INSTRUCTION IN MATHEMATICS IN THE UNITED STATES.

BY PROF. T. H. SAFFORD.

WITHIN a few years the general standard of mathematical teaching in this country has greatly risen. The history, in brief, of this subject can be easily narrated. Before the Revolution, our colleges were colonial English, and were somewhat like the smaller colleges of Oxford and Cambridge. The attempt was made to teach Newton's philosophy, with its introductory mathematics, including Euclid. These efforts were only partially successful, because of the low standard of scholarship in the colleges. Later, the country became independent; and we began to borrow ideas from our friends the French. This culminated, still later, in the mathematical text-books of Farrar, Davies, Loomis, and others; essentially based on French models, with that lack of demonstrative rigor which was permitted early in the century. The American mathematicians Bowditch, Peirce, and Chauvenet, with their disciples and followers, introduced the writings of Gauss and the Germans and other modern Continental writers, including of course Cauchy, Leverrier and other great French geometers, to this country; and the Harvard, Yale, and Baltimore schools of mathematics, together with many smaller colleges, have done much to make the study of higher subjects not only possible but necessary, at least on the part of our future professors and teachers of the subject.

The object of the present article is to give some practical hints, with especial reference to the existing state of things. In the first place, our colleges nominally require much which is practically ignored by many teachers of preparatory schools. A student is often admitted to college with many "conditions" in the mathematics. This simply means that he is ill prepared in the subject; and is required, before he can be fully matriculated, to pass a rather superficial re-examination. The teaching in the schools is based very much on the college examination papers; and hence it does not meet the ever-growing requirements of the college course. Again, it is tolerably easy to get a pupil past an entrance-examination, if he is allowed to forget as soon as may be all that is then required. The standard of teaching which entrance-examinations tend to produce is, therefore, a superficial one; and the rise of standard more nominal than real. Certain papers in algebra which the writer has seen are admirably calculated to dampen the enthusiasm of any pupil who attempts to do them; and at best to produce a kind of mathematical teaching which is adapted to the school and not at all to life.

Again, the subjects taught in schools are not rightly arranged. The "higher arithmetic" is quite useless; it is a disguised algebra, and one which is tied down to the American dollar as its main material. The moral effect of this, so far as it has any, is bad; and the results, for those who are to go into business, either bad or none at all.

Algebra, philosophically considered, is a higher arithmetic; and the pupil who is thoroughly taught algebra is quite competent to solve all business riddles much better than he can by the present rather senseless method. Here, as well as anywhere, the remark is in place, that the teachers of mathematics in this country, high and low, are too often persons who do not understand pedagogy, do not pay much attention to the mutual adaptation of the subject and the mind, and attempt to crowd utterly indigestible stuff into the rather limited intellect of the average pupil. The psychologists have long pointed out the fact that technical mathematics is not a good study for training, in so far as it is too abstract; geometry, in which the things to be learned are sensibly represented, is far better than arithmetic, which is very abstract; Euclid's geometry is rightly arranged, in that the seventh, eighth, ninth, and tenth books contain material which is dealt with superficially in our ordinary arithmetics. The neglect of geometry in our schools is a very serious defect. In order to teach the subject as well as algebra is taught, much more time must be given than at present is allowed. The early portions, the rudiments, or *Formenlehre* as the Germans call it, must be taken up in advance of the logical portion; and in fact along with quite rudimentary arithmetic. And the easy early books of demonstrative geometry should come along w it the early parts of algebra. The ordinary practice of deferring all geometry till algebra has been far advanced is altogether bad; it serves only the convenience of schoolteachers and by no means the development of the pupils' minds. Even ordinary reckoning is ill-taught. Mental arithmetic is subordinated to written; while it is an essential constituent of written, and should have priority over it in every respect. The written arithmetic is the servant and not the master.

Many of these things would be bettered if our teachers were more rigidly examined as to their preparation. One can pretty surely infer that, as a rule, they are instructing in classes too far advanced for their knowledge. Those who now teach in the so-called high schools should teach there no longer, unless they have had an education which has fitted them to completely master what they are teaching. A text-book knowledge, or a mere school knowledge of the subjects, is not enough; they should have had experience in their practical every-day application. For instance, a good teacher of logarithms and trigonometry is necessarily one who is expert with logarithms and has employed trigonometry in some of its engineering or physical applications. The best way to train an able young man or woman in these subjects up to the point required is to give him or her observatory practice up to, for instance, the calculation of a correct orbit. Then the plane trigonometry will not be an abstract and meaningless matter. The same end may, very likely, be attained by the use of a physical laboratory or in a school of surveying which practically admits that the earth is round, and that the plumb-lines at different stations are not parallel.

The higher mathematics are important, as college electives and university studies, for all who are to be college or normal-school professors; and the future high-school teacher will do well to learn at least the rudiments of the calculus.

But the pure mathematicians are a little apt to overestimate the importance of their subject as a means of education. Mathematical training is best given by rather elementary exercises, at least to most students; one principle at a time can be thus taught, and the mind will thus receive, appropriate, and apply it with more spontaneity. The heuristic method, that in which the pupil discovers a principle for himself, is far more effective than the dogmatic; and far more than that according to which large masses of mathematical formulæ are committed to memory. There are still teachers actively at work who require pupils to commit to memory a great many geometrical propositions, with corollaries and scholia, and even the numbers attached to the same in the text-books; a method which has wrought vast mischief in this country. Two subjects in which the present writer is quite certain that the heuristic method is applicable are mental arithmetic (according to Grube) and the modern geometry. The integral calculus would be included but for the difficulty (at present) which arises from the long training by another method to which all our pupils are subjected in algebra.

The great colleges have "raised the standard" in this subject rather too rapidly; so that the students come up "crammed" with it rather than trained in it. It is now quite time to insist upon a different method in preparatory schools; and to do this by adding oral examinations to those now given by printed papers. In that way it is probable the requisite thoroughness in the elements can, little by little, be attained; especially if college professors will put themselves on the pedagogic stand-point. The weakness of mere written examinations is as much a didactic commonplace as that of mere oral ones; witness the universal method of examining for the higher degrees. The college electives in mathematics ought to include subjects of moderate difficulty as well as advanced ones. Among the most valuable, which are frequently neglected, are conic sections synthetically treated, and kinematics as a branch of pure geometry. Whether the applications of the calculus contained in the ordinary text-books are sufficient is a little questionable; Messrs. Rice and Johnson have done a good work for the Naval school in preparing examples adapted to their pupils' daily work; and it is much to be wished that the other writers of our text-books in this subject would exhibit equal originality. American text-books of much value in other respects are far too largely copied from the latest English ones; and it seems a pity that so thoroughly artificial an institution as the Cambridge mathematical tripos should become as it were a model for our teachers and examiners.

In general the best principle to adopt in mathematical training of any grade is that the matter taught shall have as definite a reference as possible to other studies or pursuits of the pupil. There are few who can derive much benefit from the higher mathematics unless they are either interested in the subject for itself or for its applications. It is quite certain that those who are not interested in mathematics do not derive much benefit from its study. Any one can be taught geometry by a competent instructor; elementary algebra, although more difficult, has so many applications in practical life that it can be made interesting: plane trigonometry, too, is brought nearer educated men by its applications. Analytical geometry is less interesting to those who are not technically mathematicians or students of physics; and usually cannot be carried much beyond conic sections: these curves can be best taught to average college students by the synthetic method, by which they are rendered more tangible and interesting. All subjects beyond this should be elective. The "modern mathematics" are in the main useful only to those who are preparing themselves to be teachers of the science or of physics or astronomy; but for these they are indispensable.

We may then sum up what has been said in a few words. Mathematical preparation for college must be more thorough. The courses for this purpose must be better organized. The method must call upon the pupil to originate when possible; and must not encourage cramming. The teachers must be better prepared for their work, especially in mastery of the subject and its applications. College professors must not require the average students to do technical work. They must remember that if the power of abstraction fails their pupils they are often tempted to superficial and insincere study. They must also be willing to teach the higher subjects to small classes only.

Practical applicability in physics and astronomy must be the test by which it is decided what can be demanded of a large majority of their scholars; for it is these sciences which render the abstractions of pure mathematics not only intelligible but interesting to many who have not the utter disregard of the outside world which is characteristic of the pure mathematician.

ON THE TEACHING OF ELEMENTARY GEOMETRY.

Plane Geometry. On the heuristic plan. By G. I. HOPKINS. Boston, D. C. Heath, 1891.

Elementary Synthetic Geometry. By N. F. DUPUIS. New York, Macmillan, 1889.

Introductory Modern Geometry. By W. B. SMITH. New York, Macmillan, 1893.

Elementary Synthetic Geometry. By G. B. HALSTED. New York, Wiley & Sons, 1892.

NOWHERE has the conflict between the forces of conservatism and radicalism waged hotter than in the domain of geometry. The nature of the axioms, the character of the reasoning employed, the method in which the science shall be taught, have each given occasion for many a battle. Peace is not yet, but progress toward it is discernible.

To begin with, it is coming to be generally admitted that geometry is a physical science and that the truth of certain of its axioms, instead of being necessary and self-evident, is dependent upon the nature of space and our means of observation. Space being an hypothesis that the mind makes to explain phenomena, the character of space depends upon the character of the phenomena observed. That the phenomena that give our space-conceptions should be observed, and carefully too, the struggle for existence has inexorably compelled.

Then, too, it is here and there perceived that the reasoning of geometry, of which the characteristic is to spin out as many conclusions from as few data as possible, is not ideal. If observation, it is said, without our scarcely being aware of it, has given us our data, may it not equally have been playing a part in all our reasoning? Do we not reason rightly