

Equating the two expressions for $(Y_\mu Y_\nu)$ it is found possible by purely algebraical operations to determine the various sets of $h_{\mu,\rho}$ in terms of the $c_{\mu,jk}$ and thus to determine all the sub-groups of a given r -branch group.

We shall make no attempt here to follow the author in those investigations of special properties of groups which occupy a large part of the remainder of the book. The great and increasing importance of his subject would be sufficient of itself to call the attention of mathematical readers to Lie's volumes, but we hope, even at the risk of doing a work of supererogation, to supplement this sketch in the near future with an account of the second part of the Theory of Transformation Groups, which deals with the transformations of contact.

C. H. CHAPMAN.

MILWAUKEE, November 1, 1892.

THE THEORY OF FUNCTIONS OF A REAL VARIABLE.

ULISSE DINI, *Grundlagen für eine Theorie der Functionen einer veränderlichen reellen Grösse*. Deutsch bearbeitet von Dr. JACOB LÜROTH, Professor zu Freiburg i. B., und ADOLF SCHEFF, Premier Lieutenant A. D. zu Wiesbaden. Leipzig Teubner, 1892. 8vo, pp. xviii + 554.

THIS German version will be welcomed by many mathematicians who have been debarred from a study of Dini's classical treatise by unfamiliarity with the language. In the preface the translators state that they have not attempted to recast the materials, although they feel that some investigations might be transferred with advantage from the later to the earlier sections. Several paragraphs have been added, and the second half of the book has been divided into short chapters. The principal novelties are the introduction of Cantor's definition of the irrational number in place of Dedekind's, the proof that a continuum of points cannot be a mass of points of the first power, an account of Cantor's method for the condensation of singularities, and some additional theorems on integration. The usefulness of the book has been greatly increased by the insertion of numerous references to original sources, and by a list, at the end, of 45 authors whose works are referred to in the text. The translators have performed their difficult task with skill and judgment. We must call attention, however, to the numerous misprints which disfigure the text; for instance, § 8* is, as it stands, very obscure.

Such a work as this of Dini's raises the question as to how far refinements in analysis ought to be carried. Does science gain anything from researches on progressive and regressive differential quotients, on assemblages of points in close and loose order, on continuous functions with infinitely many maxima and minima in a region of finite extent? Or are these, and other curiosities, freely scattered through 500 pages mere mathematical toys of a somewhat higher order of excellence than the magic squares which used to please our ancestors? Eminent mathematicians are disagreed upon this point. Investigations which relate to such matters are, according to some writers, simply studies of disease, or contributions to the "morbid pathology of functions"; for they can have no bearing upon truly important problems of pure or applied mathematics. This view is, we believe, a mistaken one. It is worthy of note that some of the theorems which must have been startling in their novelty, when delivered in lectures at Pisa in 1871, have now become current mathematical coin; as, for example, those on limits, on continuous functions without differential quotients, on uniform convergence of series, on the integration and differentiation of series, etc. It is equally worthy of note that these theorems have either been discovered by, or occupied the attention of, many distinguished living mathematicians.

The first three chapters are preliminary to the discussion of the continuity and discontinuity of functions; they deal with irrational numbers, masses of points, limits, and oscillations. In chs. IV. to VI. functions are considered with regard to their continuity, or discontinuity; in ch. VII., with regard to the character of their differential quotients. Ch. VIII. treats of infinite series; ch. IX. of the condensation of singularities; ch. X. of continuous functions without differential quotients; ch. XI. of progressive and regressive differential quotients. In ch. XII. some important theorems are given, relating to the existence of differential quotients. As a specimen we may instance the following:

If a function $\varphi(x)$ is given, with finite and continuous differential quotients at the points of a given interval (a, b) , to which x_0 belongs, then Taylor's series

$$\varphi(x_0) + (x-x_0)\varphi'(x_0) + \frac{(x-x_0)^2}{2!}\varphi''(x_0) + \dots$$

is, generally, convergent throughout some interval $(x_0 - \rho, x_0 + \rho)$. This series represents a function $\psi(x)$ which is finite and continuous, together with all its differential quotients, at all points of the interval (with the possible exception of the terminal points). The functions $\varphi(x)$, $\psi(x)$ have

the same values and the same differential quotients at x_0 , but may differ in these respects at all other points of the region common to (a, b) and $(x_0 - \rho, x_0 + \rho)$. It is instructive to contrast this theorem with Cauchy's theorem, that a monogenic function $\varphi(x)$ of a complex variable x , which is holomorphic in the domain of x_0 , is equal to

$$\varphi(x_0) + (x - x_0) \varphi'(x_0) + \frac{(x - x_0)^2}{2!} \varphi''(x_0) + \dots$$

throughout that domain.

Ch. XIII. to XX. are concerned with integration. Free use is made in them of the theory of masses of points. Any system of real numbers constitutes a mass P of real numbers; or, what amounts to the same thing, a mass of points on a straight line. This mass gives rise to derived masses $P^{(1)}$, $P^{(2)}$, $P^{(3)}$, . . . The first derived mass consists of all the limiting points of the mass P . For instance, if

$$P = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{2} + \frac{1}{3}, \frac{1}{4}, \frac{1}{2} + \frac{1}{4}, \frac{1}{5}, \frac{1}{2} + \frac{1}{5}, \dots),$$

the limiting points are $0, \frac{1}{2}$. In this example there is no derived mass $P^{(2)}$; and the mass P is said to be of the first kind. If the number of derived masses is finite and $= k$, the original mass is said to be of the k th kind and of the first species; but if the number of derived masses is infinite, the mass is said to be of the second species. At each point of $P^{(1)}$ there is an infinite cluster of P -points, but the points of $P^{(1)}$ may be distinct from those of P . The points of $P^{(2)}$, $P^{(3)}$, . . . are all included among those of $P^{(1)}$. When P is identical with $P^{(1)}$ the mass is perfect. The mass which consists of all the rational numbers between 0 and 1 is imperfect. In this example the mass of points fills completely the interval $(0, 1)$; and the points are said to be in close order, since one point at least of the mass lies in any partial interval however small. If the discontinuities of a finite function $f(x)$ fill completely a certain interval, the function is not integrable for that interval. It may happen that a mass of infinitely many points is such that no segment of an assigned interval, such as $(0, 1)$, is filled completely. In this case the points are said to be in loose order. When the places of discontinuity of a finite function $f(x)$, for the assigned interval, are in loose order integration *may* be possible (Hankel fell into an error when he asserted that it *must* be possible); but if they are in close order, integration, in the sense usually accepted, is impossible. The editors mention in the preface that they omit the extension of the integral-concept to this case, because it would carry them too far into the calculus of the transfinite num-

bers. In ch. XIII. the passage to the new definition of a definite integral is skilfully effected by the use of the old definition

$$\int_{\alpha}^{\beta} \varphi'(x) dx = \varphi(\beta) - \varphi(\alpha).$$

It is shown that the integral of a finite function $f(x)$ in a finite interval (α, β) , where $\beta > \alpha$, is the limit to which $\sum_1^n \delta_i f(x_{i-1} + \varepsilon_i \delta_i)$ tends, when d , the norm of the division, decreases below any assigned number. Here the x 's are points interpolated between α, β ; the sub-intervals δ are less than d , and the numbers ε lie between 0 and 1, but depend upon the numbers x_1, x_2, \dots, x_{n-1} . The second step in the argument is the proof that the ε 's can be chosen arbitrarily between 0 and 1; and next the oscillations D_1, D_2, \dots, D_n in the partial intervals are introduced. Finally, it is shown that the necessary and sufficient condition that the finite function $f(x)$ be integrable in the interval (α, β) is the following: to every arbitrarily small number σ there must correspond some system of partial intervals $\delta_1, \delta_2, \dots, \delta_n$ such that the corresponding sum $\sum \delta_i D_i < \sigma$. Professor H. J. S. Smith has pointed out that the proof of this theorem is more difficult than might at first sight appear. It is not enough to show that

$$\delta_1 D_1 + \delta_2 D_2 + \dots + \delta_n D_n$$

tends to a limit when all the δ 's tend to 0, whatever be the law of division into sub-intervals. In the proof account must be taken of the fact that the greatest value of $\sum \delta_i D_i$ arises from a set of sub-intervals distinct from the set which makes $\sum \delta_i D_i$ a minimum. Many theorems are given which relate to the integration of finite functions whose discontinuity-points form a mass of the first species. For example, let $S = \sum_{i=1}^n \delta_i p_i$, where p_i is 1 or 0, according as a point of discontinuity falls in δ_i or not. If S has a lower limit 0, $f(x)$ is integrable in (α, β) . There are still other divisions of masses. A mass is said to be numerable (*abzählbar*) when the points of the mass correspond, one-to-one, with the system of numbers 1, 2, 3, . . . ; such a mass is of the first power.

That this theory of masses is not without applications can be proved by a striking example. Jacobi's famous dictum with regard to functions of one variable "*functio tripliciter periodica non datur*" has led to many misconceptions; its accuracy has been disputed by Göpel and Casorati, and it has been shown that there exist infinitely many-valued functions of one variable which have more than two periods. Jacobi's

argument, as presented by Clebsch and Gordan, is briefly this : the absolute value of $m_1\omega_1 + m_2\omega_2 + \dots + m_{2p}\omega_{2p}$ can be made less than any assigned number ε , however small, and therefore the Abelian integral of the first kind, with assigned upper and lower limits, can be made to take every value by a suitable choice of periods. This amounts to saying that the points $m_1\omega_1 + m_2\omega_2 + \dots + m_{2p}\omega_{2p}$ form a continuum. But it can be shown that the points just mentioned form a numerable mass, whereas the points of a continuum cannot be made to correspond one-to-one with the system $(1, 2, 3, \dots)$. Thus the conclusion drawn by Clebsch and Gordan is unwarranted ; and the upper limit of an Abelian integral of the first kind may be regarded, legitimately, as an infinitely many-valued function of the integral itself. As Casorati has pointed out,* Fuchs makes the following assertion solely on the strength of Jacobi's theorem : " Es ist ein für die Grundlagen der Theorie der Differentialgleichungen wesentlicher Umstand, dass es auch unter den Differentialgleichungen jeder Ordnung und jeden Grades, welche nicht nur Transformationen von (α) sind, Classen solcher Art gibt, welche zwischen der unabhängigen und abhängigen Veränderlichen keine functionale Beziehung im gewöhnlichen Sinne des Wortes festsetzen, so lange jene Veränderlichen complexe Werthe annehmen dürfen " [here (α) refers to a differential equation $dx/du = \sqrt{R(x)}$]. This example shows that the theory of masses of points is not of purely theoretic interest. It would not be difficult to bring forward other examples. The theory of automorphic functions, recently developed by Poincaré and Klein, affords instances of analytic functions with natural boundaries, and attention in such cases is directed to the character of the mass of singular points. The importance of the analytic function of the complex variable is well known, and it is a matter of some consequence to have a method for the construction of analytic functions with linear barriers. In this connection the continuous function without a differential quotient becomes of use. M. Painlevé, in a memoir of marked originality,† has constructed such functions by means of the series

$$\sum_{n=1}^{\infty} \frac{\cos a_n \theta}{a_n}, \quad \sum_{n=1}^{\infty} \frac{\sin a_n \theta}{a_n},$$

which, when $a_n = n^n$, or $n!$ are continuous functions of θ without differential quotients. The associated series

$$\sum_{n=1}^{\infty} \frac{z^{a_n}}{a_n},$$

* *Acta Mathematica*, vol. VIII., Les fonctions d'une variable à un nombre quelconque de périodes, p. 347.

† *Sur les lignes singulières des fonctions analytiques*. Paris, 1887.

represents an analytic function of z in a circle whose centre is 0 and radius 1 ; but the series is not continuable beyond the rim of the circle. Other examples of the use of Dini's theorems might be selected from Painlevé's memoir, from recent researches on Fourier's series, and from Dirichlet's problem in the theory of the potential.

May we, in conclusion, express the hope that some reader of the BULLETIN will follow the good example of Dr. Lüroth and Lieutenant Schepp, and translate this highly interesting work into English. After a student has become familiar with the ordinary processes of the infinitesimal calculus, it is highly desirable that he should have easy access to a special treatise in which attention is paid to fundamental principles rather than to details. Dini's treatise fulfils these requirements, and is at the same time flawless as regards rigour of proof and clearness of explanation.

J. HARKNESS.

BRYN MAWR, *December 7, 1892.*

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, December 3, at half-past three o'clock, the vice-president, Professor Fine, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Professor Fabian Franklin, Johns Hopkins University; Dr. George W. Hill, West Nyack, N. Y. It was announced that the annual meeting would be held on Thursday afternoon, December 29, at four o'clock. A committee of three, consisting of Dr. Pierson, Dr. Stabler, and Mr. Maclay, was elected to report at the annual meeting nominations for the officers and other members of the council for the coming year. A paper by Professor J. W. Nicholson on "The expression of the n th power of any number in terms of the n th powers of other numbers, n being any positive integer," was read. Professor Fine called attention to a purely algebraic method, not involving the notion of continuity, for treating the theory of contact of algebraic curves.

THE recent circular issued by the committee on the proposed joint memorial at Göttingen to Gauss and Weber, contains the following remarks upon these two great investigators: "What both accomplished in the service of science is not the property of their pupils alone, but an inheritance of all mankind, which has already been, and which still in the future