Chauvenet. The reason given by Merriman ("Text-Book on the Method of Least Squares," p. 93) for the correction of equation (5) is based on the consideration of the mean value of the excess of $\Sigma[e]$ over $\Sigma[v]$, and assumes that this excess has the same relative value as that arising in the ε-method. But it would rather appear that, while the mean excess in the value of ε (in accordance with the law of probability of δ) happens to agree with the ratio of the true to the apparent value of r as rigorously established by Peters, there is no reason to suppose that this would be the case with regard to the mean excess in the value of η. Moreover, as before remarked, if we could obtain this mean excess, the correction founded upon it would not give so satisfactory a formula as that of Peters.

THE THEORY OF TRANSFORMATION GROUPS.

Theorie der Transformationsgruppen. Erster Abschnitt. Unter Mitwirkung von Dr. FRIEDRICH ENGEL bearbeitet von SOPHUS LIE, Professor der Geometrie an der Universität Leipzig. Leipzig, B. G. Teubner, 1888. 8vo, pp. viii + 632.

THERE is probably no other science which presents such different appearances to one who cultivates it and to one who does not, as mathematics. To this person it is ancient, venerable, and complete; a body of dry, irrefutable, unambiguous reasoning. To the mathematician, on the other hand, his science is yet in the purple bloom of vigorous youth, everywhere stretching out after the "attainable but unattained," and full of the excitement of nascent thoughts; its logic is beset with ambiguities, and its analytic processes, like Bunyan's road, have a quagmire on one side and a deep ditch on the other and branch off into innumerable by-paths that end in a wilderness.

Among the most important of the newer ideas in mathematics is that of the *group.* In its nature it is essentially dynamic, involving the notion of operating with one thing upon another. Thus, if x and y be two of the entities of the group we shall derive new entities of the same kind by operating with y upon x and with x upon y. Entities failing of this virtue are by that fact excluded from the group.

The individuals of the group may be finite or infinite in number, but mere population does not suffice to classify them; we must consider whether the entities are separated by finite intervals or whether they succeed each other continuously. For instance, granting that the interval between the conditions A and B is finite, let a transformation T change an object from A to B; then in classifying the group to which T belongs we must notice whether or not there is a continuous succession of conditions between A and B through each of which the object may be passed by the transformations of the group. In this way the ideas of continuous and discontinuous groups arise. Discontinuous groups containing a finite number of individuals are sufficiently familiar to modern readers in the theory of substitutions; but there are also partly discontinuous groups, in which the finite intervals separate, not individuals, but hosts* of individuals, the entities of each host being in continuous succession. All possible transformations of rectangular coördinates in a plane form such a group; for they fall into two distinct categories, those where the new and old axes are taken in the same sense and those where they are not. For the first host we have the analytical expression:

$$x' = x \cos \alpha - y \sin \alpha, \quad y' = x \sin \alpha + y \cos \alpha;$$

while for the second,

$$x' = x \cos \alpha + y \sin \alpha, \quad y' = x \sin \alpha - y \cos \alpha.\dagger$$

The number of transformations in a continuous group is infinite from the nature of the case, but still it is perfectly rational to speak of "finite" and "infinite" continuous groups; in the first case the number of *infinities of transformations* is finite, in the second case it is not. To express these distinctions analytically let the variables $x'_1, \ldots x'_n$ be given as functions of $x_1, \ldots x_n$ by the n solvable equations:

$$1) \qquad x'_i = f_i\,(x_1 \ldots x_n) \qquad [i = 1 \ldots n]$$

These equations represent a transformation between the variables x and x'.‡ It is not necessary to the validity of many results that the f_i be analytic functions of their arguments, though this is assumed to be the case unless the contrary is stated. After saying this, it is unnecessary to add that the word "solvable" is not used above in the sense in which it is used in speaking of algebraic equations; it here means merely (1) that the Jacobian

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$$

* The word "Schaar" is tentatively translated by "host."

† See p. 7. ‡ See p. 1.

shall not identically vanish, and (2) that when in the region* of a certain point x_i^0 $[i = 1 \ldots n]$ we can expand the functions $f_i(x)$ in convergent series proceeding by positive powers of $x_i - x_i^0$, then so long as x_i remains within that region we can expand x_i in a convergent series proceeding by positive powers of $x'_i - x'^0_i$, where x'^0_i is a certain point. It is unnecessary to add that the transformations

$$x'_i = f_i\,(x)$$

$$x_i = F_i\,(x')$$

are inverse to one another, and that when performed in succession they produce the identical transformation ; but in this connection we may notice the interesting fact that not all groups contain the identical transformation. In his first investigations † Lie endeavored to prove that all groups contained the identical transformation ; but he soon recognized his fallacy, and afterwards (1884) Engel, who was of the greatest assistance to Lie in the preparation of this book, discovered a group which actually did not contain it. This group is represented by the equation

$$x' = ax$$

with the arbitrary parameter a ; mod $a < 1$. Lie finally found that the equations of every group may, by changing the parameters, and by the process of analytical propagation, be derived from those of a group containing the identical transformation and all of whose transformations can be arranged in pairs of "inverses."

The parameters play a great part in the theory ; it is by varying them continuously that we obtain the individuals of the group. In fact the equations 1) completely represent the transformation group only when written in the form :

$$1) \qquad x'_i = f_i\,(x_1, \ldots x_n, a_1, \ldots a_r)$$

with the arbitrary parameters put in evidence. When the number of parameters is finite we have a "finite continuous group." By assigning a new set of values to the a_k we obtain another transformation

$$x''_i = f_i\;(x', b)$$

* For this use of "region" cf. Craig's Linear Differential Equations, where I believe "Bereich" found an English equivalent for the first time in America.

† See p. 165.

and if in these equations we replace the x'_i by their values from 1), we must, by the definition of a group, obtain a third transformation

$$x''_i = f_i(x,c)$$

where the c_k are independent of the variables and depend only upon the a_k and b_k.

To reach a useful definition of an infinite continuous group, Lie remarks that we may differentiate the equations

$$x'_i = f_i(x,a)$$

a sufficient number of times with respect to the x_i, eliminate the parameters, and thus obtain a system of differential equations whose most general system of solutions is represented by the equations

$$x'_i = f_i(x,a) \quad [i = 1 \ . \ . \ . \ n].$$

Combining this remark with the definition of a group, we see that a *finite* continuous group is defined by a system of differential equations, such that (1) from any two systems of solutions we can always derive a third, just as above from $x'_i = f_i(x,a)$, and $x''_i = f_i(x,b)$, we derive the equations $x''_i = f_i(x,c)$; and (2) the most general solution contains only a finite number of parameters. Granted now that we have a system of differential equations whose most general solution depends not merely upon a finite number of arbitrary constants but upon arbitrary elements of a higher order, say arbitrary functions, then the host of transformations satisfying this system will form an *infinite* continuous group. For example the equations $\dfrac{\partial F_i}{\partial x_k} = 0 \quad [i \gtrless k\,;\, i, k = 1 \ . \ . \ . \ n]$ define an infinite continuous group,

$$x'_i = \Pi_i(x_i) \quad [i = 1 \ . \ . \ . \ n]$$

the functional symbol Π being clearly perfectly arbitrary.

Upon the foundation of these principles and definitions Lie constructs his theory. The first part of his work, which is now under consideration, is devoted to questions of an entirely general character; the derivation of the fundamental differential equations, the infinitesimal transformations, systems of equations which admit infinitesimal transformations, determination of all the sub-groups of a given group, and other problems of wide generality. The second part treats of the "contact transformations" and will not be specially considered in the present article. Lie's* "Elementarlehrbuch" on differential equations is devoted to ordinary equations of the first, second and third orders and to linear partial ones in three and four variables, which admit known infinitesimal transformations. He shows that those general classes of differential equations which were integrated by the older mathematicians can be characterized as the most general differential equations admitting certain groups. Owing to its elementary character and to its numerous examples it is likely that Lie's theories can be most easily approached through this work by a person entirely unacquainted with them. Reserving, for the present, further remarks upon this most interesting volume, we return to the general theory as set forth in part I.

The literary style of the book deserves mention on account of its perfect simplicity and clearness; the author has carefully avoided adding to the difficulties of the subject by obscure and involved sentences. A patient and persistent effort to make things understood, so far as careful statements, frequent *résumés,* and abundant illustration can do it, is felt throughout the volume.

The r parameters $a_1, \ldots a_r$ contained in the equations of the group may or may not all be essential. To determine this point let the $f_i(x, a)$ be expanded in powers of $x_i - x_i^0$ $[i = 1 \ldots n]$; the coefficients in the series are analytical functions of $a_1, \ldots a_r$, and if r of them are independent, then clearly by giving to $a_1, \ldots a_r$ all possible values we shall obtain ∞^r different transformations. In this case the parameters are all essential; but it is possible that no r of the coefficients may be independent, all of them being expressible in terms of a less number. We can not then obtain ∞^r transformations by varying the $a_1, \ldots a_r$ and may, by introducing functions of the $a_1, \ldots a_r$, diminish by at least unity the number of parameters in evidence. Let the new parameters be $A'_1, \ldots A'_{r-m}$. We can clearly construct at least one linear partial differential equation

$$2) \qquad \sum_1^r{}_k \, \chi_k(a_1 \ldots a_r) \frac{\partial f}{\partial a_k} = 0$$

which shall be identically satisfied by $A'_1, \ldots A'_{r-m}$; it will also be satisfied by $f_1, \ldots f_n$ since $f_1, \ldots f_n$ regarded as functions of the a_k depend only on $A'_1, \ldots A'_{r-m}$. Conversely, if the $f_1, \ldots f_n$ do satisfy one or more equations of the type 2) the parameters are not all essential. This characteristic property of the functions $f_1, \ldots f_n$ is of course of

* This book was arranged and prepared for publication by Scheffers.

enormous importance in the theory, since the number of independent infinitesimal transformations of a group depends upon the number of its essential parameters.*

The parameters being all essential, we now observe that within the scope† of all values of the a_k and the x_i $[k = 1 \ldots r,\ i = 1 \ldots n]$ we may choose regions (a) and (x) of such a nature that within both regions the $f_1, \ldots f_n$ shall be uniform functions of the $n+r$ variables x, a, and shall be developable in positive integral powers of $x_i - x_i^0$, $a_k - a_k^0$, x_i^0 and a_k^0 being arbitrary points in the regions (x) and (a); while the Jacobian

$$\Sigma \pm \frac{\partial f_1}{\partial x_1} \cdots \frac{\partial f_n}{\partial x_n}$$

vanishes for no possible combination of values within those regions. We notice further that now for any two distinct systems of values of the x_i, the equations $x'_i = f_i(x, a)$ yield two distinct values of the x'_i. Under these conditions the x'_i are confined to a certain region (x') and if we assign to the x'_i any system of values within this region, say x'^0_i, and to the a_k any system of values within (a), say a_k^0, then within the region of x'^0_i and a_k^0 the x_i are developable in ordinary power series and are uniform functions of the x'^0_i; in fact they are uniform throughout the entire region (x'). Now in order to make the substitution of the x'^{ν} from $x'_\nu = f_\nu(x, a)$ allowable in $x''^{\nu} = f_\nu(x', b)$, so as to obtain the equation $x''_\nu = f_\nu(x, c)$, we must clearly confine x and a to certain sub-regions $((x))$ and $((a))$ such that x'_ν lies within (x).

The differential equations satisfied by the group are so very important in the theory, since they lead to the infinitesimal transformations, that I shall venture to reproduce a part of the analysis which gives them. We differentiate the equations

$$f_i(x', \ldots x'_n, b_1, \ldots b_r) = f_i(x_1, \ldots x_n, c_1, \ldots c_r)$$

partially with respect to a_k, taking the x_i, c_k and a_k $[i = 1 \ldots n,\ k = 1 \ldots r]$ for independent variables. This is allowable since the only relations existing between the quantities are:

$$x'_i = f'_i(x, a)\,;\ c_k = \varphi_k(a, b)\ \ [i = 1 \ldots n\,;\ k = 1 \ldots r].$$

* A group with r essential parameters is called r-branch.
† Gebiet.

This gives

$$\sum_1^n {}_j \frac{\partial f'_i}{\partial x'_j}\frac{\partial x'_j}{\partial a_k} + \sum_1^r {}_\mu \frac{\partial f'_i}{\partial b_\mu}\frac{\partial b_\mu}{\partial a_k} = 0 \; [k = 1 \ldots r, \; i = 1 \ldots n],$$

where f'_i stands for $f_i(x', b)$. The Jacobian by hypothesis does not vanish, hence we can solve for the $\frac{\partial x'_j}{\partial a_k}$ and obtain,

$$\frac{\partial x'_\nu}{\partial a_k} = \sum_1^r {}_j \Phi_{j\nu}(x', b) \frac{\partial b_j}{\partial a_k} \; [\nu = 1 \ldots n; \; k = 1 \ldots r].$$

It is important to notice that the $\Phi_{j\nu}$ are quite independent of the index k. Further, by using the relations

$$c_\mu = \varphi_\mu(a, b),$$

we may express the $\frac{\partial b_j}{\partial a_k}$ as functions of the a_k and b_k alone, and thus reach the expression

$$\frac{\partial x'_\nu}{\partial a_k} = \sum_1^r {}_j \Psi_{jk}(a, b) \; \Phi_{j\nu}(x', b) \; [\nu = 1 \ldots n; \; k = 1 \ldots r].$$

Whatever values the b_k may have, these equations are always identically satisfied by the substitution $x'_i = f_i(x, a)$; let us then assign fixed numerical values to the b_k so as to be rid of arbitrary constants. Writing $b_k = \omega_k$ and taking

$$\Phi_{j\nu}(x', \omega) = \xi_{j\nu}(x'_1, \ldots x'_n),$$

$$\Psi_{j\nu}(a, \omega) = \psi_{j\nu}(a'_1, \ldots a'_r),$$

we may write:

$$3) \qquad \frac{\partial x'_\nu}{\partial a_k} = \sum_1^r {}_j \psi_{jk}(a) = \xi_{j\nu}(x'_1, \ldots x'_n).$$

The solution of these equations clearly has the form:

$$\xi_{ki}(x'_1, \ldots x'_n) = \sum_1^r {}_j \alpha_{kj}(a) \frac{\partial x_i}{\partial a_j}.$$

Equations 3) are the fundamental differential equations of the group; the functions ξ which appear in them possess the property that they satisfy no n relations $\sum_1^r {}_k e_k \xi_{ki} = 0$ $[i = 1 \ldots n]$, where the e_k are different from 0 and free from $x'_1, \ldots x'_n$. If then we form the r expressions $X_k(f) = \sum_1^n {}_i \xi_{ki}(x) \frac{\partial f}{\partial x_i}$, it is quite clear that the $X_k(f)$ satisfy no relation $\Sigma e_k X_k = 0$.

The beauty and success of Lie's analysis lies in his use of the expressions X_k,* which are the *infinitesimal transformations* of the group. We see from the above that from every r-branch † group we may obtain r infinitesimal transformations as soon as we have formed equations 3).

It is interesting to trace the course of thought which led Lie to designate the expressions as he did, though it is very clear and simple. In the case of a 1-branch group containing the identical transformation, equations 3) take the form

$$4) \qquad \frac{dx'_i}{dt} = \xi'_i(x'_1, \ldots x'_n),$$

t being the parameter, and the equations of the group may be obtained in the form

$$y'_1 = y_1, \ldots \, y'_{n-1} = y_{n-1}, \; y'_n = y_n + t$$

by integrating 4). If now this integration be performed with the initial condition that for $t = 0$, $x' = x$, Taylor's theorem gives

$$x'_i = x_i + \frac{t}{1} \xi_i(x) + \frac{t^2}{1.2} \sum_1^n {}_k \xi_k \frac{\partial \xi_i}{\partial x_k} + \ldots \; [i = 1 \ldots n].$$

Denoting $\sum_1^n {}_i \xi_i \frac{\partial f}{\partial x_i}$ by $X(f)$ and observing that $X(x_i) = \xi_i$, we may write:

$$x'_i = x_i + \frac{t}{1} X(x_i) + \frac{t^2}{1.2} X(X(x_i)) + \ldots$$

We pass from one transformation of the group to another by assigning different values to t, and if t be infinitesimal the value of x' will differ but little from that of x; in other words the transformation will be infinitesimal. We shall have in that case

---

* Of course Lie carefully calls attention to the previous use of similar symbols by Jacobi and Poisson. Abschnitt II., Kap. 7.

† Having r essential parameters.

$$x'_i = x_i + X(x_i)\,dt.$$

It is possible to express any function $f(x'_1, \ldots x'_n)$ in the same way

$$f(x'_1, \ldots x'_n) = f(x) + \frac{t}{1} X(f) + \ldots.$$

so that $X(f)\,dt$ really denotes the infinitesimal increment of the function when the parameter takes an increment dt. The connection between infinitesimal transformations and 1-branch groups is briefly this, that every infinitesimal transformation generates a definite 1-branch group, and every 1-branch group containing the identical transformation is generated by a definite infinitesimal transformation. There is no difficulty in showing* that when the group does not contain the identical transformation we can obtain all the transformations, at least within a certain region, by first effecting a particular, fixed, transformation of the group, and then a transformation of a group generated by infinitesimal transformations.

In a review article it is impossible to do more than point out with hasty gestures the landmarks which must guide the traveler who would explore the vast intellectual territory which Lie has opened up and minutely described; but it would be a wrong to the reader who has yet before him the pleasure of a first perusal of this book, to omit to mention the passages in which the author demonstrates the generation of the r-branch by the 1-branch groups. There are passages in mathematical works which possess all the facility, the grace, and harmony of construction which characterize purely literary productions of the highest order; but add to this charm the dramatic element which comes from the ordered movement of the demonstration, and we shall, perhaps, partially account for the reader's feeling as he peruses these pages.† It is sufficient here merely to state that the r-branch group containing the identical transformation and whose transformations can be arranged in pairs of "inverses," is generated by ∞^{r-1} 1-branch groups; and that other r-branch finite continuous groups can be obtained within a certain region by first performing a fixed transformation of the group and then a transformation of a certain 1-branch group whose infinitesimal transformation has the form $\sum_1^r k\, \lambda_k X_k(f)$, where $X_1, \ldots X_k$ are the infinitesimal transformations given by equations 3).

It is not difficult to foresee that important relations must

---

* See p. 50. † See pp. 61–81.

exist between the group whose infinitesimal transformations are $X_1, \ldots X_r$ and the complete system of linear partial differential equations formed by equating to zero $X_1, \ldots X_r$ and such expressions of the form $(X_i X_j)$,* as are not linearly expressible in terms of $X_1, \ldots X_r$; in fact the solutions of the complete system are the invariants of the group. Let $\omega\,(x_1, \ldots x_n)$ be such a solution; then, as we have seen,

$$\omega\,(x'_1, \ldots x'_n) = \omega\,(x) + \frac{t}{1} \sum_1^r{}^k \lambda_k X_k(\omega) + \frac{t^2}{1.2} \sum_1^r{}^k \lambda_k X_k \left(\sum_1^r{}^k \lambda_k X_k(\omega) \right) + \ldots$$

But by hypothesis $X_k(\omega) = 0$, and, therefore, $X_i\,(X_k(\omega)) = 0$, and so on; hence, $\omega\,(x') = \omega\,(x)$, or, as it may be expressed, *the solutions of the complete system admit* all the transformations of the group.

When $X_1, \ldots X_r$ equated to zero form a complete system, it is well known that we have always

$$(X_j X_k) = \sum_1^r{}^\mu \psi_{\mu jk}\,(x_1, \ldots x_n)\, X_\mu,$$

but it is a remarkable fact that when $X_1, \ldots X_r$ are the infinitesimal transformations of an r-branch group the $\psi_{\mu jk}$ become absolute constants and we have

$$(X_j X_k) = \Sigma c_{\mu jk}\, X_\mu.$$

Certain relations evidently exist among the $c_{\mu jk}$, since $(X_j X_k) = -(X_k X_j)$ and since the Jacobian identity gives $[X_i(X_j X_k)] + [X_j(X_k X_i)] + [X_k(X_i X_j)] = 0$; it is therefore not true that any set of constants chosen at random will correspond to some group. The array of the constants $c_{\mu jk}$ is called by Lie the setting† of the group; when they are known it is possible to determine all the continuous sub-groups.

Let $Y_1, \ldots Y_m$ be the infinitesimal transformations of a sub-group of $X_1, \ldots X_k$. Then we must have $(Y_\mu\, Y_\nu) = l_{\mu\nu\pi}\, Y_\pi$, by definition of a sub-group. But also $(Y_\mu\, Y_\nu) = \sum_{\rho\sigma}^{1 \ldots \nu} h_{\mu\rho}\, h_{\nu\sigma}\,(X_\rho\, X_\sigma)$ since

$$Y_\mu = \sum_1^r{}^\rho h_{\mu\rho} X_\rho.$$

* This well-known symbol needs no explanation.
† Zusammensetzung.

Equating the two expressions for $(Y_\mu\ Y_\nu)$ it is found possible by purely algebraical operations to determine the various sets of $h_{\mu\rho}$ in terms of the $c_{\mu jk}$ and thus to determine all the subgroups of a given r-branch group.

We shall make no attempt here to follow the author in those investigations of special properties of groups which occupy a large part of the remainder of the book. The great and increasing importance of his subject would be sufficient of itself to call the attention of mathematical readers to Lie's volumes, but we hope, even at the risk of doing a work of supererogation, to supplement this sketch in the near future with an account of the second part of the Theory of Transformation Groups, which deals with the transformations of contact. C. H. CHAPMAN.

MILWAUKEE, *November* 1, 1892.

THE THEORY OF FUNCTIONS OF A REAL VARIABLE.

ULISSE DINI, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse.* Deutsch bearbeitet von Dr. JACOB LÜROTH, Professor zu Freiburg i. B., und ADOLF SCHEPP, Premier Lieutenant A. D. zu Wiesbaden. Leipzig Teubner, 1892. 8vo, pp. xviii + 554.

THIS German version will be welcomed by many mathematicians who have been debarred from a study of Dini's classical treatise by unfamiliarity with the language. In the preface the translators state that they have not attempted to recast the materials, although they feel that some investigations might be transferred with advantage from the later to the earlier sections. Several paragraphs have been added, and the second half of the book has been divided into short chapters. The principal novelties are the introduction of Cantor's definition of the irrational number in place of Dedekind's, the proof that a continuum of points cannot be a mass of points of the first power, an account of Cantor's method for the condensation of singularities, and some additional theorems on integration. The usefulness of the book has been greatly increased by the insertion of numerous references to original sources, and by a list, at the end, of 45 authors whose works are referred to in the text. The translators have performed their difficult task with skill and judgment. We must call attention, however, to the numerous misprints which disfigure the text; for instance, § 8* is, as it stands, very obscure.