erous with his time and thoughts, loving to talk to an appreciative listener of some favorite doctrine, or of the famous mathematicians with whom he had been associated.

He was a man of rare genius, a mathematician of the first rank in this century of great mathematicians.

HENRY B. FINE.

PRINCETON COLLEGE, *April* 20, 1892.

MULTIPLICATION OF SERIES.

BY PROF. FLORIAN CAJORI.

THE salient feature of the new era which analysis entered upon during the first quarter of this century is vividly illustrated in the history of infinite series. Extending from that time back to Newton we have a *formal* period which gave rise to general theorems, the validity of which was not thoroughly tested. Thus, in series, there were put forth during that epoch the binomial theorem, the theorems of Taylor, Maclaurin, John Bernoulli, and Lagrange. Infinite series were used by Newton, Leibnitz, and Euler in the study of transcendental functions. As a rule, the convergency of expressions was not ascertained, and the confusion which prevailed in the theory of series gave rise to curious paradoxes. But with the advent of Gauss, Cauchy, and Abel, began the new era which combined dexterity in form with *rigor of demonstration.*

In the multiplication of series, mathematicians of the earlier period considered simply the form of the products and hardly ever thought of inquiring further into the validity of the operation. Reliable tests for convergency were unknown. The product of any two infinite series was accepted with nearly the same degree of confidence as was the product of finite expressions. Thus, De Moivre * extended the binomial formula to infinite series and deduced the following formula:

$$(az + bz^2 + \ldots)^m$$

$$= a^m z^m + \frac{m}{1} a^{m-1} b z^{m+1} + \frac{m}{1} \cdot \frac{m-1}{2} a^{m-2} b^2 z^{m+2} + \ldots$$

This was accepted as true without any limitations whatever.

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* A method of raising an infinite multinomial to any given power, or extracting any given root of the same. *Philosophical Transactions*, No. 230, 1697.

The first to cry "halt" to these reckless proceedings was Baron Cauchy. He instituted for the first time a painstaking examination of the principles of series and strove to introduce absolute rigor. He is the founder of the theory of convergency and divergency. He pointed out that if two series are convergent, their product is not necessarily so.

Thus,

$$1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$$

is convergent, but its square

$$1 - \frac{2}{\sqrt{2}} + \left(\frac{2}{\sqrt{3}} + \frac{1}{2}\right) - \left(\frac{2}{\sqrt{4}} + \frac{2}{\sqrt{6}}\right) + \ldots$$

is divergent. Not only did he discriminate between convergent and divergent series, but also between what we now call "absolutely convergent series" which are convergent even if all the terms are made positive, and "semi-convergent series" which cease to be convergent when the terms are all made to have like signs. In his *Cours d'analyse algébraique* (1821) Cauchy proved rigorously the following celebrated theorem: *If* Σu_n *and* Σv_n *converge* ABSOLUTELY *to values* U *and* V *respectively, then the series* $\Sigma(u_0v_n + u_1v_{n-1} + \ldots + u_nv_0)$ *converges to the value* UV. So far as the researches of Cauchy went, two absolutely convergent series appeared to be the only ones which could be multiplied by one another with absolute safety. This same theorem was proved also by Abel in course of his demonstration of the binomial formula,* but in the same article he took a giant step in advance by establishing the following theorem: *If the series* Σu_n *and* Σv_n *converge to the limits* U *and* V *respectively, then if the series* $\Sigma(u_0v_n + u_1v_{n-1} + \ldots + u_nv_0)$ *be convergent, it will converge to the product* UV. The beauty of this theorem lies in the fact that all three series in question may be semi-convergent. Strange to say, this result, so remarkable for its simplicity and generality and put forth by so prominent a mathematician as Abel, was for nearly half a century almost universally overlooked. Schlömilch's *Compendium der höheren Analysis* knows it not, nor does Bertrand's *Traité de calcul différentiel.*

Abel's theorem would dispose of the whole problem of multiplication of series, if we had a universal practical criterion of convergency for semi-convergent series. Since we do not

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* Crelle's *Journal*, Bd. 1, 1827; also *Œuvres complètes de N. H. Abel*, Tome 1, p. 66 *et seq.*

possess such a criterion, theorems have been established which remove in certain cases the necessity of applying tests of convergency to the product-series. Such a theorem is that of Mertens * who in 1875 demonstrated that Cauchy's theorem still holds true if, of the two convergent series to be multiplied together, only one is absolutely convergent. Thus, if the absolutely convergent series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

is multiplied by the semi-convergent series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots,$$

the product will surely converge to the value 2 log 2. A still more comprehensive but more complex theorem was given by Mr. A. Pringsheim, in 1882 : † *If* $U = \sum_0^\infty{}^\nu u_\nu$, $V = \sum_0^\infty{}^\nu v_\nu$ *be convergent series of which one, say* U, *has the property that its terms, arranged in certain groups containing always a finite number of terms, constitute an absolutely convergent series, i.e. that*

$$U = (u_0 + u_1 + \ldots + u_{m_1}) + (u_{m_1+1} + \ldots + u_{m_2}) + \ldots$$

be absolutely convergent, then we have

$$UV = \sum_0^\infty{}^\nu w_\nu = W, \textit{ where } w_\nu = \sum_0^\nu{}^x u_x v_{\nu-x},$$

provided that the series $\sum_0^\infty{}^\nu u_\nu v_\nu$ *be absolutely convergent and remain so when any number of factors* u_ν, v_ν *is replaced by other factors of higher indices.* That is, in any number of terms, $u_\nu v_\nu$, the factors v_ν (or u_ν) may be erased and any other factors $v_{\nu+n}$ (or $u_{\nu+n}$) put in their places, with the single restriction that none of the indices be repeated in the series. Whenever applicable, the above theorem excels Cauchy's in this, that the often difficult determination of the convergency of the product-series is replaced by the easier determination of the absolute convergency of $\sum_0^\infty{}^\nu u_\nu v_\nu$. In illustration of this theorem I give the following example. Let

$$U = \sum_0^\infty{}^\nu \left\{\frac{1}{4\nu+1} - \frac{1}{4\nu+2} + \frac{1}{\sqrt{4\nu+3}} - \frac{1}{\sqrt{4\nu+4}}\right\}.$$

* Crelle's *Journal*, Bd. 79. Proofs of this theorem and of Abel's theorem will be found in Chrystal's Algebra, Part II., p. 127 and p. 135.

† *Mathematische Annalen*, Bd. 21, p. 327.

This semi-convergent series becomes absolutely convergent when its terms are grouped thus,

$$U = \sum_0^\infty{}_\nu \left\{ \frac{1}{4\nu + 1} - \frac{1}{4\nu + 2} \right\} + \sum_0^\infty{}_\nu \left\{ \frac{1}{\sqrt{4\nu + 3}} - \frac{1}{\sqrt{4\nu + 4}} \right\}.$$

The series

$$V = \sum_0^\infty{}_\nu \left\{ \frac{1}{(4\nu + 1)^{\frac{3}{2}}} - \frac{1}{(4\nu + 2)^{\frac{3}{2}}} + \frac{1}{4\nu + 3} - \frac{1}{4\nu + 4} \right\}$$

is semi-convergent. Each fraction in the first series stands for a term u_ν and each fraction in the second for a term v_ν. Hence

$$\sum_0{}_\nu u_\nu v_\nu = \sum_0^\infty{}_\nu \left\{ \frac{1}{(4\nu + 1)^{\frac{3}{2}}} + \frac{1}{(4\nu + 2)^{\frac{5}{2}}} + \frac{1}{(4\nu + 3)^{\frac{3}{2}}} + \frac{1}{(4\nu + 4)^{\frac{3}{2}}} \right\},$$

which is absolutely convergent and remains so if any number of terms, u_ν or v_ν, be replaced by others occurring later in the series. Hence the product, W, of the two series converges toward UV.

The importance of inquiring whether $\sum_0^\infty{}_\nu u_\nu v_\nu$ remains absolutely convergent after the substitution of higher terms in place of the lower, is brought out by Pringsheim in the following example. Take the series U, given above, and

$$V' = \sum_0^\infty{}_\nu \left\{ \frac{1}{\sqrt{4\nu + 1}} - \frac{1}{\sqrt{4\nu + 2}} + \frac{1}{4\nu + 3} - \frac{1}{4\nu + 4} \right\}.$$

In this case

$$\sum_0^\infty{}_\nu u_\nu v_\nu = \sum_0^\infty{}_\nu \frac{1}{(\nu + 1)^{\frac{3}{2}}}$$

which is absolutely convergent, while

$$\sum_0^\infty{}_\nu u_\nu v_{\nu+2} = \sum_0^\infty{}_\nu \left\{ \frac{1}{(4\nu + 1)(4\nu + 3)} + \frac{1}{(4\nu + 2)(4\nu + 4)} + \frac{1}{\sqrt{(4\nu + 3)(4\nu + 6)}} + \frac{1}{\sqrt{(4\nu + 4)(4\nu + 6)}} \right\}$$

is divergent! It is indeed found that in this case the product UV' cannot be represented by the series W.

In proving his theorem, Pringsheim shows in the first place that an obviously *necessary* condition for the convergency of W, namely $\lim_{\nu=\infty} w_\nu = 0$, is satisfied. His theorem, like that of Cauchy and of Mertens, offers *sufficient* conditions for the applicability of the rule of multiplication, but they are not at the same time *necessary* conditions. He shows that Cauchy's and Mertens' theorems are included in his own.

Pringsheim then considers the multiplication and convergency of special classes of semi-convergent series, of which we shall mention one. He shows that if U and V are convergent series and one of them, say U, is so constituted that

$$U = \tfrac{1}{2}u_0 + \tfrac{1}{2}\sum_0^\infty{}^\nu (u_\nu + u_{\nu+1})$$

is absolutely convergent (as is the case when the terms u_ν never increase and have alternating signs), then

$$\lim_{\nu=\infty} w_\nu = 0$$

is a necessary and sufficient condition for the convergency of W. Mr. A. Voss* has treated similarly the more general case when the series U, expressed in the form

$$U = (u_0 + u_1) + (u_2 + u_3) + \cdots$$

is absolutely convergent and has shown that in this case the necessary and sufficient condition for the convergency of W lies in the two following relations:

$$\lim_{n=\infty} (u_0 v_{2n} + u_2 v_{2n-2} + \ldots + u_{2n} v_0) = 0;$$

$$\lim_{n=\infty} (u_1 v_{2n-1} + u_3 v_{2n-3} + \ldots + u_{2n-1} v_1) = 0.$$

Mr. Pringsheim reaches the following interesting conclusions: The product of two semi-convergent series can never converge absolutely, but a semi-convergent series, or even a divergent series, multiplied by an absolutely convergent series *may* yield an absolutely convergent product. Thus, the product of the absolutely convergent series

$$2 \log 2 = 1 + \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \ldots$$

and the semi-convergent series

$$\log 2 = \tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \ldots \text{ is}$$

$$2\,(\log 2)^2 = 1 + 2\sum_1^\infty{}^\nu \left\{ (-1)^\nu \cdot \frac{1}{(\nu+2)(\nu+3)} \sum_1^\nu{}^x \frac{1}{1+x} \right\},$$

which series converges absolutely. Again, the absolutely convergent series

$$-1 + \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots,$$

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* *Math. Annalen*, Bd. 24, p. 42.

multiplied by the divergent series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots$$

gives an absolutely convergent product. The strangeness of this last conclusion is removed when we consider that the series

$$-1 + \frac{1}{1.2} + \frac{1}{2.3} + \ldots$$
$$= -1 + (1 - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + \ldots = 0.$$

Since one of the factor-series is zero, we may well have a product-series with a definite limiting value. This value in this case is itself zero, as is seen from the following expression for the product-series

$$W = -c_1 + \sum_1^\infty{}_\nu (c_\nu - c_{\nu+1}), \text{ where } c = \sum_1^\nu{}_x \frac{1}{x(\nu + 1 - x)}.$$

COLORADO COLLEGE, *March* 23, 1892.

ON EXACT ANALYSIS AS THE BASIS OF LANGUAGE.*

BY A. MACFARLANE, SC.D., LL.D.

Abstract.

A SCHEME for an artificial language was published in the Philosophical Transactions of the Royal Society for 1668 by Bishop Wilkins. Since, however, it presupposes a complete enumeration of all that is or can be known, it would be overthrown by every considerable advance in knowledge. The mathematician and philosopher Leibnitz devoted much thought to what he called a *specieuse générale,* which he hoped would be an aid in reasoning and invention; but he died without publishing even an outline of his system. The new universal language Volapük, invented by J. M. Schleyer of Constance, is built upon a purely linguistic basis, being derived from a comparative study of the chief natural languages. In this paper it is proposed to show that the proper and necessary basis for an artificial language is scientific analysis and classification, and two specimens of language

* Abstract of a paper presented to the Society at the meeting of March 5, 1892.