This product consequently denotes a function of unique character possessing all the essential properties of an ordinary theta-function.

The special case given by the formula *

$$u = x\,\Pi\,\Pi\left(1 - \frac{x}{w}\right) e^{\frac{x}{w} + \frac{1}{2}\left(\frac{x}{w}\right)^2},$$

in which

$$w = 2\mu\omega + 2\mu'\omega',$$

has been called by Weierstrass the sigma-function $\sigma(x)$, and is the basis of his beautiful theory of elliptic functions.

EARLY HISTORY OF THE POTENTIAL.

BY PROF. A. S. HATHAWAY.

THE object of the present article is to correct an error that occurs in Todhunter's "History of the Theories of Attraction" (vol. II., arts. 789, 1007, and 1138), and that is repeated, doubtless on Todhunter's authority, in various encyclopædias. This error consists in assigning to Laplace, instead of Lagrange, the honor of the introduction of the Potential into dynamics, an honor that the Encyclopædia Britannica makes the basis of a eulogy to Laplace (art. *Laplace*) in the words: "The researches of Laplace and Legendre on the subject of attractions derive additional interest and importance from having introduced two powerful engines of analysis for the treatment of physical problems, Laplace's Coefficients and the Potential Function. The expressions for the attraction of an ellipsoid involved integrations which presented insuperable difficulties; it was, therefore, with pardonable exultation that Laplace announced his discovery that the attracting force in any direction could be obtained by the direct process of differentiating a single function. He thereby translated the forces of nature into the language of analysis and laid the foundations of the mathematical sciences of heat, electricity, and magnetism."

The announcement here referred to was made by Laplace

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* BIERMANN, *Theorie der analytischen Functionen*, Leipzig, 1887, p. 334.

SCHWARZ. *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen*, Göttingen, 1885.

in the course of a memoir by Legendre between 1783 and 1785 : Encyclopædia Britannica (art. *Laplace*),—"* * * Legendre in a celebrated paper entitled *Recherches sur l'attraction des sphéroïdes homogènes,* printed in the tenth volume of the *Divers Savans,* 1783, * * * *"; Todhunter, Hist. Th. Attr., vol. II., p. 20, "A very important memoir by Legendre is contained in the tenth volume of the *Mémoires* * * * *présentés par divers Savans* * * * . The date of publication of the volume is 1785. The memoir, however, must have been communicated to the Academy at an earlier period ; for, in the treatise *De la Figure des Planètes,* which was published in 1784, Laplace refers to the researches of Legendre, which constitute the present memoir : see p. 96 of Laplace's treatise."

Todhunter continues, in art. 789 : "In this memoir we meet for the first time the function V which we now call the *Potential,* and which denotes the sum of the elements of a body divided by their distances from a fixed point. The introduction of this function Legendre expressly assigns to Laplace. The following are the circumstances :

A point is situated outside a solid of revolution. Legendre has to determine the attractions of the solid at the point, along the radius vector which joins the point to the centre of the solid, and at right angles to this direction. He has found a series for the former ; and he says the latter might be determined by similar investigations ; then he adds : '* * * * *mais on y parvient bien plus facilement à l'aide d'un Théorême que M. de la Place a bien voulu me communiquer: voici en quoi il consiste.'*

Then follows the theorem, which is enunciated and immediately demonstrated. The theorem is that the attraction along the radius vector is $-\frac{dV}{dr}$, and the attraction at right angles to the radius vector is $-\frac{dV}{rd\theta}$; where r is the radius vector and θ the angle which it makes with the axis of the solid : these attractions being estimated towards the centre, and the pole respectively."

As V is the notation used by Laplace in this announcement, it is plain, I think, where he found this method of differentiation to get the forces ; for that is the notation used by Lagrange in the last of several memoirs previous to 1783 in which he made use of the Potential. On this account it may be well to notice this last memoir first : *Theorie de la Libration de la Lune. Mémoires de l'Académie royale des Sciences et Belles-Lettres de Berlin, Année* 1780. *Œuvres,* t. V., p. 5.

"The memoir is divided into five sections. The first is designed for the exposition of a general analytical method for resolving all the problems of dynamics. This method, which I employed in my first memoir on the libration of the moon, has the singular advantage of requiring no construction and no geometrical or dynamical reasoning, but only analytical operations subjected to a process that is simple and uniform. * * * *

Let $m, m', m'', \ldots$ be the masses of the bodies, $P, Q, R, \ldots$ the accelerative forces that attract the body m towards centres whose distances are $p, q, r, \ldots, P', Q', R', \ldots$ the accelerative forces that attract the body m' towards centres whose distances are $p', q', r', \ldots$. * * * *

Taking into consideration the mutual disposition of the bodies, one will have several equations of condition among the variables x, y, z, etc. All these are expressed in terms of some one or more variables $\varphi, \psi, \ldots$ that are independent. By substitution and differentiation, one will have the general equation $\Phi d\varphi + \Psi d\psi + \ldots = 0$: thus $\Phi = 0$, $\Psi = 0, \ldots$, give as many equations as there are undetermined variables, by means of which these variables are determined. We shall show how to abridge the calculations necessary to reduce * * * * to functions of $\varphi, \psi, \ldots$. * * * * In regard to the terms $P\delta p + Q\delta q + R\delta r + \ldots$ and similar terms, we note that in the case of nature the forces $P, Q, R, \ldots$ are ordinarily functions of the distances $p, q, r, \ldots$, so that the terms of which they consist are all integrable. This also furnishes a means of simplifying very much the calculation of these terms; for it is only necessary, in the first place, to integrate the quantity $P\delta p + Q\delta q + R\delta r + \ldots$ in the ordinary way, and then differentiate it according to the characteristic δ. * * * *

Put for abridgment

$$T = \frac{1}{2}\left\{ m\,\frac{dx^2 + dy^2 + dz^2}{dt^2} + m'\,\frac{dx'^2 + dy'^2 + dz'^2}{dt^2} + \ldots \right\};$$

$$V = m\int(P\delta p + Q\delta q + R\delta r + \ldots) + m'\int(P'\delta p' + Q'\delta q' + R'\delta r' + \ldots) + \ldots,$$

and suppose $x, y, z, x', y', z', \ldots$ expressed in terms of other variables $\varphi, \psi, \ldots$; then substituting these values in T and V and differentiating according to the characteristic δ, regarding $\varphi, \psi, \ldots, d\varphi, d\psi, \ldots$, as the corresponding variables (d referring to the time) the above equation becomes

$$\left(d\frac{\delta T}{\delta d\varphi}-\frac{\delta T}{\delta\varphi}+\frac{\delta V}{\delta\varphi}\right)\delta\varphi+\left(d\frac{\delta T}{\delta d\psi}-\frac{\delta T}{\delta\psi}+\frac{\delta V}{\delta\psi}\right)\delta\psi+\ldots=0,$$

wherein $\frac{\delta T}{\delta\varphi}$ denotes the coefficient of $\delta\varphi$ in the differential of T, and $\frac{\delta T}{\delta d\varphi}$ the coefficient of $\delta d\varphi$ in the same differential, and so for the rest."

This investigation appears also in the *Méchanique Analytique;* but, as we shall see by another example, Todhunter did not recognize that the *Méchanique Analytique,* like the *Méchanique Céleste* of Laplace, was largely a compilation from preceding memoirs. Theories of Attraction, vol. II., p. 153, art. 994: "The first edition of a famous work by Lagrange, appeared in 1788 in one volume, entitled *Méchanique Analytique.* There is nothing in this edition which bears explicitly on our subject. But on his page 474 Lagrange gives, in fact, an integral in the form of a series of the partial differential equation

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=0;$$

and from this integral, as we shall see hereafter, Biot drew important inferences with respect to the attraction of a body." The solution here referred to was given by Lagrange in 1781: *Œuvres,* t. IV., p. 695, *Théorie du Mouvement des Fluides.*

The idea of differentiating in order to obtain the forces first appeared in Lagrange's memoir of 1763: *Œuvres,* t. VI., p. 5, *Recherches sur la Libration de la Lune, Prix de l'Académie Royale des Sciences de Paris,* t. IX., 1764. The kinetic energy is differentiated to obtain the accelerations, forming the first part of Lagrange's celebrated generalized equations of motion given first in complete form in 1780. The potential is used to obtain the forces for the first time by Lagrange in the memoir *Sur l'Equation Séculaire de la Lune; L'Académie Royale des Sciences de Paris,* t. VII., 1773; *Prix pour l'année* 1774; *Œuvres,* t. VI., p. 335.

"If a point A attract another point B with any force whatever F, and if Δ be the distance between the two bodies and $d\Delta$ the increment of this distance in supposing that A attracts B an infinitely small space $d\alpha$, then $-F\frac{d\Delta}{d\alpha}$ is that part of the force F which acts in the direction $d\alpha$; and if one proposes to decompose this force in three mutually perpendicular directions $d\alpha$, $d\beta$, $d\gamma$, $-F\frac{d\Delta}{d\beta}$, $-F\frac{d\Delta}{d\gamma}$ are the remaining components. If F is proportional to $\frac{1}{\Delta^2}$, which is the case of celestial attraction, then

$$Fd\Delta = \frac{d\Delta}{\Delta^2} = -d\frac{1}{\Delta},$$

and consequently, the three forces are represented by the coefficients of $d\alpha$, $d\beta$, $d\gamma$, in the differential of $\frac{1}{\Delta}$. In short, it suffices to find the value of $\frac{1}{\Delta}$ and differentiate it by ordinary methods.

If the point B is attracted at the same time towards different points A, A', A'', . . , whose distances from B are Δ, Δ', Δ'', . . . , and if the attractions are $\frac{M}{\Delta^2}$, $\frac{M'}{\Delta'^2}$, $\frac{M''}{\Delta''^2}$, . . . , it is plain that one has only to seek the value of the quantity

$$\frac{M}{\Delta} + \frac{M'}{\Delta'} + \frac{M''}{\Delta''} + \ldots$$

and to differentiate it as a function of α, β, γ, when the coefficients of $d\alpha$, $d\beta$, $d\gamma$, in this differential immediately give the forces sought.

In general, if the point B is attracted by a body of any figure whatever, whose mass is M, then, considering each element, dM, of the body as an attracting point, it is only necessary to find the sum of all the quantities $\frac{dM}{\Delta}$, found by making the quantities that relate to the position of dM vary and regarding α, β, γ as constant; then, denoting this sum by Σ, and making it vary as to the quantities α, β, γ, that relate to the position of B, one has $\frac{d\Sigma}{d\alpha}$, $\frac{d\Sigma}{d\beta}$, $\frac{d\Sigma}{d\gamma}$ for the three forces in the directions $d\alpha$, $d\beta$, $d\gamma$, to which the total attractive force of the body M on B reduces." Lagrange then goes on to apply this method to his discussion of the moon.

In October, 1777, Lagrange read a paper that is devoted wholly to the potential and its applications to the dynamics of a system of bodies : *Remarques générales sur le Mouvement de plusieurs corps qui s'attirent mutuellement en raison inverse des carrés des distances. L'Académie royale des Sciences et Belles-Lettres de Berlin, année* 1777. *Œuvres*, t. IV., p. 402.

"Let M, M', M'', . . . be the masses of bodies which compose a given system, x, y, z the rectangular coordinates of the body M in space, x', y', z' those of the body M', and so on.

Put

$$\Omega = \frac{MM'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$
$$+ \frac{MM''}{\sqrt{(x-x'')^2 + (y-y'')^2 + (z-z'')^2}}$$
$$+ \frac{M'M''}{\sqrt{(x'-x'')^2 + (y'-y'')^2 + (z'-z'')^2}} + \ldots,$$

and let $\frac{d\Omega}{dx}, \ldots, \frac{d\Omega}{dx'}, \ldots$ denote, as usual, the coefficients of $dx, \ldots, dx', \ldots$ in the differential of Ω, regarded as a function of $x, \ldots, x', \ldots$.

One has $\frac{1}{M}\frac{d\Omega}{dx}, \frac{1}{M}\frac{d\Omega}{dy}, \frac{1}{M}\frac{d\Omega}{dz}$, for the forces with which the body M is attracted by the other bodies M', M'', in the directions of the coordinates x, y, z, and so on. It is easy to be convinced of this by performing the indicated differentiation : for that will give the same expressions as the decomposition of the forces that act upon each body in virtue of the attraction of each of the other bodies, supposed proportional to the mass divided by the square of the distance. This manner of representing the forces is, as one sees, extremely convenient, both for its simplicity and for its generality; and it has the further advantage that one distinguishes by it, clearly, the terms due to the different attractions of the bodies, for each of the attractions gives in the quantity Ω a term consisting of the product of the masses of the two bodies divided by their distance apart."

Lagrange goes on to give the equations of motion

$$M\frac{d^2x}{dt^2} = \frac{d\Omega}{dx}, \quad M\frac{d^2y}{dt^2} = \frac{d\Omega}{dy}, \quad M\frac{d^2z}{dt^2} = \frac{d\Omega}{dz}, \ldots,$$

multiplies them by $dx, dy, dz, \ldots$, adds and integrates, finding the equation of conservation of energy,

$$\Omega = \tfrac{1}{2} M \left\{ \frac{dx^2 + dy^2 + dz^2}{dt^2} \right\}$$
$$+ \tfrac{1}{2} M' \left\{ \frac{dx'^2 + dy'^2 + dz'^2}{dt^2} \right\} + \ldots + \text{constant}.$$

Since Ω does not change when the x-coordinates change by equal increments, he finds

$$\frac{d\Omega}{dx}+\frac{d\Omega}{dx'}+\frac{d\Omega}{dx''}+\ldots=0$$

with similar equations in the y- and z-coordinates.

Substituting

$$\frac{d\Omega}{dx}=M\frac{d^2x}{dt^2},\quad \frac{d\Omega}{dy}=M\frac{d^2y}{dt^2},\quad \frac{d\Omega}{dz}=M\frac{d^2z}{dt^2},\ldots,$$

and using

$$X=\frac{Mx+M'x'+\ldots}{M+M'+\ldots},\ Y=\frac{My+M'y'+\ldots}{M+M'+\ldots},\ldots,$$

he has

$$\frac{d^2X}{dt^2}=0,\ \frac{d^2Y}{dt^2}=0,\ \frac{d^2Z}{dt^2}=0.$$

In words, the centre of gravity moves uniformly in a straight line. The equations of motion are then shown to be unchanged when the centre of gravity is taken as the origin.

Since Ω does not change to the first order in α when

$$\frac{dy}{z}=\frac{dy'}{z'}=\ldots=-\frac{dz}{y}=-\frac{dz'}{y'}=\ldots\ldots=\alpha,$$

Lagrange concludes that

$$\left(y\frac{d\Omega}{dz}-z\frac{d\Omega}{dy}\right)+\left(y'\frac{d\Omega}{dz'}-z'\frac{d\Omega}{dy'}\right)+\ldots=0$$

with similar results for the axes of y, z.

Substituting the accelerations for the forces according to the equations of motion, and integrating, he finds the equation of conservation of areas

$$M\frac{ydz-zdy}{dt}+M'\frac{y'dz'-z'dy'}{dt}+\ldots=\text{constant};$$

and so on.

The article closes as follows:

"These theorems upon the movement of the centre of gravity have already been given in part by D'Alembert; but the manner in which I have demonstrated them is new, and, it appears to me, merits the attention of geometers by the utility with which it can be used. One perceives by the same principles that these theorems will be equally true if the bodies act upon each other by forces mutually proportional to any function whatever of the distance ; for, calling $f(x)$ the force of attraction at the distance x, and putting

$$F(x) = \int f(x)\,dx,$$

one has only to change the value of Ω above into

$$\Omega = -MM'F\left(\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\right)$$
$$-MM''F\left(\sqrt{(x-x'')^2+(y-y'')^2+(z-z'')^2}\right)-\ldots,$$

to easily obtain the same results."

The next memoir in which Lagrange uses the potential is that of 1780, already referred to, in which he completes his generalized equations of motion, and uses the notation V for the potential, which Laplace adopts.

Todhunter was not without warning of these facts, for he says, vol. II., p. 221 : "I must cite another sentence from Biot's memoir ; he says on page 208, after introducing the function V,

M. Lagrange a démontré que les coefficients différentiels

$$\frac{dV}{da},\ \frac{dV}{db},\ \frac{dV}{dc},$$

pris négativement expriment les attractions exercées par le sphéroïde sur ce même point, parallèlement aux trois axes rectangulaires. M. Laplace a fait voir ensuite que la fonction V est assujetie à l'équation différentielle partielle

$$\frac{d^2V}{da^2}+\frac{d^2V}{db^2}+\frac{d^2V}{dc^2}=0.$$

I do not know on what authority the above expressions for component attractions are assigned to Lagrange ; to me they appear due to Laplace : see art. 789, and also pages 70 and 133 of Laplace's *Figure des Planètes*."

Todhunter attempts a defense, vol. II., p. 160 : "Lagrange now proceeds to consider the attraction of the ellipsoid on an external particle. He introduces what we call the potential function, and denotes it by V. If f, g, h, denote the coordinates of the attracted particle, the attractions in the corresponding directions are $\frac{dV}{df}, \frac{dV}{dg}, \frac{dV}{dh}$. Lagrange does not claim these expressions for himself; and we know that they are really due to Laplace: see art. 789."

The argument is equally good if it be made to refer to Laplace's first announcement, given in art. 789, with "Laplace" and "Lagrange" interchanged. Moreover, Lagrange had claimed these expressions in the Berlin memoir of 1777, twenty years previous to the memoir Todhunter is describing.

Todhunter knew that Laplace constantly embodied the work of others in his own without credit (see preface vol. I.); and he cites a breach of etiquette towards Legendre in these very memoirs in the matter of Legendre's Coefficients, vol. II., p. 43:

"We will first reproduce a note bearing on the history of the subject which occurs at the beginning of the memoir. * * * * Legendre says:

'*La proposition qui fait l'objet de ce mémoire, étant démontrée d'une manière beaucoup plus savante et plus générale dans un mémoire que M. de la Place a déjà publié dans le volume de* 1782, *je dois faire observer que la date de mon mémoire est antérieure, et que la proposition qui paroît ici, telle qu'elle a été lûe en juin et juillet* 1784, *a donné lieu à M. de la Place, d'approfondir cette matière, et d'en présenter aux Géomètres, une théorie complète.*'"

This refers to Laplace's memoir *Figure des Planètes*, contained in the Paris *Mémoires* for 1782, published in 1785, and is the one of which Todhunter says (p. 56) "in this article we have for the first time the partial differential equation with respect to the coordinates of the attracted particle which the potential V must satisfy: it is expressed by means of polar coordinates," etc.

Nor did Todhunter neglect foreign memoirs alone, bearing on his subject; for if he had read the valuable Report on Dynamics by Cayley, *Brit. Ass. Rep.*, 1862, p. 184, he would have found the potential function properly credited to Lagrange, with a reference to the memoir, *Sur l'Équation Séculaire de la Lune*, of 1773.

In conclusion I ought to say that a sentence in Sir William Thomson's Baltimore lectures (1884), led me to investigate this subject, Lectures, p. 112: "I took the liberty of asking Professor Ball two days ago whether he had a name for this symbol ∇^2; and he has mentioned to me *nabla*, a humorous suggestion of Maxwell's. It is the name of an Egyptian harp which was of that shape. I do not know that it is a bad name for it. *Laplacian* I do not like for several reasons both historical and phonetical."

ROSE POLYTECHNIC INSTITUTE,
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