+t)^{-\frac{s}{2}-n-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\beta+1,\frac{s}{2}+n+1\\ \alpha+\beta+2\end{matrix}\,\middle|\,\frac{1}{1+t}\right).$$

From standard transformations for hypergeometric functions (see [43, Section 2.4.1]) we obtain the asymptotic equivalent as $t\to 0$

$$\begin{aligned}&(1+t)^{-\frac{s}{2}-n-1}\,{}_2\mathrm{F}_1\left(\begin{matrix}\beta+1,\frac{s}{2}+n+1\\ \alpha+\beta+2\end{matrix}\,\middle|\,\frac{1}{1+t}\right)\\ &\qquad\qquad\sim\frac{\Gamma(\alpha+\beta+2)\Gamma(\frac{s}{2}+n-\alpha)}{\Gamma(\beta+1)\Gamma(\frac{s}{2}+n+1)}t^{\alpha-\frac{s}{2}-n}\end{aligned}$$

valid for $n>\alpha-\frac{s}{2}$. Thus we choose n as the smallest integer with this property and obtain as $\delta\to 0$

$$\int_0^{\delta} t^n\int_0^{\frac{\pi}{2}}\frac{d\nu^{(\alpha,\beta)}(\vartheta)}{(t+\sin(\vartheta)^2)^{\frac{s}{2}+n+1}}\,dt\sim\frac{\Gamma(\alpha+\beta+2)\Gamma(\frac{s}{2}+n-\alpha)\delta^{\alpha+1-\frac{s}{2}}}{\Gamma(\beta+1)\Gamma(\frac{s}{2}+n+1)(\alpha+1-\frac{s}{2})}.$$

Choosing $\delta=N^{-\frac{1}{\alpha+1}}$ gives

Theorem 4.12. *Let $0<s<D$, then there is a constant $C_{s,D}>0$ such that*

$$E_{K_s}(\omega_N)\geq I_{K_s}(\sigma)N^2-C_{s,D}N^{1+\frac{s}{D}}.$$

For $s=0$ there is a constant $C_{0,D}>0$ such that

$$E_{K_0}(\omega_N)\geq I_{K_0}(\sigma)N^2-\frac{1}{D}N\log N-C_{0,D}N.$$

4.4. **Lower bounds for the Green energy.** In this section we compute lower estimates for Green energy on each of the projective spaces. For $\mathbb{RP}^2$ the lower bound follows immediately from the lower bound on the logarithmic energy.

Theorem 4.13. *There exists some constant $C_G > 0$ such that the Green energy of every point configuration ω_N on $\mathbb{RP}^2$, with $N \geq 2$, satisfies*

$$E_G(\omega_N) \geq -\frac{1}{2} N \log(N) - C_G N.$$

Proof. This follows immediately from Theorem 4.12 and the fact that

$$G(x,y) = K_0(x,y) - I_{K_0}(\sigma)$$

on $\mathbb{RP}^2$. □

For the other spaces, we employ a method developed in [41, Chapter VI, § 5], which makes use of Lemma 2.20.

Theorem 4.14. *If $\alpha = 1/2$ (i.e. $\Omega = \mathbb{RP}^3$) , the Green energy of any collection of distinct points $\{x_1, \ldots, x_N\} \subset \Omega$ is bounded below by*

$$E_G(\omega_N) \geq -\frac{3}{4} \pi^{\frac{1}{3}} N^{2-2/3} + \mathcal{O}(N \log(N)).$$

For $\alpha > 1/2$ (i.e. all projective spaces except $\mathbb{RP}^2$ and $\mathbb{RP}^3$),

$$E_G(\omega_N) \geq -\frac{1+\alpha}{4\alpha} \left(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \right)^{\frac{1}{\alpha+1}} N^{2-\frac{2}{D}} + \mathcal{O}(N^{2-\frac{3}{D}}).$$

In order to prove Theorem 4.14, we need the following two lemmas. The proof of the lemmas are given after the proof of Theorem 4.14.

Lemma 4.15. *For $\delta, c, d \geq 0$, and as $t \to 0$*

$$\sum_{k=1}^{\infty} (ck+d)^{\delta} e^{-2(ck+d)^2 t} = \frac{(2t)^{-\frac{\delta+1}{2}}}{2c} \Gamma\left(\frac{\delta+1}{2}\right) + \mathcal{O}(t^{-\delta/2}).$$

Lemma 4.16. *For $d \geq 0$, $\sum_{k=1}^{\infty} (2k+d)^{-1} e^{-2(2k+d)^2 t} = \mathcal{O}(\log(t))$, as $t \to 0$.*

Proof of Theorem 4.14. For some $t > 0$ and any collection of distinct points

$$\{x_1, \ldots, x_N\} \subset \Omega,$$

we get

$$\sum_{j \neq i}^{N} G(x_j, x_i) + N(N-1)2t \geq \sum_{j \neq i}^{N} G_{2t}(x_j, x_i).$$

Moreover from (2.18) it follows

$$
\begin{aligned}
\sum_{j\neq i}^{N} G_{2t}(x_j,x_i) =& \sum_{j\neq i}^{N}\sum_{k=1}^{\infty}\frac{e^{-2\lambda_k t}}{\lambda_k}\sum_{\ell=1}^{m_k} Y_{k,\ell}(x_j)Y_{k,\ell}(x_i)\\
=& \sum_{k=1}^{\infty}\frac{1}{\lambda_k}\sum_{\ell=1}^{m_k}\left(\left|\sum_{j=1}^{N} e^{-\lambda_k t}Y_{k,\ell}(x_j)\right|^2 - \sum_{j=1}^{N} e^{-2\lambda_k t}|Y_{k,\ell}(x_j)|^2\right)\\
\geq& -\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\sum_{\ell=1}^{m_k}\sum_{j=1}^{N} e^{-2\lambda_k t}|Y_{k,\ell}(x_j)|^2\\
=& -\sum_{j=1}^{N} G_{2t}(x_j,x_j) = -NG_{2t}(x,x).
\end{aligned}
$$

For all $t\geq 0$ and $x,y\in\Omega$

$$
G_t(x,y) = \sum_{k=1}^{\infty}\frac{m_k}{\lambda_k}\frac{P_k^{(\alpha,\beta)}(\cos(2\vartheta(x,y)))}{P_k^{(\alpha,\beta)}(1)}e^{-\lambda_k t},
$$

therefore $G_{2t}(x,x)$ is equal to

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{2k+\alpha+\beta+1}{4k(k+\alpha+\beta+1)}\frac{(\alpha+\beta+2)_{k-1}(\alpha+1)_k}{(\beta+1)_k k!}e^{-8k(k+\alpha+\beta+1)t}\\
&=\frac{\gamma_{\alpha,\beta}}{2\Gamma(\alpha+1)^2}\\
&\times\sum_{k=1}^{\infty}\frac{2k+\alpha+\beta+1}{k(k+\alpha+\beta+1)}\frac{\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(k+\beta+1)}e^{-8k(k+\alpha+\beta+1)t}.
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&\frac{2k+\alpha+\beta+1}{k(k+\alpha+\beta+1)}\frac{\Gamma(k+\alpha+\beta+1)\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(k+\beta+1)}\\
&\qquad = 2\left(k+\frac{\alpha+\beta+1}{2}\right)^{2\alpha-1}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right)
\end{aligned}
$$

we obtain for $0<t\ll 1$, $G_{2t}(x,x)$ equals

$$
\begin{aligned}
&\frac{\gamma_{\alpha,\beta}}{\Gamma(\alpha+1)^2}\sum_{k=1}^{\infty}\left(k+\frac{\alpha+\beta+1}{2}\right)^{2\alpha-1}e^{-8k(k+\alpha+\beta+1)t}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right)\\
&=\frac{\gamma_{\alpha,\beta}e^{2(\alpha+\beta+1)^2 t}}{2^{2\alpha-1}\Gamma(\alpha+1)^2}\sum_{k=1}^{\infty}\frac{e^{-2(2x+\alpha+\beta+1)^2 t}}{(2k+\alpha+\beta+1)^{1-2\alpha}}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right).
\end{aligned}
$$

Applying Lemma 4.15 for $\alpha > 1/2, \delta = 2\alpha - 1, c = 2$ and $d = \alpha + \beta + 1$, we obtain

$$\begin{aligned} G_{2t}(x,x) =& \frac{e^{2(\alpha+\beta+1)^2 t}\gamma_{\alpha,\beta}}{\Gamma(\alpha+1)^2 2^{2\alpha-1}}\left(\frac{\Gamma(\alpha)}{2^{\alpha+2}}t^{-\alpha} + \mathcal{O}(t^{-\alpha+1/2})\right) + \mathcal{O}(t^{-\alpha+1/2}) \\ =& \frac{\Gamma(\alpha)\gamma_{\alpha,\beta}}{\Gamma(\alpha+1)^2 2^{3\alpha-1}}\left(1+\mathcal{O}(t)\right)\left(t^{-\alpha} + \mathcal{O}(t^{-\alpha+1/2})\right) \\ =& \frac{\Gamma(\beta+1)}{\alpha\Gamma(\alpha+\beta+1)^2 2^{3\alpha+2}}t^{-\alpha} + \mathcal{O}(t^{-\alpha+1/2}). \end{aligned}$$

By choosing $t = \frac{1}{8}\left(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\right)^{\frac{1}{\alpha+1}} N^{-\frac{1}{\alpha+1}}$ in order to obtain a maximal lower bound and applying (3.2), we get

$$E_G(\omega_N) \geq -\frac{1+\alpha}{4\alpha}\left(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\right)^{\frac{1}{\alpha+1}} N^{2-\frac{1}{\alpha+1}} + \mathcal{O}(N^{2-\frac{3}{2(\alpha+1)}}).$$

Applying Lemma 4.16 for $\alpha = 1/2, \delta = 0, c = 2$ and $d = \alpha + \beta + 1$, we obtain

$$G_{2t}(x,x) = \frac{\gamma_{\alpha,\beta}e^{2t}}{\Gamma(\alpha+1)^2}\sum_{k=1}^{\infty}\frac{e^{-2(2x+\alpha+\beta+1)^2 t}}{(2k+\alpha+\beta+1)^{1-2\alpha}}\left(1+\mathcal{O}\left(\frac{1}{k}\right)\right).$$

Due to Lemma 4.16,

$$\sum_{k=1}^{\infty} e^{-2(2x+\alpha+\beta+1)^2 t}\mathcal{O}\left(\frac{1}{k}\right) = \mathcal{O}(\log(t)).$$

Furthermore, applying Lemma 4.15,

$$\sum_{k=1}^{\infty} e^{-2(2x+\alpha+\beta+1)^2 t} = 2^{-3/2}t^{-1/2}\sqrt{\pi} + \mathcal{O}(1).$$

Hence, we obtain

$$G_{2t}(x,x) = \frac{\gamma_{\alpha,\beta}}{\Gamma(\alpha+1)^2}2^{3/2}t^{-1/2}\sqrt{\pi} + \mathcal{O}(\log(t)).$$

By choosing $t = \frac{1}{8}\left(\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\right)^{\frac{1}{\alpha+1}} N^{-\frac{1}{\alpha+1}}$ with $\alpha = 1/2$ and $\beta = -1/2$, we get

$$E_G(\omega_N) \geq -\frac{3}{4}\pi^{\frac{1}{3}}N^{2-2/3} + \mathcal{O}(N\log(N)).$$

$\square$

Proof of Lemma 4.15. Since

$$\sum_{k=1}^{\infty}(ck+d)^{\delta}e^{-2(ck+d)^2 t} - \int_1^{\infty}(cx+d)^{\delta}e^{-2(cx+d)^2 t}dx$$

is lower bounded by $e^{-2(c+d)^2 t}(c+d)^{\delta} - e^{-\delta/2}\left(\frac{\delta}{4t}\right)^{\delta/2}$ and upper bounded by $e^{-\delta/2}\left(\frac{\delta}{4t}\right)^{\delta/2}$, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty}(ck+d)^{\delta}e^{-2(ck+d)^2 t} &= \int_1^{\infty}(cx+d)^{\delta}e^{-2(cx+d)^2 t}dx + \mathcal{O}(t^{-\delta/2}) \\ &= \int_0^{\infty}(cx+d)^{\delta}e^{-2(cx+d)^2 t}dx + \mathcal{O}(t^{-\delta/2}). \end{aligned}$$

In the last equation we used the fact that

$$0 \leq \int_0^1 (cx+d)^\delta \, e^{-2(cx+d)^2 t} dx \leq e^{-\delta/2} \left(\frac{\delta}{4t}\right)^{\delta/2}.$$

Substituting y by $(cx+d)\sqrt{2t}$ lead to the following equation

$$\begin{aligned}
&\sum_{k=1}^{\infty} (ck+d)^\delta \, e^{-2(ck+d)^2 t} \\
&\quad = \int_{d\sqrt{2t}}^{\infty} y^\delta e^{-y^2} (2t)^{-\delta/2} dy + \mathcal{O}(t^{-\delta/2}) \\
&\quad = \frac{1}{c}(2t)^{-\frac{\delta+1}{2}} \left(\int_0^\infty y^\delta e^{-y^2} dy - \int_0^{d\sqrt{2t}} y^\delta e^{-y^2} dy \right) + \mathcal{O}(t^{-\delta/2}) \\
&\quad = \frac{1}{2c}(2t)^{-\frac{\delta+1}{2}} \Gamma\left(\frac{\delta+1}{2}\right) + \mathcal{O}(t^{-\delta/2}).
\end{aligned}$$

□

Proof of Lemma 4.16. The following sum

$$\sum_{k=1}^{\infty} (2k+d)^{-1} \, e^{-2(2k+d)^2 t} - \int_1^\infty (2x+d)^{-1} \, e^{-2(2x+d)^2 t} dx$$

is lower bounded by 0 and upper bounded by $\frac{1}{2+d} e^{-2(2+d)^2 t}$. Therefore, we obtain

$$\sum_{k=1}^{\infty} (2k+d)^{-1} \, e^{-2(2k+d)^2 t} = \int_1^\infty (2x+d)^{-1} \, e^{-2(2x+d)^2 t} dx + \mathcal{O}(1).$$

Let $y = (2x+d)\sqrt{2t}$, then

$$\begin{aligned}
\int_1^\infty \frac{e^{-2(2x+d)^2 t}}{(2x+d)} dx &= \int_{(2+d)\sqrt{2t}}^\infty \frac{\sqrt{2t}}{y} e^{-y^2} (2\sqrt{2t})^{-1} dy \\
&= \frac{1}{2} \int_{(2+d)\sqrt{2t}}^\infty \frac{1}{y} e^{-y^2} dy \\
&= \frac{\Gamma(0, (2+d)^2 2t)}{4} \\
&= \frac{1}{4} \left(-\gamma - \log((2+d)^2 2t) - \sum_{k=1}^{\infty} \frac{(-(2+d)^2 2t)^k}{k! k} \right).
\end{aligned}$$

□

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