A FRENCH ANALYTICAL GEOMETRY.

Leçons de Géométrie Analytique. Par MM. BRIOT et BOUQUET. Revue et annotée par M. APPELL, professeur à la Faculté des Sciences. Paris, Ch. Delagrave, 1890. 8vo, pp. iii. + 722.

THIS popular French text-book reached its fourteenth edition in 1890. At that time, as we learn from the preface, changes in the programmes of the schools and improved methods of teaching had made a revision of the book advisable. This piece of work was done by M. Appell, a mathematician, whose name is as familiar to American students as to Frenchmen. The bare list of the articles in the book which he has touched covers a page and a half, and it is safe enough to say that "*nihil quod tetigit non ornavit.*" A treatise of this kind is of course more interesting to teachers of elementary mathematics than to any one else; to them even a slight account of a school book which has achieved great and lasting popularity in a nation where pure mathematics has flourished so splendidly and so long, can not fail to prove interesting by virtue of its subject.

The book opens with a concise notice of the different systems of plane coordinates, beginning with rectilinear coordinates in general and the particular case of rectangular axes; then passing rapidly over polar and bi-polar systems, and finally giving a notion of coordinates in general. These notions are all simple enough when presented in the transparent style of the authors; in fact plane coordinates are so much simpler than curves drawn on a sphere that it is a wonder that school books on geography should not give an account of them before taking up the subject of latitude and longitude which almost always proves difficult to young pupils. The writer was once explaining rectangular coordinates at a teachers' institute when one of the members rose and thanked him for *inventing* them; he had been trying to teach latitude and longitude without any of the preliminary ideas necessary to an understanding of the matter. At the close of the first chapter we read, "The representation of figures by equations is the basis of analytic geometry; it allows us to apply the processes of algebra to the study of figures. In analytic geometry we are concerned with three fundamental questions: when a figure is defined geometrically, to find its equation; reciprocally, when the equation is given, to construct the figure; finally, to study the relations which exist between the geometrical properties of the figures and the analytical properties of the equations."

Chapter II. takes up the first problem; various loci, including nearly all the simple curves whose names are familiar, are defined geometrically and their equations written down directly, as a mere statement of the definition in the language of algebra. The curves are drawn and sufficiently described. In this way the student not only gets a notion of what a locus is, but, what is far from easy, he comes to see how an equation, so different in its nature and belonging to quite another realm of thought, can represent a geometrical figure, and to look upon equations in x and y as brief statements of the truths of geometry. The mind is put in a condition to understand why the manipulation of an equation may lead to new facts. Without some such preparation it can hardly be very profitable to try to prove geometrical theorems with equations; there can be nothing in the student's mind corresponding to them, and a gulf which he can not bridge will exist between the proof and the conclusion. That the radius of a circle has a constant length is expressed algebraically by the equation

$$\rho = r;$$

that the sum of the distances of any point on the ellipse from the foci is constant, by

$$u + v = 2a,$$

and so on; the polar equations from their simplicity being first written down. The chapter closes with a list of exercises of which this is a specimen: "To construct the curve whose equation in bi-polar coordinates is $uv = a^2$; the distance between the poles being $2a$." Of course this lemniscate would be constructed by drawing circles with their centers at the poles. The student has been told previously how to find points on the ellipse in the same way. The problems are mostly too difficult for a beginner.

Chapter III. treats of the fundamental idea of homogeneity. A function $f(a, b, c, \ldots)$, we are told, is homogeneous and of degree m when $f(ka, kb, \ldots) = k^m f(a, b, \ldots)$; the sum, difference, product or quotient of any two homogeneous functions is homogeneous; and the same is true of any power, root or transcendental function of $f(a, b, c, \ldots)$; but the transcendental function must be of degree 0. Thus $\sin\left(\dfrac{ab}{a^2 + b^2}\right)$ is homogeneous, while $\sin(a + \sqrt{bc})$ is not. All this is sufficiently clear, and what follows shows its vital importance at the threshold of analytic geometry.

"When we seek the relations which exist between the lengths of the various lines A, B, C, ... of a figure, we imagine these lines referred to a unit of length which is usually not specified and remains quite arbitrary." Hence the reasoning which leads to a relation among the lengths of these lines is independent of any particular unit, and the relation must subsist whatever be the unit. In particular it subsists if the unit be divided by k, that is if the number expressing each length is multiplied by k; hence the relation is homogeneous, or at any rate, if not, then it must break up into several relations which are each homogeneous. An apparent exception occurs when some line of the figure is taken as the unit of length, but the exception is explained and the homogeneity reestablished. "The equations which the theorems of elementary geometry lead to directly, are homogeneous. . . . The principle of homogeneity can be used at every step to verify the algebraic transformations which have been effected."

The above is a too brief account of a part of this very elementary and most interesting chapter. No part of it is difficult even for a young student, while it opens up to him a line of thought which he must follow throughout his scientific studies in whatever direction he turns. This is the kind of work which makes mathematicians and scientists, while the student whose analytic geometry consists only in manipulating a few equations of which the meaning is but dimly seen, finds it a barren and useless subject.

Some considerations follow, still of the simplest kind, which lead to the conclusion that "all rational expressions, and all irrational expressions containing only square roots, can be constructed by means of a limited number of right lines and circles." It is added, but not proved, that no others can. A little reading of this character would turn the attention of a goodly number of bright young men who are still at work upon some of the impossible problems of antiquity, to subjects more worthy of their abilities.

Book II. opens the study of the right line and circle. The treatment of these loci is similar to that in our familiar text books and is limited to the needs of beginners. The next chapter treats of geometric loci in general. In Book I. the equations of many loci were obtained simply by writing down the definition in the language of algebra; here we obtain the equations of loci by eliminating one or more variable parameters. The coordinates of a point may be explicitly given as functions of the parameter a, but more usually they are only implicitly given in two equations

$$(1) \quad f_1(x, y, a) = 0,$$
$$(2) \quad f_2(x, y, a) = 0.$$

Each value of a gives a pair of curves intersecting in a point of the locus, and "the equation of the locus is obtained by eliminating the parameter a between the equations (1) and (2)."

Why this should give the equation of the locus is a difficult thing for students to see; but it is less difficult when properly stated. The result is not a single equation in x and y, but a pair of equations

$$(3)\quad f(x, y, a) = 0,$$
$$(4)\quad F'(x, y) = 0.$$

which are equivalent to (1) and (2). Any system of values of x, y, a, which satisfies (1) and (2) must satisfy (3) and (4); hence equation (4) is satisfied by the coordinates of every point on the locus. Conversely, every system of values of x, y, a which satisfies (3) and (4) must satisfy (1) and (2); it follows that (4) is the equation of the locus.

The teacher who reads this book will be everywhere delighted by the careful way in which the raw edges of thought are hemmed down; no loose threads are left to ravel out and destroy the fabric. It can hardly be doubted that this brave honesty in their elementary school books has much to do with that precision of thought and clearness of expression which makes the works of French mathematicians a perpetual refreshment to the reader. It would be an inquiry worth making, whether students in high schools and normal schools who never intend to enter college do not get more benefit from the study of geometry than any others. With them it is not merely a thing to be crammed for an entrance examination, but a subject to be studied for its educational value. It seems especially disastrous to make analytic geometry, a subject where the preliminary notions are so delicate and beautiful, a thing to be asked questions about when entering college; inasmuch as the amount of knowledge required can at best be but small, and will almost certainly be acquired under conditions likely to blight future results.

The remainder of the book, while deeply interesting, is more advanced; and it is not the purpose of this sketch to do more than call attention to those parts whose study may possibly be useful to teachers of classes which are beginning the subject.

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