ON PETERS'S FORMULA FOR PROBABLE ERROR.*

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THERE are three quantities which may be used as measures of the risk of error in an observation, each of which, according to the law of facility, bears a fixed ratio to the reciprocal of the measure of precision; namely, the probable error r, the mean error ε, and the mean absolute error η. Their values, in terms of h, namely

$$r = \frac{\rho}{h}, \qquad \varepsilon = \frac{1}{h\sqrt{2}}, \qquad \eta = \frac{1}{h\sqrt{\pi}},$$

may be called their *theoretic values*. By definition these would, in an infinite number of observations of the same precision, be respectively: the error which in order of absolute magnitude stands in the middle of the series, that whose square is the mean of the squares of the errors, and the mean of the absolute values of the errors.

The quantities similarly defined with reference to a series of n given observations may be called the *observational values* of r, ε and η. The assumption of the equality of the theoretic and observational value of either of the measures of the risk of error will assign a value to h and to each of the other measures. With respect to r, such an assumption would obviously be unsatisfactory, except when n is very large; but with respect to ε and η, whose observational values are

$$\sqrt{\frac{\Sigma e^2}{n}} \qquad \text{and} \qquad \frac{\Sigma [e]}{n},$$

the assumptions give the methods of determining h and r which are actually in use. Thus the ε-method gives

$$r = \rho\sqrt{2}\sqrt{\frac{\Sigma e^2}{n}} = .6745\sqrt{\frac{\Sigma e^2}{n}}; \quad . \quad . \quad . \ (1)$$

and the η-method gives

$$r = \rho\sqrt{\pi}\frac{\Sigma [e]}{n} = .8453\frac{\Sigma [e]}{n}. \quad . \quad . \quad . \ (2)$$

The first formula is preferred because we can prove otherwise that the corresponding value of h is the most probable

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* Abstract of a Paper read before the Society, June 4, 1892.

value, but the second gives very good results, and is much easier of application, especially when n is large.

Each of these formulæ however gives the value of r on the hypothesis that we are in possession of the true value of the magnitude observed: taking the arithmetical mean as the final value, they give the value of r', which we may call the *apparent probable error;* thus the first formula gives

$$r' = \rho\sqrt{2}\sqrt{\frac{\Sigma v^2}{n}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The correction in order to produce r is usually based on the consideration that, since Σv^2 is the minimum value of Σe^2, we shall do the best we can, if we add to it the mean value of the excess $\Sigma e^2 - \Sigma v^2$ due to the unknown error δ in the arithmetical mean. This is shown to be ε^2, so that $\dfrac{\Sigma e^2}{n}$ is taken to be $\dfrac{\Sigma v^2}{n-1}$;

and
$$r = \rho\sqrt{2}\sqrt{\frac{\Sigma v^2}{n-1}}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

To correct the value of the apparent probable error by the η-method, namely

$$r' = \rho\sqrt{\pi}\,\frac{\Sigma[v]}{n}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

by a similar consideration of mean values would be very difficult, nor would the result be satisfactory. If we construct the value of ε as an ordinate to the supposed "true value" as an abscissa, we shall have a parabola with vertex downward, the minimum ordinate corresponding to the arithmetical mean of the n observed values. The mean value of the ordinate according to the law of probability of δ (the divergence of the true value from the arithmetical mean) is then a plausible value to adopt for ε when δ is unknown. But, if we make a like construction for η, we have a polygonal line with an angle corresponding to each observed value as an abscissa, the minimum value occurring not at the arithmetical mean, but at a point which has an equal number of observed values on each side of it.

The concavity of the polygonal line is upward, and it is true that the mean value of the ordinate, according to the law of probability of δ, somewhat exceeds the value corresponding to $\delta = 0$, but its value depends upon the distribution of the observed values.

Sir George Airy, in his "Theory of Errors of Observation," published in 1861, takes for η the mean between the absolute values of the mean of the positive and the mean of the negative errors, and holds that no correction should be made on the ground that this value of η does not change for small values of δ. But this is fallacious, for Airy's value of η changes abruptly when the value of δ passes through the value of any one of the residuals, and the probability that δ should exceed one or more of the residuals is considerable. Moreover, there is no justification whatever for Airy's gratuitous departure from the usual definition of η.

Five years earlier, Dr. C. A. F. Peters had given in the *Astronomische Nachrichten* for 1856, vol. XLIV., pp. 29–32, the formula*

$$r = \rho \sqrt{\pi} \frac{\Sigma [v]}{\sqrt{[n(n-1)]}} \quad . \quad . \quad . \quad . \quad (6)$$

which has been generally accepted (although ignored by Airy), and is known as Peters's formula. Comparing this with equation (5), it is seen to give to the ratio $r : r'$ the value

$$r : r' = \sqrt{n} : \sqrt{(n-1)} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

which is the same as that derived from the correction for mean value in the ε-method,—compare equations (4) and (3). But it seems to have been overlooked by writers on Least Squares that Dr. Peters's method establishes directly the ratio between the real and apparent probable errors, and affords a much more satisfactory method of deriving the usual formula

$$r = \rho \sqrt{2} \sqrt{\frac{\Sigma v^2}{n-1}}$$

from equation (3) than does the method of correction for mean error of Σv^2.

Peters's method of establishing equation (7) is substantially as follows:

Denoting by $e_1, e_2, \ldots e_n$ the true errors whose probable error is r, the error of the arithmetical mean is

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* Peters states that previously the formula generally used had been

$$r = \rho \sqrt{\pi} \frac{\Sigma [v]}{n-1},$$

"which gives to r a value somewhat too great."

I do not know the origin of this method of modifying formula (5). It may have been due to the feeling that the indeterminate form should be assumed when $n = 1$, as in the case of formula (4).

$$\delta = \frac{e_1 + e_2 + \ldots + e_n}{n};$$

and the divergences from the arithmetical mean, whose probable error is denoted by r', are

$$v_1 = e_1 - \delta = \frac{n-1}{n} e_1 - \frac{1}{n} e_2 - \ldots - \frac{1}{n} e_n,$$

$$v_2 = e_2 - \delta = -\frac{1}{n} e_1 + \frac{n-1}{n} e_2 - \ldots - \frac{1}{n} e_n,$$

etc., etc.

Therefore by the formula for the probable error of a linear function of independent quantities we have, from any one of the above equations,

$$r' = \sqrt{\left[\left(\frac{n-1}{n}\right)^2 r^2 + (n-1)\frac{1}{n^2} r^2\right]} = r\sqrt{\frac{n-1}{n}}.$$

In the proof of Peters's formula given by Chauvenet in his Appendix on Least Squares ("Spherical and Practical Astronomy," vol. ii, p. 497) the above is replaced by a piece of erroneous reasoning. He says: "Each observed N may be supposed to be the result of observing the mean quantity x_0 increased by an observed error v. The probable error of $N = x_0 + v$ is therefore

$$r = \sqrt{(r_0^2 + r'^2)} = \sqrt{\left(\frac{r^2}{n} + r'^2\right)}$$

whence,

$$r = r' \sqrt{\frac{n}{n-1}}$$

This amounts to treating the equation $e_1 = v_1 + \delta$, as if e_1 were made up of *independent* observed parts v_1 and δ. It would be more natural to write $v_1 = e_1 - \delta$, whence we should have, by similar reasoning, the erroneous conclusion $r'^2 = r^2 + r_0^2 = r^2 \frac{n+1}{n}$, the error arising from the fact that e_1 and δ are *not* independent; and, if we write with Peters $v_1 = e_1 - \frac{\Sigma e}{n}$, we see the necessity of expanding Σe and expressing v_1 in terms of the independent quantities e_1, e_2, $\ldots e_n$.

Watson, in his "Theoretical Astronomy," p. 374, copies

Chauvenet. The reason given by Merriman ("Text-Book on the Method of Least Squares," p. 93) for the correction of equation (5) is based on the consideration of the mean value of the excess of $\Sigma[e]$ over $\Sigma[v]$, and assumes that this excess has the same relative value as that arising in the ε-method. But it would rather appear that, while the mean excess in the value of ε (in accordance with the law of probability of δ) happens to agree with the ratio of the true to the apparent value of r as rigorously established by Peters, there is no reason to suppose that this would be the case with regard to the mean excess in the value of η. Moreover, as before remarked, if we could obtain this mean excess, the correction founded upon it would not give so satisfactory a formula as that of Peters.

THE THEORY OF TRANSFORMATION GROUPS.

Theorie der Transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Dr. Friedrich Engel bearbeitet von Sophus Lie, Professor der Geometrie an der Universität Leipzig. Leipzig, B. G. Teubner, 1888. 8vo, pp. viii + 632.

There is probably no other science which presents such different appearances to one who cultivates it and to one who does not, as mathematics. To this person it is ancient, venerable, and complete ; a body of dry, irrefutable, unambiguous reasoning. To the mathematician, on the other hand, his science is yet in the purple bloom of vigorous youth, everywhere stretching out after the "attainable but unattained," and full of the excitement of nascent thoughts ; its logic is beset with ambiguities, and its analytic processes, like Bunyan's road, have a quagmire on one side and a deep ditch on the other and branch off into innumerable by-paths that end in a wilderness.

Among the most important of the newer ideas in mathematics is that of the *group.* In its nature it is essentially dynamic, involving the notion of operating with one thing upon another. Thus, if x and y be two of the entities of the group we shall derive new entities of the same kind by operating with y upon x and with x upon y. Entities failing of this virtue are by that fact excluded from the group.

The individuals of the group may be finite or infinite in number, but mere population does not suffice to classify them ; we must consider whether the entities are separated by finite intervals or whether they succeed each other continuously. For instance, granting that the interval between the condi-