It is to be distinctly observed that in this process we do not require the functions fx and $\sum\limits_0^{n-1} A_r\phi_r x$ to have a contact of the $(n-1)$th order at $x = a$ in order that we may equate their first $n - 1$ derivatives when $x = a$. What we require is merely that the functions fx and $\varphi_r x$ $(r = 1 \ldots n - 1)$ shall each have a determinate derivative at $x = a$, up to the $(n - 1)$th operation. Of course, if fx and $\sum\limits_0^{n-1} A_r\phi_r x$ have an $(n - 1)$th contact at $x = a$, then our value for R holds true as well; but it is not dependent on such a relation: it simply includes it.

If now the successive functions $\phi_r x$ $(r = 1 \ . \ . \ . \ n)$ may be formed in succession indefinitely according to a given law so that we may make r in $\phi_r x$ as great as we choose, then if it can be shown that R has for its limit zero, as r becomes infinite and at the same time the A's have limiting values such that $\sum\limits_0^{\infty} A_r\phi_r x$ is a converging series, then we may write

$$fx = A_0 + A_1\phi_1 x + A_2\phi_2 x + \ . \ . \ . \ \text{ad. inf.}$$

The value of R has been shown to be

$$\frac{\begin{vmatrix} 1, & \phi_1 x & \ldots & \phi_n x \\ 1, & \phi_1 a & \ldots & \phi_n a \\ 0, & \phi_1' a & \ldots & \phi_n' a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \phi_1^{n-1} a & \ldots & \phi_n^{n-1} a \end{vmatrix} \left(\dfrac{d}{dx}\right)^n_{x=u} \begin{vmatrix} fx, & 1, & \phi_1 x & \ldots & \phi_{n-1} x \\ fa, & 1, & \phi_1 a & \ldots & \phi_{n-1} a \\ f'a, & 0, & \phi_1' a & \ldots & \phi_{n-1}' a \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f^{n-1}a, & 0, & \phi_1^{n-1} a & \ldots & \phi_{n-1}^{n-1} a \end{vmatrix}}{\begin{vmatrix} \phi_1' a & \ldots & \phi_{n-1}' a \\ \cdot & \cdot & \cdot \\ \phi_1^{n-1} a & \ldots & \phi_{n-1}^{n-1} a \end{vmatrix} \left(\dfrac{d}{dx}\right)^n_{x=u} \begin{vmatrix} 1, & \phi_1 x & \ldots & \phi_n x \\ 1, & \phi_1 a & \ldots & \phi_n a \\ 0, & \phi_1' a & \ldots & \phi_n' a \\ \cdot & \cdot & \cdot & \cdot \\ 0, & \phi_1^{n-1} a & \ldots & \phi_n^{n-1} a \end{vmatrix}} \qquad (11)$$

in which u is some unknown value of x lying between x and a.

ON THE EARLY HISTORY OF THE NON-EUCLIDIAN GEOMETRY.

BY EMORY McCLINTOCK, LL.D.

IT has until recently been supposed that the earliest work on non-euclidian geometry was Lobatschewsky's.* A much earlier production (1733) has been brought into notice by

* See BULLETIN of November, 1892, vol. II, No. 2, "On the Non-Euclidian Geometry."

Beltrami,* the author of which was Geronimo Saccheri, a Jesuit father of Milan, where he had charge of the Collegio di Brera. Under the title "Euclides ab omni Nævo vindicatus," this writer produced a work which, when we recall the many later futile discussions of the parallel-axiom, must be counted as marvellous. It is to be hoped that the society of which he was an ornament, and to which the many scientific achievements of its members are a just cause for pride, may reproduce for the benefit of future readers a treatise which may otherwise be known only at second-hand.

With our present knowledge, which leads to distinctions of distance-measurement, we may designate that (euclidian) geometry in which the fourth side of an attempted rectangle is equal to the side opposite as the geometry of equal distance; that of Lobatschewsky, in which it is greater, as the geometry of greater distance; and that third system, in which it is less, as the geometry of smaller distance. Saccheri dealt with the angles adjacent to the fourth side, deriving thence three possible systems, which he named respectively "hypothesis anguli recti," "hypothesis anguli acuti," and "hypothesis anguli obtusi," the first, of course, being euclidian, and proving as to each system "si vel in uno casu sit vera, semper in omni casu illa sola est vera." Other propositions were successively established, as for instance, "Ex quolibet triangulo, cujus tres simul anguli æquales sint, aut majores, aut minores, duobus rectis, stabilitur respective hypothesis aut anguli recti, aut anguli obtusi, aut anguli acuti," and "Esto quodvis triangulum AHD rectangulum in H. Tum in AD continuata sumatur portio DC æqualis ipsi AD, demittaturque ad AH productam perpendicularis CB: dico stabilitum hinc iri hypothesim aut anguli recti aut anguli obtusi aut anguli acuti, prout portio HB æqualis fuerit aut major aut minor ipsa AH."

Saccheri proved the "hypothesis anguli obtusi" untrue, as being incompatible with customary axioms; but the "hypothesis anguli acuti" caused him much greater difficulty, just as it subsequently did to Legendre. He confessed to a distracting heretical tendency on his part in favor of the "hypothesis anguli acuti," a tendency against which, however, he kept up a perpetual struggle (diuturnum prœlium). After yielding so far as to work out an accurate theory anticipating Lobatschewsky's doctrine of the parallel-angle, he appears to have conquered the internal enemy abruptly, since, to the surprise of his commentator Beltrami, he proceeded to announce dogmatically that the specious "hypothesis anguli acuti" is positively false.

* "Un Precursore Italiano di Legendre e di Lobatschewsky": *Rendiconti, R. Accad. dei Lincei,* 1889, I, 441.

The relation of Gauss to the work of the other non-euclidian pioneers, Lobatschewsky, Bolyai, Riemann, is so obscure as to constitute a historical problem.* Jacobi, writing to Legendre in 1828 on another subject, accuses Gauss of spreading a veil of mystery over his work. On this subject Gauss himself wrote to Bessel in 1829 that he had busied himself with it for nearly forty years, had made very extensive researches, and meant to publish nothing. In 1831 he wrote to Schumacher giving various details of what he called the non-euclidian geometry, including the important formula for the circumference of a circle published later by the younger Bolyai. In his youth he had discussed the subject actively with the elder Bolyai. Another, and according to Houël the close, friend of Gauss's youth was Bartels, who in 1807 went to Kazan, where subsequently Lobatschewsky became his chief pupil, his assistant, and his successor. When Bolyai's work appeared, Gauss wrote approvingly to his father, implying that the results agreed with his own, which would not be published; and when Lobatschewsky's treatise in German came out in 1840, Gauss expressed his hearty approval, though saying he had already obtained substantially the same theory. A little later he secured for Lobatschewsky an election to the Royal Society of Göttingen. There is no question as to Lobatschewsky's priority of publication, beginning with a lecture at Kazan in 1826 followed by repeated essays in Russian, or as to the originality of his work as a whole, whatever suggestions may have come from Bartels at the beginning. And there is no proof that whatever is common between Gauss and the Bolyais was uniformly original with Gauss. He was, however, the great genius who was for many years intent upon the subject, and between whom and all others interested in it during his lifetime there are plain lines of connection. His illustrious pupil Riemann took a different point of view, starting with Gauss's formula for the measure of curvature, and must have had at least some slight acquaintance with the Gauss-Lobatschewsky trigonometry valid for his suggested sur-

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*In the paper already cited I followed Beez in stating too strongly the probable connection between Gauss and Lobatschewsky. I am indebted for my first knowledge of Beltrami's account of Saccheri to a letter from Professor Beez, in which he admits his mention of Gauss as the teacher of Lobatschewsky to be partly inferential, and not to be taken literally. Klein (Lectures, published 1892) says of Gauss: "Aber es ist auch keinem Zweifel unterworfen, dass er durch seinen Einfluss die Untersuchungen von Lobatscheffsky und Bolyai angeregt hat. . . . Von Bartels wurde Lobatscheffsky ganz ausführlich mit den Gaussischen Schriften bekannt gemacht, und es kann kaum bezweifelt werden, dass er von diesem auch eingeweiht wurde in die Fragestellung der Nicht-Euklidischen Geometrie."

faces of negative curvature,* although even of this, were any one to dispute it, there is probably no extant evidence which would be available in a court of law.

MORRISTOWN, *February* 18, 1893.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, February 4, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons, having been duly nominated and being recommended by the council, were elected to membership: Professor Heinrich Maschke, University of Chicago; Lieutenant C. De Witt Willcox, U. S. A., U. S. Military Academy, West Point; Mr. J. N. James, U. S. Naval Observatory, Washington; Mr. Abraham Cohen, Johns Hopkins University. The council announced the adoption of the following resolution: "That any member of the society in good standing who is connected with an educational institution may order one extra copy of the BULLETIN for the use of such institution at the price of $2.50 a year."

Professor Thomas Craig read a paper entitled "Some of the developments in the theory of ordinary differential equations between 1878 and 1893." This paper appears in the present number of the BULLETIN, p. 119.

Dr. McClintock mentioned his having recently devised the following continued products, wherein $y = x - x^2$:

$$\frac{\sin(x\pi)}{\pi} = y(1+y)\left(1+\frac{y^2}{2^2[1+y]}\right)\left(1+\frac{y^2}{3^2[2+y]}\right)\cdots\left(1+\frac{y^2}{r^2[r-1+y]}\right)\cdots$$

$$= y\frac{2+y}{2-y}\left(1-\frac{y^2}{2^2[3-y]}\right)\left(1-\frac{y^2}{3^2[4-y]}\right)\cdots\left(1-\frac{y^2}{r^2[r+1-y]}\right)\cdots$$

$$= x\frac{1-x^2}{1+x^2}\left(1+\frac{x^2-x^4}{1.2[2+x^2]}\right)\left(1+\frac{x^2-x^4}{2\ 3[3+x^2]}\right)\cdots\left(1+\frac{x^2-x^4}{r[r-1][r+x^2]}\right)\cdots$$

If $x = \frac{1}{2}$, these become

$$\frac{1}{\pi} = \frac{5}{16}\left(1+\frac{1}{4^2.5}\right)\left(1+\frac{1}{6^2.9}\right)\cdots = \frac{5}{16}\left(1+\frac{1}{80}\right)\left(1+\frac{1}{324}\right)\cdots$$

$$= \frac{1}{4}\cdot\frac{1^2.5}{2^2.1}\cdot\frac{3^2.9}{4^2.5}\cdot\frac{5^2.13}{6^2.9}\cdots = \frac{9}{28}\left(1-\frac{1}{4^2.11}\right)\left(1-\frac{1}{6^2.15}\right)\cdots$$

$$= \frac{9}{28}\left(1-\frac{1}{176}\right)\left(1-\frac{1}{540}\right)\cdots = \frac{1^2}{3}\cdot\frac{3^2.3}{2^2.7}\cdot\frac{5^2.7}{4^2.11}\cdots$$

$$= \frac{3}{10}\left(1+\frac{1}{24}\right)\left(1+\frac{1}{104}\right)\left(1+\frac{1}{272}\right)\left(1+\frac{1}{560}\right)\left(1+\frac{1}{1000}\right)\cdots$$

*** Riemann hat freilich schon 1854 die Beziehung sehr wohl gekannt." —Klein, Vorlesung, i. 191.**