DUALITY THEOREMS FOR CURVES OVER LOCAL FIELDS

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Abstract. We prove duality theorems for the étale cohomology of split tori on smooth curves over a local field of positive characteristic. In particular, we show that the classical Brauer–Manin pairing between the Brauer and Picard groups of smooth projective curves over such a field extends to arbitrary smooth curves over the field. As another consequence, we obtain a description of the Brauer group of the function fields of curves over local fields in terms of the characters of the idele class groups.

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1. INTRODUCTION

The duality theorems of Tate (see [54], [55]) and Lichtenbaum [31] for the étale cohomology of split tori over $p$-adic fields and smooth projective curves over such fields are fundamental results in arithmetic geometry. In recent years, these duality theorems have been extended by Scheiderer and van Hamel [47] (see also [15] and [57]) to more general settings of smooth affine curves over $p$-adic fields. These generalizations have found many applications, especially in the study of local-global principles for cohomology of commutative group schemes over the function fields of curves over $p$-adic fields.

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The analogue of Tate’s duality over local fields of positive characteristics was proven by Milne [36] while the analogue of Lichtenbaum duality over such fields was proven by Saito [44] (see also [36]). However, the analogues of the results of Scheiderer and van Hamel for smooth affine curves and their function fields over local fields of positive characteristics are currently unknown. This paper is an attempt to fill this gap.

To obtain these generalizations, we prove a new duality theorem for the logarithmic Hodge–Witt cohomology on smooth projective curves over a local field of positive characteristic. We introduce Brauer group with modulus, and extend the classical Brauer–Manin pairing to the setting of 0-cycles and Brauer groups with moduli over an arbitrary local field. The duality theorem for the logarithmic Hodge–Witt cohomology on smooth projective curves over a local field of positive characteristics are currently unknown. This paper is an attempt to fill this gap.

1.1. Duality for cohomology of $\mathbb{G}_m$. We fix a local field (i.e., a complete discrete valuation field with finite residue field) $k$ of characteristic $p > 0$. Let $X$ be a geometrically connected smooth projective curve over $k$ and let $j: X^o \hookrightarrow X$ be a dense open immersion. Let $i: D \to X$ be the inclusion of an effective Cartier divisor whose support is the complement of $X^o$. We let $\mathbb{G}_m$ denote the rank one split torus over $k$. We let $H^q_{et}(X^o, \mathbb{G}_m) = \lim_{\rightarrow} H^q_{et}(X, \mathbb{G}_m,(X,nD))$, where $\mathbb{G}_m,(X,nD) = \text{Ker}(\mathbb{G}_m,X \to \iota_*(\mathbb{G}_m,nD))$. This is an enriched version of the usual cohomology with compact support $H^q_{et,c}(X^o, \mathbb{G}_m)$ of $X^o$ in positive characteristic. One easily checks that the canonical map $H^q_{et,c}(X^o, \mathbb{G}_m) \to H^q_{et}(X, \mathbb{G}_m)$ factors through $H^q_{cc}(X^o, \mathbb{G}_m)$.

In characteristic zero, the duality theorem for $\mathbb{G}_m$ on $X^o$ (e.g., see [47]) is essentially insensitive to the topology of the underlying étale cohomology groups because it holds true if we simply endow these groups with discrete topology and pass to their profinite completions. This makes the proof of duality relatively simpler. In contrast, endowing each cohomology group with correct topology is a challenging part of the proof of the duality theorem in positive characteristic. We briefly explain the topologies that we use and refer to the body of the paper for details.

Recall that $\text{Pic}(X)$ is the group of $k$-points of the Picard scheme $\text{Pic}(X)$. Since the latter is locally of finite type $k$-scheme, $\text{Pic}(X)$ is endowed with the adic topology induced by the valuation topology of $k$. Since $\text{Pic}(X^o)$ is a quotient of $\text{Pic}(X)$, it is equipped with the quotient topology. As a key step in our proofs, we show in this paper that the relative Picard group $\text{Pic}(X|D)$ is also equipped with the adic topology such that the canonical map $\text{Pic}(X|D) \to \text{Pic}(X)$ is a topological quotient. We endow $H^1_{cc}(X^o, \mathbb{G}_m)$ with the inverse limit topology using a canonical isomorphism $\text{Pic}(X|D) \cong H^1_{et}(X, \mathbb{G}_m,(X,D))$. All other cohomology groups will be endowed with the discrete topology.

For a topological abelian group $G$ with the topology $\tau$, we let $G^{\text{pf}}$ denote the profinite $\tau$-completion of $G$. That is, $G^{\text{pf}}$ is the inverse limit $\lim_{\leftarrow} G/U$ with the inverse limit topology, where $U$ runs through $\tau$-open subgroups of finite index in $G$. It is clear that $G^{\text{pf}}$ is a profinite abelian group. We refer to [36] for the definitions of nondegenerate and perfect pairings of topological abelian groups. In this paper, we prove the following duality theorem for étale cohomology.
Theorem 1.1. For every integer \( q \neq 0 \), there is a bilinear pairing
\[
H^q(X^o, \mathbb{G}_m) \times H^{3-q}_{cc}(X^o, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}
\]
which induces perfect pairings of topological abelian groups
\[
H^1(X^o, \mathbb{G}_m)^{pf} \times H^2_{cc}(X^o, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z},
\]
\[
H^2(X^o, \mathbb{G}_m) \times H^1_{cc}(X^o, \mathbb{G}_m)^{pf} \to \mathbb{Q}/\mathbb{Z},
\]
and
\[
H^3(X^o, \mathbb{G}_m) \times H^0_{cc}(X^o, \mathbb{G}_m)^{pf} \to \mathbb{Q}/\mathbb{Z}.
\]

For smooth curves over \( p \)-adic fields, Theorem 1.1 is due to Lichtenbaum [31] in the proper case, and to Scheiderer–van Hamel [47] (see also [15]) in the open case. In positive characteristic, the \( q = 1 \) case of the theorem is due to Saito [44] (see also [36]) in the proper case.

Remark 1.2. One can construct the pairing of Theorem 1.1 for \( q = 0 \) as well but it will not be perfect (not even nondegenerate).

1.2. Duality for cohomology of \( \mathbb{G}_{m,K} \). Let \( X \) be as above and let \( K \) denote the function field of \( X \). We let \( \overline{\mathcal{I}}(X) = \prod_{x \in \mathcal{X}(k)} \overline{\mathcal{R}}_x \) denote the reduced product with respect to the subgroups \((\mathcal{O}_{X,x})^\times \), where \( \overline{\mathcal{R}}_x \) is the quotient field of \( \mathcal{O}_{X,x} \). The idele class group \( C(K) \) is the cokernel of the canonical inclusion \( K^\times \to \overline{\mathcal{I}}(X) \). This group is endowed with the inverse limit of the adic topologies of the relative Picard groups of \( X \) (cf. §12.3). We let \( Br(K) \) have the discrete topology. As an application of Theorem 1.1, we prove the following.

Theorem 1.3. There is a perfect pairing of topological abelian groups
\[
Br(K) \times C(K)^{pf} \to \mathbb{Q}/\mathbb{Z}.
\]
This result provides an explicit description of \( Br(K) \) in terms of the characters of the idele class group. In fact, we prove a duality theorem for all étale cohomology groups of \( \mathbb{G}_{m,K} \) in this paper (cf. Theorem 12.8). When \( K \) is the quotient field of an excellent normal two-dimensional complete local domain with finite residue field, an analogue of Theorem 1.3 was shown by Saito [44].

1.3. Duality for logarithmic Hodge–Witt cohomology. The proof of Theorem 1.1 is based on the duality theorem for the \( p \)-adic étale motivic cohomology groups of smooth projective curves over \( k \), which we now describe. We fix an integer \( n \geq 1 \). Let \( X \) be as above and let \( W_n \Omega^\bullet_{X,\log} \) denote the logarithmic Hodge–Witt complex on \( X_{\text{ét}} \) à la Bloch–Deligne–Illusie.

For smooth projective schemes (of arbitrary dimensions) over a finite field, Milne showed that the logarithmic Hodge–Witt cohomology (i.e., the étale cohomology of \( W_n \Omega^\bullet_{X,\log} \)) groups satisfy Poincaré duality. However, the analogous result over local fields (of positive characteristics) is presently unknown. The following result settles this problem for curves.

To state the duality theorem, we need to recall that over finite fields, Milne’s duality yields a perfect pairing of finite groups. However, the logarithmic Hodge–Witt cohomology groups over local fields are generally not finite. To offset this problem, we need to equip these groups with suitable nondiscrete topologies. In
fact, dealing with the infinitude of cohomology groups and choosing correct topology on them are perhaps the major part of proving a duality theorem over local fields. For the topologies that we endow these groups with, the reader is referred to §3.2.

Theorem 1.4 was proven by Kato–Saito [24, Prop. 4] when \( j = 0 \) by a different method.

**Theorem 1.4.** Let \( X \) be as in Theorem 1.1 and let \( q, j \) be any integers. Then the cup product on the étale cohomology of the logarithmic Hodge–Witt sheaves induces a perfect pairing of topological abelian groups

\[
H^q(X, W_n \Omega^j_{X, \log}) \times H^{2-q}(X, W_n \Omega^{2-j}_{X, \log}) \to \mathbb{Q}/\mathbb{Z}.
\]

**Remark 1.5.** We actually prove Theorem 1.4 under weaker assumptions on \( X \). Namely, we need to assume that \( X \) is regular but do not require it to be smooth over \( k \). We work under the weaker hypothesis that \( X \) is generically smooth over \( k \).

The validity of the theorem under generic smoothness is useful for some concrete applications. For instance, we hope that it will allow one to prove the duality theorem for the \( p \)-adic étale motivic cohomology groups with compact support for smooth but nonproper curves. Such a result will have applications to the class field theory of smooth open curves over local fields. This is the topic of [30].

1.4. **Brauer–Manin pairing for modulus pairs.** Let \( \iota : D \hookrightarrow X \) be the inclusion of an effective Cartier divisor as above. We shall refer to the pair \((X, D)\) as a (one-dimensional) modulus pair. Let \( \text{CH}_0(X \mid D) \) be the Chow group of 0-cycles for the modulus pair \((X, D)\). We shall show in this paper that \( \text{CH}_0(X \mid D) \) is naturally endowed with an adic topology such that the canonical map \( \text{CH}_0(X \mid D) \to \text{CH}_0(X) \) is a topological quotient. In order to extend the classical Brauer–Manin pairing to the modulus setting, we introduce the Brauer group of the modulus pair \((X, D)\) which we denote by \( \text{Br} \). This is a certain subgroup of \( \text{Br}(X) \) which is defined in terms of Kato’s filtration on the étale motivic cohomology group \( H^2(K, \mathbb{Q}/\mathbb{Z}(1)) \), where \( K \) is the function field of \( X \). We endow \( \text{Br}(X \mid D) \) with the discrete topology. As a key step for proving Theorem 1.1, we establish the following new Brauer–Manin pairing.

**Theorem 1.6.** There is a continuous pairing of topological abelian groups

\[
\text{Br}(X \mid D) \times \text{CH}_0(X \mid D) \to \mathbb{Q}/\mathbb{Z}
\]

such that the induced pairing

\[
\text{Br}(X \mid D) \times \text{CH}_0(X \mid D)^\text{pf} \to \mathbb{Q}/\mathbb{Z}
\]

is perfect.

The case \( D = \emptyset \) of Theorem 1.6 is due to Lichtenbaum [31] in characteristic zero and Saito [44] (see also [36, Thm. III.7.8]) in positive characteristic. In this paper, we provide an independent and short proof of the harder part of Saito’s result as an immediate corollary of Theorem 1.4 (see Corollary 9.6).

1.5. **Brauer–Manin pairing for singular curves.** Recall that the Brauer group of a singular variety \( Y \) does not satisfy the Brauer injectivity property, i.e., the canonical map \( \text{Br}(Y) \to \text{Br}(K) \) fails to be injective in general, for \( K \) the function field of \( Y \). This defect is analogous to the one the classical Chow group (one defined in [29]) \( \text{CH}_0(Y) \) suffers from. In this paper, we introduce a refined Brauer group of \( Y \) which we denote by \( \text{Br}^{\text{lw}}(Y) \). We refer to this as the Levine–Weibel Brauer...
group. The refined Brauer group satisfies some nice properties, including the Brauer injectivity.

Let \( \text{CH}_0^{lw}(Y) \) denote the Levine-Weibel Chow group of \( Y \). One may recall that this coincides with \( \text{Pic}(Y) \) if \( \dim(Y) = 1 \). We show in this paper that \( \text{CH}_0^{lw}(Y) \) is naturally endowed with an adic topology such that the pull-back map \( \text{CH}_0^{lw}(Y) \to \text{CH}_0(Y_n) \) is continuous if \( Y \) is a geometrically integral projective curve over \( k \) with smooth normalization \( Y_n \). We endow \( \text{Br}^{lw}(Y) \) with the discrete topology and prove the following extension of the Brauer–Manin pairing to singular curves.

**Theorem 1.7.** Let \( Y \) be a geometrically integral projective curve over a local field \( k \) whose normalization is smooth over \( k \). Then there is a continuous pairing of topological abelian groups

\[
\text{Br}^{lw}(Y) \times \text{CH}^{lw}_0(Y) \to \mathbb{Q}/\mathbb{Z}
\]

such that the induced pairing

\[
\text{Br}^{lw}(Y) \times \text{CH}^{lw}_0(Y)_{pt} \to \mathbb{Q}/\mathbb{Z}
\]

is perfect.

**1.6. A brief outline of proofs.** The main ingredients for proving Theorem 1.1 are Theorems 1.4 and 1.6. A key step in the proof of Theorem 1.4 is a duality theorem of Zhao [58]. In §2 we recall this duality theorem. In §3 we equip the logarithmic Hodge–Witt cohomology of a smooth projective curve \( X \) over a local field \( k \) of positive characteristic with a suitable topology using some deductions. We construct a trace homomorphism for the top logarithmic Hodge–Witt cohomology of \( X \) in §4 and complete the proof of Theorem 1.4. A trace map in this case was also constructed by Kato–Saito [24, §3, Prop. 4]. But our trace homomorphism is, a priori, different. It has an advantage that one can easily show its compatibility with the trace homomorphisms for closed points under the Gysin homomorphisms. This compatibility is crucial in our proofs. In the next three sections, we equip the relative Picard group of modulus pairs \((X,D)\) with the adic topology. We also prove some key properties of the Albanese map for \( X \) and the Pontryagin dual of the relative Picard group.

We introduce the Brauer group of modulus pairs in §8 and establish the Brauer–Manin pairing between the Brauer and Chow groups of a modulus pair in §9. One of the key steps in the proof of Theorem 1.6 is a duality theorem for the cohomology of \( G_m \) on \( X \) and its compatibility with Kato’s duality for two-dimensional local fields obtained by completing the function field of \( X \) at closed points. This is done in §10. In §11 we complete another step for proving Theorem 1.6, namely, we show that the Brauer–Manin paring for a modulus pair \((X,D)\) is continuous. We complete the proof of Theorem 1.6 in §12 and then apply it to complete the proofs of Theorems 1.1 and Theorem 1.3. In §13 we introduce a refined version of the Brauer group of singular varieties and prove Theorem 1.7 as an application of Theorem 1.6.

**1.7. Notations.** We shall work over a field \( k \) of characteristic \( p \geq 0 \) throughout this paper. We shall let \( k_s \) (resp. \( \bar{k} \)) denote a fixed separable (resp. algebraic) closure of \( k \). We let \( \text{Sch}_k \) denote the category of separated Noetherian \( k \)-schemes and \( \text{Sm}_k \) the category of smooth (in particular, finite type) \( k \)-schemes. A \( k \)-scheme will mean an object of \( \text{Sch}_k \) and a finite type \( k \)-scheme will mean an object of \( \text{Sch}^{\text{ft}}_k \).
product $X \times_{\text{Spec}(k)} Y$ in $\text{Sch}_k$ will be written as $X \times Y$. We let $X^{(q)}$ (resp. $X_{(q)}$) denote the set of points on $X$ having codimension (resp. dimension) $q$. We shall let $X_{\text{sing}}$ denote the singular locus of $X$ with the reduced closed subscheme structure. We shall let $X_{\text{reg}}$ denote the regular locus of $X$. We let $Z_0(X)$ denote the free abelian group of 0-cycles on $X$.

We let $\text{Sch}_{k/z}$ (resp. $\text{Sch}_{k/\text{nis}}$, resp. $\text{Sch}_{k/\text{ét}}$) denote the Zariski (resp. Nisnevich, resp. étale) site of $\text{Sch}_k$. Unless we mention the topology specifically, all cohomology groups in this paper will be considered with respect to the étale topology. We shall let $\text{cd}_p(X)$ denote the étale $p$-cohomological dimension of $X \in \text{Sch}_k$ if $\text{char}(k) = p > 0$. We shall let $\text{cd}(X)$ denote the étale cohomological dimension of torsion sheaves on $X$. We shall let $G_k$ denote the abelian absolute Galois group of $k$.

For an abelian group $A$, we shall write $\text{Tor}^1_A(Z/nA)$ as $nA$ and $A/nA$ as $A/n$. We shall let $A\{p\}$ denote the subgroup of elements of $A$ which are torsion of order prime to $p$. We let $A\{p\}$ denote the subgroup of elements of $A$ which are torsion of order some power of $p$. The tensor product $A \otimes Z B$ will be written as $A \otimes B$. We let $\text{Ab}$ denote the category of abelian groups and $\text{Tab}$ denote the category of topological abelian groups with continuous homomorphisms. For $A, B \in \text{Tab}$, we shall let $\text{Hom}_T(A, B) = (\text{Hom}_{\text{Tab}}(A, B))_{\text{tor}}$. Unless a specific topology is mentioned, we shall assume all finite abelian groups to be endowed with the discrete topology. In this paper, we shall use the general notation $\{A_n\}$ for pro-abelian or ind-abelian group indexed by $\mathbb{N} = \{1, 2, \ldots\}$. However, we shall also use specific notations $\overset{\longrightarrow}{\lim} A_n$ for an ind-abelian group and $\overset{\longleftarrow}{\lim} A_n$ for a pro-abelian group if we need to make a distinction between them.

2. Recollection of Zhao's duality

In this section, we recall Brauer group, Milnor $K$-theory, Hodge–Witt sheaves and the duality theorem of Zhao [58] for the logarithmic Hodge–Witt cohomology. This will be the key ingredient in the proof of Theorem 1.4. We also recall the Pontryagin duality for a special class of locally compact Hausdorff topological abelian groups.

In this paper, we shall consider $\mathbb{Z}$ and $\mathbb{Q}/\mathbb{Z}$ as topological abelian groups with discrete topologies. For $G \in \text{Ab}$, we shall let $G^\vee = \text{Hom}_{\text{Ab}}(G, \mathbb{Q}/\mathbb{Z})$. If $G \in \text{Tab}$, we shall let $G^* = \text{Hom}_{\text{Tab}}(G, \mathbb{Q}/\mathbb{Z})$. We let $\mathbb{T}$ be the circle group with the usual Euclidean topology and let $\text{Hom}_{\text{Tab}}(G, \mathbb{T})$ denote the classical Pontryagin dual of $G$. The identification $\mathbb{Q}/\mathbb{Z} = \mathbb{T}_{\text{tor}}$ yields a continuous homomorphism $\mathbb{Q}/\mathbb{Z} \to \mathbb{T}$ of topological abelian groups. This induces a canonical homomorphism $t_G: G^* \to \text{Hom}_{\text{Tab}}(G, \mathbb{T})$. Recall here that the discrete topology of $\mathbb{Q}/\mathbb{Z}$ is different from its subspace topology from $\mathbb{T}$.

2.1. Pontryagin duality for torsion-by-profinite groups. Let $G$ be a locally compact Hausdorff topological abelian group. We shall say that $G$ is ‘torsion-by-profinite’ if there is an exact sequence

$$0 \to G_{\text{pf}} \to G \to G_{\text{dt}} \to 0,$$

where $G_{\text{pf}}$ is an open and profinite subgroup of $G$ and $G_{\text{dt}}$ is a torsion group with the quotient topology (necessarily discrete). We shall call (2.1) a torsion-by-profinite presentation of $G$. Note that such a presentation of $G$ is not unique.
Recall that for a torsion-by-profinite group $G$, the map $t_G$ above is an isomorphism (e.g., see [42, Lem. 2.9.2]). In particular, $G^*$ is endowed with the structure of a locally compact topological abelian group with the compact-open topology. Furthermore, the evaluation map $G \to (G^*)^*$ is an isomorphism of locally compact topological abelian groups (e.g., see [42, §2.9] or [8, Thm. 4.32]). In other words, letting $\text{Pfd}$ denote the category of torsion-by-profinite topological abelian groups with continuous homomorphisms, we have the following.

**Lemma 2.1.** The category $\text{Pfd}$ is closed under taking Pontryagin dual such that $G \in \text{Pfd}$ is profinite if and only if $G^*$ is a discrete torsion group. Moreover, the map $t_G: G^* \to \text{Hom}_{\text{Tab}}(G, \mathbb{T})$ is an isomorphism of topological abelian groups for every $G \in \text{Pfd}$.

**Lemma 2.2.** If $G \in \text{Pfd}$ and $G' \subseteq G$ is a closed subgroup, then $G' \in \text{Pfd}$.

*Proof.* This is elementary and we skip the proof. $\square$

**Lemma 2.3.** Let $\beta: G \to G''$ be a continuous surjective homomorphism in $\text{Pfd}$ such that $G''_p \subseteq \beta(G_p)$ with respect to some torsion-by-profinite presentations of $G$ and $G''$ as in (2.1). Then the induced map $\beta': G/Ker(\beta) \to G''$ is an isomorphism of topological abelian groups if $G/Ker(\beta)$ is endowed with the quotient topology.

*Proof.* If $G''$ is profinite, the claim is an easy application of the fact that a continuous bijective homomorphism from a compact Hausdorff topological abelian group to a Hausdorff topological abelian group is a topological isomorphism. For the general case, we let $H = \beta^{-1}(G''_p)$. Then $H \subseteq G$ is an open subgroup and $G/H$ is discrete. Furthermore, there is a surjective continuous homomorphism $\beta: H \to G''_p$ such that $(G_p \cap H) \to G''_p$ by our assumption. Lemma 2.2 implies that $\beta: H \to G''_p$ is a continuous surjective homomorphism from a torsion-by-profinite group to a profinite group. In particular, this is a topological quotient. Since $G''_p$ is open in $G''$, one deduces that $\beta$ is an open map on $G$. $\square$

**Lemma 2.4.** Let

$$0 \to G' \xrightarrow{\alpha} G \xrightarrow{\beta} G'' \to 0$$

be a short exact sequence in $\text{Pfd}$. Assume that with respect to some torsion-by-profinite presentations of $G$ and $G''$ as in (2.1), one has $G''_p \subseteq \beta(G_p)$. Then the sequence

$$0 \to (G'')^* \xrightarrow{\beta^*} G^* \xrightarrow{\alpha^*} (G')^* \to 0$$

is exact.

*Proof.* The sequence

$$0 \to (G'')^* \xrightarrow{\beta^*} G^* \xrightarrow{\alpha^*} (G')^*$$

is exact by Lemma 2.3 and $\alpha^*$ is surjective by Lemma 2.1 and [8, Cor. 4.42]. $\square$

Recall from §1 that for a topological abelian group $G$, the profinite completion of $G$ (with respect to its topology) is the profinite topological abelian group $G^{\text{pf}} = \varprojlim_U G/U$ with the inverse limit topology, where the limit is taken over all open subgroups of finite index in $G$. The following result is elementary whose proof is left to the reader.
Lemma 2.5. Let $G$ be a topological abelian group. Then the canonical map $(G^{\text{pf}})^* \to G^*$ induces an isomorphism $(G^{\text{pf}})^* \cong (G^*)_{\text{tor}}$.

Suppose that $\{G_i\}_{i \in I}$ is a pro-object in $\text{Tab}$ such that $I$ is cofiltered. We let $G = \varprojlim G_i$ and endow it with the inverse limit topology. Let $\pi_i: G \to G_i$ be the projection map. If the topology of each $G_i$ is generated by open subgroups, then the same holds for $G$ as well. More precisely, $G$ admits a fundamental system of neighborhoods of the identity of the form $\pi_i^{-1}(U_i)$, where $U_i \subset G_i$ is an open subgroup. This happens, for instance, when each $G_i$ is a Hausdorff, totally disconnected and locally compact topological abelian group. If $k$ is a local field and $X$ is a commutative group scheme over $k$, then $X(k)$ is endowed with a topology of this kind (cf. §6.3). We shall use the following result in this paper.

Lemma 2.6. Let $\{G_i\}_{i \in I}$ be as above such that the topology of each $G_i$ is generated by open subgroups. Then the canonical map $\varprojlim G_i^* \to G^*$ is surjective. This map is bijective if $\pi_i: G \to G_i$ is surjective for each $i \in I$.

Proof. Let $\chi: G \to \mathbb{Q}/\mathbb{Z}$ be a continuous character and let $H = \text{Ker}(\chi)$. Then our hypothesis implies that there exist $i \in I$ and an open subgroup $U_i \subset G_i$ such that $V_i := \pi_i^{-1}(U_i) \subset H$. It follows that $\chi$ factors through $\chi^*: G/V_i \to \mathbb{Q}/\mathbb{Z}$. We thus get a diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\pi_i} & G_i/U_i \\
\downarrow & & \downarrow \\
G/V_i & \xrightarrow{\chi} & \mathbb{Q}/\mathbb{Z},
\end{array}
\]

whose left square is commutative. Since $G/V_i$ and $G_i/U_i$ are discrete, we get a continuous character $\phi^*: G_i/U_i \to \mathbb{Q}/\mathbb{Z}$ such that the left triangle commutes. Letting $\phi = \phi^* \circ \pi_i$, we see that $\phi$ is continuous and $\phi \circ \pi_i = \chi$. The second part of the lemma is clear. \qed

2.2. Milnor $K$-theory and logarithmic Hodge–Witt sheaves. For a commutative ring $A$, we let $K_r(A)$ denote the Quillen $K$-theory of finitely generated projective $A$-modules. We let $\tilde{K}_r^M(A)$ denote the graded commutative ring defined as the quotient of the tensor algebra $T_r(A^\times)$ by the two-sided ideal generated by the homogeneous elements $a_1 \otimes a_2$ such that $a_1 + a_2 = 1$. We let $K_r^M(A)$ denote the improved Milnor $K$-theory of $A$ à la Gabber–Kerz [26]. There are natural multiplicative homomorphisms $\tilde{K}_r^M(A) \xrightarrow{\alpha_A} K_r^M(A) \xrightarrow{\beta_A} K_r(A)$ such that $\alpha_A$ is surjective (resp. an isomorphism) if $A$ is a local ring (resp. a field). Furthermore, $\alpha_A$ is an isomorphism if $A$ is local with infinite residue field and $\beta_A$ is an isomorphism for $r \leq 2$ if $A$ is local [26 Prop. 10]).

For a scheme $X$, we let $K_r^M_{r,X}$ (resp. $\tilde{K}_r^M_{r,X}$, resp. $K_r^M_{r,X}$) denote the sheaf (in Zariski, Nisnevich or étale topology) on $X$ associated to the presheaf $U \mapsto K_r^M(O(U))$ (resp. $\tilde{K}_r^M(O(U))$, resp. $K_r(O(U))$). For an ideal $I \subset A$ in a local ring, we let $\tilde{K}_r^M(A, I) = \text{Ker}(\tilde{K}_r^M(A) \to \tilde{K}_r^M(A/I))$. Given a closed immersion $\nu: Y \to X$, we let $K_r^M_{r,(X,Y)} = \text{Ker}(K_r^M_{r,X} \to \nu_*\tilde{K}_r^M_{r,Y})$. The group $K_r^M(A, I)$ and the sheaf $K_r^M_{r,(X,Y)}$ are defined in a similar manner.

Let $k$ be a field of exponential characteristic $p > 1$. Recall from [15] that for a $k$-scheme $X$, $\{W_m\Omega^*_X\}_{m \geq 1}$ denotes the pro-complex of de Rham-Witt (Nisnevich)
sheaves on $X$. This is a pro-complex of sheaves of differential graded algebras with the structure map $R$ and the differential $d$. Let $[-]_m:\mathcal{O}_X \to W_m\mathcal{O}_X$ be the multiplicative Teichmüller homomorphism. Recall that the pro-complex $\{W_m\Omega^r_X\}_{m \geq 1}$ is equipped with the Frobenius homomorphism of graded algebras $F:W_m\Omega^r_X \to W_{m-1}\Omega^r_X$ and the additive Verschiebung homomorphism $V:W_m\Omega^r_X \to W_{m+1}\Omega^r_X$. We let $ZW_m\Omega^r_X = \text{Ker}(d: W_m\Omega^r_X \to W_m\Omega^{r+1}_X)$ and $BW_m\Omega^r_X = \text{Image}(d: W_m\Omega^{r+1}_X \to W_m\Omega^r_X)$. We write $\mathcal{H}^r(W_m\Omega^r_X) = ZW_m\Omega^r_X/BW_m\Omega^r_X$.

Recall from [18] that $W_m\Omega^r_X$, locally given by $\text{dlog}(\{a_1, \ldots, a_r\}) = \text{dlog}[a_1]_m \wedge \cdots \wedge \text{dlog}[a_r]_m$. It is easily seen that this map exists (e.g., see [32, Rem. 1.6]). Moreover, it is an isomorphism in any of the above topologies if $X$ is regular (e.g., see [13, Lem. 2.3]). Equivalently, there is a short exact sequence of (Zariski, Nisnevich or étale) sheaves

\begin{equation}
0 \to \mathcal{K}^M_{r,X} \xrightarrow{p^n} \mathcal{K}^M_{r,X} \to W_n\Omega^r_{X,\log} \to 0.
\end{equation}

The multiplicative structure of $\mathcal{K}^M_{r,X}/p^m$ and $W_m\Omega^r_X$ together with the multiplicativity of $\text{dlog}$ induces a graded commutative ring structure on $W_m\Omega^r_{X,\log}$.

We end this subsection by recalling the Brauer group. For any Noetherian scheme $X$, recall that $\text{Br}(X)$ denotes the cohomological Brauer group $H^2(X,\mathcal{O}^*_X)$. A result of Gabber (e.g., see [4, Thm. 4.2.1]) says that $\text{Br}(X)_{\text{tor}}$ coincides with the Azumaya Brauer group $\text{Br}^r(X)$ if $X$ is quasi-projective over an affine scheme. Since all schemes in this paper will satisfy this condition, we shall make no distinction between the Azumaya and cohomological Brauer groups. Brauer group of curves over local fields will be one the main objects of study in this paper.

2.3. The setup. The setup for our duality theorem will be the following. We fix an equicharacteristic complete discrete valuation ring (cdvr) $R$ with finite residue field $\mathfrak{f}$. We let $\mathfrak{m}$ denote the maximal ideal of $R$ and $k$ denote the quotient field of $R$. It is well known that $R$ is canonically isomorphic to the formal power series ring $\mathbb{F}_q[[\pi]]$, and $k = \mathbb{F}_q((\pi))$, where $q = p^n$ for some prime number $p \geq 2$ and some positive integer $n \geq 1$. We shall fix this isomorphism throughout our discussion. We let $S = \text{Spec}(R)$ and let $\eta$ (resp. $s$) denote the generic (resp. closed) point of $S$.

We let $\mathcal{X}$ be a connected Noetherian regular scheme with a flat and projective morphism $\mathbb{F}:\mathcal{X} \to S$ of relative dimension $d \leq 1$. We let $X$ denote the generic fiber and $\mathcal{X}_s$ the (scheme-theoretic) closed fiber of $\mathbb{F}$. We shall assume that $Y := (\mathcal{X}_s)_{\text{red}}$ is a simple normal crossing divisor on $\mathcal{X}$. A morphism $\mathbb{F}$ satisfying these properties will be called a semistable model for the $k$-scheme $X$. We let $w: X \to \mathcal{X}$ and $v: Y \to \mathcal{X}$ be the inclusions. We let $f: X \to \text{Spec}(k)$ and $g: \mathcal{X}_s \to \text{Spec}(\mathfrak{f})$ denote the structure maps. We shall assume that $X$ is geometrically connected and generically smooth over $k$. Recall that the generic smoothness of a morphism of schemes $f: Z \to W$ means that $f$ is smooth in a neighborhood of each generic point of $Z$. All results in the remainder (and also in the next two sections) will be proven under this setup. We let $K$ denote the function field of $\mathcal{X}$.

The following elementary observation will be useful in this paper. We skip its proof.
Lemma 2.7. \( \mathcal{X}_s \) (equivalently, \( Y \)) is geometrically connected over \( \mathfrak{f} \).

2.4. Zhao’s duality theorem. We now recall the \( d = 1 \) case of the duality theorem of Zhao (see [58, Cor. 1.4.10, Thm. 3.1.1]) which will be a key ingredient in the proof of Theorem 1.4. We fix an integer \( n \geq 1 \).

Theorem 2.8.

1. There is a canonical isomorphism of \( \mathbb{Z}/p^n \)-modules

\[
\text{Tr}_\mathcal{X} : H^3_Y(\mathcal{X}, W_n \Omega^2_{\mathcal{X}, \log}) \cong \mathbb{Z}/p^n.
\]

2. The graded commutative ring structure of \( H^*(\mathcal{X}, W_n \Omega^*_{\mathcal{X}, \log}) \) gives rise to a pairing

\[
H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \times H^{3-i}(\mathcal{X}, W_n \Omega^{2-j}_{\mathcal{X}, \log}) \to \mathbb{Z}/p^n
\]

such that the induced map

\[
\Phi_{ij}^{ij} : H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \to \text{Hom}_{\text{Ab}}(H^{3-i}(\mathcal{X}, W_n \Omega^{2-j}_{\mathcal{X}, \log}), \mathbb{Z}/p^n)
\]

is an isomorphism.

3. The map

\[
\Psi^{ij} : H^i_Y(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \to \text{Hom}_{\text{Ab}}(H^{3-i}(\mathcal{X}, W_n \Omega^{2-j}_{\mathcal{X}, \log}), \mathbb{Z}/p^n)
\]

is a topological isomorphism if \( H^i_Y(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \) is endowed with the discrete topology and \( H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \) is endowed with the profinite topology by virtue of \( \Phi_{ij}^{ij} \) and the Pontryagin duality.

Proof. This result was proven by Zhao in [58] except that he additionally assumed that \( \mathcal{X}_s \) is reduced and \( X \in \text{Sm}_k \). But these assumptions are unnecessary as we shall explain by outlining Zhao’s proof.

Recall that \( \iota : Y \hookrightarrow \mathcal{X} \) is the inclusion of the reduced special fiber, which is assumed to be a simple normal crossing divisor in \( \mathcal{X} \). The proof of (1) goes as follows. Using a duality theorem of Sato for normal crossing projective schemes over \( \mathfrak{f} \), the proof of (1) is reduced to showing a purity isomorphism (i.e., Corollary 1.4.9 of op. cit.). The proof of this corollary has the following ingredients. The first is Proposition 1.2.4 of op. cit. for the case of the duality theorem. The second is Corollary 1.3.14 of op. cit., which is a result of Shiho that holds for any normal crossing scheme. The third is Proposition 1.4.6 of op. cit., which is a result of Moser that holds for any finite type \( \mathfrak{f} \)-scheme. The final step is Theorem 1.4.4 of op. cit., whose proof also requires only the above three ingredients.

To prove (2), we let \( \iota' : \mathcal{X}_s \hookrightarrow \mathcal{X} \) and \( \iota'' : Y \hookrightarrow \mathcal{X}_s \) be the inclusions. We then have a cup product pairing (see §3.1 of op. cit.)

\[
H^i(Y, \iota'^*(W_n \Omega^j_{\mathcal{X}, \log})) \times H^{3-i}(Y, \iota''(W_n \Omega^{2-j}_{\mathcal{X}, \log})) \to H^3(Y, \iota''(W_n \Omega^2_{\mathcal{X}, \log})).
\]

On the other hand, there are canonical maps

\[
H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \xrightarrow{\iota'^*} H^i(\mathcal{X}_s, \iota'^*(W_n \Omega^j_{\mathcal{X}, \log}))
\]

\[
H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \xrightarrow{\iota'^*} H^i(\mathcal{X}, \iota'^*(W_n \Omega^j_{\mathcal{X}, \log}))
\]

\[
\cong H^i(\mathcal{X}, \iota''(W_n \Omega^j_{\mathcal{X}, \log}))
\]

\[
\cong H^i(Y, \iota''(W_n \Omega^j_{\mathcal{X}, \log})).
\]

(2.4)
The first arrow is an isomorphism by the proper base change theorem for the étale cohomology of torsion sheaves (e.g., see [34, Cor. VI.2.7]). The second arrow is the canonical counit of adjunction map which is an isomorphism since the closed immersion \( \iota'' \) has empty complement and \( \iota''_* \) is exact. Since \( H^i(Y, R^i(\mathcal{W}_n^j,_{\log X}) \cong H^i_Y((\mathcal{W}, \mathcal{W}_n^j,_{\log X}), \mathcal{W}_n^j,_{\log X})) \), the above pairing can be written as

\[
H^i(X, \mathcal{W}_n^j,_{\log X}) = H^i(Y, R^i(\mathcal{W}_n^j,_{\log X})) \xrightarrow{\text{Tr}_X} \mathbb{Z}/p^n.
\]

The proof of the perfectness of (2.5) is easily reduced to the case when \( n = 1 \) using the exact sequence (see [50, Prop. 2.8, 2.12])

\[
0 \to \mathcal{W}_m^j,_{\log X} \to \mathcal{W}_n^j,_{\log X} \to \mathcal{W}_m^j,_{\log X} \to 0
\]

and compatibility of (2.5) with respect to maps

\[
H^i(\mathcal{X}, \mathcal{W}_n^j,_{\log X}) \xrightarrow{p^n} H^i(\mathcal{X}, \mathcal{W}_{n+1}^j,_{\log X})
\]

and

\[
H^3(\mathcal{X}, \mathcal{W}_{n+1}^j,_{\log X}) \xrightarrow{R^j} H^3(\mathcal{X}, \mathcal{W}_n^j,_{\log X}).
\]

Here, \( p^n \) is induced by the multiplication map \( \mathbb{k}_{j,\mathcal{X}}/p^{n-m} \to \mathbb{k}_{j,\mathcal{X}}/p^n \) via the isomorphism dlog.

The proof of the case \( n = 1 \) is a direct consequence of Cor. 2.5.2 (which is the coherent duality) and Prop. 3.1.2 (which proves the compatibility between the trace maps of Sato and that of the coherent duality) of op. cit. The latter result is a statement only about \( \mathcal{S} \). The proof of the former result does not depend on the special fiber at all, but it requires us to know that the locally free sheaf \( \Omega_{\mathcal{X}}^1 \) has rank two (e.g., see [50, Prop. 2.1.2]). However, this can be checked by restricting to the open subscheme of \( \mathcal{X} \) where \( \mathcal{X} \) is smooth over \( k \). The generic smoothness of \( \mathcal{X} \) over \( k \) suffices for this purpose.

\[\square\]

3. The topology of logarithmic Hodge–Witt cohomology

As we mentioned in §1, we need to endow the logarithmic Hodge–Witt cohomology of complete curves over local fields with suitable topology in order to extend Milne’s duality for logarithmic Hodge–Witt cohomology over finite fields to local fields. This is the goal of the present section. We shall continue to work under the setup of §2.3

3.1. Some finiteness results. In this subsection, we prove some technical lemmas regarding certain logarithmic Hodge–Witt cohomology groups as preparation for proving Theorem 1.1. We let \( n \geq 1 \) be any integer. We begin with the following vanishing statement.

Lemma 3.1. We have \( \mathcal{W}_n^j,_{\log X} = 0 \) for \( n \geq 1 \) and \( j \geq 3 \). Furthermore,

\[
H^i(\mathcal{X}, \mathcal{W}_n^j,_{\log X}) = 0
\]

for \( i \geq 3 \) and \( j \geq 0 \).
Proof. To prove the first claim, we can assume, using (2.6), that \( n = 1 \). By the exact sequence (where \( \bar{\theta} \) is the canonical quotient map and \( F \) is induced by \( F \), see [13 §5.2])

\[
(3.1) \quad 0 \to W_n \Omega_X^1 \to \Omega_X^1 \xrightarrow{d} W_n \Omega_X^1 \to 0,
\]

it suffices to show that \( \Omega_X^1 = 0 \) for \( j \geq 3 \). Since \( X \) is connected and \( \Omega_X^1 \) is locally free (e.g., see [58 Prop. 2.1.2]), it suffices to show that \( \Omega_X^1 \) is generically zero. We can thus replace \( X \) by the largest affine open subscheme \( X_{sm} \subset X \) such that \( X_{sm} \) is smooth over \( k \). Since \( X_{sm} \neq \emptyset \) by our assumption, the claim is therefore reduced to showing that \( \Omega_{U_{ij}}^1 = 0 \) for \( j \geq 3 \) if \( U \) is any smooth curve over \( k \).

Now, we look at the exact sequence (e.g., see [33 Thm. 25.1])

\[
(3.2) \quad 0 \to \Omega_k^1 \otimes_k \mathcal{O}_U \xrightarrow{d} \Omega_U^1 \to \Omega_{U/k}^1 \to 0.
\]

Taking the higher exterior powers, we get \( \Omega_{U_{ij}}^1 \cong \Omega_{U/k}^1 \) for \( j \geq 2 \). Hence, it suffices to show that \( \Omega_{U_{ij}}^1 = 0 \) for \( j \geq 2 \). But this is well known because \( R = \mathbb{F}_q[[t]] \) and a direct computation shows that \( R \cong \Omega_{U_{ij}}^1 \), which implies that \( \Omega_{U_{ij}}^1 = 0 \) for \( j \geq 2 \). Note that this also shows that \( \Omega_X^1 = \Omega_{U/k}^1 \).

To prove the second claim, it suffices to show using (2.4) that

\[
H^i(Y, \mathcal{O}^j(W_n \Omega_X^1)) = 0
\]

for \( i \geq 3 \). But this is clear since \( cd_p(Y) \leq 2 \) (e.g., see [34 Chap. VI, Rem. 1.5(b)]).

Lemma 3.2. The boundary map of the localization sequence

\[
\partial: H^2(X, W_n \Omega_X^1) \to H^3_Y(X, W_n \Omega_X^1)
\]

is an isomorphism.

Proof. The localization sequence in question is the exact sequence

\[
(3.3) \quad H^2(X, W_n \Omega_X^1) \to H^2(X, W_n \Omega_X^2) \to H^3_Y(X, W_n \Omega_X^1) \to H^3(X, W_n \Omega_X^1).
\]

It suffices therefore to show that \( H^i(X, W_n \Omega_X^1) = 0 \) for \( i \geq 2 \). By (2.4), we have

\[
H^i(X, W_n \Omega_X^1) \cong H^i(Y, \mathcal{O}^j(W_n \Omega_X^1)).
\]

But the latter group is zero for \( i \geq 3 \) since \( cd_p(Y) = 2 \), as we observed before.

To prove the vanishing of \( H^2(X, W_n \Omega_X^1) \), it suffices to show, using Theorem [2.8] that \( H^1(Y, \mathbb{Z}/p^n) = 0 \). To that end, we look at the exact sequence

\[
(3.4) \quad 0 \to H^0(X, \mathbb{Z}/p^n) \xrightarrow{u^*} H^0(X, \mathbb{Z}/p^n) \to H^1(Y, \mathbb{Z}/p^n) \to H^1(X, \mathbb{Z}/p^n).
\]

Since \( X \) is integral and normal, the map \( u^* \) on the left is an isomorphism and the map \( u^* \) on the right is injective. It follows that the middle group is zero.

Lemma 3.3. The group \( H^i_Y(X, W_n \Omega_X^1) \) is finite unless \( i \in \{2, 3\} \) and \( j \in \{0, 1\} \).

Proof. In view of Lemma 3.1, we can assume \( j \leq 2 \). We have seen in the proof of Lemma 3.2 that \( H^i_Y(X, W_n \Omega_X^1) = 0 \) for \( i \leq 1 \). We have \( H^i_Y(X, W_n \Omega_X^1) = 0 \) for every \( j \geq 0 \) by the Gersten injectivity \( W_n \Omega_X^1 \to u^*(W_n \Omega_X^1) \) (see [50 Thm. 4.1]). The finitude claim for \( H^i_Y(X, W_n \Omega_X^1) \) follows from Theorem 2.8.
since \( H^i(X, \mathbb{Z}/p^n) \cong H^i(Y, \mathbb{Z}/p^n) \) (see (2.4)), and the latter group is finite by Thm. 1.2.2. We are left with showing the finitude of \( \text{Coker}(H) \) since

We look at the exact (localization) sequence

\[
\begin{align*}
(3.5) \quad H^0(X, W_n \Omega^1_{X, \log}) &\xrightarrow{u^*} H^0(X, W_n \Omega^1_{X, \log}) \\
&\rightarrow H^1(X, W_n \Omega^1_{X, \log}) \xrightarrow{u_1^*} H^1(X, W_n \Omega^1_{X, \log}).
\end{align*}
\]

It suffices to show that \( \text{Coker}(u_0^*) \) and \( \text{Ker}(u_1^*) \) are finite. To prove the finiteness of \( \text{Coker}(u_0^*) \), note that (2.3) yields the commutative diagram of exact sequences

\[
\begin{align*}
(3.6) \quad 0 &\rightarrow H^0(X, \mathcal{O}_X^*)/p^n \rightarrow H^0(X, W_n \Omega^1_{X, \log}) \rightarrow \mathcal{P}(X) \rightarrow 0.
\end{align*}
\]

We next note that there are exact sequences

\[
(3.7) \quad 0 \rightarrow H^0(X, \mathcal{O}_X^*) \xrightarrow{u^*} H^0(X, \mathcal{O}_X^*) \xrightarrow{u_0^*} \mathbb{Z} \rightarrow 0
\]

and

\[
(3.8) \quad 0 \rightarrow \mathbb{Z} \xrightarrow{\iota'_s} \bigoplus_{\mathcal{P}(X)} \mathbb{Z} \xrightarrow{\text{cyc}} \mathcal{P}(X) \xrightarrow{u^*} \mathcal{P}(X) \rightarrow 0,
\]

where the first sequence follows from the geometric integrality condition (cf. Lemma 2.7). The second sequence is well known (e.g., see 49) if we let \( \iota'_s(1) = [X_s] \), the class of the Weil divisor associated to the special fiber \( X_s \) and cyc denotes the cycle class map. It easily follows from these exact sequences that the kernels and cokernels of the left and right vertical arrows in (3.6) are finite. Hence, so are the kernel and cokernel of the middle vertical arrow (the assertion about the kernel will be used only in Lemma 3.5).

To prove the finitude of \( \text{Ker}(u_1^*) \), we look at the commutative diagram with exact rows (following (2.3))

\[
(3.9) \quad 0 \rightarrow \mathcal{P}(X)/p^n \rightarrow H^1(X, W_n \Omega^1_{X, \log}) \rightarrow \mathcal{P}(X) \rightarrow 0.
\]

It follows from Thm. 5.6.1(v), 10.3.1(ii)] that

\[
(3.10) \quad \mathcal{P}(X) = \mathcal{P}(X_s) = \mathcal{P}(Y) = 0.
\]

On the other hand, it follows from (3.8) that the kernel of the left vertical arrow in (3.9) is finite. We conclude that \( \text{Ker}(u_1^*) \) is finite. We have thus shown that \( H^i_Y(X, W_n \Omega^1_{X, \log}) \) is finite. This concludes the proof.

\section*{Remark 3.4}
It can be shown that each of the groups \( H^i_Y(X, W_n \Omega^1_{X, \log}) \) is infinite in general if \( i \in \{2, 3\} \) and \( j \in \{0, 1\} \).

\section*{Lemma 3.5}
For all \( i, j \geq 0 \), the kernel of \( u^*: H^i(X, W_n \Omega^j_{X, \log}) \rightarrow H^i(X, W_n \Omega^j_{X, \log}) \) is finite.
3.2. Logarithmic Hodge–Witt cohomology as topological groups. We shall now endow the logarithmic Hodge–Witt cohomology of $X$ with torsion-by-profinite topology. For $i,j \geq 0$, we consider the localization sequence

\[
\cdots \to H^i_Y(X, \Omega^j_{X, \log}) \xrightarrow{\iota_*} H^i(X, \Omega^j_{X, \log}) \xrightarrow{u^*} H^i(X, W_n\Omega^j_{X, \log}) \xrightarrow{\partial} H^{i+1}_Y(X, \Omega^j_{X, \log}) \to \cdots.
\]

By Lemma 3.3 it breaks into exact sequences

\[
H^i_Y(X, \Omega^j_{X, \log}) \xrightarrow{\iota_*} H^i(X, \Omega^j_{X, \log}) \xrightarrow{u^*} F_{ij} \to 0, \tag{3.12}
\]

\[
0 \to F_{ij} \to H^i(X, \Omega^j_{X, \log}) \xrightarrow{\partial} H^{i+1}_Y(X, \Omega^j_{X, \log}) \tag{3.13}
\]

such that $\text{Ker}(u^*)$ is finite.

Recall from Theorem 2.8 that each $H^i_Y(X, \Omega^j_{X, \log})$ is a discrete torsion group of exponent $p^n$ and $H^i(X, \Omega^j_{X, \log})$ is a profinite topological abelian group. It follows that $\text{Image}(\iota_*)$ is a finite closed subgroup of $H^i(X, \Omega^j_{X, \log})$. In particular, its subspace topology induced from $H^i(X, \Omega^j_{X, \log})$ as well as its quotient topology induced from $H_Y^i(X, \Omega^j_{X, \log})$ is discrete.

For $i,j \geq 0$, we endow $F_{ij}$ with the quotient topology via (3.12). Then $F_{ij}$ becomes a quotient of the profinite group $H^i(X, \Omega^j_{X, \log})$ by a closed subgroup. It follows that $F_{ij}$ is a profinite abelian group and (3.12) is an exact sequence of topological abelian groups.

We endow $H^i(X, \Omega^j_{X, \log})$ with the unique topology for which $F_{ij}$ is an open subgroup of $H^i(X, \Omega^j_{X, \log})$ and $H^i(X, \Omega^j_{X, \log})/F_{ij} = \text{Ker}(\iota_*)$ is a discrete quotient of $H^i(X, \Omega^j_{X, \log})$. This endows $H^i(X, \Omega^j_{X, \log})$ with the structure of a topological abelian group such that (3.13) is an exact sequence of topological abelian groups. If we combine this with everything we have shown in §3.1 and §3.2 we get the following.

**Proposition 3.6.** For each $i,j \geq 0$, the group $H^i(X, \Omega^j_{X, \log})$ is equipped with the structure of a torsion-by-profinite topological abelian group satisfying the following.

1. (3.11) is an exact sequence in $\text{Pfd}$ for every $j \geq 0$.
2. $H^i(X, \Omega^j_{X, \log}) = 0$ unless $0 \leq i,j \leq 2$.
3. $H^i(X, \Omega^j_{X, \log})$ is profinite unless $i \in \{1, 2\}$ and $j \in \{0, 1\}$.
4. $H^2(X, \Omega^j_{X, \log})$ is a discrete torsion group.
5. $H^1(X, \Omega^0_{X, \log})$ is an infinite discrete torsion group.
6. $H^1(X, \Omega^1_{X, \log})$ is neither a profinite nor a discrete torsion group.

**Proof.** We only need to explain (4) and (6). But the former follows from Thm. 1.2.2 and the isomorphism $H^2(X, \mathbb{Z}/p^n) \cong H^2(Y, \mathbb{Z}/p^n)$ while the latter follows from the perfectness of the Brauer–Manin pairing for $X$ (e.g., see §9). □
Using Lemma \textit{2.4} Proposition \textit{3.6} and considering the Pontryagin duals associated to \textit{(3.11)}, we get a chain complex in $\text{Pfd}$:

\[(3.14) \quad \cdots \rightarrow H^{i+1}_Y(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \xrightarrow{\partial^i} H^i(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \xrightarrow{(u^*)^*} H^i(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \rightarrow \cdots.\]

\textbf{Lemma 3.7.} The sequence (3.14) is exact.

\textit{Proof.} We look at the exact sequences

\[(3.15) \quad 0 \rightarrow \text{Ker}(u^*) \rightarrow H^i_Y(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \rightarrow \text{Ker}(u^*) \rightarrow 0;\]

\[(3.16) \quad 0 \rightarrow \text{Ker}(u^*) \rightarrow H^i(\mathcal{X}, W_n\Omega^j_{X,\text{log}}) \rightarrow F_{ij} \rightarrow 0;\]

\[(3.17) \quad 0 \rightarrow F_{ij} \rightarrow H^i(\mathcal{X}, W_n\Omega^j_{X,\text{log}}) \rightarrow \text{Ker}(u^*) \rightarrow 0.\]

It follows from Proposition \textit{3.6} and the definition of the topologies of the various groups in \textit{(3.11)} that the above three are exact sequences in $\text{Pfd}$ and they satisfy the hypothesis of Lemma \textit{2.4}. We can therefore apply this lemma to get short exact sequences

\[(3.18) \quad 0 \rightarrow \text{Ker}(u^*) \rightarrow H^i_Y(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \rightarrow \text{Ker}(u^*) \rightarrow 0,\]

\[(3.19) \quad 0 \rightarrow \text{Ker}(u^*) \rightarrow H^i(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \rightarrow (F_{ij})^* \rightarrow 0,\]

\[(3.20) \quad 0 \rightarrow (F_{ij})^* \rightarrow H^i(\mathcal{X}, W_n\Omega^j_{X,\text{log}})^* \rightarrow \text{Ker}(u^*)^* \rightarrow 0.\]

The desired result follows by piecing these exact sequences together. \hfill \Box

4. Duality for logarithmic Hodge–Witt cohomology

We continue to work under the setup of \textit{2.3}. In this section, we shall prove the duality theorem for the logarithmic Hodge–Witt cohomology of the proper curve $X$ over $k$. We begin by constructing the trace map. For a scheme $Z$, we let $\mathcal{D}_{\text{et}}(Z)$ denote the derived category of étale sheaves of abelian groups on $Z$. We let $\mathcal{D}_{\text{et}}(Z, \mathbb{Z}/m)$ denote the derived category of étale sheaves of $\mathbb{Z}/m$-modules on $Z$ with $m \in \mathcal{O}^\times(Z)$. For a local ring $A$, we shall let $A^h$ (resp. $A^{sh}$, resp. $\widehat{A}$) denote the Henselization (resp. strict Henselization, resp. completion) of $A$ with respect to its maximal ideal. For a closed point $x \in X$, we shall let $K_x$ (resp. $K_x^{sh}$, resp. $\widehat{K}_x$) denote the quotient field of the Henselization $\mathcal{O}^h_{X,x}$ (resp. strict Henselization $\mathcal{O}^{sh}_{X,x}$, resp. completion $\widehat{\mathcal{O}}_{X,x}$) of $\mathcal{O}_{X,x}$. We fix an integer $n \geq 1$.

4.1. The trace maps. We let $j: \text{Spec}(K) \hookrightarrow X$ be the inclusion of the generic point of $X$ and let $t_x: \text{Spec}(k(x)) \hookrightarrow X$ be the inclusion of a closed point. By the Gersten resolution of the logarithmic Hodge–Witt sheaves, one has an exact sequence of étale sheaves

\[(4.1) \quad 0 \rightarrow W_n\Omega^2_{X,\text{log}} \rightarrow j_* (W_n\Omega^2_{K,\text{log}})^{\text{res}} \bigoplus_{x \in X_{(0)}} (t_x)_* (W_n\Omega^1_{k(x),\text{log}}) \rightarrow 0,\]

where res is the sum of residue maps in Kato’s complex (see \textit{[50 Thm. 5.2]}). It follows that there is an exact triangle

\[(4.2) \quad \bigoplus_{x \in X_{(0)}} (t_x)_* (W_n\Omega^1_{k(x),\text{log}})[-1] \rightarrow W_n\Omega^2_{X,\text{log}} \rightarrow j_* (W_n\Omega^2_{K,\text{log}}) \]

\end{document}
in $\mathcal{D}_{\text{et}}(X)$. We let $\nu_{n,X}^2$ denote the complex

$$
j_* (W_n \Omega_{K,\log}^2) \xrightarrow{\text{res}} \bigoplus_{x \in X(0)} (t_x)_* (W_n \Omega_{k(x),\log}^1).
$$

For every $x \in X(0)$, we have the canonical trace map $\epsilon_x : (f_x)_* (W_n \Omega_{k(x),\log}^1) \to W_n \Omega_{k,\log}^1$, induced by the norm $N_{x^*} : (f_x)_* (\mathcal{O}_{k(x)}^1) \to \mathcal{O}_k^1$ via the dlog map, where $f_x : \text{Spec} (k(x)) \to \text{Spec} (k)$ is the structure map. We let $\epsilon_X := \sum_x \epsilon_x$. By the reciprocity law for Milnor $K$-theory (e.g., see [21 Lem. 4]), it is clear that the composite map

$$g_* (W_n \Omega_{K,\log}^2) \xrightarrow{\text{res}} \bigoplus_{x \in X(0)} (f_x)_* (W_n \Omega_{k(x),\log}^1) \xrightarrow{\epsilon_X} W_n \Omega_{k,\log}^1
$$

is zero if we let $g = f \circ j$, where recall that $f : X \to \text{Spec} (k)$ is the structure map. In other words, we have a morphism of complexes

$$(4.3) \quad f_* : f_* (\nu_{n,X}^2) \to W_n \Omega_{k,\log}^1[-1]$$

whose composition with $(f_x)_* (W_n \Omega_{k(x),\log}^1)[-1] \to f_* (\nu_{n,X}^2)$ is $\epsilon_x [-1]$ for every $x \in X(0)$. We now prove some key lemmas.

**Lemma 4.1.** One has $R^q j_* (W_n \Omega_{K,\log}^2) = 0$ for $q \geq 1$.

**Proof.** The stalk of $R^q j_* (W_n \Omega_{K,\log}^2)$ at a closed point $x \in X$ is $H^q (K_x^{sh}, W_n \Omega_{K^{sh,\log}}^2)$. To show that this cohomology is zero, we can assume $q = 1$ because $c_{dp} (K_x^{sh}) \leq 1$. By [23 Lem. 21], we can pass to the completion $\tilde{K}_x^{sh}$ of $K_x^{sh}$. We let $A_x$ denote the completion of $\mathcal{O}_{X,x}^{sh}$, $F_x = \tilde{K}_x^{sh}$ and consider the exact sequence

$$(4.4) \quad H^1 (A_x, W_n \Omega_{A_x,\log}^2) \to H^1 (F_x, W_n \Omega_{F_x,\log}^2) \to H^2_{\pi} (A_x, W_n \Omega_{A_x,\log}^2),$$

where $\pi$ is the closed point of $\text{Spec} (A_x)$. By the proper base change theorem, we have $H^1 (A_x, W_n \Omega_{A_x,\log}^2) \cong H^1 (k(\overline{x}), t^* (W_n \Omega_{A_x,\log}^2))$, where $t : \text{Spec} (k(\overline{x})) \to \text{Spec} (A_x)$ is the inclusion of the closed point. But the latter group vanishes since $k(\overline{x})$ is separably closed (e.g., see [34 Thm. VI.1.1]). It remains to show that $H^2_{\pi} (A_x, W_n \Omega_{A_x,\log}^2) = 0$.

Since $k(\overline{x})$ is separably closed, we have the well-known equality $[k(\overline{x}) : k(\overline{x})^p] = p$ (e.g., see [24 §3]). Since $F_x \cong k(\overline{x})(T)$, it is an elementary exercise that $[F_x : (F_x)^p] = p^2$ (e.g., see the proof of [58 Lem. 1.4.5]). We therefore conclude from [59 Thm. 3.2] that there is a purity isomorphism $t_* : H^1 (k(\overline{x}), W_n \Omega_{k(\overline{x}),\log}^1) \cong H^2_{\pi} (A_x, W_n \Omega_{A_x,\log}^2)$. This proves our claim because $H^1 (k(\overline{x}), W_n \Omega_{k(\overline{x}),\log}^1)$ is clearly zero. To finish the proof of the lemma, we note that the stalk of $R^q j_* (W_n \Omega_{K,\log}^2)$ at the generic point of $X$ is $H^q (K_s, W_n \Omega_{K_s,\log}^2)$, which is zero if $q \geq 1$.

**Lemma 4.2.** The canonical map $f_* (\nu_{n,X}^2) \to Rf_* (\nu_{n,X}^2)$ is an isomorphism in $\mathcal{D}_{\text{et}}(k)$.

**Proof.** In view of Lemma 4.1, it suffices to show that $R^q f_* (j_* (W_n \Omega_{K,\log}^2)) = 0$ and $R^q f_* ((t_x)_* (W_n \Omega_{k(x),\log}^1)) = 0$ for all $q \geq 1$ and $x \in X(0)$. Using the exactness of $(t_x)_*$ for $x \in X(0)$ and Lemma 4.1 for the generic point of $X$, it suffices to show that $R^q g_* (W_n \Omega_{K,\log}^2) = 0$ and $R^q (f_x)_* (W_n \Omega_{k(x),\log}^1) = 0$ for all $q \geq 1$ and $x \in X(0)$. This latter claim for $x \in X(0)$ is obvious since $f_x$ is a finite morphism.
On the other hand, we have $R^n g_*(W_n \Omega^2_{X, \log}) \cong H^n(K', W_n \Omega^2_{K', \log})$, where $K'$ is the function field of $X_s := X \times_{\text{Spec}(k)} \text{Spec}(k_s)$. We are now done because $H^n(K', W_n \Omega^2_{K', \log})$ is clearly zero if $q \geq 2$, and its vanishing for $q = 1$ was shown by Kato–Saito [24] (see the proof of their Lemma 1(2) on p. 252). We remark that even though the cited lemma of Kato–Saito assumes that $X$ is smooth over $k$, the proof of the vanishing of $H^1(K', W_n \Omega^2_{K', \log})$ only requires $X$ to be generically smooth over $k$. This concludes the proof. □

**Lemma 4.3.** There exists a canonical morphism $\text{tr} X Rf_*(W_n \Omega^2_{X, \log}) \rightarrow W_n \Omega^1_{k, \log}[-1]$ in $\mathcal{D}_{\text{ét}}(k)$ whose composition with the canonical morphisms

$$(f_*)_*(W_n \Omega^1_{k(x), \log}[-1]) \rightarrow f_*(W_n \Omega^2_{X, \log}) \rightarrow Rf_*(W_n \Omega^2_{X, \log})$$

is $\epsilon_x[-1]$ for every $x \in X(0)$.

**Proof.** Combine the isomorphism $W_n \Omega^2_{X, \log} \rightarrow \nu_{n, X}^2$ in $\mathcal{D}_{\text{ét}}(X)$ with Lemma 4.2 and the morphism $f_*$ in (4.3). □

Recall that there is a canonical isomorphism $\text{inv}_k : \text{Br}(k) \rightarrow \mathbb{Q}/\mathbb{Z}$. If $k'/k$ is a finite field extension of degree $m$ and $v : \text{Spec}(k') \rightarrow \text{Spec}(k)$ is the projection, then the norm map $N_{k'/k} : v_* (\mathcal{O}'_{k'}) \rightarrow \mathcal{O}_k$ between the étale sheaves induces the pull-back and push-forward maps $v^* : \text{Br}(k) \rightarrow \text{Br}(k')$ and $v_* : \text{Br}(k') \rightarrow \text{Br}(k)$ such that $\text{inv}_{k'} \circ v^* = m(\text{inv}_k)$ on $\mathbb{Q}/\mathbb{Z}$ and $\text{inv}_k \circ v_* = \text{inv}_{k'}$.

Identifying $\rho^* \text{Br}(k')$ with $H^1(k', W_n \Omega^1_{k', \log})$, we get a commutative diagram

$$
\begin{array}{ccc}
H^1(k', W_n \Omega^1_{k', \log}) & \xrightarrow{\text{inv}_{k'}} & \mathbb{Q}/\mathbb{Z} \\
\downarrow v_* & & \downarrow \text{inv}_k \\
H^1(k, W_n \Omega^1_{k, \log}) & \xrightarrow{\text{inv}_k} & \mathbb{Z}/p^n
\end{array}
$$

in which the horizontal and the diagonal arrows are isomorphisms. It follows that $v_*$ is an isomorphism such that $\text{inv}_k \circ v_* = \text{inv}_{k'}$. We shall therefore identify $v_*$ with the identity map of $\mathbb{Z}/p^n$. Similarly, we shall identify $v_* : \text{Br}(k') \rightarrow \text{Br}(k)$ with the identity map of $\mathbb{Q}/\mathbb{Z}$ throughout this paper. In the sequel, we shall write $v_* = \epsilon_x$ if $k' = k(x)$ for some closed point $x$ on a curve over $k$.

**Proposition 4.4.** There exists a unique homomorphism $\text{Tr}_X : H^2(X, W_n \Omega^2_{X, \log}) \rightarrow \mathbb{Z}/p^n$ such that $\text{Tr}_X \circ (\epsilon_x)_*$ is the identity map for every $x \in X(0)$. Furthermore, $\text{Tr}_X$ is bijective.

**Proof.** By (4.1) and Lemma 4.1 we have an exact sequence

$$
(4.6) \quad H^1(K, W_n \Omega^2_{K, \log}) \xrightarrow{\text{res}} \bigoplus_{x \in X(0)} H^1(k(x), W_n \Omega^1_{k(x), \log}) \rightarrow H^2(X, W_n \Omega^2_{X, \log}) \rightarrow 0.
$$

This implies that $H^2(X, W_n \Omega^2_{X, \log})$ is generated by the images of $(\epsilon_x)_*(1)$ as $x$ varies in $X(0)$. The uniqueness assertion follows immediately from this. The existence follows directly from Lemma 4.3 and (4.3). The claim that $\text{Tr}_X$ is bijective follows from the fact that its source and target are finite of equal cardinality by Theorem 2.8 and Lemma 3.2 while $\text{Tr}_X \circ (\epsilon_x)_*$ is bijective. □
Remark 4.5. The assertion that $H^2_\et(X, W_n\Omega^2_{X,\log})$ is isomorphic to $\mathbb{Z}/p^n$ is not new (at least when $X$ is smooth over $k$) as it was already shown by Kato–Saito in [24 Prop. 4]. It also follows from Theorem 2.8 and Lemma 3.2. However, Proposition 4.4 provides an explicit isomorphism which has the advantage that it commutes with the trace maps for all closed points of $X$. This is unclear in the construction of [24]. We shall need this explicit nature of $\text{Tr}_X$ later in the paper in a critical way. Another advantage is that it does not require $X$ to be smooth everywhere. Although this flexibility is not important in this paper, it will be vital in the study of duality and class field theory for open smooth curves over $k$ which do not admit smooth compactifications. We were unable to prove if $\text{Tr}_X$ coincides with the isomorphism of Kato–Saito, whose construction, we believe, is quite intricate.

Given $x \in X(0)$, we now look at the canonical maps of étale sheaves

$$(\iota_x)_* (W_n\Omega^1_{k(x),\log}) \to (\iota_x)_* R(\iota_x)^1(W_n\Omega^2_{X,\log}) \to W_n\Omega^2_{X,\log},$$

on $X$, where $[t_x]$ is the Teichmüller image of the chosen uniformizer $t_x \in \mathcal{O}_{X,x}$ and the first arrow is obtained by taking cup product with $d\log([t_x])$. This arrow is an isomorphism in $D_\et(X)$ by [58, Cor. 1.3.14]. The same is also true if we replace $X$ by $\text{Spec} \,(\mathcal{O}_{X,x})$.

Furthermore, the diagram of the induced maps on cohomology

$$(4.7) \quad H^1(k(x), W_n\Omega^1_{k(x),\log}) \xrightarrow{\cup [t_x]} H^2(X, W_n\Omega^2_{X,\log}) \xrightarrow{z} H^2(X, W_n\Omega^2_{X,\log})$$

is commutative, where the left vertical arrow $\cup [t_x]$ is the isomorphism of [20 §3.2, Lem. 3]. It follows from the proper base change theorem that the group on the bottom right corner is zero. Since the other two groups in the bottom row are isomorphic to $\mathbb{Z}/p^n$ (e.g., see [20 Prop. 3.1]), it follows that $\partial_x$ is bijective.

Using Proposition 4.4 and (4.7), we get a commutative diagram

$$(4.8) \quad H^1(k(x), W_n\Omega^1_{k(x),\log}) \xrightarrow{(\iota_x)_*} H^2(X, W_n\Omega^2_{X,\log})$$

where $h_x := \cup [t_x]$ is an isomorphism.

Letting $\text{Tr}_X := \text{inv}_k \circ \epsilon_x \circ (h_x)^{-1}$, we get the following.
Corollary 4.6. For every $x \in X(0)$, there is a commutative diagram

$$
\begin{align*}
H^1(k(x), W_n \Omega^1_{k(x), \log}) & \cong H^2(X, W_n \Omega^2_{X, \log}) \\
h_x \downarrow & \downarrow \\
H^1(K_x, W_n \Omega^2_{K_x, \log}) & \cong H^2(X, W_n \Omega^2_{X, \log})
\end{align*}
$$

in which all arrows are isomorphisms.

4.2. Duality theorem for $H^*(X, W_n \Omega^*_X, \log)$. Recall that a pairing between locally compact Hausdorff topological abelian groups $\lambda: A \times B \to \mathbb{Q}/\mathbb{Z}$ is called continuous if $\lambda$ is continuous with respect to the product topology of $A \times B$. Equivalently, either (and hence both) of the maps $A \to B^\vee$ and $B \to A^\vee$ is continuous and factors through the continuous dual (e.g., see [17 Prop. A.14]). $\lambda$ is called nondegenerate on the left (resp. right) if the induced map $A \to B^\vee$ (resp. $B \to A^\vee$) is injective. One says that $\lambda$ is perfect (in particular, continuous) if it induces isomorphisms $A \cong B^\vee$ and $B \cong A^\vee$ of topological abelian groups.

We now return to the setup of \S 2.3. Recall from Proposition 4.4 that there is a canonical isomorphism of $\mathbb{Z}/p^n$-modules

$$
\text{Tr}_X: H^2(X, W_n \Omega^2_{X, \log}) \cong \mathbb{Z}/p^n.
$$

Using Lemma 3.2, we get a unique isomorphism $\text{Tr}_X: H^3_X(\mathcal{X}, W_n \Omega^2_{\mathcal{X}, \log}) \cong \mathbb{Z}/p^n$ such that $\text{Tr}_X \circ \partial = \text{Tr}_X$ (cf. Theorem 2.8). The cup product pairing

$$
\begin{align*}
H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) \times H^{3-i}(\mathcal{X}, W_n \Omega^{2-j}_{\mathcal{X}, \log}) & \to H^3(\mathcal{X}, W_n \Omega_{\mathcal{X}, \log}^{2-j}) \\
& \xrightarrow{\text{Tr}_X} \mathbb{Z}/p^n
\end{align*}
$$

gives rise to maps

$$
\begin{align*}
H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) & \xrightarrow{\Phi^j} H^{3-i}(\mathcal{X}, W_n \Omega^{2-j}_{\mathcal{X}, \log})^*, \\
H^i(\mathcal{X}, W_n \Omega^j_{\mathcal{X}, \log}) & \xrightarrow{\Phi^j} H^{3-i}(\mathcal{X}, W_n \Omega^{2-j}_{\mathcal{X}, \log})^*
\end{align*}
$$

in Pfd. Furthermore, it follows from Theorem 2.8 that these maps are topological isomorphisms. In other words, (4.11) is a perfect pairing in Pfd.

We shall now prove our duality theorem for the logarithmic Hodge–Witt cohomology of $X$. This will prove Theorem 1.4. This duality recovers as well as generalizes [24 Prop. 4]. Let the notations and hypotheses be as in \S 2.3. In particular, $X$ is a geometrically connected, regular, and generically smooth curve over the local field $k$ of exponential characteristic $p > 1$. $\mathcal{X}$ is a projective and flat regular semistable model of $X$ over Spec($\mathcal{O}_k$).

Theorem 4.7. For every pair of integers $i, j \geq 0$, the cup product on logarithmic Hodge–Witt cohomology induces a perfect pairing of topological abelian groups

$$
\begin{align*}
H^i(X, W_n \Omega^j_{X, \log}) \times H^{2-i}(X, W_n \Omega^{2-j}_{X, \log}) & \to H^2(X, W_n \Omega^2_{X, \log}) \\
& \xrightarrow{\text{Tr}_X} \mathbb{Z}/p^n.
\end{align*}
$$

Proof. The existence of the bilinear pairing is clear. We shall prove its perfectness in several steps. In order to save space, we shall write all cohomology groups by suppressing the underlying schemes. We shall also use the short hand $\mathcal{F}^j_X$ for $W_n \Omega^j_{X, \log}$ and $\mathcal{F}^j_X$ for $W_n \Omega^j_{X, \log}$.
We now fix \(i, j \geq 0\) and we look at the diagram

\[
\begin{array}{cccccc}
H^i(Y) & \to & H^i(F_{X}^j) & \to & H^i(F_{X}^{j+1}) & \to H^{i+1}(F_{X}^{j+1}) \\
\Phi^i \downarrow & & \Phi^i \downarrow & & \Phi^i \downarrow & \\
H^3(F_{X}^{j-1})^* & \to & H^3(F_{X}^{j-1})^* & \to & H^3(F_{X}^{j-1})^* & \to H^{3+1}(F_{X}^{j-1})^*
\end{array}
\]

where the vertical arrows on the lower floor are the canonical inclusions. The top and the bottom rows are clearly exact while the exactness of the middle row is shown in Lemma 3.7. All squares on the lower floor clearly commute. We let the composite vertical arrow in the middle be the map shown in Lemma 3.7. All squares on the lower floor clearly commute. We let the composite vertical arrow in the middle be the map \((-1)^{i} \Phi_{X}^{i}\), where \(\Phi_{X}^{i}: H^i(F_{X}^j) \to H^{2-i}(F_{X}^{j-1})^\vee\) is induced by (4.12). We now show that the composition of each of the squares on the upper floor with the one below it on the lower floor in (4.13) commutes.

The composite squares on the left and right corners commute because they are induced by the canonical commutative diagram

\[
\begin{array}{cccccc}
\tau_* R^j \mathcal{F}^{j}_X & \times & \mathcal{F}^{j}_X & \to & \tau_* R^j \mathcal{F}^{j+1}_X \\
\tau_* \downarrow & & \tau_* \downarrow & & \tau_* \downarrow \\
\mathcal{F}^{j}_X & \times & \tau_* R^j \mathcal{F}^{j}_X & \to & \tau_* R^j \mathcal{F}^{j+1}_X
\end{array}
\]

For the two middle composite squares, note that if we apply the classical formula relating the cup product and the boundary maps in sheaf cohomology (e.g., see [3 Thm. 7.1] or [53 Lem. 3.2]) to the cohomology sequences associated to the (derived) tensor product of \(\mathcal{F}^{j}_X\) with the exact triangle

\[
\tau_* R^j \mathcal{F}^{2-j}_X \to \mathcal{F}^{2-j}_X \to \sigma_* \mathcal{F}^{2-j}_X
\]

for \(0 \leq j \leq 2\), we get that the diagram

\[
\begin{array}{cccccc}
H^i(F_{X}^{j}) \times H^{3-i}(F_{X}^{j-1}) & \to & H^3(F_{X}^{j}) \\
\phi^i \downarrow & & \phi^i \downarrow \\
H^i(F_{X}^{j}) \times H^{2-i}(F_{X}^{j-1}) & \to & H^2(F_{X}^{j})
\end{array}
\]

is commutative up to multiplication by \((-1)^i\). Since \(\text{Tr}_X \circ \partial = \text{Tr}_X\), it follows that the two middle composite squares commute.

Since all vertical arrows on the lower floor of (4.13) are injective and all lower squares are commutative, it follows that the squares on the left and the right corners on the upper floor are commutative. Next, an easy diagram chase shows that the square labeled (1) is Cartesian. This implies that the image of the composite middle vertical arrow lies in the subgroup \(H^{2-i}(F_{X}^{j-1})^\vee\). Furthermore, if we let \(\Phi_{X}^{i}: H^i(F_{X}^j) \to H^{2-i}(F_{X}^{j-1})^\vee\) denote the induced map, then all squares on the upper floor commute. Since all vertical arrows on this floor, except possibly the middle one, are isomorphisms, it follows that \(\Phi_{X}^{i}\) is also an isomorphism.
To finish the proof of the theorem, it remains to show that $\Phi_{ij}^*$ is a continuous and open homomorphism. To show this, recall that $F_{ij}$ is a quotient of $H^i(F_X^j)$ and is an open subgroup of $H^i(F_X^j)$ such that $H^i(F_X^j)/F_{ij}$ is discrete. Hence, it suffices to show in the square labeled (2) in (4.13) that the composite map $H^i(F_X^j) \to H^i(F_X^{2-j})^* \to H^{2-j}(F_X^{2-j})^*$ is continuous and open. Let us call this composite map $\theta$. Now, we have already seen above that $\Phi_{ij}^*$ is a topological isomorphism. On the other hand, $H^{2-i}(F_X^{2-j})^* \to H^{2-i}(F_X^{2-j})^*$ is clearly continuous because the Pontryagin dual is an auto-functor in the category of locally compact Hausdorff topological abelian groups. This proves that $\theta$ is continuous.

To show that $\theta$ is open, it suffices to show that the map $H^{i+1}_V(F_X^j)^* \to H^i(F_X^j)^*$ is open for any $i, j \geq 0$. From (3.18) and (3.19), we see that this map has a factorization

$$H^{i+1}_V(F_X^j)^* \to \text{Im}(\partial^*) = \text{Ker}(\iota_*)^* \to H^i(F_X^j)^*.$$  

The first arrow in this factorization is clearly open because it is a quotient map in $\textbf{Pfd}$ by Lemma (2.3) and the other hand, $(F_{ij})^*$ is discrete and $H^i(F_X^j)^* \to (F_{ij})^*$ is a continuous surjective homomorphism. It follows that its kernel $\text{Im}(\partial^*) = \text{Ker}(\iota_*)^*$ is open in $H^i(F_X^j)^*$. We have thus shown that the composite map in (4.16) is open. This concludes the proof.

For integers $m, n \geq 1$, we have a diagram of bilinear parings

$$\begin{align*}
\mathcal{K}_{i,X}^M/p^m \times \mathcal{K}_{j,X}^M/p^m & \rightarrow \mathcal{K}_{i+j,X}^M/p^m \\
p^n & \downarrow \text{can} \\
\mathcal{K}_{i,X}^M/p^{m+n} \times \mathcal{K}_{j,X}^M/p^{m+n} & \rightarrow \mathcal{K}_{i+j,X}^M/p^{m+n},
\end{align*}$$

where ‘can’ indicates the canonical surjection. It easily follows from the bilinearity of the product in Milnor $K$-theory that this diagram is commutative. Passing to the cohomology and using (2.3), we get a bilinear pairing

$$\lim_n H^i(X, W_n \Omega^j_{X,\log}) \times \lim_n H^{2-i}(X, W_n \Omega^{2-j}_{X,\log}) \to \lim_n \mathbb{Z}/p^n$$

between ind-abelian and pro-abelian groups.

We endow $\lim_n H^i(X, W_n \Omega^j_{X,\log})$ with the direct limit (i.e., the weak) topology and $\varprojlim_n H^i(X, W_n \Omega^j_{X,\log})$ the inverse limit topology. Taking the limits in (4.18) and using Lemma (2.6), we get the following.

**Corollary 4.8.** There is a pairing of topological abelian groups

$$\lim_n H^i(X, W_n \Omega^j_{X,\log}) \times \lim_n H^{2-i}(X, W_n \Omega^{2-j}_{X,\log}) \to \mathbb{Q}_p/\mathbb{Z}_p$$

which induces an isomorphism of abelian groups

$$\lim_n H^i(X, W_n \Omega^j_{X,\log}) \to \left(\lim_n H^{2-i}(X, W_n \Omega^{2-j}_{X,\log})\right)^*,$$

a continuous epimorphism of topological abelian groups

$$\lim_n H^i(X, W_n \Omega^j_{X,\log}) \to \left(\lim_n H^{2-i}(X, W_n \Omega^{2-j}_{X,\log})\right)^*.$$
5. The relative Picard scheme

After the duality theorem for the logarithmic Hodge–Witt cohomology, the next key ingredients in the proof of Theorem \[1\] and the Brauer–Manin pairing for modulus pairs are the representability and certain topological properties of the relative Picard group. These results are of independent interest and we shall establish them in the next two sections using \[28\] and \[48\].

We fix an arbitrary field \( k \) of exponential characteristic \( p \geq 1 \). We let \( X \) be a smooth projective geometrically connected curve over \( k \) and let \( D \subset X \) be an effective Cartier divisor whose support is a nonempty finite closed subset \( S = \{x_1, \ldots, x_r\} \) of \( X \). We shall use \( m \) as another notation for \( D \) and call it the modulus divisor, following the terminology of \[48\]. We let \( v: D \to X \) be the inclusion.

We shall call \((X, D)\) a modulus pair (of dimension one). We write \( D = \sum_{i=1}^{r} n_i[x_i] \) as a Weil divisor and let \( \deg(D) = \sum_{i=1}^{r} n_i[k(x_i): k] \) denote the degree of \( D \). Note that \( \deg(D) = 1 \) if and only if \( D = \text{Spec}(k(x)) \), where \( x \in X(k) \). We let \( X^0 = X \setminus S \). We let \( A = \mathcal{O}_{X,S} \) and \( I \subset A \) the ideal defining \( D \) in \( \text{Spec}(A) \). We write \( A_D = A/I \).

We let \( K \) denote the function field of \( X \). For any \( k \)-scheme \( Z \), we let \( Z_0(Z) \) denote the free abelian group on \( Z_0 \). We shall let \( \mathcal{O}(Z) \) (resp. \( \mathcal{O}^*(Z) \)) denote the group \( H^0(Z, \mathcal{O}_Z) \) (resp. \( H^0(Z, \mathcal{O}_Z^*) \))

5.1. Relative Picard and 0-cycles with modulus. Recall that for an integral quasi-projective scheme \( Y \) over \( k \) and an effective Cartier divisor \( E \subset Y \), the Chow group of 0-cycles with modulus \( \text{CH}_0(Y|E) \) is defined to be the quotient of \( Z_0(Y \setminus E) \) by the subgroup \( R_0(Y|E) \) generated by the 0-cycles \( \nu_* (\text{div}(f)) \), where \( \nu: C_n \to Y \) is the canonical morphism from the normalization of an integral curve \( C \subset Y \) not contained in \( E \) and \( f \in \text{Ker}(\mathcal{O}_{C_n, \nu^{-1}(E)} \to \mathcal{O}^*(\nu^*(E))) \). The relative Picard group \( \text{Pic}(Y|E) \) (usually written as \( \text{Pic}(Y,E) \) in the literature) is the set of isomorphism classes of pairs \((\mathcal{L}, u: \mathcal{O}_E \xrightarrow{\cong} \mathcal{L}|_E)\), where \( \mathcal{L} \) is an invertible sheaf on \( Y \) and \( \mathcal{L}|_E := \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{O}_E \). One says that \((\mathcal{L}, u) \cong (\mathcal{L}', u')\) if there is an isomorphism \( w: \mathcal{L} \xrightarrow{\cong} \mathcal{L}' \) such that \( u' = w|_E \circ u \). \( \text{Pic}(Y|E) \) is an abelian group under the tensor product of invertible sheaves and their trivializations along \( E \) whose identity element is \((\mathcal{O}_Y, \text{id})\). There is a canonical homomorphism \( \text{Pic}(Y|E) \to \text{Pic}(Y) \) which is surjective if \( \dim(Y) = 1 \).

We now specialize to the one-dimensional modulus pair \((X, D)\) over \( k \) that we fixed above. Let \( \text{Pic}^0(X|D) \) denote the subgroup of pairs \((\mathcal{L}, u)\) such that \( \deg(\mathcal{L}) = 0 \). We let \( \text{CH}_0(X|D)_0 \) be the kernel of the composite map \( \text{deg}: \text{CH}_0(X|D) \to \text{CH}_0(X) \). There is a degree preserving cycle class map \( \text{cyc}_{X|D}: \text{CH}_0(X|D) \to \text{Pic}(X|D) \) (e.g., see \[29\] Lem. 3.1) such that \( \text{cyc}_{X|D}([Q]) = (\mathcal{O}_X(Q), u(1) = f^-1) \), where \( Q \in X_{(0)}^o \) and \( f \in \mathcal{O}_{X,S,0}(Q) \) is the ideal of \( Q \).

Lemma 5.1. The homomorphism \( \text{cyc}_{X|D} \) is bijective.

Proof. This is a combination of \[52\] Lemmas 2.1, 2.3, 2.4].

From the above definitions, it is clear that there is an exact sequence

\[
0 \to \mathcal{O}^*(X) \xrightarrow{\iota^*} \mathcal{O}^*(D) \xrightarrow{\partial_{X|D}} \text{Pic}(X|D) \xrightarrow{\partial_{X|D}} \text{Pic}(X) \to 0,
\]
where $i^*$ is the canonical inclusion and $\partial_{X,D}(u) = (O_X, uO_D \cong O_X|_D)$. In §7 we shall study the exactness of the sequence obtained by taking the Pontryagin dual of (5.1). Here, we note the following consequence.

**Lemma 5.2.** The canonical map $\text{Pic}(X|D) \to \text{Pic}(X)$ is an isomorphism if $\deg(D) = 1$.

### 5.2. The singular curve attached to $(X, D)$.

Assume now that $\deg(D) \geq 2$. In order to prove the representability of $\text{Pic}(X|D)$, the strategy is to replace it with the ordinary Picard group of a singular curve (without modulus). The following result from [48, Chap. IV, §1, no. 3, 4] helps us in achieving this.

**Proposition 5.3.** There exists a unique geometrically integral $k$-scheme $X_m$ together with a finite morphism $\psi_m: X \to X_m$ such that the following hold.

1. $\psi_m(S)$ is the unique singular point $P_0 \in X_m$ which is $k$-rational.
2. $\psi_m: X^o \cong X_m \setminus \{P_0\}$.
3. If we let $I_m \subset O_{X_m, P_0}$ denote the maximal ideal, then the pull-back map $\psi_m^*: O_{X_m, P_0} \to O_{X, S}$ induces a bijection $I_m \cong I$.

**Proof.** Since we do not assume any condition on $k$ while the construction of $X_m$ given in [48] assumes $k$ to be algebraically closed, we sketch the proof of this proposition to explain that this additional hypothesis is not necessary.

Since $S$ has an affine neighborhood in $X$, one easily observes that it suffices to show the existence of $X_m$ when $X = \text{Spec}(R)$ is affine. We let $B = k + I = \{a + b| a \in k, b \in I\} \subset A$. Then $B$ is a $k$-subalgebra of $A$ and there is a commutative diagram of short sequences of $B$-modules:

\[
\begin{array}{cccccc}
0 & \to & I & \to & B & \to & k & \to & 0 \\
| & & | & & | & & | & & | \\
0 & \to & I & \to & A & \to & A_D & \to & 0 \\
| & & | & & | & & | & & | \\
A/B & \cong & A_D/k \\
| & & | & & | & & | & & | \\
0 & & 0 & & & & & & .
\end{array}
\]

Since $A_D$ is finite over $k$, it follows from the above diagram that there is a finite-dimensional $k$-vector space $V \subset A$ such that $A = V + I$. It follows that there is a surjective map of $B$-modules $(k \oplus V) \otimes_k B \to A$. In particular, $A$ is a finite $B$-module. An application of the Eakin–Nagata theorem (e.g., see [33, Thm. 3.7]) shows that $B$ is Noetherian. Using [11, Thm. 5.10], we conclude furthermore that $B$ is a one-dimensional local ring with maximal ideal $I$. 


We now let $L$ be the fraction field of $B$ and take the tensor product of the middle row in (5.2) with $L$ over $B$ to get a commutative diagram of short exact sequences

\begin{equation}
0 \to B \longrightarrow A \longrightarrow A/B \longrightarrow 0
\end{equation}

\[\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\begin{equation}
0 \to L \to A \otimes_B L \to (A_D/k) \otimes_B L \to 0.
\end{equation}

Since $A_D/k$ is a torsion $B$-module, we get $(A_D/k) \otimes_B L = 0$. Since $A \otimes_B L$ is a localization of $A$, the map $A \to A \otimes_B L$ is injective. It follows that $A \subset L$. This implies that the inclusion $L \hookrightarrow K$ is a bijection. Putting everything together, we conclude that the conditions (1) to (3) of the proposition hold if we replace $X$ by $\text{Spec}(A)$. We now let $B'$ be the intersection (in $K$) of the localizations of $R$ at all maximal ideals except those supported on $S$ and let $R' = B \cap B'$. It is then easy to check that $X_m := \text{Spec}(R')$ is the desired singular curve with the conductor ideal $I_m := I$. Since $R_{\overline{K}}' \to R_{\overline{K}}$ and the latter ring is an integral domain, it follows that $X_m$ is geometrically integral. Note also that $P$ has to be a singular point of $X_m$ since $\deg(k(P)) = 1$ while $\deg(D) \geq 2$, by our assumption. \hfill \Box

Let $X_m$ be the singular curve with the unique singular point $P_0$ associated to the modulus pair $(X,D)$ as obtained in Proposition 5.3. We shall call this ‘the $m$-contraction of $X$’ in the sequel. We shall identify $X_m \setminus \{P_0\}$ with $X^0$ via $\psi_m$ throughout our discussion.

**Lemma 5.4.** Let $k'/k$ be any field extension and let $m' := D_{k'} \subset X_{k'}$. Then the $m'$-contraction of $X_{k'}$ is canonically isomorphic to $(X_m)_{k'}$.

**Proof.** This is a direct consequence of the construction of $X_m$ using its property that $P_0 \in X_m(k)$.

Recall that the Levine-Weibel Chow group of 0-cycles $\text{CH}_0^\text{lw}(Y)$ of an integral curve $Y$ over $k$ is the quotient of $Z_0(Y_{\text{reg}})$ by the subgroup generated by the divisors of those rational functions on $Y$ which lie in $\mathcal{O}^*_Y \cdot Y_{\text{sing}}$. It is well known that assigning each closed point of $Y_{\text{reg}}$ its class in $\text{Pic}(Y)$ induces a canonical cycle class isomorphism $\text{cyc}_Y : \text{CH}_0^\text{lw}(Y) \cong \text{Pic}(Y)$ (e.g., see [2, Lem. 3.12]). The following is a key lemma.

**Lemma 5.5.** Let $k'/k$ be any field extension. Then the identity map of $Z_0(X^0)$ induces via $\psi_m^*$, a degree preserving isomorphism

$$\psi_m^* : \text{Pic}(X_m)_{k'} \cong \text{Pic}(X_{k'})_{k'}$$

whose composition with $\text{Pic}(X_{k'})_{k'} \to \text{Pic}(X_{k'})$ is the usual pull-back between the Picard groups of $(X_m)_{k'}$ and $X_{k'}$.

**Proof.** Since the base field $k$ is arbitrary, we can assume $k' = k$ by virtue of Lemma 5.4. To construct $\psi_m^*$, we can replace $\text{Pic}(X|D)$ and $\text{Pic}(X_m)$ with $\text{CH}_0(X|D)$ and $\text{CH}_0^\text{lw}(X_m)$, respectively, via the cycle class isomorphisms. We first show that $\psi_m^*(\text{div}(f)) = 0$ if $f \in \mathcal{O}_m^* \cdot P_0$. To that end, we let $g$ denote the residue class of $f$ in $k(P_0)^* = k^*$ and let $\tilde{f} = g^{-1}$. Then we see that $\tilde{f} \in \mathcal{O}_m^* \cdot P_0$ is such that $\text{div}(\tilde{f}) = \text{div}(f)$ in $Z_0(X^0)$. Since $\tilde{f} \in K_1(A, I)$, we also see that $\psi_m^*(\text{div}(\tilde{f})) = 0$ in $\text{CH}_0(X|D)$. This proves the desired factorization of the pull-back map between the
Picard groups. It is clear that \( \psi^*_m \) is degree preserving (e.g., see [31, Tag 0AYU, Lem. 33.44.4]).

To complete the proof of the lemma, we look at the commutative diagram of short exact sequences

\[
\begin{array}{ccc}
0 & \to & A_D^*/k^* \\
\psi^*_m & \downarrow & \psi^*_m \\
0 & \to & A_D^*/k^* \\
\end{array}
\]

This shows that the middle vertical arrow is an isomorphism. \( \square \)

It follows from Lemma 5.5 that there is commutative diagram

\[
\begin{array}{ccc}
\text{CH}_0^w(X_m) & \xrightarrow{\psi_m} & \text{Pic}(X_m) \\
\downarrow & & \downarrow \\
\text{CH}_0(X|D) & \xrightarrow{\psi_m} & \text{Pic}(X|D),
\end{array}
\]

in which all arrows are isomorphisms and degree preserving. We can now prove the existence of the relative Picard variety of the modulus pair \((X, D)\).

**Theorem 5.6.** There exists a group scheme \( \text{Pic}(X|D) \) over \( k \) such that \( \text{Pic}(X|D)(k') \cong \text{Pic}(X_{k'}|D_{k'}) \) for every field extension \( k'/k \). The identity component \( \text{Pic}^0(X|D) \) of \( \text{Pic}(X|D) \) is a smooth and irreducible quasi-projective group scheme over \( k \) of dimension equal to \( \dim_k H^1(X_m, \mathcal{O}_{X_m}) \) such that \( \text{Pic}^0(X|D)(k') \cong \text{Pic}^0(X_{k'}|D_{k'}) \). If \( X^0(k) \neq \emptyset \), then the connected components of \( \text{Pic}(X|D) \) are parameterized by \( \mathbb{Z} \) and each component is isomorphic to \( \text{Pic}^0(X|D) \). We have \( \text{Pic}(X|D)_{k'} \cong \text{Pic}(X_{k'}|D_{k'}) \) for every field extension \( k'/k \). If \( f:(X', D') \to (X, D) \) is a strict morphism of modulus pairs (i.e., \( f^*(D) = D' \)), then there is a canonical homomorphism of group schemes \( f^* : \text{Pic}(X|D) \to \text{Pic}(X'|D') \).

**Proof.** If \( \deg(D) = 1 \), then \( D = \text{Spec}(k(x)) \) for some \( x \in X(k) \). In this case, we have \( \text{Pic}(X_{k'}|D_{k'}) \cong \text{Pic}(X_{k'}) \) for every field extension \( k'/k \) by Lemma 5.2. The theorem therefore follows directly from its known case \( D = \emptyset \). We now assume \( \deg(D) \geq 2 \) and let \( \psi_m : X \to X_m \) be the \( m \)-contraction of \( X \) (cf. Proposition 5.3).

Since \( X_m \) is geometrically integral, it follows from the flat base change property of the higher direct images of coherent sheaves that for every \( k \)-scheme \( T \), the map \( \mathcal{O}_T \to (f_T)_*(\mathcal{O}_{X_m \times T}) \) is an isomorphism of sheaves on \( T \), where \( f : X_m \to \text{Spec}(k) \) is the structure map and \( f_T : X_m \times T \to T \) is the base change of \( f \) to \( T \). Since \( P_0 \in X_m(k) \), the map \( f \) has a section.

Using the last two properties of \( X_m \), we conclude from [28, Thm. 9.2.5, 9.4.8] that there exists a group scheme \( \text{Pic}(X_m) \) over \( k \) such that \( \text{Pic}(X_m)(T) \cong \text{Pic}(X_m \times T)/\text{Pic}(T) \) for every \( k \)-scheme \( T \). In particular, we have \( \text{Pic}(X_m)(k') \cong \text{Pic}(X_{k'})(k') \) for every field extension \( k'/k \). Furthermore, the identity component \( \text{Pic}^0(X_m) \) of \( \text{Pic}(X_m) \) is a geometrically irreducible quasi-projective group scheme over \( k \) and \( \text{Pic}^0(X_m)(k') \cong \text{Pic}^0((X_m)_{k'}) \) by [28, Lem. 9.5.1]. It follows from [39, Cor. 4.2] (see also [28, Prop. 9.5.19]) and the fpf descent property of
smoothness that \( \text{Pic}^0(X_m) \) is a smooth group scheme over \( k \), i.e., it is an algebraic group over \( k \). In particular, it is geometrically integral. The assertion about \( \dim(\text{Pic}^0(X|D)) \) follows from its smoothness and [28, Cor. 9.5.13].

If \( P \in X^o(k) \), then the connected components of \( \text{Pic}(X_m) \) are parameterized by \( \mathbb{Z} \), and each connected component is isomorphic to \( \text{Pic}^0(X_m) \) via the transformation \( \mathcal{L} \mapsto \mathcal{L} \otimes \mathcal{O}_{X_m}(-mP) \) on the scheme points, where \( \deg(\mathcal{L}) = m \in \mathbb{Z} \). The theorem now follows from Lemmas 5.4 and 5.5 if we let \( \text{Pic}(X|D) \doteq \text{Pic}(X_m) \). The last claim is clear because the Picard scheme is a contravariant functor. \( \square \)

5.3. Picard scheme of singular curves. Suppose that \( Y \) is a geometrically integral singular curve with the normalization \( f: X \rightarrow Y \). Let \( Z \subset Y \) be a conductor subscheme for \( f \) with support \( Y_{\text{sing}} \) and let \( D = Z \times_Y X \). Assume that \( X \) is smooth over \( k \). Then \( (X, D) \) is a one-dimensional modulus pair, and it is easy to see from the construction in Proposition 5.3 that the \( m \)-contraction of \( X \) has a unique factorization

\[
(5.6) \quad X \xrightarrow{f} Y \xrightarrow{\psi_m} X_m.
\]

We conclude from Theorem 5.6 that there are morphisms of group schemes

\[
(5.7) \quad \text{Pic}(X|D) \xrightarrow{\psi_m^*} \text{Pic}(Y) \xrightarrow{f^*} \text{Pic}(X)
\]

whose composition is the canonical morphism \( \text{Pic}(X|D) \rightarrow \text{Pic}(X) \).

We now recall the exact sequence

\[
(5.8) \quad 0 \rightarrow k^* \rightarrow \mathcal{O}^*(Z) \rightarrow H^1_{\text{nis}}(Y, \mathcal{K}^M_{1}(Y,Z)) \rightarrow H^1_{\text{nis}}(Y, \mathcal{K}^M_{1,Y}) \rightarrow 0.
\]

Using the isomorphisms \( H^1_{\text{nis}}(Y, \mathcal{K}^M_{1}(Y,Z)) \xrightarrow{\sim} H^1_{\text{nis}}(X, \mathcal{K}^M_{1,X,D}) \) \( \xrightarrow{\sim} \text{Pic}(X|D) \) (see [29, Lem. 3.1]) for the second isomorphism), we get an exact sequence

\[
(5.9) \quad 0 \rightarrow \mathcal{O}^*(Z)/k^* \rightarrow \text{Pic}(X|D) \rightarrow \text{Pic}(Y) \rightarrow 0.
\]

Since this remains true over every algebraic extension of \( k \), it follows that the map \( \psi_m^* \) in (5.7) is surjective.

6. Relative Picard group over local fields

The goal of this section is to prove some topological properties of the relative Picard group and the Albanese map with respect to the adic topology that the schemes acquire when the base field is a local field. Before we do this, we need to study the smoothness properties of the Albanese morphism to the relative Picard scheme and of the morphism from the relative to the ordinary Picard scheme. We continue with the notations and assumptions of §5 (in particular, \( k \) is still arbitrary).

6.1. Generic smoothness of Albanese map. Until we reach Theorem 6.6, we shall assume in this subsection that \( \deg(D) \geq 2 \) and \( X^o(k) \neq \emptyset \). We fix a point \( P \in X^o(k) \). For any integer \( n \geq 1 \), let \( \text{Div}^n(X_m) \) be the quasi-projective \( k \)-scheme which parameterizes effective Cartier divisors \( E \subset X_m \) of degree \( n \). By [28, Defn. 9.4.6], there is a canonical morphism of \( k \)-schemes \( \text{Alb}_{X_m}^{n}: \text{Div}^n(X_m) \rightarrow \text{Pic}^0(X_m) \) which sends \( E \) to \( \mathcal{O}_{X_m}(E - np) \) if the support of \( E \) lies in \( X^o \). Since \( \text{Div}^1(X_m) \cong X^o \) by [28, Exc. 9.3.8], we get a \( k \)-morphism \( \text{alb}_{X|D}: X^o \rightarrow \text{Pic}^0(X_m) \) (called the Albanese
Lemma 6.1. For any Cartier divisor morphism) which has the property that \( \text{alb}_{X/D}(x) = [x] - [P] \) for any \( x \in X^o(k) \). Furthermore, it induces

\[
\text{AJ}_{X|D} : \text{CH}^m(X_m)_0 \cong \text{Pic}^0(X_m) \xrightarrow{\sim} \text{Pic}^0(X_m)(k).
\]

For \( n \geq 1 \), the \( n \)th power of the Albanese map descends to a morphism \( \text{alb}_{X|D}^n \cdot \text{Sym}^n(X^o) \rightarrow \text{Pic}^0(X_m) \), where \( \text{Sym}^n(X^o) \) is the \( n \)th symmetric power of \( X^o \). We let

\[
\varphi^n_{X|D} : (X^o)^n \xrightarrow{\sim} \text{Sym}^n(X^o) \xrightarrow{\text{alb}_{X|D}} \text{Pic}^0(X_m)
\]
denote the composite morphism. Let \( p_a(X_m) = \text{dim}_k(H^1(X_m, \mathcal{O}_{X_m})) \) denote the arithmetic genus of \( X_m \).

Let us now assume that \( E \subset X_m \) is an effective Cartier divisor disjoint from the unique singular point \( P_0 \) of \( X_m \) and let \( \mathcal{L} = \mathcal{O}_{X_m}(E) \). Equivalently, \( E \) is an effective Cartier divisor on \( X \) disjoint from \( S \) and \( \mathcal{L}' := \mathcal{O}_X(E) \cong \psi_m^*(\mathcal{L}) \). Since \( E \) is effective, there is a canonical inclusion \( \mathcal{O}_X \rightarrow \mathcal{L}' \). This yields a canonical inclusion \( k \cong H^0(X, \mathcal{O}_X) \rightarrow H^0(X, \mathcal{L}') \) such that the composite \( k \rightarrow H^0(X, \mathcal{L}') \rightarrow A/I \) is the canonical inclusion of rings \( k \rightarrow A/I \). Using this and the tensor product of the exact sequence

\[
0 \rightarrow \mathcal{O}_{X_m} \rightarrow (\psi_m)_*(\mathcal{O}_X) \rightarrow \mathcal{O}_D/\mathcal{O}_{P_0} \rightarrow 0
\]

with \( \mathcal{L} \), we get a commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(X, \mathcal{L}'(-D)) \rightarrow H^0(X, \mathcal{L}') \rightarrow A_D \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(X_m, \mathcal{L}) \rightarrow H^0(X, \mathcal{L}') \rightarrow A_D/k.
\end{array}
\]

Hence, we get an exact sequence

\[
0 \rightarrow H^0(X, \mathcal{L}'(-D)) \rightarrow H^0(X_m, \mathcal{L}) \rightarrow k \rightarrow 0.
\]

For a line bundle \( \mathcal{L} \) on \( X_m \), let \( |\mathcal{L}| \) denote the complete linear system \( \text{Proj}_k(H^0(X_m, \mathcal{L})) \). If \( \mathcal{L} = \mathcal{O}_{X_m}(E) \) for an effective divisor \( E \) on \( X^o \), we let \( U_E = |\mathcal{L}| \setminus |\mathcal{L}'(-D)| \subset |\mathcal{L}| \).

Lemma 6.1. For any Cartier divisor \( E \) on \( X_m \), the linear system \( |\mathcal{O}_{X_m}(E)| \) represents effective Cartier divisors \( F \) on \( X_m \) such that \( \mathcal{O}_{X_m}(F) \cong \mathcal{O}_{X_m}(E) \) as line bundles on \( X_m \).

Proof. One checks that the proof of a similar statement for smooth schemes given in [10] Prop. II.7.7 (see also [11] §13.13, p. 395) is based on two claims both of which are met in the present case.

Lemma 6.2. Assume that \( k \) is algebraically closed and let \( E \subset X^o \) be an effective Cartier divisor of degree \( n \geq 1 \). Then \( E \) is canonically an element of \( \text{Sym}^n(X^o)(k) \) and there is a canonical isomorphism of \( k \)-schemes \( (\text{alb}_{X|D}^n)^{-1}(\text{alb}_{X|D}(E)) \cong U_E \).
Proof. It is clear that $\text{alb}_{X|D}^n$ has factorization $\text{Sym}^n(X^o) \to \text{Div}^n(X_m) \xrightarrow{\text{AJ}_{X|D}^n} \text{Pic}^0(X_m)$. As $k$ is algebraically closed, $\text{AJ}_{X|D}^n$ is given by $E \mapsto [E] - [nP] \in \text{Pic}^0(X_m)$. Using Lemma 6.1, it is clear from this that $(\text{alb}_{X|D}^n)^{-1}(\text{alb}_{X|D}^n(E)) = |L| \cap \text{Sym}^n(X^o)$. It follows from (6.4) that, up to multiplication by elements of $k^\times$, the latter set consists of those sections of $L$ which do not die in $k$ under the map $H^0(X_m, L) \to k$ in the exact sequence (6.5). In other words, $|L| \cap \text{Sym}^n(X^o) = U_L$. This finishes the proof. □

Lemma 6.3. Assume that $k$ is algebraically closed. Then for all integers $n \gg 0$, all nonempty fibers of the morphism $\text{alb}_{X|D}^n: \text{Sym}^n(X^o) \to \text{Pic}^0(X_m)$ are affine spaces over the corresponding residue fields of $\text{Pic}^0(X_m)$, and have equal and positive dimensions.

Proof. For any effective Cartier divisor $E \subset X^o$, we let $L = O_{X_m}(E)$ and $L' = \psi_m^*(L)$. Since $|L'(-D)|$ is a hyperplane of the projective space $|L|$, it follows from Lemma 6.2 that all fibers of the morphism $\text{alb}_{X|D}^n: \text{Sym}^n(X^o) \to \text{Pic}^0(X_m)$ are affine spaces over the corresponding residue fields of $\text{Pic}^0(X_m)$. It remains to prove the independence of these fiber dimensions if $n \gg 0$.

Suppose that $E$ is such that $\deg(E) = n \geq \deg(D) + p_a(X_m) + 2g - 1$, where $g = p_a(X)$ denotes the genus of $X$. Then $\deg(L'(-D)) = \deg(L') - \deg(D) = \deg(E) - \deg(D) \geq 2g - 1$. It follows from the Serre duality and the Riemann-Roch theorem for $X$ that

$$(6.6) \quad H^1(X, L'(-D)) = 0 = H^1(X, L').$$

Using the cohomology exact sequence associated to the sheaf exact sequence

$$(6.7) \quad 0 \to L'(-D) \to L' \to O_D \to 0,$$

we get an exact sequence

$$(6.8) \quad 0 \to H^0(X, L'(-D)) \to H^0(X, L') \to H^0(D, O_D) \to 0.$$

In particular, the map $H^0(X, L') \to H^0(D, O_D)/k$ is surjective. On the other hand, by tensoring (6.3) with $L$ and passing to cohomology, we get an exact sequence

$$(6.9) \quad 0 \to H^0(X_m, L) \to H^0(X, L') \to H^0(D, O_D)/k \to H^1(X_m, L) \to H^1(X, L') \to 0.$$

We conclude that $H^1(X_m, L) = 0$.

Now, we use the Riemann-Roch theorem for $X_m$ (e.g., see [16 Exc. IV.1.9]) to get the identity $\dim_k(H^0(X_m, L)) = n + 1 - p_a(X_m)$. In other words, $\dim_k(U_E) = \dim_k(|L|) = n - p_a(X_m) \geq \deg(D) + 2g - 1 \geq 1$. Since $\deg(D) + 2g - 1$ depends only on $(X, D)$, we conclude that all fibers of $\text{alb}_{X|D}^n$ are of a fixed positive dimension. □

We now let $k$ be any field.

Proposition 6.4. For all integers $n \gg 0$, the morphism $\text{alb}_{X|D}^n: \text{Sym}^n(X^o) \to \text{Pic}^0(X_m)$ is smooth and surjective.

Proof. Via the standard descent properties of the smooth and surjective morphisms with respect to fpqc morphisms to the base scheme (e.g., see [11] Prop. 14.48, §16.C), and the commutativity of $\mathfrak{m}$-contraction and $\text{Pic}^0(X_m)$ with the field extensions, we easily reduce the proof to the case where we can assume $k$ to be algebraically closed.
Since \( k = \overline{k} \) and \( \text{CH}_0^w(X_m)_0 \xrightarrow{\sim} \text{Pic}^0(X_m) \xrightarrow{\sim} \text{Pic}^0(X_m)(k) \), it already follows from \([6]\) Defn. 7.1, Thm. 7.2 that \( \text{alb}_{X|D}^n \) is surjective for all \( n > 0 \). Using the fpqc descent, the smoothness claim now follows from Lemma \([6,3]\) and \([16, \text{Exc. III.10.9, Thm. III.10.2}]\). \(\square\)

For \( n > 0 \) and \( 1 \leq i < j \leq n \), we let \( \Delta_{ij}^n(X^o) \subseteq (X^o)^n \) be the closed subscheme (via a permutation) \( \Delta_{ij} \times (X^o)^{n-2} \subseteq (X^o)^n \), where \( \Delta_{ij} \subseteq X^o \times X^o \) is the diagonal of the product of the \( i \)th and \( j \)th factors of \( (X^o)^n \). We let \( U_n = (X^o)^n \setminus \bigcup_{1 \leq i < j \leq n} \Delta_{ij}^n(X^o) \).

Then \( U_n \) is invariant under the action of the symmetric group \( S_n \) and there is a Cartesian square

\[
\begin{array}{ccc}
U_n & \xrightarrow{j} & (X^o)^n \\
\pi_n & \downarrow & \pi_n \\
U_n/S_n & \xrightarrow{j} & \text{Sym}^n(X^o).
\end{array}
\]

Furthermore, \( \pi_n: U_n \to U_n/S_n \) is an étale \( S_n \)-torsor (e.g., see \([7, \text{Prop. 0.9}]\)). In particular, it is finite and étale. We let \( U_n(X^o) = U_n/S_n \).

**Corollary 6.5.** For all \( n > 0 \), the morphism \( \phi_{X|D}^n: U_n(X^o) \to \text{Pic}^0(X_m) \) is smooth and has dense image.

**Proof.** Since \( \phi_{X|D}^n = \text{alb}_{X|D}^n \circ \pi_n \) (see \((6.2)\)) and \( U_n(X^o) \xrightarrow{\pi_n} \text{Sym}^n(X^o) \) is étale with dense image, the corollary follows from Proposition \(6.4\). \(\square\)

We can now prove the following generic smoothness of the Albanese map for a modulus pair. We let \( k \) be any field and \( (X, D) \) a one-dimensional modulus pair over \( k \) such that \( X \) is connected and smooth over \( k \) and \( X^o(k) \neq \emptyset \). We fix a point \( P \in X^o(k) \) so that \( \text{alb}_{X|D} \) is defined with respect to the base point \( P \).

**Theorem 6.6.** For all \( n > 0 \), the Albanese map \( \phi_{X|D}^n: U_n(X^o) \to \text{Pic}^0(X|D) \) is smooth and has dense image.

**Proof.** Our assumption implies that \( X \) is geometrically integral over \( k \). If \( \deg(D) = 1 \), then the canonical map \( \text{Pic}^0(X|D) \to \text{Pic}^0(X) \) is an isomorphism of algebraic groups over \( k \) (see Lemma \(5.2\)) and one knows classically that the Albanese map \( U_n(X^o) \to \text{Pic}^0(X) \) is smooth and has dense image. If \( \deg(D) \geq 2 \), the theorem follows from Theorem \(5.6\) and Corollary \(6.5\). \(\square\)

6.2. The map from relative to ordinary Picard scheme. In order to compare the relative and ordinary Picard schemes, we need to identify \( \text{Pic}^0(X|D) \) with the generalized Jacobian variety of \((X, D)\) à la Rosenlicht–Serre \([18]\) under some additional assumptions. We fix a field \( k \) and a one-dimensional modulus pair \((X, D)\) over \( k \) such that \( X \) is connected and smooth over \( k \). We write \( D = \sum_{i=1}^{r} n_i[x_i] \). If \( \deg(D) \geq 2 \), we let \( \psi_m: X \to X_m \) be the \( m \)-contraction of \( X \). The following is the main result of Rosenlicht–Serre for which we refer to \([18, \text{Chap. V, \S 4.22, 4.23}]\).
Theorem 6.7. Assume that \( \text{Supp}(D) \subset X(k) \) and \( P \in X^0(k) \). Then there exists a quasi-projective algebraic group \( J_{X|D} \) over \( k \) satisfying the following properties.

1. There is a canonical isomorphism of group schemes \( J_{X'\mid D'} \cong (J_{X|D})_k' \) over \( k' \) for every field extension \( k'/k \).
2. The assignment \( Q \mapsto [Q] - \text{deg}(Q)[P] \) on \( X^0_\text{(o)} \) defines a morphism of \( k \)-schemes (called the Albanese morphism) \( \lambda_{X|D}: X^0 \to J_{X|D} \).
3. The Albanese morphism induces a homomorphism of abelian groups
   \[ AJ_{X|D}: \text{CH}_0(X|D)_0 \to J_{X|D}(k). \]
4. Given any commutative group scheme \( G \) over \( k \) and a \( k \)-morphism \( f: X^0 \to G \) which induces a homomorphism \( f_*: \text{CH}_0(X|D)_0 \to G(k) \), there is a unique homomorphism of group schemes \( \theta: J_{X|D} \to G \) over \( k \) such that \( f(x) = f(P) + (\theta \circ \lambda_{X|D})(x) \) for every \( x \in X^0 \).

Assume that \( \text{deg}(D) \geq 2 \). Using the Albanese map \( \text{alb}_{X|D}: X^0 \to \text{Pic}^0(X_m) \) and \( \text{Pic}^0(X|D) \), it follows from the property (4) of \( J_{X|D} \) that there exists a unique homomorphism (recall here that \( \text{alb}_{X|D}(P) = 0 \)) of group schemes \( \theta_{X|D}: J_{X|D} \to \text{Pic}^0(X_m) \cong \text{Pic}^0(X|D) \) over \( k \) such that \( \text{alb}_{X|D} = \theta_{X|D} \circ \lambda_{X|D} \).

Corollary 6.8. Under the assumptions of Theorem 6.7, the Albanese map \( \lambda_{X|D}: X^0 \to J_{X|D} \) induces an isomorphism of algebraic groups \( \theta_{X|D}: J_{X|D} \cong \text{Pic}^0(X|D) \).

Proof. If \( \text{deg}(D) = 1 \), it easily follows from [48, Chap. V, §3.13] that the canonical map \( J_{X|D} \to J_X \) is an isomorphism. We have seen above that this holds also for \( \text{Pic}^0(X|D) \). If \( \text{deg}(D) \geq 2 \), we can replace \( \text{Pic}^0(X|D) \) with \( \text{Pic}^0(X_m) \). In the latter case, we can argue as in the proof of Proposition 6.4 to reduce the proof to the case where we can assume that \( k \) is algebraically closed. But this case is classically known. \[\Box\]

We can now compare the relative and ordinary Picard schemes of a modulus pair. For any \( n \geq 1 \), let \( \mathbb{W}_n \) be the étale sheaf on \( \text{Sch}_k \) given by \( \mathbb{W}_n(Z) = \mathbb{W}_n(O(Z)) \), where \( \mathbb{W}_n(R) \) is the ring of big Witt-vectors over the ring \( R \) of length \( n \) (e.g., see [43, App. A]). One knows that \( \mathbb{W}_n \) is representable by a unipotent linear algebraic group over \( k \) (e.g., see [48, Chap. V, §3.13, Lem. 20]).

Proposition 6.9. Under the assumptions of Theorem 6.7, there is an exact sequence of algebraic groups

\[ (6.11) \quad 0 \to \text{Pic}^0(X|D)_{\text{aff}} \to \text{Pic}^0(X|D) \to \text{Pic}^0(X) \to 0 \]

over \( k \), where \( \text{Pic}^0(X|D)_{\text{aff}} \) is a linear algebraic group canonically isomorphic to \( (\mathbb{G}_m)^{r-1} \times \prod_{i=1}^r \mathbb{W}_{n_i-1} \).

Proof. Using Corollary 6.8, we can work with \( J_{X|D} \), in which case the proposition is well known (e.g., see [48, Chap. V, §3, no. 13]). \[\Box\]

Corollary 6.10. Let \( X \) be a smooth projective geometrically integral curve over \( k \) and let \( D \leq D' \) be two effective Cartier divisors on \( X \). Then the canonical map \( \text{Pic}^0(X|D') \to \text{Pic}^0(X|D) \) is smooth, affine and surjective. If \( Y \) is a singular curve over \( k \) with normalization \( X \) and \( Z \subset Y \) is a conductor subscheme for the normalization such that \( D = Z \times_Y X \), then the map \( \text{Pic}^0(X|D) \to \text{Pic}^0(Y) \) in (5.7) is smooth, affine and surjective.
Proof. To prove either of the statements, we can pass to the algebraic closure of \( k \).
In this case, the first statement follows directly by comparing (6.11) for \( \text{Pic}^0(X|D') \) and \( \text{Pic}^0(X|D) \). The second statement follows directly from [5, Prop. 5.4]. □

6.3. The case of local fields. We shall assume in this subsection that \( k \) is a
local field. Recall that \( k \) is a topological field with respect to its valuation (also
called adic) topology such that the ring of integers \( \mathcal{O}_k \) is an open subring. One also
knows that for any locally of finite type \( k \)-scheme \( Y \), the set \( Y(k) \) has a unique
structure of a totally disconnected locally compact Hausdorff topological space (see
§5). This topology is characterized by the property that if \( Y \subset \mathbb{A}_k^n \) as a
locally closed subscheme, then it coincides with the subspace topology on \( Y(k) \)
induced by the product of the adic topology of \( k \) on \( \mathbb{A}_k^n(k) \cong k^n \). This assignment
of adic topology defines a functor \( \text{Sch}_k \to \text{Top} \), where the latter is the category of
topological spaces with continuous maps. We shall call this the adic topology of
\( Y(k) \) and say that \( Y(k) \) is an adic space. For any property of the adic topology of
\( Y(k) \) that we shall use in this paper, the reader is referred to [5] and [4, Thm. 10.5.1].

Every finite-dimensional \( k \)-vector space \( M \) is equipped with the product topology
of the adic topology of \( k \), called the adic topology of \( M \). If \( A \) is a finite-dimensional
\( k \)-algebra, the adic topology of \( A^\times \) is the subspace topology induced from \( A \). These
topologies make \( A \) and \( A^\times \) into topological abelian groups. If \( k'/k \) is a finite field
extension and \( A \) is a finite-dimensional \( k' \)-vector space, then the adic topology of
\( A \) (and \( A^\times \) if \( A \) is a \( k' \)-algebra) coincides with its adic topology when considered as
a \( k \)-vector space (e.g., see [19, §7.2]). In this paper, all finite-dimensional algebras
over a local field and their unit groups will be assumed to be endowed with the adic
topology unless mentioned otherwise.

We now let \((X,D)\) be as in §6.2. We assume that \( X^0(k) \neq \emptyset \) and fix a point
\( P \in X^0(k) \) so that \( \text{alb}_{X|D} \) is defined with respect to the base point \( P \). Let \( Y \) be
a singular curve over \( k \) with normalization \( X \) and a conductor subscheme \( D \subset X \). Since
\( \text{Pic}(X|D) = \text{Pic}(X|D)(k) \) and \( \text{Pic}(Y) = \text{Pic}(Y)(k) \), we see that \( \text{Pic}(X|D) \)
and \( \text{Pic}(Y) \) are locally compact Hausdorff topological abelian groups. Furthermore,
it follows from [4, Thm. 10.5.1] that \( \text{Pic}^0(X|D) \) and \( \text{Pic}^0(Y) \) are open subgroups
of \( \text{Pic}(X|D) \) and \( \text{Pic}(Y) \), respectively. In the following result, we shall use the adic
topology on all sets.

Corollary 6.11. We have the following.

1. For all \( n \gg 0 \), the map \( \phi^n_{X|D}: U_n(X^0)(k) \to \text{Pic}^0_X(k|D) \) is a continuous open
map whose image is dense.
2. \( \text{Pic}^0(X) \) is a profinite abelian group.
3. The map \( \text{Pic}^0(X|D) \to \text{Pic}^0(X) \) is a topological quotient.
4. If \( D' \geq D \), the map \( \text{Pic}^0(X|D') \to \text{Pic}^0(X|D) \) is a topological quotient.
5. The map \( \text{Pic}^0(X|D) \to \text{Pic}^0(Y) \) is a topological quotient.

Proof. The statement (1) follows directly from Theorem 6.6 and [4, Thm. 10.5.1].
To prove (2), we note that \( \text{Pic}^0(X) \) is an abelian variety over \( k \). This implies
that \( \text{Pic}^0(X) \) is compact as an adic space (see op. cit.). Since it is also totally
disconnected, \( \text{Pic}^0(X) \) must be profinite. Since the maps \( \text{Pic}^0(X|D') \to \text{Pic}^0(X|D) \)
and \( \text{Pic}^0(X|D) \to \text{Pic}^0(Y) \) are surjective, the remaining statements of the corollary
follow directly from Corollary 6.10 and [4, Thm. 10.5.1]. □
7. Pontryagin dual of the Chow group with modulus

The goal of this section is to strengthen Corollary 6.11. More precisely, we shall show that all maps in the exact sequence (6.1) are continuous with respect to the adic topology and the resulting dual complex is partially exact. This will be a key step in the proofs of the main results. We begin by recalling the Kato topology and its relation with the adic topology.

7.1. Kato topology. Let \( R \) be an equicharacteristic excellent Henselian discrete valuation ring with maximal ideal \( m \) whose residue field \( j \) is a local field of exponential characteristic \( p \geq 1 \). Let \( L \) denote the quotient field of \( R \). In this case, one can find a two dimensional excellent normal local integral domain \( A \) whose residue field is finite such that \( R \cong (A_p)^h \) for some height one prime ideal \( p \subset A \). For any ideal \( I \subset A \) not contained in \( p \) and \( n \geq 1 \), we let \( \text{fil}^n I_1 K^M_1 (R) \) be the subgroup of \( R^\times \) generated by \( (1+m^n) \) and \( (1+I) \subset A^\times \subset K^M_1 (R) \). We let \( \text{fil}^1_0 K^M_1 (R) = R^\times \).

When \( p > 1 \), Kato defined a subgroup topology on \( R^\times \) (see [19 §7] and [44 §2.3]) for which the fundamental system of open neighborhoods of the identity is given by subgroups of the form \( \text{fil}^n I_1 K^M_1 (R) \), where \( I \) and \( n \) vary as above. It follows from [19 §7, Lem. 1] (see also [44 §2.3]) that this topology of \( R^\times \) does not depend on the choice of \( A \). We shall call this the Kato topology on \( R^\times \). The Kato topology on \( L^\times \) is the unique topology which is compatible with its group structure and for which \( R^\times \) with its Kato topology is an open subgroup.

It is easy to check that the Kato topology of \( R^\times \) coincides with the subspace topology induced from the Kato topology of \( \hat{R}^\times \) via the inclusion \( R^\times \subset \hat{R}^\times \). In particular, the same also holds for the inclusion \( L^\times \subset \hat{L}^\times \). When \( p = 1 \), we shall assume the Kato topology on \( R^\times \) and \( L^\times \) to be discrete. By choosing a uniformizer \( \pi \in m \), we get a canonical isomorphism of \( f \)-algebras \( \phi : \hat{f}[[T]] \cong \hat{R} \) which sends \( T \) to \( \pi \). In §6.3 we recalled the adic topology of \( f \) and \( O_1 \). We shall use the following description of the Kato topology in terms of the adic topology.

Lemma 7.1. When \( p > 1 \), the map \( \phi (\hat{f}[[T]])^\times \rightarrow (\hat{R})^\times \) is an isomorphism of topological abelian groups if we endow \( f[[T]] \cong f \times f^N \) with the product topology of the adic topology of \( f \) and \((f[[T]])^\times \) with the subspace topology. In particular, the quotient topology of the Kato topology on \((\hat{R}/m^n)^\times \) coincides with the adic topology of \((f[T]/(T^n))^\times \).

Proof. This is a straightforward consequence of [19 §7, Rem. 1, Lem. 3].
and \( \mathbb{W}_m(A) \) have a simple description. That is, the maps

\[
\text{(7.1)} \quad \nu_A: \mathbb{W}(A) \to \Gamma(A), \quad \nu_A((a_n)) = \prod_{n \geq 1} (1 - a_nT^n) \quad \text{and}
\]

\[
\text{(7.2)} \quad \mu_A: \Gamma(A) \to \mathbb{W}(A), \quad \mu_A(\prod_{n \geq 1} (1 - a_nT^n)) = (a_n)
\]

are group homomorphisms which are inverse to each other. These maps induce isomorphisms

\[
\text{(7.3)} \quad \nu_A: \mathbb{W}_m(A) \xrightarrow{\cong} \Gamma_m(A) := \frac{(1 + TA[[T]])^n}{(1 + T^{m+1}A[[T]])^n}; \quad \mu_A: \Gamma_m(A) \xrightarrow{\cong} \mathbb{W}_m(A).
\]

It follows from (7.2) that there are polynomials \( p_i \in A[T_1, \ldots, T_m] \) such that \( \mu_A \) in (7.3) is of the form \( \mu_A(1 + a_1T + \cdots + a_mT^m) = (p_1(a_1, \ldots, a_m), \ldots, p_m(a_1, \ldots, a_m)) \). We shall let \( \mathbb{W}_m(A) = 0 \) for \( m \leq 0 \).

Recall from (6.2) that for any field \( F \), there is a unipotent linear algebraic group \( \mathbb{W}_{m,F} \) over \( F \) such that \( \mathbb{W}_m(A) = \mathbb{W}_{m,F}(A) \) for any \( F \)-algebra \( A \). The map \( \psi_{m,F}: \mathbb{W}_{m,F} \to A^m_F \), given by \( \psi_{m,F}((a_i)) = (a_i) \) for \( (a_i) \in \mathbb{W}_m(A) \) (\( A \) any \( F \)-algebra) is an isomorphism between \( F \)-schemes (e.g., see [13] Chap. V, §3.13]). Note however that this is not an isomorphism of group schemes over \( F \). One can in fact write \( \mathbb{W}_{m,F} = \text{Spec}(A_{m,F}) \), where

\[
\text{(7.4)} \quad A_{m,F} = F\left[ X_{ij}[1 \leq i, j \leq m + 1] \right] / \left( X_{ij} \text{ for } i < j, X_{ij} - 1 \text{ for } i = j, X_{ij} - X_{i+1j+1} \text{ for } i < j \right).
\]

The isomorphism \( \psi_{m,F} \) is easily deduced from this description.

Suppose now that \( F \) is our local field \( k \). For any finite field extension \( k'/k \), the group scheme \( \mathbb{W}_{m,k'} \) then induces the adic topology on the abelian group \( \mathbb{W}_m(k') = \mathbb{W}_{m,k'}(k') \). Furthermore, the isomorphism \( \mathbb{W}_{m,k'} \cong A^m_{k'} \) as \( k' \)-schemes implies that \( \mathbb{W}_m(k') \cong k'^m \) as adic spaces, where the latter has the product of the adic topology of \( k' \). Note here that the adic topology of \( \mathbb{W}_m(k') \) does not depend on whether it has been induced by the \( k' \)-scheme structure or by the \( k' \)-scheme structure of \( \mathbb{W}_{m,k'} \) (e.g., see [13] §7.2]). We shall henceforth consider \( \mathbb{W}_{m,k'} \) as a unipotent algebraic group over \( k' \) and \( \mathbb{W}_m(k') \) as a topological abelian group with its adic topology. Using the descriptions of Kato and adic topologies, we get the following.

**Lemma 7.2.** Assume \( p > 1 \). Let \( X \) be a regular curve over \( k \) and \( \pi_x \) a uniformizer of the local ring \( \mathcal{O}_{X,x} \) at a closed point \( x \in X \), then the map

\[
\mu_{k(x)} \circ \phi^{-1}: \left( \frac{\mathcal{O}_{X,x}}{(\pi_x^{m+1})} \right)^\times \to k(x)^\times \times \mathbb{W}_m(k(x))
\]

is a continuous bijection between topological abelian groups for every \( m \geq 0 \), where the left hand side is endowed with the quotient of the Kato topology.

**Proof.** We can assume \( m \geq 1 \), else the statement is immediate from Lemma (7.1). Using Lemma (7.1) and (7.2), we only need to show that \( \mu_{k(x)}: \Gamma_m(k(x)) \to \mathbb{W}_m(k(x)) \) is continuous if \( \Gamma_m(k(x)) \) is endowed with the subspace topology from
\((k(x)[T]/(T^{m+1}))^\times\). For this, we look at the diagram

\[
\begin{align*}
\Gamma_m(k(x)) & \xrightarrow{\mu_{k(x)}} W_m(k(x)) \\
\theta_{k(x)} & \downarrow \\
k(x)^m & \xrightarrow{\gamma_{k(x)}} k(x)^m,
\end{align*}
\]

where \(\gamma_{k(x)}((a_i)) = (p_1((a_i)), \ldots, p_m((a_i)))\) (cf. (7.3)). The arrow \(\theta_{k(x)}\) is the canonical bijection induced by the homeomorphism of topological spaces \(\alpha_{k(x)}:\ (k(x)[[T]])^\times \xrightarrow{\cong} k(x)^\times \times k(x)^N\) as in Lemma 7.1. It is clear from the definition of various maps that this diagram is commutative. Since \(\psi_{k(x)}\) is a homeomorphism, it is enough to show that \(\psi_{k(x)} \circ \mu_{k(x)}\) is continuous. Equivalently, we need to show that \(\gamma_{k(x)} \circ \theta_{k(x)}\) is continuous.

Since \(\alpha_{k(x)}\) is a homeomorphism of topological spaces (cf. Lemma 7.1) and \(\theta_{k(x)}\) is the induced map on the quotients with the quotient topologies (note that the projection \(k(x)^N \to k(x)^m\) is a quotient map), it is a homeomorphism. Since each component of \(\gamma_{k(x)}\) is a polynomial map, it is clearly continuous. In particular, \(\gamma_{k(x)}\) is continuous. It follows that \(\gamma_{k(x)} \circ \theta_{k(x)}\) is continuous.

\[
\square
\]

### 7.2. Pontryagin duals of Chow groups.

Let \(k\) be a local field of exponential characteristic \(p \geq 1\). Let \((X, D)\) be a modulus pair over \(k\), where \(X\) is a geometrically connected smooth projective curve over \(k\). We let \(D^\dagger = D_{\text{red}}\). We shall assume that \(\emptyset \neq D^\dagger \subset X(k)\). We let \(X^o = X \setminus D\). We let \(j: X^o \to X\) and \(\kappa: D \to X\) be the inclusions. We write \(D = \sum n_i[x_i] \in \text{Div}(X)\). If \(U \subset X\) is any Zariski dense open, then \(U(k) \subset X(k)\) is an adically dense open subset (e.g., see [4, Thm. 10.5.1]).

Since \(X(k) \neq \emptyset\), it easily follows that \(U(k)\) must be infinite. In particular, \(X^o(k)\) is infinite. We fix a point \(P \in X^o(k)\) such that \(\text{alb}_{X|D}(P) = 0\). We have noted in 6.3 that \(\text{CH}_0(X|D)\) and \(\text{CH}_0(X)\) are adic topological abelian groups and the degree zero parts are their open subgroups.

Assume that \(p > 1\). Using the identification \(O^\times(D) \cong \prod_{x \in D^\dagger} \left(\frac{O_{X,x}}{(\pi_x)}\right)^\times\) as an adic space, we conclude from Proposition 6.9 and Lemma 7.2 that in (5.1), the map \(\partial_{X|D}\) is continuous, for it is the product of maps \(\mu_{k(x)} \circ \phi_x^{-1}\) over \(x \in D^\dagger\) followed by the topological quotient \((k^\times)^r \times (\prod_{i=1}^r W_{n_i-1}(k)) \to (k^\times)^{r-1} \times (\prod_{i=1}^r W_{n_i-1}(k))\). Since the map \(\partial_{X|D}\) and the inclusion \(k^\times \to O^\times(D)\) are clearly continuous, (5.1) gives rise to a sequence of homomorphisms

\[
(7.6) \quad 0 \to \text{CH}_0(X)^* \xrightarrow{(\partial_{X|D})^*} \text{CH}_0(X|D)^* \xrightarrow{(\partial_{X|D})^*} (O^\times(D))^* \to (k^\times)^*.
\]

**Lemma 7.3.** (7.6) is a chain complex of abelian groups which is exact at \(\text{CH}_0(X)^*\) and \(\text{CH}_0(X|D)^*\).

**Proof.** The lemma is obvious if \(\deg(D) = 1\) by Lemma 5.2. We shall therefore assume that \(\deg(D) \geq 2\). That (7.6) is a complex is a direct consequence of (5.1). The injectivity of \((\partial_{X|D})^*\) follows again from (5.1). To show the exactness at \(\text{CH}_0(X|D)^*\), we can clearly replace \(\text{CH}_0(X|D)\) and \(\text{CH}_0(X)\) by their degree zero
Lemma 7.4. We let $G = (k^*)^{r-1} \times (\prod_{i=1}^r \mathbb{W}_{n_i-1}(k))$. We need to show that the sequence
\begin{equation}
0 \to (\text{CH}_0(X)_0)^* \xrightarrow{(\vartheta_X|D)^*} (\text{CH}_0(X|D)_0)^* \to G^* \to 0
\end{equation}
is exact at the middle term.

To show the above exactness, we note that the sequence
\begin{equation}
0 \to (\text{CH}_0(X)_0)^{\vee} \xrightarrow{(\vartheta_X|D)^{\vee}} (\text{CH}_0(X|D)_0)^{\vee} \to G^{\vee} \to 0
\end{equation}
is clearly exact. Hence, given a continuous character $\chi \in (\text{CH}_0(X|D)_0)^*$ whose restriction to $G$ is zero, we get a unique character $\chi' \in (\text{CH}_0(X)_0)^{\vee}$ such that $\chi = \chi' \circ \vartheta_X|D$. It remains to show that $\chi'$ is continuous with respect to the adic topology of $\text{CH}_0(X)_0$. But this is an easy consequence of Lemma 5.1 and Corollary 6.11. □

We now assume $p = 1$ and prove the analogous result. We begin with the following.

**Lemma 7.4.** If $p = 1$, we have $\text{CH}_0(X|D)^* \cong (\text{CH}_0(X|D)^*_{\text{tor}} \cong (\text{CH}_0(X|D)^{\vee}_{\text{tor}}$. We also have
\begin{equation}
((\mathcal{O}^*(D))^{\ast})_{\text{tor}} \cong (\mathcal{O}^*(D)^{\vee})_{\text{tor}} \cong (\mathcal{O}^*(D^{\hat{}})^{\ast}) \cong (\mathcal{O}^*(D^{\hat{}})^{\vee})_{\text{tor}},
\end{equation}
where $\mathcal{O}^*(D)$ and $\mathcal{O}^*(D^{\hat{}})$ are endowed with the adic topology.

**Proof.** The isomorphisms in (7.9) are immediate consequences of [38, Prop. II.5.7] because $\text{Ker}(\mathcal{O}^*(D) \to \mathcal{O}^*(D^{\hat{}}))$ is a divisible group. To prove the first statement, we first assume $D = \emptyset$. Since $\text{CH}_0(X)_0 = \text{Pic}^0(X)(k)$ is a profinite abelian group by Corollary 6.11, it follows that $(\text{CH}_0(X)_0)^*$ is a torsion group. Furthermore, the assumption $p = 1$ implies using the Kummer sequence that $\text{CH}_0(X)/n$ is discrete (and finite) for the quotient topology for every integer $n \geq 1$. Since $\text{Pic}^0(X) \to \text{Pic}^0(X)$ is a smooth isogeny, it follows from [3] Thm. 10.5.1] that $\text{CH}_0(X) \to \text{CH}_0(X)/n$ is continuous. In particular, $\text{CH}_0(X) \to \text{CH}_0(X)/n$ is continuous.

Now, if $\chi \in \text{CH}_0(X)^*$, then $\chi(\text{CH}_0(X)_0)$ is a finite cyclic group of the type $\mathbb{Z}/n$. This yields a commutative diagram of exact sequences
\begin{equation}
0 \to \text{CH}_0(X)_0 \to \text{CH}_0(X) \to \mathbb{Z} \to 0
\end{equation}
\begin{equation}
\xrightarrow{\chi} \text{Z}/n \to \text{Q}/\mathbb{Z} \to \text{Q}/\mathbb{Z} \to 0.
\end{equation}
Since the image of $\bar{\chi}$ must be finite, it follows that $\chi$ has a finite order. Conversely, if $\chi \in (\text{CH}_0(X)^{\vee})_{\text{tor}}$, then it factors through $\text{CH}_0(X)/n$ for some $n \geq 1$ and it follows from the previous paragraph that $\chi$ is continuous on $\text{CH}_0(X)$. Suppose now that $D$ is nonempty but reduced and let $\chi \in \text{CH}_0(X|D)^*$. We have seen above that the restriction of $\chi$ to $(k^*)^{r-1}$ has finite order. We thus get an integer $n \geq 1$ and a commutative diagram of exact sequences (cf. Proposition 6.9)
\begin{equation}
0 \to (k^*)^{r-1} \to \text{CH}_0(X|D) \to \text{CH}_0(X) \to 0
\end{equation}
\begin{equation}
\xrightarrow{\chi} \text{Z}/n \to \text{Q}/\mathbb{Z} \to \text{Q}/\mathbb{Z} \to 0.
\end{equation}
Since $\text{CH}_0(X|D) \twoheadrightarrow \text{CH}_0(X)$ is a topological quotient by Corollary \ref{cor:finite-quotient} we get that $\bar{\chi}$ is continuous, and hence, its image is finite. We conclude that the image of the middle vertical arrow is finite.

Conversely, if $\chi \in (\text{CH}_0(X|D)^\vee)_\text{tor}$, then it factors through $\text{CH}_0(X|D)/n \xrightarrow{\bar{\chi}} \mathbb{Q}/\mathbb{Z}$ for some $n \geq 1$. We have shown above that $\text{CH}_0(X)/n$ is finite. It follows from the top row of (7.11) and \cite[Prop. II.5.7]{Illusie} that $\text{CH}_0(X|D)/n$ is finite. On the other hand, as $\text{Pic}^0(X|D) \xrightarrow{n} \text{Pic}^0(X|D)$ is a smooth isogeny, it follows that $\text{CH}_0(X|D)/n$ is a finite and discrete quotient of $\text{CH}_0(X|D)$. We conclude that $\chi$ is continuous.

If $D$ is not necessarily reduced, we look at the commutative diagram

\begin{equation}
\label{eq:comm-diag}
\begin{array}{ccc}
\text{CH}_0(X|D)^* & \xrightarrow{(\partial_X|D)^*} & \text{CH}_0(X|D)^* \\
\downarrow & & \downarrow \\
(\text{CH}_0(X|D)^*)_\text{tor} & \xrightarrow{(\partial_X|D)_\text{tor}} & (\text{CH}_0(X|D)^*)_\text{tor},
\end{array}
\end{equation}

in which all arrows are the canonical inclusions. Since any element of $(\text{CH}_0(X|D)^*)_\text{tor}$ must annihilate the kernel of $\text{CH}_0(X|D) \twoheadrightarrow \text{CH}_0(X|D\text{red})$, it follows that the right vertical arrow in (7.12) is a bijection. We have shown in the previous paragraph that the top horizontal arrow is a bijection. A diagram chase shows that all arrows are bijections. This concludes the proof. \hfill \Box

**Lemma 7.5.** If $p = 1$ and $D$ is reduced, then (5.4) induces a chain complex of abelian groups

\begin{equation}
\label{eq:chain-complex}
0 \rightarrow \text{CH}_0(X)^* \xrightarrow{(\partial_X|D)^*} \text{CH}_0(X|D)^* \xrightarrow{(\partial_X|D)_*} (\mathcal{O}^\times(D))^* \rightarrow (k^\times)^* 
\end{equation}

which is exact at $\text{CH}_0(X)^*$ and $\text{CH}_0(X|D)^*$.

**Proof.** At any rate, we do have an exact sequence

$0 \rightarrow (\text{CH}_0(X)^*)_\text{tor} \xrightarrow{(\partial_X|D)^*_\text{tor}} (\text{CH}_0(X|D)^*)_\text{tor} \xrightarrow{(\partial_X|D)^*_\text{tor}} ((\mathcal{O}^\times(D))^*)_\text{tor} \rightarrow ((k^\times)^*)_\text{tor}.$

The desired claim now follows by Lemma 7.4. \hfill \Box

8. **Brauer group with modulus**

The goal of this section is to define the Brauer group of a modulus pair and prove some functorial properties. We begin by recalling Kato’s ramification filtration which will play a fundamental role in our exposition.

8.1. **Étale motivic cohomology.** Let $k$ be a field of exponential characteristic $p \geq 1$ and let $X$ be a Noetherian $k$-scheme. If $n \in k^\times$ is an integer and $r \in \mathbb{Z}$, we let $\mathbb{Z}/n(r)$ be the étale sheaf on $X$ defined as the usual Tate twist of the constant sheaf $\mathbb{Z}/n$ (e.g., see \cite[p. 163]{Illusie}). If $p > 1$ and $n = p^s m$ with $s \geq 0$ and $p \nmid m$, we let $\mathbb{Z}/m(r)$ be the object $\mathbb{Z}/m(r) \oplus \mathcal{W}_s \mathcal{O}_{X,\text{log}}[-r]$ as an object of $\mathcal{D}_\mathcal{et}(X)$. We have the cup product pairing of the (hyper)cohomology of the form $H^i(X, \mathbb{Z}/n(j)) \times H^j(X, \mathbb{Z}/n(j')) \rightarrow H^{i+j}(X, \mathbb{Z}/n(j+j'))$. For $q \in \mathbb{Z}$, we let $H^q_n(X)$ denote the étale cohomology group $H^q(X, \mathbb{Z}/n(q-1))$. We let $H^q_n(X) = \lim_{\rightarrow n} H^q_n(X)$ with respect to the canonical transition maps $\mathbb{Z}/n(r) \xrightarrow{m} \mathbb{Z}/m(r)$ (see (2.6) for their definition in positive characteristic). If $X = \text{Spec} (A)$ is affine, we write $H^q_n(X)$ (resp. $H^q(X)$) as $H^q_n(A)$ (resp. $H^q(A)$).
For $m, n \in k^\times$ and $s \geq 1$, we have commutative diagrams of exact sequences

\begin{align}
(8.1) & \quad 0 \to \mathbb{Z}/n(1) \to O_X^n \to O_X^s \to 0 \quad 0 \to O_X^n \to O_X^s \to O_X^{p^s} \to 0 \\
& \quad 0 \to \mathbb{Z}/mn(1) \to O_X^{mn} \to O_X^m \to 0 \quad 0 \to O_X^{p^m} \to O_X^s \to O_X^{p^s} \to 0,
\end{align}

of étale sheaves, where the diagrams on the right make sense when $p > 1$. The arrow $\text{can}'$ is the canonical inclusion $\mu_n \to \mu_{mn}$, and the arrow $O_X/p^{s+1} \text{can} O_X/p^s$ is the canonical projection.

The commutative diagrams in (8.1) give rise to an exact sequence of ind-abelian groups

\begin{align}
(8.3) & \quad 0 \to \{\text{Br}(X)/n\} \to \{H^3(X, \mathbb{Z}/m(1))\} \to \{mH^3(X, O_X^s)\} \to 0,
\end{align}

which are indexed by $\mathbb{N}$ and whose transition maps are induced by $\mathbb{Z}/m \to \mathbb{Z}/mn$. Taking the limits and noting that $H^i(X, O_X^s)$ is a torsion group for $i \geq 2$ (e.g., see [3] Lem. 3.5.3) when $X$ is regular, we get the following.

**Lemma 8.1.** If $X$ is regular, then the canonical map

$$H^3(X, \mathbb{Q}/\mathbb{Z}(1)) \to H^3(X, O_X^s)$$

is an isomorphism.

**Proof.** This is an easy application of the above discussion once we use an elementary fact that $A \otimes B = 0$ if $A$ is a torsion abelian group and $B$ is a divisible abelian group. \qed

We also get the following folklore result which we shall use throughout this paper without giving reference.

**Lemma 8.2.** Let $A$ be an equicharacteristic regular local ring. Then one has a canonical isomorphism $H^2(A) \xrightarrow{\sim} \text{Br}(A)$.

For a local ring $A$ over $k$, we have the Norm residue map (e.g., see [12] §5.1)

\begin{align}
(8.4) & \quad \text{NR}_A: K_i^M(A)/n \to H^i(A, \mathbb{Z}/n(i)).
\end{align}

Composing this with the cup product, we see that there is a canonical bilinear pairing

\begin{align}
(8.5) & \quad \langle \cdot, \cdot \rangle: H^i(A, \mathbb{Z}/n(j)) \times K_i^M(A)/n \to H^{i+j'}(A, \mathbb{Z}/n(j+i')).
\end{align}

If $A$ is any equicharacteristic local integral domain with maximal ideal $m_A$ and quotient field $E$, we let $\text{fil}_0 K_i^M(E) = A^\times$ and $\text{fil}_n K_i^M(E) = (1 + m_A^n)$ if $n \geq 1$. For $i \geq 2$, let $\text{fil}_n K_i^M(E)$ be the image of the cup product map $\text{fil}_n K_i^M(E) \otimes K_i^{M-1}(E) \to K_i^M(E)$. We let $\text{fil}_n K_i^M(E) = K_i^M(E)$ for $n < 0$. It is clear that $\text{fil}_n K_i^M(E)$ is a decreasing filtration of $K_i^M(E)$. We shall call this ‘the logarithmic filtration’ of $K_i^M(E)$.
8.2. Kato’s ramification filtration. In order to define the Brauer group with modulus, we need to recall Kato’s ramification filtration. We let \( L \) be a Henselian discrete valuation field with the ring of integers \( \mathcal{O}_L \), the maximal ideal \( \mathfrak{m}_L = (\pi_L) \) and the residue field \( l \) such that \( \text{char}(L) = \text{char}(l) = p \geq 0 \). Let \( \hat{L} \) denote the completion of \( L \). Recall the following from [22, Cor. 2.5, Prop. 6.3], where we have shifted Kato’s filtration one place to the right.

**Definition 8.3.** Let \( q \geq 1 \) be an integer.

1. If \( p = 0 \), we let \( \text{fil}_0 H^q(L) = H^q(\mathcal{O}_L) \) and \( \text{fil}_n H^q(L) = H^q(L) \) if \( n \geq 1 \).
2. If \( p > 0 \) and \( n \geq 0 \), we let \( \text{fil}_n H^q(L) \) be the subgroup of elements \( \chi \in H^q(L) \) such that \( (\chi, 1 + \pi_L^n \mathcal{O}_L') = 0 \) for all Henselian discrete valuation fields \( L' \) such that \( \mathcal{O}_L \subset \mathcal{O}_{L'} \) and \( \mathfrak{m}_{L'} = \mathfrak{m}_L \mathcal{O}_{L'} \), where \( 1 + \pi_L^n \mathcal{O}_{L'} := \mathcal{O}_L' \).
3. We let \( \text{fil}_n H^q(L) = 0 \) for \( n < 0 \).

It follows from [22, Lem. 2.2] that

\[
H^q(L) = \bigcup_{n \geq 0} \text{fil}_n H^q(L).
\]

We shall call \( \text{fil}_n H^q(L) := \{\text{fil}_n H^q(L)\}_{n \in \mathbb{Z}} \), the ramification filtration of \( H^q(L) \). Recall that for any \( \chi \in H^q(L) \), the Swan conductor \( \text{sw}(\chi) \) is the smallest integer \( n \) such that \( \chi \in \text{fil}_n H^q(L) \). The following result shows that one can characterize \( \text{fil}_n H^q(L) \) purely in terms of characters of the Milnor \( K \)-groups of \( L \) in cases of our interest.

**Lemma 8.4.** Assume that \( p > 0 \) and \( l \) is a local field. Let \( 1 \leq q \leq 2 \) and \( n \geq 0 \) be integers. Then the following hold.

1. There are canonical isomorphisms

\[
H^3(L) \xrightarrow{\sim} H^3(\hat{L}) \xrightarrow{\sim} H^3(l) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.
\]

2. The canonical map \( H^q(\mathcal{O}_L) \to H^q(L) \) induced by the inclusion \( \mathcal{O}_L \to L \) fits into a split short exact sequence

\[
0 \to H^q(\mathcal{O}_L) \to \text{fil}_1 H^q(L) \to H^{q-1}(l) \to 0
\]

in which the splitting is canonical for a given choice of \( \pi_L \).
3. \( \text{fil}_0 H^q(L) = H^q(\mathcal{O}_L) \).
4. For \( n \geq 1 \), an element \( \chi \in H^q(L) \) lies in \( \text{fil}_n H^q(L) \) if and only if \( \langle \chi, \text{fil}_n K^M_{3,q}(L) \rangle = 0 \) under the pairing

\[
H^q(L) \times K^M_{3,q}(L) \xrightarrow{(,)} H^3(L) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}.
\]

**Proof.** Part (1) of the lemma is a special case of the general isomorphism (\( \forall q \))

\[
H^q(L) \xrightarrow{\sim} H^q(\hat{L})
\]

as shown in [23, Lem. 21], and the well-known case of complete discrete valuation fields (see [20, §3.2, Prop. 1]). For \( q = 1 \), the other parts of the lemma follow from [12, Thm. 6.3]. For \( q = 2 \), part (2) follows directly from [22, Prop. 6.1].
To prove (3), suppose first that \( \chi \in \text{fil}_0 H^2(L) \). We consider the commutative diagram of exact sequences

\[
\begin{align*}
0 \to H^2(O_L) &\to \text{fil}_1 H^2(L) \xrightarrow{\alpha} H^1(\Omega^1_L) \to 0 \\
0 \to (\mathbb{Z})^\vee &\to (L^\times)^\vee \xrightarrow{\beta} (O_L^\times)^\vee \to 0,
\end{align*}
\]

where the vertical arrows are induced by \([8.3]\). Our hypothesis says that \( \beta \circ \alpha(\chi) = 0 \). Since the right vertical arrow is injective by the class field theory of local fields, it follows that \( \chi \in H^2(O_L) \). Conversely, suppose that \( \chi \in H^2(O_L) \). Then a diagram chase of \([8.8]\) tells us that \( \chi \in \text{fil}_1 H^2(L) \) such that \( \beta \circ \alpha(\chi) = 0 \). But this is equivalent to saying that \( \chi \in \text{fil}_0 H^2(L) \). This proves (3).

We now assume \( n \geq 1 \) and \( q = 2 \). In view of \([22]\) Prop. 6.3], we only need to show that if \( \chi \in H^2(L) \) is a character such that \( \langle \chi, \text{fil}_n L^\times \rangle = 0 \), then it lies in \( \text{fil}_n H^2(L) \). At any rate, we know that \( \chi \in \text{fil}_n H^2(L) \) for some \( m \gg 0 \). We can assume that \( m > n \) (else we are done). Our hypothesis implies that as a character of \( L^\times \), \( \chi \) factors through \( L^\times / \text{fil}_n L^\times \).

We now note that the ring of integers \( \mathcal{O}_{\bar{L}} \) of \( \bar{L} \) is the \( m_L \)-adic completion \( \bar{\mathcal{O}}_L \) of \( O_L \). Since \( \mathcal{O}_{\bar{L}}^\times / \text{fil}_n L^\times \cong \mathcal{O}_{\bar{L}}^\times / \text{fil}_n \bar{L}^\times \), it follows from the commutative diagram

\[
\begin{align*}
0 \to \mathcal{O}_{\bar{L}}^\times &\to L^\times \xrightarrow{\nu_L} \mathbb{Z} \to 0 \\
0 \to \mathcal{O}_{\bar{L}}^\times &\to \bar{L}^\times \xrightarrow{\nu_{\bar{L}}} \mathbb{Z} \to 0
\end{align*}
\]

of exact sequences that \( L^\times / \text{fil}_n L^\times \cong \bar{L}^\times / \text{fil}_n \bar{L}^\times \). Hence, \( \chi \) factors through \( \bar{L}^\times / \text{fil}_n \bar{L}^\times \) as a character of \( \bar{L}^\times \). We conclude from \([22]\) Prop. 6.3, Rem. 6.6] that \( \chi \in \text{fil}_n H^2(\bar{L}) \).

By an iteration of the commutative diagram

\[
\begin{align*}
0 \to \text{fil}_{m-1} H^2(L) &\to \text{fil}_m H^2(L) \to \frac{\text{fil}_m H^2(L)}{\text{fil}_{m-1} H^2(L)} \to 0 \\
0 \to \text{fil}_{m-1} H^2(\bar{L}) &\to \text{fil}_m H^2(\bar{L}) \to \frac{\text{fil}_m H^2(\bar{L})}{\text{fil}_{m-1} H^2(\bar{L})} \to 0,
\end{align*}
\]

it remains to show that the right vertical arrow in this diagram is injective.

We now look at the diagram

\[
\begin{array}{cccc}
\text{fil}_m H^2(L) & \xrightarrow{r_{sw_L}} & \Omega^2_1 \oplus \Omega^1_1 \\
\frac{\text{fil}_m H^2(L)}{\frac{\text{fil}_{m-1} H^2(L)}} & \xrightarrow{=} & \frac{\text{fil}_m H^2(\bar{L})}{\frac{\text{fil}_{m-1} H^2(\bar{L})}} \xrightarrow{r_{sw_{\bar{L}}}} & \Omega^2_1 \oplus \Omega^1_1.
\end{array}
\]

Since \( \pi_L \) is also a uniformizer of \( \mathcal{O}_{\bar{L}} \), it follows from \([22]\) Thm. 0.1] that this diagram is commutative and the horizontal arrows are injective. It follows that the left vertical arrow is injective. This concludes the proof. \(\square\)
We shall use the following result of Kato to prove various functorial properties of the Brauer and Picard groups of modulus pairs.

**Lemma 8.5.** Let \( L'/L \) be a finite extension of Henselian discrete valuation fields of characteristic \( p > 0 \). Let \( e \) denote the ramification index of \( L'/L \). Let \( \iota \) (resp. \( \iota' \)) denote the residue field of \( L \) (resp. \( L' \)). Assume that \( \iota \) and \( \iota' \) are local fields. Then we have the following.

1. There is a commutative diagram

\[
\begin{align*}
H^2(L') \times L'^\times & \xrightarrow{\cup} H^3(L') \xrightarrow{\partial_{L'}} H^2(L) \\
\text{Cores} & \quad \text{Res} \quad \text{Cores} \quad \text{Cores} \\
H^2(L) \times L^\times & \xrightarrow{\cup} H^3(L) \xrightarrow{\partial_{L}} H^2(1) \xrightarrow{\text{inv}_{\iota'}} \mathbb{Q}/\mathbb{Z}.
\end{align*}
\]

The same holds if we interchange \text{Cores} and \text{Res} on the left side of the cup product maps.

2. \( \text{Cores}(\text{fil}_n H^2(L')) \subseteq \text{fil}_n H^2(L) \) for any \( n \geq 0 \).

3. \( \text{Res}(\text{fil}_n H^2(L)) \subseteq \text{fil}_n H^2(L') \) for any \( n \geq 0 \).

**Proof.** (1) follows from [20, §3.2, Lem. 1, Prop. 1(2)], (2) follows from (1) using that \( \text{Res}(\text{fil}_n L^\times) \subseteq \text{fil}_n L'^\times \) by definition, and (3) follows from (1) and [41, Lem. 6.19].

### 8.3. The Brauer group with modulus

Let \( k \) be a local field with ring of integers \( \mathcal{O}_k \), maximal ideal \( m = (\pi) \) and residue field \( \mathcal{f} = \mathcal{O}_k/m \cong \mathbb{F}_q \), where \( q = p^n \) for some prime \( p \geq 2 \) and integer \( n \geq 1 \). Let \( X \) be a connected regular quasi-projective scheme over \( k \) of dimension \( d \geq 1 \). Let \( D \subseteq X \) be an effective divisor (possibly empty) and \( D^\dagger = D|_{\text{red}} \). We write \( D = \sum_{x \in X^{(1)}} n_x [x] \). Clearly, \( n_x = 0 \) unless \( x \) is a generic point of \( D^\dagger \), and \( D = 0 \) if \( D^\dagger \) is empty. We let \( \text{Div}_{D^\dagger}(X) \) be the set of all effective divisors on \( X \) whose support is \( D^\dagger \). This is a filtered set under inclusion. We let \( \text{Div}(X) \) denote the filtered set of all effective Cartier (equivalently Weil) divisors on \( X \). We let \( X^o = X \setminus D \). We let \( j: X^o \hookrightarrow X \) and \( t: D \hookrightarrow X \) be the inclusions. We let \( K \) denote the function field of \( X \).

For any \( x \in X^{(1)} \), we let \( K_x \) denote the quotient field of \( \mathcal{O}_{X,x} \) and \( \mathcal{R}_x \) the quotient field of \( \mathcal{O}_{X,x} \). We shall say that an element \( \chi \in H^i(K) \) is unramified at a point \( x \in X^{(1)} \) if the image of \( \chi \) under the canonical map \( H^i(K) \to H^i(K_x) \) lies in the image of \( H^i(\mathcal{O}^h_{X,x}) \). We shall say that \( \chi \) is unramified on an open subscheme \( U \subseteq X \) if it lies in the image of the canonical map \( H^i(U) \to H^i(K) \).

**Lemma 8.6.** Let \( U \) be a connected Noetherian regular scheme with function field \( K \). Let \( \chi \in \text{Br}(K) \) be unramified at all codimension one points of \( U \). Then \( \chi \) is unramified on \( U \).

**Proof.** This is an easy consequence of the purity theorem for Brauer group, see [4, Thm. 3.7.7].

**Definition 8.7.** We let \( \text{Br}^\text{div}(X|D) \) denote the subgroup of \( \text{Br}(K) \) consisting of elements \( \chi \) such that for every \( x \in X^{(1)} \), the image \( \chi_x \) of \( \chi \) under the canonical map \( \text{Br}(K) = H^2(K) \to H^2(K_x) \) lies in \( \text{fil}_{n_x} H^2(K_x) \).
Lemma 8.8. We have the following relations between the subgroups of \( \text{Br}(K) \).

1. For every pair of effective Cartier divisors \( D \leq D' \) on \( X \), one has
   \[
   \text{Br}(X) \subseteq \text{Br}^\text{div}(X|D) \subseteq \text{Br}^\text{div}(X|D') \subseteq \text{Br}(X \setminus D') \subseteq \text{Br}(K).
   \]

2. \( \lim_{n \to \infty} \text{Br}^\text{div}(X|nD) \xrightarrow{\cong} \lim_{D' \in \text{Div}_D(X)} \text{Br}^\text{div}(X|D') \xrightarrow{\cong} \text{Br}(X^o) \).

3. \( \lim_{D' \in \text{Div}_D(X)} \text{Br}^\text{div}(X|D') \xrightarrow{\cong} \text{Br}(K) \).

4. If \( \text{char}(k) = 0 \), then \( \text{Br}^\text{div}(X|D) \xrightarrow{\cong} \text{Br}(X^o) \) for every \( D \in \text{Div}_D(X) \).

Proof. Part (1) follows easily from Lemmas 8.4 and 8.6. To prove (2), let \( w \in \text{Br}(X^o) \). For every irreducible component \( D_i^1 \) of \( D^1 \), (8.6) says that the image of \( w \) in \( \text{Br}(K_{x_i}) \) lies in \( \text{fil}_m H^2(K_{x_i}) \) for some \( m_i \geq 0 \), where \( D_i^1 = \{ x_i \} \). We now take \( D = \sum m_i D_i^1 \in \text{Div}_D(X) \). Then it is clear that \( w \in \text{Br}^\text{div}(X|D) \). Part (3) follows from (2) and (34, Lem. III.1.16), and (4) follows from the definition of \( \text{Br}^\text{div}(X|D) \).

We shall now define the Brauer group of a modulus pair. We let \( \mathcal{C}(X) \) denote the set of integral curves on \( X \) and let \( \mathcal{C}(X|D) \) denote the subset of \( \mathcal{C}(X) \) consisting of those curves which are not contained in \( D \). For any \( C \in \mathcal{C}(X) \), we let \( \nu : C_n \to X \) denote the canonical map from the normalization of \( C \) and let \( \nu^*(D) \) denote the scheme theoretic pull-back of \( D \). We let \( C_n = \nu^{-1}(X^o) \).

Definition 8.9. We let \( \text{Br}(X|D) \) denote the subgroup of \( \text{Br}(X^o) \) consisting of elements \( \chi \) such that for every \( C \in \mathcal{C}(X|D) \), the Brauer class \( \nu^*(\chi) \in \text{Br}(C_n) \) lies in the subgroup \( \text{Br}^\text{div}(C_n|\nu^*(D)) \). The group \( \text{Br}(X|D) \) will be called the Brauer group of the modulus pair \( (X,D) \).

The Brauer group of modulus pairs has the following functorial properties first of which is not clear for \( \text{Br}^\text{div}(X|D) \). Recall that an admissible (resp. coadmissible) morphism of modulus pairs \( f : (X',D') \to (X,D) \) is a morphism of schemes \( f : X' \to X \) such that \( D' \leq f^*(D) \) (resp. \( D' \geq f^*(D) \)) as Cartier divisors on \( X' \). One says that \( f \) is strict if \( D' = f^*(D) \).

Proposition 8.10. Let \( f : (X',D') \to (X,D) \) be a coadmissible morphism of modulus pairs over \( k \). Then the pull-back on étale cohomology induces a homomorphism

\[
f^* : \text{Br}(X|D) \to \text{Br}(X'|D')
\]

such that \( f^*((\text{Br}(X|E)) \subseteq \text{Br}(X'|E') \) if \( E \leq D \) and \( D' \geq E' \geq f^*(E) \). If \( g : (X'',D'') \to (X',D') \) is another coadmissible morphism of modulus pairs, then \( (f \circ g)^* = g^* \circ f^* \).

Proof. We let \( U' = f^{-1}(X^o) \) so that \( X^{o'} \subseteq U' \). We then have the pull-back map \( f^* : \text{Br}(X^o) \to \text{Br}(U') \subseteq \text{Br}(X^{o'}) \). Suppose now that \( \chi \in \text{Br}(X|D) \) and let \( C' \in \mathcal{C}(X'|D') \). If the image of \( C' \) under \( f \) is a closed point, then this closed point must lie in \( X^o \). In the latter case, it is clear that the pull-back of \( \chi \) under the composite map \( C'_n \to U' \to X^o \) lies in \( \text{Br}(C'_n) \subseteq \text{Br}(C'_n|\nu'^*(D')) = \text{Br}^\text{div}(C'_n|\nu'^*(D')) \).
Otherwise, the scheme theoretic image of $C'$ under $f$ is a curve $C$ which is necessarily integral and lies in $\mathcal{C}(X|D)$. Since $f:C'\to C$ is also dominant, we get a commutative diagram

\begin{equation}
\begin{array}{ccc}
C' & \xrightarrow{\nu'} & X' \\
\downarrow{g} & & \downarrow{f} \\
C_n & \xrightarrow{\nu} & X.
\end{array}
\end{equation}

If we let $E = \nu^*(D)$ and $E' = \nu'^*(D')$, then we get $E' \geq g^*(E)$ by our assumption. In other words, $g:(C'_n, E') \to (C_n, E)$ is a dominant morphism of one-dimensional modulus pairs. We let $L$ (resp. $L'$) be the function field of $C$ (resp. $C'$).

We now let $\omega = \nu^*(\chi)$ and let $x \in E'$ be a closed point. If $x \notin g^{-1}(E)$, then $g^*(\chi) \notin \text{fil}_0 H^2(L_x')$. If $x \in g^{-1}(E)$, we let $y = g(x)$. We then get an inclusion of Henselian discrete valuation fields $L_y \hookrightarrow L_x'$. It follows from Definition 5.3 that this inclusion induces a map $\text{Br}(L_y) \to \text{Br}(L_x')$ which preserves the ramification filtrations. In particular, if $n_y$ (resp. $n_x$) denotes the multiplicity of $E$ (resp. $E'$) at $y$ (resp. $x$), then we get $g^*(w_y) \in \text{fil}_{n_y} \text{Br}(L_y') \subset \text{fil}_{n_x} \text{Br}(L_x')$, where the latter inclusion holds because $E' \geq g^*(E)$. It follows that $g^*(\omega) \in \text{Br}(C_n'E')$. We have thus shown that $f^*(\chi) \in \text{Br}(X'|D')$. The inclusion $f^*((\text{Br}(X|E)) \subset \text{Br}(X'|E')$, as well as the composition law, is clear from the definition of the pull-back map. This concludes the proof. \hfill \qed

**Proposition 8.11.** Let $f:(X', D') \to (X, D)$ be a finite morphism of modulus pairs over $k$. Assume that $f$ is strict. Then the norm map $f_*(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ induces a homomorphism

$$f_*:\text{Br}(X'|D') \to \text{Br}(X|D)$$

such that $f_*((\text{Br}(X'|E')) \subset \text{Br}(X|E) \enspace \text{if} \enspace E \leq D \enspace \text{and} \enspace E' = f^*(E)$. If $g:(X'', D'') \to (X', D')$ is another finite and strict morphism of modulus pairs, then $(f \circ g)_* = f_* \circ g_*$. \hfill \qed

**Proof.** Since $f$ is strict, we have $X'^o = f^{-1}(X^o)$ and the norm $f_*(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ induces $f_*:\text{Br}(X^o) \to \text{Br}(X^o)$. To show that $f_*$ preserves the ramification filtration, we can assume that $\text{dim}(X) = \text{dim}(X') = 1$. We let $K'$ be the function field of $X'$. We let $w \in \text{Br}(X'|D')$ and $x \in D^1$. By [4] Prop. 3.8.1, Lem. 3.8.6, there is a commutative diagram

\begin{equation}
\begin{array}{ccc}
\text{Br}(X'^o) & \xrightarrow{f_*} & \prod_{y \notin f^{-1}(x)} \text{Br}(A'_y) \\
\downarrow{f_*} & & \downarrow{f_* \circ \text{Cores}} \\
\text{Br}(X^o) & \xrightarrow{f_* \circ \text{Cores}} & \text{Br}(K_x),
\end{array}
\end{equation}

where $\prod_{y \notin f^{-1}(x)} A'_y = X'^o \times_{X^o} \text{Spec}(K_x)$ and the horizontal arrows are the pull-back maps. We fix $y \in f^{-1}(x)$ and let $w_y$ be the restriction of $w$ to $\text{Br}(A'_y) \cong \text{Br}(K'_y)$. It suffices to show that $\text{Cores}_{K'_y/K_x}(w_y) \in \text{fil}_{n_x} \text{Br}(K_x)$. But this follows from Lemma 8.5 because $n_y = n_x e_{y/x}$, where $e_{y/x}$ is the ramification index of $K'_y/K_x$ by the definition of $f^*(D)$. \hfill \qed

It is clear that $\text{Br}^{\text{div}}(X|D) \cong \text{Br}(X|D)$ when $X$ is a curve. For future study, it will be of interest to know an answer to the following.
**Question 8.12.** Let \((X, D)\) be a modulus pair over \(k\). Is it true that \(\Br^{\text{div}}(X|D) \cong \Br(X|D)\)?


In this section, we shall define an extension of the classical Brauer–Manin pairing between the Picard and Brauer groups of smooth projective curves over local fields to setting of relative Picard and Brauer groups of modulus pairs. Throughout this section, we fix a local field \(k\) of exponential characteristic \(p \geq 1\) and a connected regular projective scheme \(X\) of dimension \(d \geq 1\) over \(k\). We also fix a divisor \(D \subset X\) and let \(D^\dagger = D_{\text{red}}\). We let \(j : X^o \hookrightarrow X\) and \(\nu : D \hookrightarrow X\) be the inclusions, where \(X^o = X \setminus D\). We let \(K\) denote the function field of \(X\). All other notations will be those of \([8, 3]\).

#### 9.1. Idele class group

For any \(C \in \mathcal{C}(X)\), we let \(k(C)\) be the function field of \(C\). Given \(C \in \mathcal{C}(X|D)\), we let \(k(C)_{\infty} = \bigoplus_{x \in \mathcal{X}^e(D)} k(C)_x\), where \(\nu : \mathcal{C}_n \to X\) is the canonical map from the normalization of \(C\). We let \(k(C)^\times_{\infty} = \bigoplus_{x \in \mathcal{X}^e(D)} k(C)_x^\times\), \(I(C_n|D) = \bigoplus_{X \in \mathcal{X}^e(D)} (1 + \mathcal{I}_D \mathcal{O}_{\mathcal{C}_n,x})\), and \(\mathbf{t}(C_n|D) = \bigoplus_{x \in \mathcal{X}^e(D)} (1 + \mathcal{I}_D \mathcal{D}_{\mathcal{C}_n,x})\), where \(\mathcal{I}_D \subset \mathcal{O}_X\) is the ideal sheaf defining \(D\). The inclusions \(k(C) \hookrightarrow k(C)_x^\times\) induce a canonical map \(\nu^\ast C : k(C)^\times \to k(C)_{\infty}/I(C_n|D)\). We also have the map \(\partial C_{\nu^\ast} : k(C)^\times \to \mathcal{Z}_0(C_n^\ast)\) \(\to \mathcal{Z}_0(X^o)\), where \(C_n^\ast = \nu^\ast(X^o)\). We let \(\mathcal{D}_{C_n^\ast, \infty} = \mathcal{D}_{C_n^\ast, \infty}^\ast\) be the function field of \(C\).

It is easy to check that the left square in the diagram

\[
\begin{array}{c}
\bigoplus_{C \in \mathcal{C}(X|D)} k(C)^\times \to \mathcal{Z}_0(X^o) \\
\bigoplus_{C \in \mathcal{C}(X|D)} k(C)^\times \to \mathcal{Z}_0(X^o) \\
\end{array}
\]

is commutative, where the left vertical arrow is the canonical inclusion, the middle vertical arrow is the identity map of \(\mathcal{Z}_0(X^o)\) and \(\mathcal{Z}_0(X|D)^\prime\) is the cokernel of the left horizontal arrow in the bottom row. A straightforward application of the weak approximation theorem (e.g., see [27, Lem. 6.3]) allows one to conclude that the left square induces a natural isomorphism \(\mathcal{Z}_0(X|D)^\prime \cong \mathcal{Z}_0(X^o)\). In other words, there is a canonical exact sequence

\[
\bigoplus_{C \in \mathcal{C}(X|D)} k(C)^\times \overset{(\text{div}, \iota_C)}{\longrightarrow} \mathcal{Z}_0(X^o) \bigoplus_{C \in \mathcal{C}(X|D)} \frac{k(C)^\times}{I(C_n|D)} \longrightarrow \mathcal{Z}_0(X|D)^\prime \longrightarrow 0.
\]

**Remark 9.1.** We remark that the canonical map \(\frac{k(C)^\times}{I(C_n|D)} \to \frac{k(C)^\times}{I(C_n|D)}\) is an isomorphism for every \(C \in \mathcal{C}(X|D)\). In particular, the exact sequence (9.2) remains unchanged if we replace \(k(C)_{\infty}^\ast\) by \(k(C)_{\infty}^\ast\) and \(I(C_n|D)\) by \(I(C_n|D)\) for \(C \in \mathcal{C}(X|D)\). This can be easily verified.

**Definition 9.2.** The idele class group \(C(X^o)\) is the cokernel of the map

\[
\bigoplus_{C \in \mathcal{C}(X|D)} k(C)^\times \overset{(\text{div}, \iota_C)}{\longrightarrow} \mathcal{Z}_0(X^o) \bigoplus_{C \in \mathcal{C}(X|D)} \frac{k(C)^\times}{I(C_n|D)}.
\]
One can show using \[14\] Prop. 5.3 that \(C(X^o)\) depends only on \(X^o\) and not on \(X\). Using Remark \[9.1\], it is clear that there is a canonical surjective morphism of pro-abelian groups \(\{C(X^o)\} \to \{\text{CH}_0(X|D)\}_{D \in \text{Div}_{D^1}(X)}\). Taking the limit, we get a homomorphism of abelian groups

\[
\theta_{X^o} : C(X^o) \to \lim_{D \in \text{Div}_{D^1}(X)} \text{CH}_0(X|D). \tag{9.3}
\]

It is well known (e.g., see the proof of \[14\] Prop. 5.3) that the sequence \(9.2\) and the one in the definition of \(C(X^o)\) are covariantly functorial for a finite morphism of modulus pairs \(f:(X', D') \to (X, D)\). Hence, they induce push-forward maps \(f_* : \text{CH}_0(X'|D') \to \text{CH}_0(X|D)\) and \(f_* : C(X'^o) \to C(X^o)\) if \(f\) is admissible. Furthermore, \(9.3\) is compatible with the push-forward maps.

9.2. The pairing between Brauer and Chow groups with modulus. Let the notations be as above. We shall now define a bilinear pairing between the Brauer group and the Chow group of the modulus pair \((X, D)\). In the following sections, we shall prove several properties of this pairing which will be key to the proofs of our main results.

Given a closed point \(x \in X^o\) with the inclusion \(\iota_x : \text{Spec}(k(x)) \to X^o\), we have the pull-back map \(\iota_x^* : \text{Br}(X^o) \to \text{Br}(k(x))\). By composing with inv\(_k(x)\), we get a canonical map \(\iota_x^* : \text{Br}(X^o) \to \mathbb{Q}/\mathbb{Z}\). We thus get a bilinear pairing \(\text{Br}(X^o) \times \mathbb{Z}_0(X^o) \to \mathbb{Q}/\mathbb{Z}\) given by \(\langle \omega, \sum_i n_i[x_i]\rangle = \sum_i n_i \iota_x^*(\omega)\).

Suppose now that \(d = 1\) and \(x \in D\), where \(D = \sum x n_x[x]\). Since \(K_x\) is a Henselian discrete valuation field whose residue field is a local field, the cup product of \(\mathbb{Z}\) and Lemma \(8.41\) (we note here that Lemma \(8.41\) holds in characteristic zero as well, see \[20\] p. 311) together induce a bilinear pairing \(\text{Br}(K_x) \times K_x^e \to \mathbb{Q}/\mathbb{Z}\). By composing with the canonical map \(\text{Br}(X^o) \to \text{Br}(K_x)\), this yields a pairing \(\text{Br}(X^o) \times K_x^e \to \mathbb{Q}/\mathbb{Z}\). Moreover, it follows from Definition \(8.3\) that the induced pairing \(\text{Br}(X|D) \times K_x^e \to \mathbb{Q}/\mathbb{Z}\) uniquely factors through \(\text{Br}(X|D) \times K_x^e/\text{fil}_x K_x^e \to \mathbb{Q}/\mathbb{Z}\). Summing over the points of \(D\), we get a pairing \(\text{Br}(X|D) \times K_{\infty}^e/\text{I}(X|D) \to \mathbb{Q}/\mathbb{Z}\).

It follows from the above discussion that for any \(d \geq 1\), there exist bilinear pairings

\[
\text{Br}(X^o) \times \left[\mathbb{Z}_0(X^o) \bigoplus \left( \bigoplus_{C \in \mathcal{C}(X|D)} k(C)^{\times}_{\infty} \right) \right] \to \mathbb{Q}/\mathbb{Z} \quad \text{and} \tag{9.4}
\]

\[
\text{Br}(X|D) \times \left[\mathbb{Z}_0(X^o) \bigoplus \left( \bigoplus_{C \in \mathcal{C}(X|D)} \frac{k(C)^{\times}_{\infty}}{\text{I}(D)} \right) \right] \to \mathbb{Q}/\mathbb{Z}. \tag{9.5}
\]

One checks (cf. Remark \(9.1\) that the pairing \(9.5\) remains unchanged if we replace \(k(C)^{\times}_{\infty}\) by \(\bar{C}^{\times}_{\infty}\) and \(I(C_n|D)\) by \(\overline{I}(C_n|D)\) for \(C \in \mathcal{C}(X|D)\).

We recall the following reciprocity law due to Kato–Saito \[25\] when \(d = 1\). Let \(m \neq 0\) be an integer. Using the localization sequence, together with the purity theorem (due to Gabber when \(p \nmid m\) and Corollary \[4.6\] when \(m = p^n\)), we get maps

\[
\text{H}^3(K, \mathbb{Z}/m(2)) \xrightarrow{\partial_x} \bigoplus_{x \in X(0)} \text{H}^2(k(x), \mathbb{Z}/m(1)) \xrightarrow{\sum_x \text{inv}(k(x))} \mathbb{Z}/m. \tag{9.6}
\]
**Lemma 9.3.** The composite map \((\sum x \inv_{k(x)}) \circ \partial_X\) in (9.6) is zero.

**Proof.** This is proven in [25, §5, p. 120].

**Proposition 9.4.** The pairings (9.4) and (9.5) descend to bilinear maps of abelian groups

\[
\begin{align*}
\text{Br}(X^o) \times C(X^o) & \to \mathbb{Q}/\mathbb{Z} \\
\text{Br}(X|D) \times \text{CH}_0(X|D) & \to \mathbb{Q}/\mathbb{Z}.
\end{align*}
\]

**Proof.** We let \(C \in C(X|D)\) and let \(\nu: C_n \to X\) be the canonical map from the normalization of \(C\). We write \(E = \nu^*(D)\). It follows from Proposition 8.10 and the remark below (9.2) that there is a diagram

\[
\begin{array}{ccc}
\text{Br}(X|D) \times Z_0(X^o) & \to & \mathbb{Q}/\mathbb{Z} \\
\nu^* & & \downarrow \text{id} \\
\text{Br}(C|E) \times Z_0(C^o) & \to & \mathbb{Q}/\mathbb{Z}
\end{array}
\]

It follows easily from the construction of (9.5) that this diagram is commutative. Using this, and a similar commutative diagram associated to the first pairing, it suffices to prove the proposition when \(d = 1\). We shall thus assume that \(X\) is a curve.

We let \(w \in \text{Br}(X^o)\) and let \(\chi_w: Z_0(X^o) \oplus K^\times_{\infty} \to \mathbb{Q}/\mathbb{Z}\) be the associated character.

We shall show that the composite map

\[
K^\times \to Z_0(X^o) \oplus K^\times_{\infty} \xrightarrow{\chi_w} \mathbb{Q}/\mathbb{Z}
\]

is zero. This will automatically imply that the composite map

\[
K^\times \to Z_0(X^o) \oplus (K^\times_{\infty}/I(X|D)) \xrightarrow{\chi_w} \mathbb{Q}/\mathbb{Z}
\]

is zero if \(w \in \text{Br}(X|D)\).

We let \(I(X) := \prod_{x \in X_{(o)}} K^\times_x\) denote the restricted product with respect to the subgroups \((\mathcal{O}^h_{X,x})^\times\). We consider the diagram

\[
\begin{array}{ccc}
K^\times & \xrightarrow{\alpha} & I(X) \\
\downarrow \beta & & \downarrow \gamma \\
Z_0(X^o) \oplus K^\times_{\infty} & \xrightarrow{\chi_w} & \text{Hom}_{\text{Ab}}(\text{Br}(X^o), \mathbb{Q}/\mathbb{Z}) \\
\downarrow \text{ev}_w & & \downarrow \text{ev}_w \\
\mathbb{Q}/\mathbb{Z} & & \\
\end{array}
\]

where \(\alpha\) is the canonical inclusion, \(\delta\) is induced by the canonical inclusion \(\text{Br}(X^o) \to \text{Br}(K)\), \(\epsilon\) is induced by the pairing (9.3), and \(\text{ev}_w\) is the evaluation by \(w\). To describe \(\beta\), we write \(I(X) = I(X^o) \oplus K^\times_{\infty}\). We let \(\beta\) to be identity on \(K^\times_{\infty}\) and \(\beta(a_x) = \nu_x(a_x)\) if \(a_x \in K^\times_x\) with \(x \in X^o\). This is well-defined because \(\nu_x((\mathcal{O}^h_{X,x})^\times) = 0\). It is clear that the upper left triangle commutes.
To define $\gamma$, we let $\chi \in H^2(K)$ and $(a_x) \in I(X)$. It follows from Lemma 8.8(3) that $\chi \in \text{Br}(U)$ for some dense open subscheme $U \subset X^o$. We conclude by [44 Thm. 2.7(4)] that $(\chi, a_x) = 0$ for all $x \in U$. In particular, the sum $\sum_{x \in X^o}(\chi, a_x)$ is finite. This shows that the cup product of Lemma 8.4(4) induces a pairing

\[(9.10) \quad H^2(K) \times I(X) \to \mathbb{Q}/\mathbb{Z},\]

which is clearly compatible with (9.4). Letting $\gamma$ denote the induced map $I(X) \to \text{Hom}_{\text{AB}}(H^2(K), \mathbb{Q}/\mathbb{Z})$, it follows that the right square in (9.9) is commutative. Hence, it suffices to show that $\gamma \circ \alpha = 0$.

We let $a \in K^\times$, $\chi \in H^2(K)$ and write $b = (\chi, a) \in H^3(K)$. We let $a_x$ (resp. $\chi_x$) be the image of $a$ (resp. $\chi$) in $K^\times_x$ (resp. $H^3(K_x)$). We need to show that $\sum_{x \in X^o}(\chi_x, a_x) = 0$. Equivalently, we need to show that $(\sum_x \text{inv}_{k(x)}) \circ \partial_X(b) = 0$ in $\text{Br}(X)$.

The following is straightforward from the construction of the pairings (9.4) and (9.5).

**Corollary 9.5.** There is a commutative diagram of bilinear pairings

\[(9.11) \quad \lim\limits_{\rightarrow n} \text{Br}(X|nD) \times \lim\limits_{\rightarrow n} \text{CH}_0(X|nD) \to \mathbb{Q}/\mathbb{Z} \quad \text{id} \quad \text{id}
\]

between $\text{ind}$ and $\text{pro}$ abelian groups.

Corollary 9.6 provides a quick proof of the Brauer–Manin duality isomorphism $\beta_X: \text{Br}(X) \xrightarrow{\cong} (\text{CH}_0(X)^*)_{\text{tor}}$ of Lichtenbaum [31] and Saito [44], assuming that $\beta_X$ is injective. Recall that the main difficulty in proving that $\beta_X$ is an isomorphism lies in showing its surjectivity.

**Corollary 9.6.** Let $X$ be a geometrically connected smooth projective curve over $k$. Then the pairing of Proposition 9.4 (with $D = \emptyset$) induces an isomorphism $\beta_X: \text{Br}(X) \xrightarrow{\cong} (\text{CH}_0(X)^*)_{\text{tor}}$.

**Proof.** We fix an integer $n \geq 1$ and look at the diagram

\[(9.12) \quad 0 \to \text{CH}_0(X)/p^n \to H^1(X, W_n \Omega^1_{X, \text{log}}) \to p^n \text{Br}(X) \to 0
\]

where $\alpha_p^n$ and $\gamma_p^n$ are induced by the pairing (9.11). The map $\beta_p^n$ is induced by the pairing of Theorem 4.7. From the definition of the torsion-by-profinite topology of $H^1(X, W_n \Omega^1_{X, \text{log}})$, one knows that the image of the map $\text{CH}_0(X)/p^n \to H^1(X, W_n \Omega^1_{X, \text{log}})$ in the top row (9.12) is open (see [4.4] and [3.2]). In particular, the quotient topology of $p^n \text{Br}(X)$ is discrete. This shows that the bottom arrow of (9.12) is defined and exact. It is an easy exercise that this diagram is commutative.

The middle arrow is bijective by Theorem 4.7. Since $\gamma_p^n$ is known to be injective (cf. Lemma 12.1), it follows that $\alpha_p^n$ is an isomorphism. Since $\text{CH}_0(X)/p^n$ is...
it follows from Lemma 2.5 and Pontryagin duality between profinite and discrete torsion groups that $\gamma_p$ maps $p^n\text{Br}(X)$ isomorphically onto $(\text{CH}_0(X)/p^n)^*$. Taking the limit, we conclude that $\beta_X: \text{Br}(X)\{p\} \sim (\text{CH}_0(X)^*)\{p\}$ is an isomorphism. An identical argument (with Theorem 4.7 replaced by Saito–Tate duality, see [14, Thm. 9.9]) shows that $\beta_X: \text{Br}(X)\{p'\} \sim (\text{CH}_0(X)^*)\{p'\}$ is also bijective. This concludes the proof.

We end this section with the following functorial property of the Brauer–Manin pairings of modulus pairs.

**Lemma 9.7.** Let $f:(X',D') \to (X,D)$ be a finite and strict morphism between modulus pairs, where $X$ and $X'$ are connected regular projective curves over $k$. Then we have a commutative diagram

$$
\begin{array}{c}
\text{Br}(X'|D') \times \text{CH}_0(X'|D') & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\
\downarrow f_* & & \downarrow \text{id} \\
\text{Br}(X|D) \times \text{CH}_0(X|D) & \longrightarrow & \mathbb{Q}/\mathbb{Z}.
\end{array}
$$

**Proof.** We first note that $f^*$ is the well-known flat pull-back map between the Chow groups with modulus and $f_*$ is defined by Proposition 8.11. To show the commutativity of (9.13), we fix a Brauer class $w' \in \text{Br}(X'|D')$ and let $w = f_*(w')$. We need to show that the right side triangle in the diagram

$$
\begin{array}{c}
\mathbb{Z}_0(X^o) & \longrightarrow & \text{CH}_0(X|D) \\
\downarrow f^* & & \downarrow \chi_w \\
\mathbb{Z}_0(X'^o) & \longrightarrow & \text{CH}_0(X'|D') \longrightarrow \mathbb{Q}/\mathbb{Z}
\end{array}
$$

is commutative. Since the left square is commutative and its top horizontal arrow is surjective, it suffices to show that the outer trapezium in (9.14) is commutative when evaluated on every free generator of $\mathbb{Z}_0(X^o)$.

Let $x \in X^o$ be a closed point and let $f^*([x]) = \sum_{i=1}^r n_i [y_i]$. Let $i_x: \text{Spec}(k(x)) \hookrightarrow X^o$ and $i_{y_i}: \text{Spec}(k(y_i)) \hookrightarrow X'^o$ be the inclusion maps. By the definition of the Brauer–Manin pairing in §9.2, we need to show that $\text{inv}_{k(x)}(i_x^*(w)) = \sum_{i=1}^r n_i \text{inv}_{k(y_i)}(i_{y_i}^*(w'))$. But this follows by a combination of [4, Prop. 3.8.1, Lem. 3.8.6] and [20] §3.2, Prop. 1(2)).

10. LOCAL VS. GLOBAL DUALITY FOR $\mathbb{G}_m$ ON COMPACT CURVE

In this section, we shall establish a duality for some cohomology groups of $\mathbb{G}_m$ on a smooth projective curve over a local field and show that it is compatible with the duality à la Kato [20] for the local cohomology of $\mathbb{G}_m$ at closed points of the curve. This compatibility will be another key step in the proof of Theorem 1.1.

---

1This is the only place where the smoothness of $X$ is used.
10.1. **Local duality for $\mathbb{G}_m$.** Let $R$ be an equicharacteristic Henselian discrete valuation ring whose residue field $\mathfrak{f}$ is a local field of exponential characteristic $p \geq 1$. Let $L$ denote the quotient field of $R$. Let $x = \text{Spec}(\mathfrak{f})$ denote the closed point of $W = \text{Spec}(R)$. We let $\hat{W} = \text{Spec}(\hat{R})$. We choose a common uniformizer $\pi$ of $R$ and $\hat{R}$, and let $W_n = \text{Spec}(R/(\pi^n))$ for $n \geq 1$. The following result is due to Kato [20, §3.5].

**Theorem 10.1.** There is a perfect pairing of topological abelian groups

$$\text{Br}(L) \times (L^\times)^{bf} \to \mathbb{Q}/\mathbb{Z},$$

where $\text{Br}(L)$ and $L^\times$ are endowed with the discrete and the Kato topologies, respectively.

We remark that Kato proved this result for $\hat{L}$. However, it is easily checked that the canonical maps $\text{Br}(L) \to \text{Br}(\hat{L})$ and $(L^\times)^{bf} \to (\hat{L}^\times)^{bf}$ are topological isomorphisms.

Let $\gamma_L: \text{Br}(L) \to (L^\times)^*$ denote the map induced by the above pairing. It follows from Theorem 10.1 (see [44, Rem. 2.14]) that there is a commutative diagram of exact sequences

\[
\begin{align*}
0 & \to \text{Br}(R) \to \text{Br}(L) \to H^3_{\text{et}, x}(W, \mathcal{O}^*_W) \to 0 \\
\gamma_R & \downarrow \quad \quad \quad \gamma_L \downarrow \quad \quad \quad \gamma_x \downarrow \\
0 & \to \mathbb{Z}^* \to (L^\times)^* \to (R^\times)^* \to 0,
\end{align*}
\]

where the duals $(L^\times)^*$ and $(R^\times)^*$ are considered with respect to the Kato topology.

In the above diagram, $\gamma_R$ is an isomorphism because it is the composition $\text{Br}(R) \xrightarrow{\pi} \text{Br}(\mathfrak{f}) \xrightarrow{\text{inv}_L} \mathbb{Q}/\mathbb{Z} = \mathbb{Z}^*$. $\gamma_L$ is an isomorphism if $p > 1$ by [20, §3.5, Rem. 4]. It maps $\text{Br}(L)$ isomorphically onto the subgroup $((L^\times)^*)_{\text{tor}}$ if $p = 1$. It follows that $\gamma_x$ is an isomorphism if $p > 1$, and maps $H^3_{\text{et}, x}(W, \mathcal{O}^*_W)$ isomorphically onto $((R^\times)^*)_{\text{tor}}$ if $p = 1$. Recall that each $\mathcal{O}^*(W_n)$ is endowed with the adic topology.

**Lemma 10.2.** For $R$ as above, we have the following.

1. There exists a unique isomorphism $\gamma_x: H^3_x(W, \mathcal{O}^*_W) \xrightarrow{\pi} ((R^\times)^*)_{\text{tor}}$ such that (10.1) is a commutative diagram.
2. If $p > 1$, each of the groups $(R^\times)^*$, $(\hat{R}^\times)^*$ and $(\mathcal{O}^*(W_n))^*$ $(n \geq 1)$ is torsion.
3. The canonical maps $H^3_x(W, \mathcal{O}^*_W) \to H^3_x(\hat{W}, \mathcal{O}^*_W)$ and $(\hat{R}^\times)^* \to (R^\times)^*$ are isomorphisms.
4. If $p > 1$, one has a canonical isomorphism

\[
\gamma_x: H^3_x(W, \mathcal{O}^*_W) \xrightarrow{\pi} \lim_{n \to \infty} (\mathcal{O}^*(W_n))^*,
\]

where the duals on the right hand side are taken with respect to the adic topology.
5. If $p = 1$, the canonical map $(\mathfrak{f}^\times) \to (\hat{R}^\times)^*$ induces an isomorphism $(\mathfrak{f}^\times)^* \cong ((\hat{R}^\times)^*)_{\text{tor}}$.

**Proof.** The statement (1) is already shown above. To prove (2), note that the map $(\mathcal{O}^*(W_n))^* \to (R^\times)^*$ is injective for all $n \geq 1$. Using the bottom row of (10.1), it suffices therefore to show that $(L^\times)^*$ and $(\hat{L}^\times)^*$ are torsion groups. Using the definition of Kato topology, it is an easy exercise that the canonical map $(\hat{L}^\times)^* \to (L^\times)^*$
is an isomorphism. But \cite{20} §3.5, Rem. 4] says that \((\widetilde{L}^*)^*\) is a torsion group if \(p > 1\). The statement (3) follows easily by comparing (10.1) for \(R\) and \(\widetilde{R}\) and using the isomorphism \((\widetilde{L}^*)^* \cong (L^*)^*\). To show (4), we note that the map \(R/(\pi^n) \to \widetilde{R}/(\pi^n)\) is bijective for \(n \geq 1\). Furthermore, the map \(\widetilde{R}^* \to \lim_{\to n} \mathcal{O}^*(W_n)\) is an isomorphism of topological abelian groups by Lemma 7.1. It is an elementary checking (cf. Lemma 2.6) that the induced map \(\lim_{\to n} (\mathcal{O}^*(W_n))^* \to (\widetilde{R}^*)^*\) is bijective.

To prove (5), we can replace \((\widetilde{R}^*)^*\) by \((\widetilde{R}^*)^\vee\) since \(\widetilde{R}^*\) is a discrete group. By Lemma 7.4 we can also replace \((f^*)^*\) by \(((f^*)^*)_{\text{tor}}\). We are now done because it is clear that \((f^*)^\vee \to (\widetilde{R}^*)^\vee\) is injective, and it is surjective at the level of torsion subgroups by \cite{14} Lem. 5.9. This concludes the proof. □

10.2. Global duality for \(\mathbb{G}_m\). Let \(k\) be a local field of exponential characteristic \(p \geq 1\) and let \(X\) be a smooth and geometrically connected projective curve over \(k\). Let \(K\) denote the function field of \(X\). The subject of this subsection is the construction of a perfect duality between \(H^0(X, \mathcal{O}_X^\vee)\) and \(H^3(X, \mathcal{O}_X^\vee)\) which is compatible with the local duality of 10.1.

One can deduce from the Hochschild–Serre spectral sequence that there is an isomorphism \(H^3(X, \mathcal{O}_X^\vee) \cong H^3(k, \text{Pic}(X_s))\), where \(X_s\) is the base change of \(X\) by \(k_s\). Using the decomposition \(\text{Pic}(X_s) \cong \text{Pic}^0(X_s) \oplus \mathbb{Z}\) as \(G_k\)-module (note that \(X(k_s) \neq \emptyset\), see \cite{40} Prop. 3.5.70, for instance) and the vanishing \(H^2(k, \text{Pic}^0(X_s)) = 0\) (see \cite{36} Thm. 7.8), one deduces an isomorphism \(\nu_X: H^3(X, \mathcal{O}_X^\vee) \cong (G_k)^* \cong (k^*)^*\). However, this isomorphism does not serve our purpose. The reason is that it is not known if \(\nu_X\) is compatible with the local duality isomorphism \(\gamma_x\) in Lemma 10.2 for a closed point \(x \in X\), in positive characteristic. As mentioned above, this compatibility is very crucial for proving our main results. The compatibility of \(\nu_X\) with the local duality seems like a challenging independent problem. The main obstacle in solving this is the failure of purity for \(H^2_\mathbb{Z}(X, W_n \Omega^1_{X, \log})\).

To serve our purpose, we shall construct a different pairing between \(H^0(X, \mathcal{O}_X^\vee)\) and \(H^3(X, \mathcal{O}_X^\vee)\). We shall show that this pairing is perfect and is compatible with the local duality isomorphism \(\gamma_x\) for \(x \in X(0)\). We do not have any guess whether the two duality maps coincide. We let \(H^i(X, \mathbb{Z}(j)) := \lim_{\leftarrow n} H^i(X, \mathbb{Z}/n(j))\), where the limit is taken over \(\mathbb{N}\). We begin with the following.

Lemma 10.3. There exists a canonical bilinear pairing

\[
(10.3) \quad H^3(X, \mathcal{O}_X^\vee) \times H^0(X, \mathcal{O}_X^\vee) \to \mathbb{Q}/\mathbb{Z}.
\]

Proof. By Lemma 8.1 we can replace \(H^3(X, \mathcal{O}_X^\vee)\) by \(H^3(X, \mathbb{Q}/\mathbb{Z}(1))\). Using (4.17) and the analogous diagram for the sheaves \(\mathbb{Z}/n(r)\) (with \(n\) prime to \(p\)), we get a bilinear pairing between ind-abelian and pro-abelian groups

\[
\text{“lim”} \ H^i(X, \mathbb{Z}/n(j)) \times \text{“lim”} \ H^{i'}(X, \mathbb{Z}/n(j')) \to \text{“lim”} \ H^{i+i'}(X, \mathbb{Z}/n(j+j')).
\]

Taking the limits, we conclude by the Saito–Tate duality \cite{45} (see also \cite{14} Thm. 9.9) in the prime-to-\(p\) case and Corollary 4.8 in the \(p\)-primary case that there is a bilinear pairing between abelian groups

\[
(10.4) \quad H^3(X, \mathbb{Q}/\mathbb{Z}(1)) \times H^1(X, \mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z}.
\]
Composing this with the canonical maps (see (8.1))
\[ H^0(X, \mathcal{O}_X^\vee) \to \lim_{\leftarrow n} H^0(X, \mathcal{O}_X^\vee)/n \to H^1(X, \mathbb{Z}(1)), \]
we get the desired pairing. \qed

In the remainder of this subsection, our goal is to prove the perfectness of \((10.3)\) with respect to the discrete topology on \(H^3(X, \mathcal{O}_X^\vee)\) and the adic topology on \(H^0(X, \mathcal{O}_X^\vee) \cong k^\times\). We begin by noting that the diagrams in (8.2) give rise to an exact sequence of pro-abelian groups
\[ 0 \to \left\{ H^0(X, \mathcal{O}_X^\vee)/n \right\} \to \left\{ H^1(X, \mathbb{Z}/n(1)) \right\} \to \{ n\text{Pic}^0(X) \} \to 0, \]
whose transition maps are induced by the canonical surjections \(\mathbb{Z}/mn \to \mathbb{Z}/n\). Taking the limits, we get an exact sequence
\[ 0 \to \lim_{\leftarrow n} H^0(X, \mathcal{O}_X^\vee)/n \to H^1(X, \mathbb{Z}(1)) \to \lim_{\leftarrow n} n\text{Pic}^0(X) \to 0. \]

Applying the dual functor in the category of abelian groups, we get an exact sequence
\[ 0 \to \lim_{\rightarrow n} (n\text{Pic}^0(X))^\vee \to \lim_{\rightarrow n} (H^1(X, \mathbb{Z}/n(1)))^\vee \to \lim_{\rightarrow n} (H^0(X, \mathcal{O}_X^\vee)/n)^\vee \to 0. \]

**Lemma 10.4.** We have \(\lim_{\rightarrow n} n\text{Pic}^0(X) = \lim_{\rightarrow n} (n\text{Pic}^0(X))^\vee = 0.\)

**Proof.** One knows that there is a canonical inclusion \(\lim_{\rightarrow n} n\text{Pic}^0(X) \to \lim_{\leftarrow n} nJ_X(k)\), where \(J_X\) is the Jacobian variety of \(X\) (e.g., see \([35\text{, Rem. 1.5}]\)). On the other hand, \(J_X(k)\) is a profinite abelian group (see the proof of Corollary 6.11 and \(\lim_{\leftarrow n} nJ_X(k)\) coincides with the Tate module of \(J_X(k)\). It follows that \(\lim_{\leftarrow n} nJ_X(k) = 0\) as the Tate module of a profinite abelian group is zero.

To show that \(\lim_{\rightarrow n} (n\text{Pic}^0(X))^\vee\) is zero, it suffices to show the stronger assertion that the pro-abelian group \(\{ n\text{Pic}^0(X) \}\) is zero as it would imply that the ind-abelian group \(\{ (n\text{Pic}^0(X))^\vee \}\) is zero. But this follows from our first assertion and the well-known fact that a pro-abelian group satisfying the Mittag-Leffler condition is zero if and only if its limit is zero (e.g., see \([51\text{, Tag 07KV, Lem. 15.86.13}]\)). \qed

**Lemma 10.5.** The map \(\delta_X: H^3(X, \mathcal{O}_X^\vee) \to H^0(X, \mathcal{O}_X^\vee)^\vee\), induced by \((10.3)\), is injective.

**Proof.** By Lemma 8.1 and (10.5), \(\delta_X\) is the same as the composite map
\[ H^3(X, \mathbb{Q}/\mathbb{Z}(1)) \to \lim_{\rightarrow n} H^1(X, \mathbb{Z}/n(1))^\vee \to \lim_{\rightarrow n} (H^0(X, \mathcal{O}_X^\vee)/n)^\vee \to H^0(X, \mathcal{O}_X^\vee)^\vee. \]
The first arrow in this sequence is injective in the \(p\)-primary case by Theorem 4.7 and isomorphism in the prime-to-\(p\) case by \([14\text{, Thm. 9.9}]\). The second arrow is injective by \((10.3)\) and Lemma 10.4. The third arrow is easily seen to be injective. This concludes the proof. \qed
The following shows the compatibility between the local and global duality maps.

**Lemma 10.6.** Let \( x \in X(\mathbb{O}) \), \( R = \mathcal{O}_X^h \) and let the notations be as in Lemma [10.2]. Then the diagram

\[
\begin{array}{c}
\mathcal{H}^3(W, \mathcal{O}_{W}^\times) \\
\downarrow \gamma_x \\
(R^x)^* \\
\downarrow \delta_x
\end{array}
\begin{array}{c}
\mathcal{H}^3(X, \mathcal{O}_X^\times) \\
\downarrow \delta_x
\end{array}
\]

is commutative if we let the top horizontal arrow be the forget support map and the bottom horizontal arrow be the dual of the canonical pull-back map \( \mathcal{H}^0(X, \mathcal{O}_X^\times) \to R^x \).

**Proof.** Using [10.1], the lemma is equivalent to showing that the diagram

\[
\begin{array}{c}
\mathcal{H}^2(K_x) \\
\downarrow \gamma_{K_x}
\end{array}
\begin{array}{c}
\mathcal{H}^3(X, \mathcal{O}_X^\times) \\
\downarrow \delta_x
\end{array}
\]

is commutative, where \( \phi_x \) is induced by the canonical pull-back map. To show this, we can replace \( \mathcal{H}^3(X, \mathcal{O}_X^\times) \) by \( \mathcal{H}^3(X, \mathbb{Q}/\mathbb{Z}(1)) \) using Lemma [8.1]. It also suffices to prove the commutativity after replacing \( \mathbb{Q}/\mathbb{Z} \) by \( \mathbb{Z}/n \) for all \( n \geq 1 \).

We now fix an integer \( n \geq 1 \). We let \( \alpha \in \mathcal{H}^2(K_x, \mathbb{Z}/n(1)) \) and \( \beta \in \mathcal{H}^0(X, \mathcal{O}_X^\times) \) be arbitrary elements. We look at the diagram

\[
\begin{array}{c}
\mathcal{H}^2(K_x, \mathbb{Z}/n(1)) \times \mathcal{H}^1(K_x, \mathbb{Z}/n(1)) \\
\downarrow \partial_x \\
\mathcal{H}^3(K_x, \mathbb{Z}/n(1)) \\
\downarrow \partial_x
\end{array}
\begin{array}{c}
\mathcal{H}^3(K_x, \mathbb{Z}/n(2)) \\
\downarrow \partial_x
\end{array}
\]

where \( \psi_X \) is the canonical map (see [10.6]).

All squares on the left are induced by the cup products, pull-back and the boundary maps in étale cohomology. In particular, these squares are known to be commutative (e.g., see [4.15]) for the commutativity of the top square on the left. The upper square on the right is commutative by the definition of \( \text{Tr} \) (see Corollary [4.6] and note that this also holds when \( p \nmid n \)). The lower square on the right commutes by [4.8] when \( n = p^n \) and by [14] Lem. 9.7 when \( p \nmid n \). The bottom right triangle clearly commutes.
We let $\theta: H^2(K_x, \mathbb{Z}/n(1)) \to H^3(X, \mathbb{Z}/n(1))$ be the composition of all vertical arrows going down on the extreme left of (10.11) and let $\theta': H^0(X, \mathcal{O}_X^\times)/n \to H^1(K_x, \mathbb{Z}/n(1))$ be the composition of all vertical arrows going up in the middle of (10.11). It follows then by a diagram chase that $\text{Tr}_x(\alpha \cup \theta'(\beta)) = \text{Tr}_X(\theta(\alpha) \cup \psi_X(\beta))$. On the other hand, it is straightforward to check that

$$\langle \phi_x \circ \gamma_{K_x}(\alpha), \beta \rangle = \text{Tr}_x(\alpha \cup \theta'(\beta)) \quad \text{and} \quad \langle \delta_X \circ \partial_x(\alpha), \beta \rangle = \text{Tr}_X(\theta(\alpha) \cup \psi_X(\beta)).$$

This proves the commutativity of (10.10) and concludes the proof of the lemma. □

We shall also need the following independent result for proving the perfectness of (10.3) as well as in the proof of Lemma 11.3.

**Lemma 10.7.** Let $k'/k$ be a finite field extension of $k$. Then the inclusion $k^\times \hookrightarrow k'^\times$ induces a surjective map $(k'^\times)^* \twoheadrightarrow (k^\times)^*$ when the duals are taken with respect to the adic topologies.

**Proof.** Let $\chi \in (k^\times)^*$. Since the inclusion $k^\times \hookrightarrow k'^\times$ is continuous and its image is closed (e.g., see [19, §7.2] or [35, Prop. II.5.7]) with respect to the adic topologies, it follows from [8, Cor. 4.42] that there exists a continuous homomorphism $\chi': k'^\times \to \mathbb{T}$ whose restriction to $k^\times$ is $\chi$. It remains to show that $\chi'$ has finite order. Since $\mathcal{O}_{k'}^\times$ is profinite, it is clear that $\chi'$ has finite order when we restrict it to $\mathcal{O}_{k'}^\times$. We let $G = (k^\times)^* \cdot (\mathcal{O}_{k'}^\times) \subset k'^\times$. Then $G$ is a closed subgroup of $k'^\times$ and $\chi'$ has finite order (say, $m$) on $G$.

We now look at the commutative diagram of exact sequences

\[
\begin{array}{ccc}
0 & \longrightarrow & G & \longrightarrow & k'^\times & \longrightarrow & \mathbb{Z}/e & \longrightarrow & 0 \\
& & \downarrow \chi' & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/m & \longrightarrow & \mathbb{T} & \longrightarrow & \mathbb{T} & \longrightarrow & 0,
\end{array}
\]

where $e$ is the ramification index of $k'/k$. It follows that $\chi'(k'^\times)$ is finite. This concludes the proof. □

We can now prove the main result of §10.

**Theorem 10.8.** The bilinear pairing of (10.3) induces a perfect pairing of topological abelian groups

\[
\delta_X: H^3(X, \mathcal{O}_X^\times) \times H^0(X, \mathcal{O}_X^\times)^{pt} \to \mathbb{Q}/\mathbb{Z}.
\]

**Proof.** We can identify $H^0(X, \mathcal{O}_X^\times)$ with $k^\times$ and $H^3(X, \mathcal{O}_X^\times)$ with $H^3(X, \mathbb{Q}/\mathbb{Z}(1))$ (cf. Lemma 8.1). In the first step, we show that $\delta_X: H^3(X, \mathcal{O}_X^\times) \to (k^\times)^\vee$ factors through $(k^\times)^*$. Since $(k^\times)^* \cong \text{Hom}_{\text{Tab}}(k^\times, \mathbb{Q}/\mathbb{Z}(p')) \oplus \text{Hom}_{\text{Tab}}(k^\times, \mathbb{Q}_p/\mathbb{Z}_p)$ and $H^3(X, \mathcal{O}_X^\times)$ is a torsion group, it suffices to consider the prime-to-$p$ and $p$-primary cases separately.

On the prime-to-$p$ torsion subgroup, $\delta_X$ is identified with the composite map

\[
H^3(X, \mathbb{Q}/\mathbb{Z}(1)) \langle p' \rangle \to \lim_{\text{nek}^x} (H^1(X, \mathbb{Z}/n(1)))^\vee \to \lim_{\text{nek}^x} (k^\times/n)^\vee \to (k^\times)^\vee.
\]

Since the groups $H^1(X, \mathbb{Z}/n(j))$ are finite for $n \in k^\times$ (e.g., see [14, Thm. 9.9]), we have $H^1(X, \mathbb{Z}/n(1))^\vee \cong H^1(X, \mathbb{Z}/n(1))^\vee$. Since $k^\times/n$ is also discrete (and finite), it
follows that \((k^x/n)^* \cong (k^x/n)^\vee\) for every \(n \geq 1\). Hence, the above sequence of maps has a factorization
\[
H^3(X, \mathbb{Q}/\mathbb{Z}(1))\{p\} \to \lim_{n \to k^x} H^1(X, \mathbb{Z}/n(1))^* \to \lim_{n \to k^x} (k^x/n)^* \to (k^x)^*.
\]

To prove the \(p\)-primary case, we can assume \(p > 1\). We choose a closed point \(x \in X\). We follow the notations of Lemma 10.2 and look at the commutative diagram
\[
\begin{array}{ccc}
H^3_x(W, \mathcal{O}_W^x)\{p\} & \longrightarrow & H^3(X, \mathcal{O}_X^x)\{p\} \\
\gamma_x \uparrow & & \downarrow \delta_x \\
\lim_{n \to n} H^0(W_n, \mathcal{O}_{W_n}^x)^* & \longrightarrow & (k^x)^* \longrightarrow (k^x)^\vee.
\end{array}
\]
Note that \(\delta_X\) is injective by Lemma 10.5. It is clear that the map \(H^0(W_n, \mathcal{O}_{W_n}^x)^* \to (k^x)^\vee\) factors through \((k^x)^*\) for every \(n \geq 1\). In particular, the bottom left arrow is defined. Moreover, the outer trapezium commutes by Lemma 10.6. To show that the image of \(\delta_X\) lies in \((k^x)^*\), it suffices therefore to show that the top horizontal arrow is surjective.

To show the last claim, we set \(U = X \setminus \{x\}\). We now recall that \(H^i(Z, \mathcal{O}_Z^x)\) is a torsion group for all \(i \geq 2\) if \(Z \in \{X, U\}\). Using the support cohomology exact sequence, this implies that \(H^i_x(W, \mathcal{O}_W^x) \cong H^i_x(X, \mathcal{O}_X^x)\) is a torsion group for \(i \geq 3\) (the case of interest \(i = 3\) also follows from Lemma 10.2). In particular, the sequence
\[
H^3_x(W, \mathcal{O}_W^x)\{p\} \to H^3(X, \mathcal{O}_X^x)\{p\} \to H^3(U, \mathcal{O}_U^x)\{p\}
\]
is exact.

On the other hand, there is a surjection \(H^2(U, W_n \mathcal{O}_{U,\log}) \to p^n H^3(U, \mathcal{O}_U^x)\) for any \(n \geq 1\), as follows from (8.1). It suffices therefore to show that \(H^2(U, W_n \mathcal{O}_{U,\log}) = 0\) for all \(n \geq 1\). For \(n = 1\), note that in the exact sequence
\[
0 \to \mathcal{O}_{U,\log} \to \mathcal{O}_U^1 \xrightarrow{\pi} \Omega_U^1 \xrightarrow{\text{deg} O_U} 0
\]
(see (3.1)), where the term on the right is a coherent \(\mathcal{O}_{U,(p)}\)-module. In particular, \(\Omega_U^1\) and \(\text{deg} O_U\) have their cohomological dimension zero as \(U\) is affine. It follows that \(H^2(U, \mathcal{O}_{U,\log}) = 0\). The general case follows by induction on \(n\) using the cohomology sequence associated to (2.6).

We shall now show that \(\delta_X: H^3(X, \mathcal{O}_X^x) \to (k^x)^*\) is bijective. It is injective by Lemma 10.5. To prove its surjectivity between the prime-to-\(p\) torsion subgroups, note that all arrows in (10.15) are isomorphisms (the first arrow is an isomorphism by the Saito-Tate duality, see [13, Thm. 9.9]). To prove the surjectivity of \(\delta_X\) between the \(p\)-primary torsion subgroups, note that this map is the same as the dotted arrow in (10.16). The map \(\gamma_x\) is bijective by Lemma 10.2(4). It suffices therefore to show that the map \(\lim_{n \to n} H^0(W_n, \mathcal{O}_{W_n}^x)^* \to (k^x)^*\) is surjective. Equivalently, the map \((\mathcal{R}^x)^* \to (k^x)^*\) is surjective, where recall that \(\mathcal{R} = \mathcal{O}_{X,x}^x\). But this latter map is induced by the canonical inclusion \(k^x \to \mathcal{R}^x\). We are now done because the composite map \((k(x))^* \to (\mathcal{R}^x)^* \to (k^x)^*\) is surjective by Lemma 10.7.

To prove the perfection of (10.13), we let \(F = (k^x)^{\text{pf}}\). We now note that the canonical map \(F^* \to (k^x)^*\) is an isomorphism, as one easily checks using [83, Prop. II.5.7] and the profiniteness of \(\mathcal{O}_X^x\). It follows that (10.3) gives rise to (10.13) and the resulting map \(\delta_X: H^3(X, \mathbb{Q}/\mathbb{Z}(1)) \to F^*\) is an isomorphism. Since this map
is clearly continuous and $F$ is profinite, it follows that (10.13) is a perfect pairing of topological abelian groups. This concludes the proof. □

Remark 10.9. We showed in the proof of Theorem [10.8] that $H^3(U, \mathcal{O}_U^*)\{p\} = 0$ if $U$ is the complement of a closed point in $X$. But the proof actually shows that $H^i(U, \mathcal{O}_U^*)\{p\} = 0$ if $U$ is any affine open in $X$ and $i \geq 3$ is any integer. This observation will be used in the proof of Theorem [1.1]

Corollary 10.10. For any nonempty effective Cartier divisor $D \subset X$, the forget support map $H^3_D(X, \mathcal{O}_X^*) \to H^3(X, \mathcal{O}_X^*)$ is surjective.

Proof. Using Lemmas [10.2, 10.6] and Theorem [10.8] it suffices to show that the map $(k(x)^*)^* \to (k^*)^*$ is surjective for any $x \in D$. But this follows from Lemma [10.7] □

11. CONTINUITY OF BRAUER–MANIN PAIRING

Let $k$ be a local field of exponential characteristic $p \geq 1$. The goal of this section is to show that the Brauer–Manin pairing for a one-dimensional modulus pair $(X, D)$ over $k$ is continuous with respect to the discrete topology of $\text{Br}(X/D)$ and the adic topology of $\text{Pic}(X|D)$. This is an important step in the proofs of the main results. We shall also prove a few more properties of this pairing which will be used in the proof of Theorem [1.1]. We begin with the following general statement about regular (but not necessarily smooth) curves.

Lemma 11.1. Let $X$ be a connected and regular projective curve over $k$ and let $D \subset X$ be a divisor with the complement $X^o$. Then the map $\theta_{X^o}: C(X^o) \to \lim_{D' \in \text{Div}_{D_{\text{red}}}(X)} \text{CH}_0(X|D')$ is bijective. In particular, the map $C(X^o) \to \text{CH}_0(X|D')$ is surjective for every $D' \in \text{Div}_{D_{\text{red}}}(X)$.

Proof. We can replace the right hand side by $\lim_{\Sigma} \text{CH}_0(X|nD)$. Using Remark [9.1] we can replace $k(X)^*_\infty$ (resp. $I(X|nD)$) by $\overline{k(X)}^*_\infty$ (resp. $\overline{T(X|nD)}$) in (9.2). It suffices now to show the stronger statement that the canonical map $Z_0(X^o) \oplus \overline{k(X)}^*_\infty \to \lim_n \left( Z_0(X^o) \oplus \lim_{\Sigma} \frac{k(X)^*_{\infty}}{I(X|nD)} \right)$ is bijective. It is clear that the right hand side of the latter map is $Z_0(X^o) \oplus \left( \lim_{\Sigma} \frac{k(X)^*_{\infty}}{I(X|nD)} \right)$. It remains therefore to show that for every $x \in D$, the map $\overline{k(X)}^*_x \to \lim_{\Sigma} \frac{k(X)^*_{\infty}}{I(X|nD)}$ is bijective. But this can be easily checked using the completeness of $\mathcal{O}_{X,x}$ and the strict exact sequence of pro-abelian groups

$$0 \to \left\{ \frac{\mathcal{O}_{X,x}}{\text{fil}_n k(X)^*_{x}} \right\}_n \to \left\{ \frac{\overline{k(X)}^*_x}{\text{fil}_n k(X)^*_{x}} \right\}_n \to 0.$$

To see that $C(X^o) \to \text{CH}_0(X|D')$ is surjective, we only have to note that $\{ \text{CH}_0(X|nD') \}_n$ is an inverse system indexed by $\mathbb{N}$ with surjective transition maps whose limit is $C(X^o)$. □

We now let $X$ be a smooth projective geometrically integral curve over $k$ and $D \subset X$ an effective divisor with support $D^\dagger$. We write $D = \sum_{x \in D^\dagger} n_x [x] \in \text{Div}(X)$. Let $j: X^o = X \setminus D \hookrightarrow X$ be the inclusion. We shall consider $C(X^o) = \lim_{\Sigma} \text{CH}_0(X|nD)$
as a topological abelian group, endowed with the inverse limit of the adic topologies of $\text{CH}_0(X|nD)$ using Lemma 11.1.

**Proposition 11.2.** The pairings

$$\text{Br}(X|D) \times \text{CH}_0(X|D) \to \mathbb{Q}/\mathbb{Z}, \quad \text{and} \quad \text{Br}(X^0) \times C(X^0)^{pf} \to \mathbb{Q}/\mathbb{Z}$$

are continuous.

**Proof.** Let $\beta^\vee_{X|D}: \text{Br}(X|D) \to \text{CH}_0(X|D)^\vee$ and $\beta_{X^0}: \text{Br}(X^0) \to C(X^0)^\vee$ be the induced homomorphisms. As mentioned in §4.2 the proposition is equivalent to the statement that the image of $\beta^\vee_{X|D}$ (resp. $\beta_{X^0}$) lies in $\text{CH}_0(X|D)^\ast$ (resp. $C(X^0)^\ast$).

Note that this uses the discreteness of $\text{Br}(X^0)$. Since $\lim \limits_n \text{CH}_0(X|nD)^\ast \cong C(X^0)^\ast$, it suffices to show that $\beta^\vee_{X|D}(\text{Br}(X|D)) \subset \text{CH}_0(X|D)^\ast$. To prove the latter statement, we fix $w \in \text{Br}(X|D)$ and let $\chi_w: \text{CH}_0(X|D) \to \mathbb{Q}/\mathbb{Z}$ be the induced character.

Assume first that $p = 1$. As $nw = 0$ for some $n \geq 1$, the map $\chi_w: \text{CH}_0(X|D) \to \mathbb{Q}/\mathbb{Z}$ must factor through $\text{CH}_0(X|D)/n$. On the other hand, $\text{CH}_0(X|D)/n$ is a finite discrete group, as one easily checks using [5.1, 35 Prop. II.5.7] and the Kummer sequence. In particular, the map $\chi_w: \text{CH}_0(X|D)/n \to \mathbb{Q}/\mathbb{Z}$ is continuous. It follows that $\beta^\vee_{X|D}(w) \in \text{CH}_0(X|D)^\ast$. We shall now assume in the rest of the proof that $p > 1$.

We begin by proving the special case of the proposition when $D$ and $X^0(k)$ are not empty. We fix a closed point $P \in X^0(k)$. As $\chi_w$ is a group homomorphism, it will be enough to show that it is continuous at some point of $\text{Pic}(X|D)$. We now note that the composition $\chi^1_w: X^0(k) \hookrightarrow Z_0(X) \to \text{Pic}(X|D) \xrightarrow{\chi_w} \mathbb{Q}/\mathbb{Z}$ (see (9.14)) is given by $\chi^1_w(x) = \text{inv}_{k(x)}(\nu^x_{k}(w))$, where $\nu^x_k: \text{Spec}(k(x)) \to X^0$ is the inclusion. It follows from [4 Prop. 10.5.2] (see also [40 Prop. 8.2.9]) that $\chi^1_w$ is continuous. In particular, the map $\chi^r_w: (X^0(k))^r \to \mathbb{Q}/\mathbb{Z}$, given by $\chi^r_w(x_1, \ldots, x_r) = \sum_{i=1}^r \chi^1_w(x_i)$, is continuous for every $r \geq 1$.

We let $\text{alb}_{X|D}: X^0(k) \to \text{Pic}^0(X|D)$ be the Albanese map, given by $\text{alb}_{X|D}(x) = [x]-[P]$ (cf. (6.2)). It follows from [4 Thm. 10.5.1] and Theorem 6.6 that for all $r \gg 0$, there exists a Zariski open subscheme $U \subset (X^0)^r$ such that the composite map $\phi^r_{X|D}: U(k) \to (X^0(k))^r \xrightarrow{\text{Pic}^0(X|D)/(k)} \text{Pic}^0(X|D)$ is open and has a dense image. We now look at the commutative diagram

$$\begin{array}{ccc}
U(k) & \xrightarrow{\phi^r_{X|D}} & \text{Pic}^0(X|D) \\
\downarrow{\chi^r_w} & & \downarrow{\chi_w} \\
\mathbb{Q}/\mathbb{Z} & \underset{=r\chi_w (P)}{\longrightarrow} & \mathbb{Q}/\mathbb{Z}.
\end{array}$$

We showed above that the left vertical arrow is continuous. Since the bottom horizontal arrow is clearly continuous, it follows that $\chi_w \circ \phi^r_{X|D}$ is continuous. Since the image of $\phi^r_{X|D}$ is open dense and the map $U(k) \to \phi^r_{X|D}(U(k))$ is open, it follows that $\chi_w$ is continuous on the nonempty open subset $\phi^r_{X|D}(U(k))$. This proves the proposition when $X^0(k)$ and $D$ are not empty.

We assume next that $D \neq \emptyset$ but $X(k) = \emptyset$. We let $G^\ast(X^0) = \lim \limits_n \text{CH}_0(X|nD)^\ast$ and $G^\vee(X^0) = \lim \limits_n \text{CH}_0(X|nD)^\vee$. Recall from Lemma 8.8 that $\lim \limits_n \text{Br}(X|nD)^\ast \cong \text{Br}(X^0)$. Suppose we know that the map $\beta_{X^0}: \text{Br}(X^0) \to G^\vee(X^0)$ factors through
G^*(X^o)$. Then it follows from \cite{9,11} that $\beta_{X|D}(\chi) \in G^*(X^o) \cap CH_0(X|D)^\vee$. In particular, the composite map $CH_0(X|nD) \rightarrow CH_0(X|D) \xrightarrow{\chi_w} \mathbb{Q}/\mathbb{Z}$ is continuous for $n \gg 0$. This implies that $\chi_w$ is continuous because the first arrow is a quotient map of adic spaces by Corollary 6.10 and \cite[Thm. 10.5.1]{4}. It remains therefore to show that $\beta_{X^o}(Br(X^o)) \in G^*(X^o)$.

We fix an element $w \in Br(X^o)$ and let $\chi_w = \beta_{X^o}(w)$. By \cite[Thm. 4.3.1]{56} (see also \cite[Thm. 2.2]{10}), we can find a finite field extension $k'/k$ and a geometrically connected smooth projective curve $Y$ over $k'$ together with a finite morphism $g: Y \rightarrow X_{k'}$ such that the degrees of $k'/k$ and $g$ are powers of $p$ and $Y$ admits a strict semistable reduction over $k'$. That is, there exists a projective and flat morphism $\phi: Y \rightarrow \text{Spec}(O_{k'})$ such that $Y$ is regular whose generic fiber is $Y$ and whose (scheme theoretic) closed fiber $Y_s$ is a reduced curve with only double point singularities.

We let $Y^o$ (resp. $E^\dagger$) be the inverse image of $X^o$ (resp. $D^\dagger$) under the composite map $Y \rightarrow X_{k'} \rightarrow X$ and let $Y^o$ be the complement of the closure of $E^\dagger$ in $Y$. By an application of Weil conjectures over the residue field $\mathbb{F}_q$ of $k'$, we can find a closed point $P \in Y^o \cap (Y_s)_{\text{reg}}$ whose degree over $\mathbb{F}_q$ is a power of $p$. It is easy to see that there exists a closed point $P \in Y^o$ whose degree over $k'$ is same as that of $P_s$ over $\mathbb{F}_q$. We let $F = k'(P)$ and let $X = Y_F$. Let $f: X' \rightarrow X$ be the composite finite surjective map. We let $X^o = f^{-1}(X^o)$ and $E = f^*(D)$. It follows then that $X'$ is a smooth and geometrically connected curve over the local field $F$ such that $P \in X^o(F)$ and $\deg(f) = p^n$ for some $n \geq 1$.

We have the canonical maps $\text{Br}(X^o) \xrightarrow{f^*} \text{Br}(X^o) \xrightarrow{f_*} \text{Br}(X^o)$ whose composition is multiplication by $p^n$ (e.g., see \cite[§3.8]{4}). In particular, $\text{Coker}(f_*)$ is torsion of exponent $p^n$. On the other hand, $\text{Coker}(f_*)$ is a p-divisible group by \cite[Thm. 3.2.3]{4}. It follows that $f_*$ is surjective. In particular, $w = f_*(w')$ for some $w' \in \text{Br}(X^o)$. We can assume that $w' \in \text{Br}(X'|nE)$ for some $n \gg 0$. Since $X^o(F) \neq \varnothing$, the previous case of the lemma shows that $\beta_{X'|nE}(w') \in CH_0(X'|nE)^\vee$. We let $\chi_{w'} = \beta_{X'|nE}(w')$.

We now look at the diagram

\begin{equation}
CH_0(X|nD) \xrightarrow{\chi_w} \mathbb{Q}/\mathbb{Z} \xrightarrow{\chi_{w'}} \text{CH}_0(X'|nE).
\end{equation}

It follows from Lemma \cite[9,7]{9} that this diagram is commutative. Since $f^*: \text{Pic}(X|nD) \rightarrow \text{Pic}(X'|nE)$ is a morphism between locally of finite type $k$-schemes by Theorem \cite[5.6]{5}, it follows from \cite[Prop. 5.4]{5} that the left vertical arrow is continuous. We deduce that $\chi_w$ is continuous. This concludes the proof when $D \neq \varnothing$.

To prove the proposition for $(X, \varnothing)$, we choose a nonempty effective divisor $D \subset X$ and let $w \in \text{Br}(X) \subset \text{Br}(X|D)$. We have shown above that the composite map $CH_0(X|D) \rightarrow CH_0(X) \xrightarrow{\chi_w} \mathbb{Q}/\mathbb{Z}$ is continuous. On the other hand, it follows from Corollary \cite[6.10]{9} and \cite[Thm. 10.5.1]{4} that the first arrow is a topological quotient map. We deduce that $\chi_w$ is continuous. \hfill $\Box$

\textbf{Lemma 11.3.} Let $k'/k$ be a finite field extension. Let $X' = X_{k'}$, $D' = D_{k'}$ and $X'^o = X' \setminus D'$. Let $f: X' \rightarrow X$ be the projection map. Then the following hold.

1. The map $f^*: \text{Pic}(X|D) \rightarrow \text{Pic}(X'|D')$ is a closed embedding of adic spaces.
2. $f^*$ induces a surjective homomorphism $(f^*)^*: \text{Pic}(X'|D')^* \rightarrow \text{Pic}(X|D)^*$.
Proof. By Theorem 5.6 the pull-back map $f^* : \text{Pic}(X|D) \to \text{Pic}(X'|D')$ is the canonical inclusion $\text{Pic}(X|D)(k) \subset \text{Pic}(X|D)(k')$. To prove (1), we therefore need to show that the inclusion $\text{Pic}(X|D)(k) \subset \text{Pic}(X|D)(k')$ of adic spaces is closed. But this follows from [5 Prop. 5.11(3)] since the inclusion $k \subset k'$ is easily seen to be closed (e.g., see [38 Prop. II.5.7]).

Since $\mathbb{Q}/\mathbb{Z}$ is a divisible group, one easily reduces the proof of (2) to showing that the map $(f^*)^* : \text{Pic}^0(X'|D')^* \to \text{Pic}^0(X|D)^*$ is surjective. Let $\chi : \text{Pic}(X|D) \to \mathbb{Q}/\mathbb{Z}$ be a continuous character. By part (1) of the lemma and [8 Cor. 4.42], $\chi$ extends to a continuous character $\chi : \text{Pic}^0(X'|D') \to \mathbb{T}$. It remains to show that the image of $\chi'$ lies in $\mathbb{Q}/\mathbb{Z}$. We write $F = F_1 \times F_2$, where $F_1$ is the cokernel of the diagonal inclusion $k^x \hookrightarrow \prod_{i=1}^r k(x_i)^x$ and $F_2 = \prod_{i=1}^s \mathbb{W}_{n_i-1}(k(x_i))$. We then note using (5.1) and the proof of Lemma 7.1 (see also Proposition 6.9) that there is an exact sequence of topological groups

$$0 \to F \to \text{Pic}^0(X|D) \to \text{Pic}^0(X) \to 0. \tag{11.3}$$

We claim that $\chi(F)$ is finite.

Since $F_2$ is a torsion group of bounded exponent, $\chi(F_2)$ must be finite. For every $1 \leq i \leq r$, we have an exact sequence of topological groups

$$0 \to \mathcal{O}^x_{k(x_i)} \to k(x_i)^x \to \mathbb{Z} \to 0. \tag{11.4}$$

Since $\mathcal{O}^x_{k(x_i)}$ is a profinite group, its image under $\chi$ must be finite. In particular, it is a cyclic subgroup of $\mathbb{Q}/\mathbb{Z}$ of the type $\mathbb{Z}/n$ for some integer $n \geq 1$. We now look at the commutative diagram of exact sequences

$$0 \to \chi(\mathcal{O}^x_{k(x_i)}) \to \mathbb{Q}/\mathbb{Z} \to 0. \tag{11.5}$$

Since the image of the right vertical arrow must be finite, it follows that $\chi(k(x_i)^x)$ is finite. Summing over $1 \leq i \leq r$, we get that $\chi(F_1)$ is finite. This proves the claim.

We let $D'_\text{red} = \{y_1, \ldots, y_s\}$ and $D' = \sum_i m_i[y_i]$. We let $F' = F'_1 \times F'_2$, where $F'_1$ is the cokernel of the diagonal inclusion $k^x \hookrightarrow \prod_{i=1}^s k(y_i)^x$ and $F'_2 = \prod_{i=1}^s \mathbb{W}_{m_i-1}(k(y_i))$. We then have an exact sequence similar to (11.3). Since $F'_2$ is a torsion group of bounded exponent, $\chi'(F'_2)$ must be finite. On the other hand, each $\chi'(k(y_i)^x)$ is finite by Lemma 10.7. Summing over $1 \leq i \leq s$, we get that $\chi'(F'_1)$ is finite. We deduce that $\chi'(F')$ is finite.

Finally, we look at the commutative diagram of exact sequences

$$0 \to F' \to \text{Pic}^0(X'|D') \xrightarrow{\alpha} \text{Pic}^0(X') \to 0 \tag{11.6}$$

It follows from Corollary 6.10 that $\alpha$ is a quotient map. Since $m' \circ \chi'$ is continuous, it follows that $\overline{\chi'}$ is a continuous homomorphism. On the other hand, $\text{Pic}^0(X')$ is profinite. It follows that the image of $\overline{\chi'}$ is finite. We conclude that the image of
Pic\(^0\)(\(X'|D'\)) under \(\chi'\) is finite. We have thus shown that \(\chi' \in \text{Pic}^0(X'|D')^*\). This proves (2) and concludes the proof of the lemma.

12. Perfectness of Brauer–Manin pairing for modulus pairs

The goal of this section is to prove the perfectness of the Brauer–Manin pairing for a one-dimensional modulus pair. We shall also prove Theorem 1.1 using the perfectness of the Brauer–Manin pairing. We let \(k\) be a local field of exponential characteristic \(p \geq 1\). Let \(X\) be a geometrically integral smooth projective curve over \(k\) and let \(D \subset X\) be an effective divisor. We let \(j: X^\circ \to X\) be the inclusion of the complement of \(D\). We write \(D = \sum_i n_i[x_i]\), where \(D^\dagger = \{x_1, \ldots, x_r\}\) is the support of \(D\) with reduced closed subscheme structure. We let \(K\) denote the function field of \(X\). For \(x \in X_0\), we let \(X_x = \text{Spec}(\mathcal{O}_{X,x})\) and \(X^h_x = \text{Spec}(\mathcal{O}_{X,x}^h)\).

12.1. Proof of Theorem 1.6. By Proposition 11.2 the Brauer–Manin pairing with modulus (cf. Proposition 9.3) induces continuous homomorphisms \(\beta_{X|D}\): \(\text{Br}(X|D) \to \text{CH}_0(X|D)^*\) and \(\beta_{X^\circ}\): \(\text{Br}(X^\circ) \to \text{CH}(X^\circ)^*\). We begin by showing the injectivity of these homomorphisms.

Lemma 12.1. \(\beta_{X|D}\) is a monomorphism.

Proof. Let \(w \in \text{Br}(X|D)\) be such that \(\chi_w := \beta_{X|D}(w) = 0\). For \(x \in X_0\), let \(\iota_x: \text{Spec}(k(x)) \to X\) be the inclusion. We let \(w_x\) be the image of \(w\) under the canonical composite map \(\text{Br}(X^\circ) \to \text{Br}(K) \to \text{Br}(K_x)\). Let \(\chi^1_w: \mathbb{Z}_0(X^\circ) \to \mathbb{Q}/\mathbb{Z}\) and \(\chi^2_w: K_x^\times \to \mathbb{Q}/\mathbb{Z}\) be the two components of \(\chi_w\) (see §9.2). Our assumption implies that \(\chi^i_w = 0\) for \(i = 1, 2\).

Suppose \(x \in D\). Since \(\chi^2_w = 0\), it follows from [19] §6, Thm. 1] that \(w_x = 0\). The commutative diagram of exact sequences

\[
\begin{align*}
0 & \to \text{Br}(X_x) \to \text{Br}(K) \to H^3_x(X_x, \mathcal{O}^\times_{X_x}) \\
0 & \to \text{Br}(X^h_x) \to \text{Br}(K_x) \to H^3_x(X^h_x, \mathcal{O}^\times_{X^h_x})
\end{align*}
\]

implies that as an element of \(\text{Br}(K_x)\), the Brauer class \(w\) lies in \(\text{Br}(X_x)\) and it dies in \(\text{Br}(X^h_x)\). In particular, \(\iota^*_x(w) = 0\). Since \(\chi^1_w = 0\), we also have \(\iota^*_x(w) = 0\) for every \(x \in X^o_0\). It follows that \(w\) is a class in \(\text{Br}(X)\) such that \(\iota^*_x(w) = 0\) for \(x \in X_0\). We conclude from [41] Thm. 9.2] that \(w = 0\). This finishes the proof.

Lemma 12.2. We have the following.

1. If \(p > 1\), the map \(\beta_{X|D}: \text{Br}(X|D) \to \text{CH}_0(X|D)^*\) is an isomorphism.
2. If \(p = 1\), the map \(\beta_{X|D}: \text{Br}(X|D) \to (\text{CH}_0(X|D)^*)_{\text{tor}}\) is an isomorphism.

Proof. In view of Lemma 12.1 we only need to show that \(\beta_{X|D}\) is surjective. We first prove this surjectivity under the assumption that \(D^\dagger \subset X(k)\). It follows from Lemma 8.8 that \(\text{Br}(X|D^\dagger) = \text{Br}(X|D) = \text{Br}(X^\circ)\) when \(p = 1\). Using Lemma 11.4 we can assume in this case that \(D\) is reduced. Hence, \(D\) will be assumed to be reduced (and \(n = 1\)) in the following argument when \(p = 1\).
We let \( G^*(X^o) = \lim \to_n \text{CH}_0(X|nD)^* \). We shall first show that the map \( \beta_{X^o}: \text{Br}(X^o) \to G^*(X^o) \) is surjective. We consider the diagram

\[
\begin{array}{ccccccccc}
0 & \to & \text{Br}(X) & \xrightarrow{j^*} & \text{Br}(X^o) & \xrightarrow{\partial_X} & \bigoplus_{x \in D} H^3_x(X, \mathcal{O}^x_X) & \xrightarrow{\eta_{X|D}} & H^3(X, \mathcal{O}^x_X) & \to & 0 \\
& & \downarrow{\beta_X} & & \downarrow{\beta_{X^o}} & & \downarrow{\gamma_{X|D}} & & \downarrow{\delta_X} & & \\
0 & \to & \text{CH}_0(X)^* & \xrightarrow{i^*} & G^*(X^o) & \xrightarrow{\partial^*} & \bigoplus_{x \in D} H^0(X^o_x, \mathcal{O}^x_{X^o}) & \xrightarrow{(t^*)^*} & (k^*)^* & \to & 0,
\end{array}
\]

where \( \gamma_{X|D} \) is the direct sum of maps \( \gamma_x \) given by Lemma \[10.5\] and \( \delta_X \) is given by Lemma \[10.5\]. The top row is the localization sequence for the étale sheaf \( \mathcal{O}^x_X \) and hence is exact except that we need to explain the surjectivity of \( \beta_{X^o} \).

Using Definition \[8.9\], this forces

\[
\text{(12.2)} \quad 0 \to \text{Br}(X) \xrightarrow{j^*} \text{Br}(X^o) \xrightarrow{\partial_X} \bigoplus_{x \in D} H^3_x(X, \mathcal{O}^x_X) \xrightarrow{\eta_{X|D}} H^3(X, \mathcal{O}^x_X) \to 0,
\]

which is commutative by the definition of \( \gamma_{X|D} \). Taking the limit over \( n \), we see that the middle square in \( (12.2) \) commutes.

The left square in \( (12.2) \) is commutative by Corollary \[9.5\] and the right square is commutative by Lemma \[10.6\]. To show the commutativity of the middle square, we look at the diagram

\[
\begin{array}{cccccc}
\text{Br}(X|nD) & \xrightarrow{\partial_{X|nD}} & \bigoplus_{x \in D} \text{Br}(K_x) & \xrightarrow{\Sigma_x \gamma_K} & \text{CH}_0(X|nD)^* & \xrightarrow{\partial^*} & \bigoplus_{x \in D} (K_x^*)^*
\end{array}
\]

for \( n \geq 1 \), where the dual of \( K_x^* \) is taken with respect to its Kato topology. This diagram is commutative by the definition of \( \beta_{X|nD} \), where \( \gamma_K \) is as in \[10.1\]. When we restrict \( \beta_{X^o} \) to \( \text{Br}(X|nD) \), the middle square in \( (12.2) \) commutes because it is the composition of \( (12.3) \) with the sum (over \( x \in D \)) of right squares in \[10.1\]. Taking the limit over \( n \geq 1 \), we see that the middle square in \( (12.2) \) commutes.

To show that \( \beta_{X^o} \) is an isomorphism, we note that \( \beta_X \) is an isomorphism by \[31\] Thm. 4 and \[44\] Thm. 9.2. The arrow \( \gamma_{X|D} \) is an isomorphism by Lemma \[10.2\] and \( \delta_X \) is an isomorphism by Theorem \[10.8\]. An easy diagram chase shows that the bottom row of \( (12.2) \) is exact and \( \beta_{X^o} \) is an isomorphism. This also proves that \( \beta_{X|D} \) is surjective when \( p = 1 \). To pass from \( \beta_{X^o} \) to \( \beta_{X|D} \) when \( p > 1 \), we fix a character \( \chi \in \text{CH}_0(X|D)^* \). It follows from the surjectivity of \( \beta_{X^o} \) that there exists a class \( w \in \text{Br}(X^o) = \bigcup_n \text{Br}(X|nD) \) such that \( \chi = \chi_w := \beta_{X^o}(w) \). It follows from \[9.2\] and the definition of \( \beta_{X|D} \) that the map \( \chi_w: K_x^* \to \mathbb{Q}/\mathbb{Z} \) annihilates \( I(X|D) \).

Using Definition \[8.9\] this forces \( w \) to lie in \( \text{Br}(X|D) \). In this case, we must have \( \chi = \beta_{X|D}(w) \). This finishes the proof of surjectivity of \( \beta_{X|D} \) when \( D^1 \subset X(k) \).

To prove the general case, we choose a finite field extension \( k'/k \) such that \( \text{Supp}(D_{k'}) \subset X(k') \). We let \( X' = X_{k'} \) and \( D' = D_{k'} \). We let \( f: (X', D') \to (X, D) \) be the projection and consider the diagram

\[
\begin{array}{cccc}
\text{Br}(X'|D') & \xrightarrow{\beta_{X'|D'}} & \text{CH}_0(X'|D') & \xrightarrow{(f^*)^*} \\
\downarrow{f_*} & & \downarrow{f_*} & \\
\text{Br}(X|D) & \xrightarrow{\beta_{X|D}} & \text{CH}_0(X|D)^* & 
\end{array}
\]
This diagram is commutative by Lemma 9.7. We have shown above that $\beta_{X'|D'}$ is surjective, and Lemma 11.3 says that the right vertical arrow is surjective. It follows that $\beta_{X|D}$ is surjective. This concludes the proof.

\[\text{End of the proof of Theorem 1.6. Combine Lemmas 9.7, 11.3 and Proposition 11.2.}\]

As an application of Theorem 1.6 we get the following result about the norm map between the Brauer groups of regular curves over local fields.

**Theorem 12.3.** Let $X$ be a geometrically connected regular quasi-projective curve over a local field $k$ and let $k'$ be a finite field extension of $k$. Let $f: X_{k'} \to X$ be the projection. Assume that $X$ is either smooth or affine. Then the map $f_*: \text{Br}(X_{k'}) \to \text{Br}(X)$ is surjective.

**Proof.** We choose an open embedding $j: X \to \overline{X}$, where $\overline{X}$ is a connected regular projective curve. Then $\overline{X}$ is necessarily geometrically connected. We let $D = \overline{X} \setminus X$ with the reduced closed subscheme structure. We let $\overline{X}' = \overline{X}_{k'}$, $X' = X_{k'}$, and $D' = D_{k'}$. If $\overline{X}$ is smooth (e.g., when $p = 1$), we conclude the proof by combining Proposition 8.11, Lemma 11.3 and Theorem 1.6. We shall now assume that $p > 1$ and $X$ is affine.

If $k'/k$ is purely inseparable, then $\text{Coker}(f_*)$ is annihilated by some power of $p$ (e.g., see [4, §3.8]). On the other hand, $\text{Coker}(f_*)$ is $p$-divisible by [4, Thm. 3.2.3]. It follows that $\text{Coker}(f_*) = 0$. In general, we can get a factorization $X_{k'} \to X_{k''} \to X$, where $k''/k$ is separable and $k'/k''$ is purely inseparable. Since $X_{k''}$ is a geometrically connected regular affine curve, the map $\text{Br}(X_{k'}) \to \text{Br}(X_{k''})$ is surjective. We can thus assume that $k'/k$ is separable.

As in the proof of Proposition 11.2 we can find a finite field extension $l/k$, a geometrically connected smooth projective curve $\overline{Y}$ over $l$, and a finite surjective morphism of $k$-schemes $g: \overline{Y} \to \overline{X}$ such that $\text{deg}(g)$ and $[l : k]$ are some powers of $p$. We let $Y = g^{-1}(X)$ and $E = g^*(D)$. We let $A = l \otimes_k k'$ and $\overline{Y}_A = \overline{Y} \times_{\text{Spec}(l)} \text{Spec}(A)$. Since $k'/k$ is separable, we see that $A = \prod_{i=1}^r l_i$, where each $l_i/l$ is a finite separable field extension.

We consider the diagrams

\[
\begin{array}{ccc}
\prod_{i=1}^r Y_{l_i} & \xrightarrow{g'} & \overline{X}' \\
\downarrow h & & \downarrow f \\
Y & \xrightarrow{g} & X \\
\end{array}
\quad
\begin{array}{ccc}
\text{Br}(Y_{l_i}) & \xrightarrow{g'_*} & \text{Br}(X') \\
\downarrow h_* & & \downarrow f_* \\
\text{Br}(Y) & \xrightarrow{g_*} & \text{Br}(X), \\
\end{array}
\]

where the left square is Cartesian whose all arrows are finite. The right square is commutative by Proposition 8.11.

We now proceed as follows. Since $\overline{Y}$ is a geometrically connected smooth projective curve over $l$, we have argued previously that $h_*: \text{Br}(Y_{l_i}) \to \text{Br}(Y)$ is surjective for each $i$. Since $\text{deg}(g) = p^n$ for some $n \geq 1$, we have also seen previously that $g_*: \text{Br}(Y) \to \text{Br}(X)$ is surjective. In particular, $f_* \circ g'_* = g_* \circ h_*$ is surjective. It follows that $f_*$ is surjective. This concludes the proof.
12.2. Proof of Theorem 1.1. We let the notations and assumptions be as stated in the beginning of §12. We shall divide the proof of Theorem 1.1 into several cases. We begin with the following case. Recall that

\[ H^2_{cc}(X, \mathbb{G}_m) = \lim_{n} H^3(X, \mathbb{G}_m(X, nD)) \cong \lim_{n} H^2(X, \mathcal{K}_M^{1}(X, nD)). \]

**Lemma 12.4.** One has \( H^0_{cc}(X^o, \mathcal{O}_{X^o}^*) = H^i(X^o, \mathcal{O}_{X^o}^*) = 0 \) for all \( i \geq 3 \).

**Proof.** It is easy to see that \( H^0(X, \mathcal{K}_M^{1}(X, nD)) = 0 \) for every \( n \geq 1 \). In particular, \( H^0_{cc}(X^o, \mathcal{O}_{X^o}^*) = 0 \). To show that \( H^i(X^o, \mathcal{O}_{X^o}^*) = 0 \) for \( i \geq 3 \) is equivalent to show that \( H^3(X^o, \mathcal{O}_{X^o}^*)_{\text{tor}} = 0 \) for all \( i \geq 3 \) (e.g., see [14, Lem. 3.5.3]). It already follows from Remark 10.9 that \( H^i(X^o, \mathcal{O}_{X^o}^*)\{p\} = 0 \) for all \( i \geq 3 \).

We now show the prime-to-\( p \) case. For \( i \geq 4 \), we have a surjection

\[ H^i(X^o, \mathbb{Q}/\mathbb{Z}\{p\}^*(1)) \rightarrow H^i(X^o, \mathcal{O}_{X^o}^*)\{p\}. \]

On the other hand, the term on the left of this surjection is zero because \( cd(X^o) \leq 3 \). For \( i = 3 \), note that there is an inclusion \( H^3(X^o, \mathcal{O}_{X^o}^*)\{p\} \rightarrow H^3_D(X, \mathcal{O}_{X^o}^*)\{p\} \) by Corollary 10.10. On the other hand, there is a canonical surjection

\[ H^3_D(X, \mathbb{Q}/\mathbb{Z}\{p\}^*(1)) \rightarrow H^3_D(X, \mathcal{O}_{X^o}^*)\{p\}. \]

Using Gabber’s purity, the term on the left of this surjection is isomorphic to

\[ \bigoplus_{x \in D} H^3(k(x), \mathbb{Q}/\mathbb{Z}\{p\}). \]

We are thus reduced to showing that \( H^2(k', \mathbb{Q}/\mathbb{Z}\{p\}) = 0 \) for \( i \geq 2 \) if \( k'/k \) is any finite field extension. By [14, Thm. 9.9], we have an isomorphism \( H^2(k', \mathbb{Q}/\mathbb{Z}\{p\}) \cong \lim_{\rightarrow p^m} \text{Hom}_{\text{Ab}}(\mu_n(k'), \mathbb{Q}/\mathbb{Z}) \), where the limit is taken with respect to the maps \( \mu_{mn}(k') \rightarrow \mu_n(k') \). But this limit is zero since \( (k'^n)\{p\} \) is finite. This concludes the proof. \( \square \)

Recall that there is an exact sequence

\[ Z^r \rightarrow \text{CH}_0(X) \xrightarrow{j^*} \text{CH}_0(X^o) \rightarrow 0, \]

where the first arrow is the sum of cycle class maps for points in \( D \). We endow \( \text{CH}_0(X^o) \) with the quotient of the adic topology of \( \text{CH}_0(X) \) via this exact sequence. Note that this topology of \( \text{CH}_0(X^o) \) is independent of any choice of a regular compactification of \( X^o \). We shall consider \( H^2_{cc}(X^o, \mathcal{O}_{X^o}^*) \) to be a discrete abelian group.

**Lemma 12.5.** There is a perfect pairing of topological abelian groups

\[ \text{CH}_0(X^o)^{\text{pf}} \times H^2_{cc}(X^o, \mathcal{O}_{X^o}^*) \rightarrow \mathbb{Q}/\mathbb{Z}. \]

**Proof.** We look at the diagram

\[ \begin{array}{cccccc}
0 & \rightarrow & H^2_{cc}(X, \mathcal{O}_{X^o}^*) & \rightarrow & \text{Br}(X) & \rightarrow & \text{Br}(D^1) \\
& & \downarrow{\beta_X} & & \downarrow{\beta_{D^1}} & & \\
0 & \rightarrow & \text{CH}_0(X^o)^* & \xrightarrow{(j^*)^*} & \text{CH}_0(X)^* & \rightarrow & (\mathbb{Q}/\mathbb{Z})^r,
\end{array} \]

where the right vertical arrow is the sum (over \( x \in D^1 \)) of the maps \( inv_{k(x)} \). It follows from the exactness of the top row that the canonical map \( H^2_{cc}(X, \mathcal{O}_{X^o}^*) \rightarrow H^2(X, \mathcal{K}_M^{1}(X, nD)) \) is an isomorphism.
It is immediate from the construction of the Brauer–Manin pairing (9.5) (with \( D = \emptyset \)) that the right square in (12.7) is commutative. The middle and the right vertical arrows are isomorphisms. It follows that there is a unique isomorphism \( \beta^1_{X^0}: H^2_{cc}(X^0, O_{X^0}) \rightarrow \text{CH}_0(X^0)^* \) such that (12.7) is commutative. Since \( \beta^1_{X^0} \) is automatically continuous, we get a continuous pairing \( \text{CH}_0(X^0)^f \times H^2_{cc}(X^0, O_{X^0}) \rightarrow \mathbb{Q}/\mathbb{Z} \). Since \( \text{CH}_0(X^0)^* \) is a torsion group, \( \text{CH}_0(X^0)^* \) must also be torsion. We can now apply Lemma 12.5 and the Pontryagin duality between profinite and discrete torsion groups to conclude the proof.

\[ \square \]

**End of the proof of Theorem 1.1** Combine Theorem 1.6 with Lemmas 2.5, 8.8, 12.4 and Lemma 12.5

**Corollary 12.6.** There is a perfect pairing of topological abelian groups

\[ \text{Br}(X^0) \times C(X^0)^f \rightarrow \mathbb{Q}/\mathbb{Z}. \]

**Proof.** Combine Theorem 1.4 and Lemma 11.1

\[ \square \]

**12.3. Duality for the cohomology of \( \mathbb{G}_{m,K} \).** Let the notations and assumptions be as stated in the beginning of §12. Recall that \( K \) denotes the function field of \( X \). We let \( H^q_{cc}(K, \mathbb{G}_m) := \lim_{\rightarrow} H^q(X, \mathbb{G}_m(X,D)) \). We let \( C(K) \) be the cokernel of the canonical map \( K^\times \rightarrow \hat{\text{Pic}}(X) := \prod_{x \in X^{(0)}} \hat{K}_x^\times \), where \( \alpha \) is the canonical inclusion (cf. (9.9)). We endow \( H^q_{cc}(K, \mathbb{G}_m) \) with the inverse limit of the adic topologies of \( \text{Pic}(X|D) \) for \( D \in \text{Div}(X) \), and \( \text{Br}(K) \) with the discrete topology.

To prove the duality theorem for the étale cohomology of \( \mathbb{G}_{m,K} \), we need the following.

**Lemma 12.7.** For every effective divisor \( D \subset X \) with complement \( X^0 \), the canonical maps \( H^1_{cc}(K, \mathbb{G}_m) \rightarrow H^1_{cc}(X^0, \mathbb{G}_m) \rightarrow H^1(X, \mathbb{G}_m(X,D)) \) are surjective.

**Proof.** Using Lemma 11.1 it suffices to show that for every reduced effective divisor \( D \subset X \) with \( X^0 = X \setminus D \), the map \( C(K) \rightarrow C(X^0) \) is surjective, and the map \( C(K) \rightarrow \lim_{\leftarrow} C(X^0) \) is bijective, where the limit is over all reduced effective divisors \( D \subset X \).

We have a commutative diagram of exact sequences

\[ (12.8) \quad 0 \rightarrow K^\times \rightarrow \alpha \rightarrow \hat{\text{Pic}}(X) \rightarrow C(K) \rightarrow 0 \]

where we let \( B(D) := \hat{K}_x^\times = \prod_{x \in D} \hat{K}_x^\times \). We let \( A(D) = \mathbb{Z}_0(X^0) \oplus B(D) \). It is straightforward to check that each \( \beta_D \) is surjective. In particular, each transition map of the cofiltered system \( \{ A(D) \} \) is surjective. It remains to show that the map \( \beta := \lim_{\leftarrow} \beta_D: \hat{\text{Pic}}(X) \rightarrow \lim_{\leftarrow} A(D) \) is bijective.

It is easy to see that \( \prod_{x \in X^{(0)}} \hat{K}_x^\times \rightarrow \lim_{\leftarrow} B(D) \). We let \( \pi_D: A(D) \rightarrow B(D) \) denote the projection. For \( D' > D \), the transition map \( \Phi_{D' > D}: A(D') \rightarrow A(D) \) has the property that if \( (a_1, a_2, a_3) \in A(D') \) with \( a_1 \in \mathbb{Z}_0(X \setminus D') \), \( a_2 \in \prod_{x \in D} \hat{K}_x^\times \) and \( a_3 \in \prod_{x \in D \cap D'} \hat{K}_x^\times \), then

\[ (12.9) \quad \Phi_{D' > D}(a_1) = a_1, \quad \Phi_{D' > D}(a_2) = a_2 \quad \text{and} \quad \Phi_{D' > D}(a_3) = \prod_{x \in D \cap D'} v_x(a_3). \]
It follows that $\pi_D \circ \Phi_{D,D'} = \Phi_{D',D} \circ \pi_{D'}$. That is, the projection $\{\pi_D\}: \{A(D)\} \to \{B(D)\}$ is a strict morphism of pro-abelian groups. Taking the limits, we get a commutative diagram

\[
\begin{array}{ccc}
\lim_D A(D) & \xrightarrow{\pi} & \prod_{x \in X(0)} \hat{R}_x^* \\
\beta & \downarrow & \\
\hat{T}(X),
\end{array}
\]

where $\beta'$ is the canonical inclusion $\hat{T}(X) \to \prod_{x \in X(0)} \hat{R}_x^*$. In particular, $\beta$ is injective.

We claim that $\pi$ is injective. Indeed, if $(a_D) \in \lim_D A(D)$ is an element indexed by reduced divisors $D \subset X$ with $(a_D) = (a_0, a_1)$ (where $a_0 \in Z_0(X \setminus D)$ and $a_1 \in B(D)$) such that $\pi((a_D)) = 0$, then we must have $a_1 = 0$ for each reduced divisor $D \subset X$. That is, $(a_D) \in \lim_D Z_0(X \setminus D) = \cap_D Z_0(X \setminus D)$. The claim now follows because it is not hard to see that $\cap_D Z_0(X \setminus D) = 0$.

To prove that $\beta$ is surjective, let $(a_D) \in \lim_D A(D) \subset \prod_{x \in X(0)} \hat{R}_x^*$. Let $b = \pi((a_D))$ and write $b = (b_x)_{x \in X(0)}$. If $v_x(b_x) \neq 0$ for infinitely many $x \in X(0)$, then it follows from \eqref{12.9} that the projection map $\lim_D A(D) \to A(D)$ is not defined at $b$ for any reduced divisor $D \subset X$. But this is absurd. We conclude that $b = \pi((a_D)) \in \hat{T}(X)$. This shows that $\beta$ is surjective, and concludes the proof.

Using Lemma \ref{12.7} we endow $C(K)$ with the inverse limit of the adic topologies of $\text{Pic}(X|D)$ for $D \in \text{Div}(X)$. We can now prove the duality theorem for the cohomology of $\mathbb{G}_{m,K}$.

**Theorem 12.8.** For every integer $q \neq 0$, there is a bilinear pairing

\[
H^q(K, \mathbb{G}_m) \times H^{3-q}(K, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z}
\]

which induces perfect pairings of topological abelian groups

\[
H^1(K, \mathbb{G}_m)^{\text{pf}} \times H^2_{cc}(K, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z},
\]

\[
H^2(K, \mathbb{G}_m) \times H^1_{cc}(K, \mathbb{G}_m)^{\text{pf}} \to \mathbb{Q}/\mathbb{Z},
\]

and

\[
H^3(K, \mathbb{G}_m) \times H^0_{cc}(K, \mathbb{G}_m)^{\text{pf}} \to \mathbb{Q}/\mathbb{Z}.
\]

**Proof.** The existence of the pairing is an immediate consequence of Theorem \ref{1.1} and a limit argument (over $D \in \text{Div}(X)$). The perfectness of the third pairing follows from Lemma \ref{12.4} using a limit argument. The perfectness of the second pairing follows by applying Theorem \ref{1.6} Lemmas \ref{2.5} \ref{2.6} \ref{8.8} and \ref{12.7} and taking limit over $\text{Div}(X)$. To prove the perfectness of the first pairing, we only need to show that $H^2_{cc}(K, \mathbb{G}_m) = 0$. But this follows by taking limit over $\text{Div}(X)$ of the top exact sequence in \eqref{12.7} and observing that the map $\text{Br}(X) \to \prod_{x \in X(0)} \text{Br}(k(x))$ is injective by Lemma \ref{12.1} (with $D = \emptyset$).

By \eqref{12.9}, we have a bilinear pairing $\text{Br}(K) \times \hat{T}(X) \to \mathbb{Q}/\mathbb{Z}$, and the proof of Proposition \ref{9.4} shows that this factors through $\text{Br}(K) \times C(K) \to \mathbb{Q}/\mathbb{Z}$. The following result is the global version of \cite[Thm. 2.10, Cor. 2.12]{44} and provides an explicit description of $\text{Br}(K)$.
Corollary 12.9. The above pairing induces a perfect pairing of topological abelian groups

$$\text{Br}(K) \times C(K)^{\text{pf}} \to \mathbb{Q}/\mathbb{Z}.$$  

In particular, \(\text{Br}(K) \cong (C(K)^*)_{\text{tor}}\).

13. BRAUER–MANIN PAIRING FOR SINGULAR CURVES

In this section, we shall apply Theorem [1.6.6] to extend the Brauer–Manin pairing for smooth projective curves à la Lichtenbaum–Saito to singular curves over local fields. To achieve this, we shall introduce a refined version of the Brauer group for singular varieties. We shall then show that this refined group is directly related to the Picard group for singular curves. We fix a local field \(k\) of exponential characteristic \(p \geq 1\).

13.1. Levine-Weibel Brauer group of singular varieties. Before we introduce the refined Brauer group for singular quasi-projective varieties, we recall some known facts about the completions of local rings. For any semilocal ring \(R\), let \(\hat{R}\) denote the completion of \(R\) with respect to its Jacobson radical. Let \(A\) be a local integral domain which is essentially of finite type over a field and let \(B\) denote the integral closure of \(A\). We let \(\mathfrak{m}_A\) (resp. \(\mathfrak{m}_B\)) denote the maximal ideal of \(A\) (resp. the Jacobson radical of \(B\)). It is an elementary fact (e.g., see [1, Chap. 10]) that \(\hat{A} \twoheadrightarrow \hat{B} \cong B \otimes_A \hat{A} \cong \prod_i \hat{B}_{\mathfrak{m}_i}\), where the product is over all maximal ideals of \(B\). Furthermore, \(\hat{B}\) coincides with the normalization of \(\hat{A}\) if \(\dim(A) = 1\). In particular, it is a product of discrete valuation rings \(\hat{B}_{\mathfrak{m}_i}\). In the latter case, it is also easy to see using the faithfully flatness property of \(A \to \hat{A}\) that \(\hat{A} \cap K = A\), where \(K\) is the quotient field of \(A\).

We let \(X\) be an integral quasi-projective curve over \(k\). Let \(\pi: X' \to X\) be the normalization of \(X\) and let \(K\) denote the function field of \(X\). For every closed point \(x \in X\) and every \(y \in \pi^{-1}(x)\), let \(U^0_{x,y} \subset \hat{K}_y\) be the image of \((\hat{O}_{X,x})^*\) under the composite map \(\hat{O}_{X,x} \to \hat{O}_{X',y} \to \hat{K}_y\).

Definition 13.1. We let \(\text{Br}^{lw}(X)\) be the subgroup of \(\text{Br}(X_{\text{reg}})\) consisting of elements \(w\) such that for every singular point \(x \in X\) and every \(y \in \pi^{-1}(x)\), the image of \(w\) under the canonical map \(\text{Br}(X_{\text{reg}}) \to \text{Br}(K) \to \text{Br}(\hat{K}_y)\) has the property that the associated character (under Kato’s pairing) \(\chi_w: \hat{K}_y^* \to \mathbb{Q}/\mathbb{Z}\) annihilates \(U^0_{x,y}\).

Let \(X\) be an integral quasi-projective variety over \(k\). Let \(C^{l.c.i.}(X)\) be the set of all integral curves \(C \subset X\) such that \(C\) is not contained in \(X_{\text{sing}}\) and the inclusion \(C \to X\) is a local complete intersection at every point of \(C \cap X_{\text{sing}}\).

Definition 13.2. We let \(\text{Br}^{lw}(X)\) be the subgroup of \(\text{Br}(X_{\text{reg}})\) consisting of elements \(w\) such that for every \(C \in C^{l.c.i.}(X), x \in X_{\text{sing}} \cap C\) and \(y \in \nu^{-1}(x)\), the image of \(w\) under the canonical composite map \(\text{Br}(X_{\text{reg}}) \to \text{Br}(\nu^{-1}(X_{\text{reg}})) \to \text{Br}(\hat{k(C)^*}_y)\) has the property that the associated character (under Kato’s pairing) \(\chi_w: \hat{k(C)^*}_y \to \mathbb{Q}/\mathbb{Z}\) annihilates \(U^0_{x,y}\), where \(\nu: C_n \to X\) is the canonical map.

It easily follows from Definitions [13.1] and [13.2] that \(\text{Br}^{lw}(X) \subset \text{Br}^{lw}(U)\) if \(U \subset X\) is dense open. This is in contrast with the classical Brauer group of singular varieties. It is also clear that \(\text{Br}^{lw}(X) = \text{Br}(X)\) if \(X\) is regular. We shall refer to \(\text{Br}^{lw}(X)\) as the ‘Levine-Weibel Brauer group’ of \(X\).
13.2. The Brauer–Manin pairing. For the rest of §13 we shall work with the following setup. Let $X$ be a geometrically integral projective curve over $k$ and $\pi: X_n \to X$ the normalization of $X$. It is clear that $X_n$ is geometrically integral. We shall assume that $X_n$ is smooth over $k$ (this is automatic if $p = 1$). We let $K$ denote the function field of $X$. We let $S$ be the singular locus of $X$ with the reduced closed subscheme structure and let $E$ be the scheme theoretic inverse image of $S$ in $X_n$. We let $X^o = X_{\text{reg}}$ be the regular locus of $X$. We represent this datum in the Cartesian squares

$$
\begin{array}{c}
E \ar{r}{i'} \ar{d}{\pi'} & X_n \ar{d}{\pi} & X^o \\
S \ar{r}{i} & X & X^o.
\end{array}
$$

We have recalled the definition of the Levine-Weibel Chow group $CH^l_w(X)$ and the isomorphism $\text{cyc}_X: CH^l_w(X) \cong \text{Pic}(X)$ after Lemma 5.4. We let $CH^l_w(X)_0 = \text{Ker}(\text{deg}: CH^l_w(X) \to \mathbb{Z}) \cong \text{Pic}^0(X)$. We shall use $\text{Pic}(X)$ and $CH^l_w(X)$ interchangeably. In order to construct a duality between the Levine-Weibel Brauer group and the Picard group of $X$, we need to give a new description of the latter group.

For a point $x \in S$, we let $E_x = \pi^{-1}(x)$ and $R_{x,\infty} = \prod_{y \in E_x} R_y$. We let $U_{x,\infty}^0 = \prod_{y \in E_x} U_{x,y}^0$. We have the canonical maps $K^x \hookrightarrow \overline{R}_{x,\infty}^x \rightarrow \overline{R}_{x,\infty}^x / U_{x,\infty}^0$. We let $\delta_x$ be the composite map. Let $\delta = \prod_{y \in E_x} \delta_y$.

**Lemma 13.3.** There is a canonical exact sequence

$$
0 \to k^x \to K^x \xrightarrow{(\text{div},\delta)} Z_0(X^o) \bigoplus \left( \prod_{x \in S} \overline{R}_{x,\infty}^x / U_{x,\infty}^0 \right) \to CH^l_w(X) \to 0.
$$

**Proof.** We consider the diagram

$$
\begin{array}{ccc}
\mathcal{O}_{X,S} \ar{r}{\text{div}} \ar{d} & Z_0(X^o) \ar{d} \ar{r} & CH^l_w(X) \ar{d} \ar{r} & 0 \\
K^x \ar{r} & Z_0(X^o) \bigoplus \left( \prod_{x \in S} \overline{R}_{x,\infty}^x / U_{x,\infty}^0 \right) \ar{r} & CH^l_w(X) \ar{r} & 0,
\end{array}
$$

where the vertical arrows are the canonical inclusions and $\overline{CH^l_w(X)}$ is defined to make the bottom row exact. The top row is exact by definition of $CH^l_w(X)$. It is clear that the left square is commutative. This yields a unique homomorphism $\psi_X: CH^l_w(X) \to \overline{CH^l_w(X)}$ such that the right square also commutes.

To prove that $\psi_X$ is bijective, it is equivalent to show that

$$
0 \to \mathcal{O}_{X,S} \ar{r} & K^x \ar{r}{\delta} \ar@{..>}[d] & \prod_{x \in S} \overline{R}_{x,\infty}^x / U_{x,\infty}^0 \ar{r} & 0
$$

is exact. Now, it is clear that if $f \in K^x$ is such that $\delta_x(f) = 0$ for every $x \in S$, then $f \in \bigcap_{x \in S} \mathcal{O}_{X,x}$ (see the first paragraph of §13.1). But it is not hard to see that
Let \( \mathcal{O}^*_X = \bigcap_{x \in S} \mathcal{O}^*_X, \) using the fact that \( \mathcal{O}^*_X, \) is the set of rational functions on \( X \) which are regular without a zero or a pole in a neighborhood of \( x. \) It remains to show that \( \delta \) is surjective.

We let \( I_x \subset \mathcal{O}^*_X, \) be a conductor ideal for the normalization \( \mathcal{O}^*_X \to \mathcal{O}^*_X, E_x = \prod_{y \in E_x} \mathcal{O}^*_X, y. \) Then \( I_x \to I_x \mathcal{O}^*_X, y \subset \mathfrak{m}_y \) for every \( y \in E_x, \) where the latter is the maximal ideal of \( \mathcal{O}^*_X, y. \) It is also clear that \( \sqrt{I_x \mathcal{O}^*_X, y} = \mathfrak{m}_y. \) As \( I_x \to \prod_{y \in E_x} I_x \mathcal{O}^*_X, y, \) it follows that \( (1 + I_x) \to \prod_{y \in E_x} (1 + I_x \mathcal{O}^*_X, y). \) In particular, \( \prod_{y \in E_x} (1 + \mathfrak{m}_y) \subset U^0_{x, \infty} \) for all \( n \gg 0. \) In other words, we have a factorization

\[
(13.5) \quad K^x \to \prod_{x \in S} \widehat{\mathcal{R}}^x_{\infty, \infty} \to \prod_{x \in S} \widehat{U}^0_{x, \infty}
\]

for all \( n \gg 0, \) where we let \( U^0_{x, \infty} = \prod_{y \in E_x} (1 + \mathfrak{m}_y)^n. \) We are now done because the left arrow is well known to be surjective as an application of the approximation lemma in number theory (e.g., see [27, Lem. 6.3]). This proves the exactness of (13.4).

To conclude the proof of the lemma, it remains to show that \( k^x = \ker(\text{div}) \cap \ker(\delta). \) But this is easy because we have shown that \( \ker(\text{div}) \cap \ker(\delta) = \ker(\text{div}) \cap \mathcal{O}^*_X, S \) and the latter is \( k^x \) because \( X \) is geometrically integral. \( \square \)

We now define the Brauer–Manin pairing for \( X \) as follows. Let \( w \in \text{Br}^{lw}(X). \) We define the associated character \( \chi_w: Z_0(X^o) \oplus \bigoplus_{x \in S} \widehat{\mathcal{R}}^x_{\infty, \infty} \to \mathbb{Q}/\mathbb{Z} \) exactly as we did to define the pairing (9.4) in 9.2. It follows from the definition of \( \text{Br}^{lw}(X) \) that \( \chi_w \) factors through \( Z_0(X^o) \oplus \bigoplus_{x \in S} \widehat{\mathcal{R}}^x_{\infty, \infty}. \) The proof of the claim that \( \chi_w \circ \text{div}, \delta = 0 \) is identical to the proof of Proposition 9.4 mutatis mutandis. We have thus shown the following.

**Proposition 13.4.** There exists a canonical bilinear pairing

\[
(13.6) \quad \text{Br}^{lw}(X) \times \text{CH}^{lw}_0(X) \to \mathbb{Q}/\mathbb{Z}.
\]

13.3. **Perfectness of the pairing.** We now state and prove the main result of 11.3. We shall continue to work with the setup described in 13.2. Since \( X \) is geometrically integral, it follows from [25, Thm. 9.2.5, 9.4.8] that there is a group scheme \( \text{Pic}(X) \) over \( k \) which is locally of finite type and whose identity component \( \text{Pic}^0(X) \) is a smooth quasi-projective group scheme over \( k, \) and \( \text{Pic}(X)(k) \neq \text{Pic}(X). \) One also has that \( \text{Pic}(X_{k'}) = \text{Pic}(X)_{k'} \) for every field extension \( k'/k. \) It follows that \( \text{Pic}(X) \) is equipped with its canonical adic topology and \( \text{Pic}^0(X) \) is its open subgroup. We shall let \( \text{Br}^{lw}(X) \) have the discrete topology.

**Lemma 13.5.** The pairing (13.6) is continuous.

**Proof.** It suffices to show that the image of the induced homomorphism \( \beta_X: \text{Br}^{lw}(X) \to \text{Pic}(X)^v \) lies in \( \text{Pic}(X)^* \) (cf. 11.2). To that end, we choose a conductor ideal sheaf \( \mathcal{I}_Z \subset \mathcal{O}_X \) associated to \( \pi \) so that we have an isomorphism \( (1 + \mathcal{I}_n \pi) \to \pi_* \left((1 + \mathcal{I}_n \pi, D) \right) \) for every \( n \geq 1, \) where we let \( Z \subset X \) be the closed subscheme defined by \( \mathcal{I}_Z \) and \( D \) the scheme theoretic inverse image of \( Z \) under \( \pi. \) It follows from Corollary 6.10 that the canonical map \( \text{Pic}(X_n|D) \to \text{Pic}(X_n) \) has
a factorization $\text{Pic}(X_n|D) \twoheadrightarrow \text{Pic}(X) \twoheadrightarrow \text{Pic}(X_n)$ in which all maps are topological quotient maps between adic spaces.

We now let $w \in \text{Br}^{\text{lw}}(X)$ and let $\chi_w: \text{Pic}(X) \rightarrow \mathbb{Q}/\mathbb{Z}$ be the associated character. Since $w \in \text{Br}(X^o)$, it lies in $\text{Br}(X_n|D)$ for some conductor closed subscheme $Z \subset X$ and $D = Z \times_X X_n$ as above. It is easily seen that the composite map $\chi_w: \text{Pic}(X_n|D) \hookrightarrow \text{Pic}(X) \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$ is same as the one in (9.8) (see Remark 9.1). Proposition 11.2 then implies that this composite map is continuous. Since $\text{Pic}(X_n|D) \hookrightarrow \text{Pic}(X)$ is a topological quotient, it follows that $\chi_w$ is continuous on $\text{Pic}(X)$. This concludes the proof.

Let the notations be as in the proof of Lemma 13.5. From (13.5), one also deduces for $|D| > \pi^{-1}(S)$ the following.

**Lemma 13.6.** For all $m > 0$, we have $\text{Br}^{\text{lw}}(X) \subseteq \text{Br}(X_n|mD)$.

**Theorem 13.7.** The pairing (13.6) induces a perfect pairing of topological abelian groups

$$\text{Br}^{\text{lw}}(X) \times \text{CH}_0^{\text{lw}}(X)^{\text{pf}} \rightarrow \mathbb{Q}/\mathbb{Z}.$$ 

**Proof.** The continuity of the pairing follows from Lemmas 2.3 and 13.5. As in the proof of Theorem 1.6 it suffices to show that the map $\beta_X: \text{Br}^{\text{lw}}(X) \rightarrow F$ is bijective if we let $F = \text{CH}_0^{\text{lw}}(X)^*$ when $p > 1$ and $F = (\text{CH}_0^{\text{lw}}(X)^*)^{\text{tor}}$ when $p = 1$.

Using Lemma 13.6 we choose conductor subschemes $Z \subset X$ and $D = Z \times_X X_n \subset X_n$ such that $\text{Br}^{\text{lw}}(X) \subseteq \text{Br}(X_n|D)$. We let $G = \text{CH}_0(X_n|D)^*$ (resp. $(\text{CH}_0(X_n|D)^*)^{\text{tor}}$) if $p > 1$ (resp. $p = 1$). We consider the commutative diagram

$$(13.7) \quad \begin{array}{ccc} \text{Br}^{\text{lw}}(X) & \xrightarrow{\beta_X} & F, \\ \downarrow & & \downarrow, \\ \text{Br}(X_n|D) & \xrightarrow{\beta_{X_n|D}} & G. \end{array}$$

The right vertical arrow is injective by Corollary 6.10. Since $\beta_{X_n|D}$ is injective by Lemma 12.1 it follows that $\beta_X$ is injective. To prove its surjectivity, we let $\chi \in F$. If we consider $\chi$ as an element of $G$, then Lemma 12.2 says that $\chi = \beta_{X_n|D}(w)$ for some $w \in \text{Br}(X_n|D)$. In this case, it follows directly from the definition of $\text{Br}^{\text{lw}}(X)$ and Lemma 13.5 that $w \in \text{Br}^{\text{lw}}(X)$. This implies subsequently that $\beta_X(w) = \chi$. This concludes the proof of the theorem.

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