THE ANNUAL MEETING OF GERMAN MATHEMATICIANS.

THROUGH the courtesy of the secretary, Dr. H. Wiener, of the University of Halle, who kindly sent us advance sheets of the *Proceedings*, we are now enabled to give a more detailed account of the papers read before the section for mathematics and astronomy of the German *Naturforscher-Versammlung* held at Halle, September 21 to 25, 1891. The meetings of this section constitute at the same time the annual meeting of the German Mathematical Union (*Deutsche Mathematiker-Vereinigung*). The section had seven meetings; the total number of members registered as present was 70.

1. The first paper read was a report by Prof. FELIX KLEIN, of Göttingen, *On recent English investigations in mechanics.* The following abstract of this paper is translated from the *Proceedings.*

"The distinguishing characteristic of the English work in mechanics in comparison with that of continental writers lies in its being based on a thorough grasp of physical reality and in the resulting graphical lucidity (*durchgängige Anschaulichkeit*) of the investigations. For this very reason the English work in mechanics proves particularly interesting and instructive to the mathematician accustomed to a purely abstract train of reasoning. The usual lack of that methodical treatment and mathematical rigor which the continental mathematician is wont to expect cannot be regarded as a serious objection ; in fact, it adds to the interest.

Among the matters of detail discussed by the speaker, his remarks on the history of the discovery of Hamilton's method of integrating the equations of dynamics may be of general interest. The matter seems to be entirely unknown, although Hamilton distinctly states the facts at various places in his writings, in particular in his first paper on systems of rays (1824). At the time when Hamilton began writing, the emission theory was still prevalent so that the determination of a ray of light passing through any non-homogeneous (but isotropic) medium was considered as a special case of the ordinary mechanical problem as to the motion of a material particle. It may be noticed in passing that the distinction between this special case and the general problem is not an essential one : by proceeding to higher spaces, any mechanical problem may be reduced to the determination of a ray of light traversing a properly selected medium. Now *Hamilton's discovery,* according to which *the integration of the differential equations of dynamics is made to depend upon the integration of a certain* *partial differential equation of the first order,* was simply the result of the fact that Hamilton, following the great movement just then taking place in physics, undertook to derive, from the point of view of the undulatory theory, the results in geometrical optics already known in the form of the corpuscular theory. Hamilton's method for integrating the differential equations of dynamics is, primarily, nothing but the general analytical expression for the relation between *ray* and *wave,* a distinction which in its physical form was well known at the time. Considered in this new light it is readily understood why Hamilton gave to his investigations that unnecessarily specialized form in which he published them and which was removed only later by Jacobi. In his investigations on systems of rays Hamilton had originally in view certain entirely practical questions relating to the construction of optical instruments. This is the reason why he operates throughout with waves of light issuing from single *points.* The real meaning of Jacobi's generalization is that *any other waves of light may be used to determine a ray.* The general wave is constructed in optics from the special waves by means of the so-called principle of Huygens. This construction is an exact equivalent to the analytical process by which we ascend in the theory of partial differential equations of the first order from any 'complete' solution to the 'general' solution."

2. The paper of Mr. PAPPERITZ, of Dresden, *On the system of the mathematical sciences,* it is announced, will be published elsewhere.

3. Mr. MAX SIMON, of Strassburg, read a paper *On the axiom of parallels.*

4. Mr. FRANZ MEYER, of Clausthal, presented an elaborate report *On the progress of the projective theory of invariants during the last twenty-five years,* which will probably be published *in extenso* by the German Mathematical Union.

5. Mr. FINSTERWALDER, of Munich, read a paper *On the images of dioptric systems of somewhat large aperture and field* which is published in the Transactions of the Bavarian Academy of Sciences (*Abhandlungen,* Class II, Vol. 17, Abth. 3, pp. 517–588).

6. Mr. ROHN, of Dresden, spoke *On rational twisted quartics,* illustrating his remarks with the aid of models.

7. Dr. H. WIENER, of Halle, read a paper *On the foundations and the systematic development of geometry.* The following abstract is given in the *Proceedings.*

"To be rigorous we may demand that the proof of a mathematical proposition should make use of those assumptions only on which the proposition really depends. The simplest conceivable assumptions are the existence of certain objects and the possibility of certain operations by which said objects may be connected. If it be possible, without further assumptions, to connect such objects and operations so as to produce propositions, these propositions will form a *self-sustaining* (*in sich begründet*) *domain* of science. Such, for instance, is algebra.

In geometry it is of interest to go back to the simplest objects and operations, since starting from these it is possible to build up an abstract science which will be independent of the axioms of geometry while its propositions run parallel to those of geometry.

The projective geometry of the plane offers an example. Let the objects be *points* and *lines*, the operations those of *joining* and *cutting*, and let objects as well as operations be restricted to a finite number. Throwing off the geometrical dress we shall have elements of two kinds and two kinds of operations such that the connection of any two elements of the same kind produces an element of the other kind. The geometrical propositions obtained on these assumptions (apart from combinatory propositions involving the number of elements) are *closing propositions* (*Schliessungssätze*), if this term be taken to mean propositions about certain lines and points such that every one of the lines contains at least three of the points and every one of the points lies at least on three of the lines. Such are for instance : (1) Desargues's theorem of perspective triangles, and (2) Pascal's hexagram theorem applied to two lines.

The proof of such propositions cannot be obtained from the given objects and operations ; in other words, this domain of geometry is not self-sustaining. If however the proof for any one such proposition (or for several) be taken from some other domain, then, by its repeated application a closed domain of plane geometry may be obtained. Thus, proving Desargues's theorem by means of solid geometry, we obtain the domain embracing all propositions usually derived by means of geometrical addition of vectors or points. The attempts at deriving proposition (2) above from (1) have not been successful. Another possibility would be its derivation by projection from a space of three or more dimensions, or else (which is easily done) by introducing the idea of continuity. *These two "closing propositions,"* however, *are sufficient to prove, without further considerations of continuity or infinite processes, the fundamental proposition of projective geometry*, and thus to develop the whole domain of linear projective plane geometry.

Similarly it is possible to build up a solid geometry resting on the point, line, and plane as fundamental elements, or objects. But in this case we obtain a self-sustaining domain.

These considerations can be extended to higher spaces. It will however be more important to descend from the plane to the *geometry of the line.* The only element we here have is the point; there can be neither joining nor cutting. It thus becomes necessary to borrow an operation from another domain; as such we may use constructions executed in the plane but concerning only points lying on our line, especially constructions of projective, involutory, and harmonic groups of points. It appears that the construction of harmonic groups is sufficient as the following proposition can be proved: *If in a line two pairs of points of an involution, or three pairs of corresponding points of a projective system be given, it is possible to construct the corresponding point to any other given point by a* FINITE *number of constructions of harmonic points.*

Other domains are obtained by introducing other assumptions. Thus the *geometry of order* presupposes the proposition that on a closed line four points can be divided in a definite way into two pairs that separate each other. Still other domains depend on the assumption of the *continuity* of the elements, which may be either the analytical continuity of the method of limits, or the geometrical continuity that finds its expression in the necessary meeting of points moving in a certain way in a line."

8. Mr. SCHUBERT, of Hamburg, read a paper *On the enumerative geometry of p-dimensional spaces of the first and second degrees.*

9. Mr. EBERHARD, of Königsberg: *Elements of the theory of forms of polyhedra.* An elaborate work by the author on this subject has just been published (*Zur Morphologie der Polyëder*, Leipzig, Teubner, 1891).

10. Mr. BOLTZMANN, of Munich: *On some points in Maxwell's theory of electricity;* will be published in the *Proceedings* of the section for physics.

11. Mr. HENSEL, of Berlin: *On the fundamental problem of the theory of algebraic functions;* see the same author's paper in the *Journal für Mathematik,* vol. 108, p. 142.

12. Mr. FELIX MÜLLER, of Berlin: *On literary enterprises adapted to facilitate the study of mathematics.* The speaker pointed out the desirability of an introduction to the bibliography of mathematics; complained of the want of subject-indexes in the most prominent mathematical journals; gave an account of the progress of recent bibliographical works; and expressed a regret that the continuation to Poggendorf's Dictionary of Authors has not yet appeared. He also laid before the Section a plan for a new Mathematical Dictionary for which he has been collecting the material for the last 20 years; it contains about 4000 mathematical terms and over 1200 names.

13. Mr. DYCK, of Munich: *On the forms of the systems of curves defined by a differential equation of the first order, in particular on the arrangement of the curves of principal tangents to an algebraic surface;* will be published elsewhere.

14. Mr. DAVID HILBERT, of Königsberg: *On full systems of invariants.*

"Let $J_1, J_2, \ldots, J_{n-2}$ be integral rational invariants of a binary ground-form of the nth order, of the degrees $\nu_1, \nu_2, \ldots \nu_{n-2}$, respectively, in the coefficients of the ground-form; and let these invariants be so selected that all other integral rational invariants of the ground-form are *integral* algebraic functions of those $n-2$ invariants. Then the integral rational invariants of the ground-form form the integral functions of a body (*Körper*) of algebraic functions; let g be the degree of this body. Then the following formula can be shown to hold:

$$\frac{g}{\nu_1\,\nu_2\ldots\nu_{n-2}} = -\frac{1}{2.\,n\,!}\left\{\binom{n}{2}^{n-3} - \binom{n}{1}\left(\frac{n}{2}-1\right)^{n-3} + \ldots \pm \binom{n}{\frac{1}{2}n-1}\right\} \text{ for even } n,$$

$$= -\frac{1}{4.\,n\,!}\left\{\binom{n}{2}^{n-3} - \binom{n}{1}\left(\frac{n}{2}-1\right)^{n-3} + \ldots \pm \binom{n}{\frac{1}{2}n-1}\right\} \text{ for odd } n.$$

15. Mr. SCHOENFLIES, of Göttingen: *On Configurations that can be derived from given space-elements by the operations of cutting and joining alone.* Referring to Dr. Wiener's paper (7) the speaker showed that prop. (2) (Pascal's hexagram applied to two lines) cannot be derived by the operations of joining and cutting alone.

16. Mr. MINKOWSKI, of Bonn: *On the geometry of numbers.* The author gives the name *number-frame* (*Zahlengitter*) to the totality of all those points of space whose rectangular coordinates are three *integral* numbers and considers certain solids and their relation to the frame. The two most important cases are as follows. (1) Solids having the origin of coordinates as centre and bounded by a surface that appears at no point concave from without. For such solids it can be shown that if the volume be $\geqq 2^3$ the solid must contain other points of the frame besides the origin. (2) Solids containing the origin and bounded by a surface which as seen from the origin shows no double point. If the

volume of such a solid be $\leqq 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots$, it is always possible to indicate deformations of the solid for which the volume remains constant, the origin remains fixed and all straight lines of the solid remain straight while all points of the frame excepting the origin are found outside the solid after deformation.

17. Mr. FRITZ KÖTTER, of Berlin: *On the problem of rotation treated by Mrs. Kovalevsky.* The paper develops somewhat farther the formulæ given by Mrs. Kovalevsky in the 12th volume of the *Acta Mathematica* for a certain integrable case of the problem of rotation of a heavy body about a fixed point.

18. Mr. PILITZ, of Jena: *A question in the theory of numbers.* After an introductory discussion of the necessity for a new calculus, or at least of a new way of conceiving of the combination of elements in the problems of the theory of numbers and the theory of functions, the speaker gave a proof of the proposition announced by Riemann as probably true: that the complex 0-points of the function $\zeta(s)$ all have $\frac{1}{2}$ as their real part.

19. Mr. F. STÄCKEL, of Halle: *On the bending of curved surfaces under certain conditions.*

20. Mr. A. WANGERIN, of Halle: *On the development of surfaces of rotation with constant negative curvature on each other.*

21. Mr. WILTHEISS, of Halle: *On some differential equations of the theta functions of two variables.*

22. Mr. G. CANTOR, of Halle: *On an elementary question in the theory of manifoldnesses.*

23. Mr. GORDAN, of Erlangen: *Remarks on a proposition of Mr. Hilbert.*

ALEXANDER ZIWET.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, December 5, at half-past three o'clock, the president in the chair. Mr. Wiley, Mr. Snook, and Dr. Pupin were appointed a committee to report at the annual meeting, on December 30, nominations for the officers and other members of the council for the calendar year 1892.

Dr. Pupin read an original paper entitled "On a peculiar family of complex harmonics," in which he deduced several