A BIT OF MATHEMATICAL HISTORY.

BY DR. MAXIME BÔCHER.

IN the transactions of the Academy of St. Petersburg for the year 1764 is a memoir by Leonhard Euler entitled : *De Motu Vibratorio Tympanorum,* which is remarkable in several respects. Certainly the last part of this memoir has been lost sight of for many years by mathematicians. Whether the same can be said of the first part I do not know,* but I will venture to give a brief account of the contents of the whole paper.

During the second third of the eighteenth century a number of mathematicians, among whom Daniel Bernoulli and Euler deserve special mention, had given much attention to problems concerning the vibration of such one dimensional bodies as strings, rods and columns of air, but the paper now under consideration is the first in which a two dimensional problem of the same nature was taken up; the question chosen being the problem of the vibration of a stretched elastic membrane. The now familiar equation of motion of the membrane :

$$\frac{1}{e^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}$$

is first obtained, and then follows a brief discussion of the simple harmonic vibrations of a rectangular membrane, but although the square membrane is chosen as a simple case in which to give numerical results the peculiar interest of this case is not mentioned.

Failing to find a general method for the discussion of the vibrations of membranes of *any* shape, Euler next takes up the question of the vibration of a circular membrane. The first step towards the solution of this problem is the introduction of polar in place of rectangular coördinates in the equation of motion, and even this step is worthy of notice as being apparently the first case of a change of independent variables in a partial differential equation.†

* Certainly Poisson did not know of the existence of this paper when he wrote his celebrated memoir on elasticity (*Mémoires de l'Institut,* 1829) which is frequently referred to as containing the first investigation of the vibration of membranes.

† Euler himself says: Modus autem, quo hanc æquationem elicuimus, novam quandam algorithmi speciem constituit, quæ omni attentione digna videtur.

Euler then proceeds to find an infinite number of solutions of this equation in the form of products of three factors, functions respectively of the three independent variables; a step which although perhaps small in itself is important as being in the line of development of a great theory.

One of these three factors turned out of course to be what is now known as a Bessel's function,* and the fact that these functions occur in this paper is the most important point I have to bring out in this note. Until recently it was thought that Fourier had introduced the Bessel's function of the zeroth order (1812), and that the other functions of integral orders were first made use of by Bessel (1818). A few years ago, however, Maggi called attention to papers by Daniel Bernoulli (1732) and Euler on the vibration of a heavy string suspended from one end, in which Bessel's functions of the zeroth order occur.† That further than this *all* the Bessel's functions of the first kind and of integral orders go back like so many other things to the great mathematicians of the eighteenth century seems to have remained unnoticed.

The second solution of Bessel's equation is known to contain a logarithmic term. Euler, however, appears to find a second solution free from this term. The explanation of this peculiar mistake is that Euler's second solution really vanishes identically.‡ Fortunately he decides from quite different considerations that this second solution is not applicable to the problem in hand.

Finally I will quote in full a passage by the same author which, although not occurring in the paper just discussed, bears upon the same questions. It will be found in the transactions of the Academy of St. Petersburg for 1781 (p. 164) in a paper on the above mentioned problem of the vibration of a heavy string hanging from one end. The expression $\Pi : \frac{a}{b}$ which occurs in the first line is essentially a Bessel's function of the first kind and zeroth order:

"Quodsi ergo æquationis $\Pi : \frac{a}{b} = 0$ omnes radices assignare liceret, ex iis utique solutionem tam generalem deducere possemus, quæ sine ullo dubio omnes plane motus, qui in fune

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* Euler mentions the connection between Bessel's equation and Riccati's equation, and hence determines the cases, which he does not further consider, where it can be solved in finite form.

† For references see HEINE, Handbuch, vol. II., p. 343.

‡ Euler avoids this mistake in a subsequent paper in which he introduces the Bessel's functions of the second kind although only of the zeroth order; see the *Transactions of the Academy of St. Petersburg* for 1781, p. 187, where a problem similar to Bernoulli's problem of the swinging string is considered.

locum habere queant, in se complectatur. Neque vero idcirco problema principale resolvere liceret, quo pro statu initiali quocunque ejus motus secuturus requiritur; ad hoc enim necesse esset, infinitas illas constantes arbitrarias pro statu initiali dato debite determinare, quod certe opus omnes vires analyseos longe esset superaturum."

In order to see how remarkable this passage is it is necessary to recall the celebrated triangular contest between D'Alembert, Euler and Daniel Bernoulli which began at about the middle of the eighteenth century.* We have here to deal with the question under discussion between Bernoulli and Euler, the former maintaining that the most general vibration of a stretched string could be expressed as a trigonometric series, the latter that this could give only a particular class of solutions, since (a fact no one doubted at that time) a function which would be graphically represented by a broken line could not possibly be expressed as a trigonometric series, or in fact by any analytic formula.

Now the above quotation refers to an entirely analogous problem, the principal difference being that the trigonometric series is replaced by a series involving Bessel's functions, and here we find Euler taking Bernoulli's position and asserting that without any doubt this series expresses the most general motion possible! I am at a loss to explain this passage, for we must apparently either assume that Euler had forgotten the previous discussion which had lasted so long and formed such an important episode in his life and in which, considering the mathematical knowledge of the time, his position had been unassailable; or that he had anticipated Fourier's wonderful idea that *any* function can be developed in a trigonometric series, or a series involving Bessel's functions. That he should have arrived at this point without the method of determining the coefficients seems incredible.

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NOTES.

THE annual meeting of the NEW YORK MATHEMATICAL SOCIETY was held Thursday afternoon, December 29, at four o'clock, the president, Dr. McClintock, in the chair. The council announced that Professor Henry B. Fine had been appointed to act as delegate of the Society to the meetings of the American Philosophical Society on the occasion of the

* See RIEMANN, Works, pp. 214–218 (sec. ed. pp. 227–231).