REDUCTION OF THE RESULTANT OF A BINARY QUADRIC AND n-IC BY VIRTUE OF ITS SEMI-COMBINANT PROPERTY.*

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THE resultant of two binary quantics is comparatively simple when written either in the Eulerian or in the Bezoutian form as a determinant. Its degree in the coefficients of each quantic, however, is generally high.

Hence the problem arises, when invariants are arranged according to their degrees in the coefficients: *To reduce the resultant to a sum of terms, each of which shall contain the simplest possible invariant factors.* This problem is completely solved in papers by Clebsch, Gordan, and E. Pascal. But much may yet be done in the way of simplifying the processes by which their results are deduced. I propose here to discuss the partial problem which Clebsch solved; namely, *to write in symbolic form the resultant of a binary quadric and a binary quantic of arbitrary order n.*

By a method different from his I shorten the labor involved and at the same time illustrate the utility of the theory of *conjugate forms.*

Let the quantics be denoted by

$$\phi = \alpha_x^2 = \alpha_1^2 x_1^2 + 2\alpha_1\alpha_2 x_1 x_2 + \alpha_2^2 x_2^2,$$

$$f = a_x^n = a_1^n x_1^n + n \,.\, a_1^{n-1} a_2 x_1^{n-1} x_2 + \ldots,$$

according to the Clebsch-Aronhold symbolism.

Then the resultant is written in Sylvester's formula :

$$R = \begin{vmatrix} a_1^n & n a_1^{n-1} a_2 & \frac{n(n-1)}{1\,.\,2} a_1^{n-2} a_2^2 & \ldots & a_2^n & 0 \\ 0 & a_1^n & n a_1^{n-1} a_2 & \ldots & n a_1 a_2^{n-1} & a_2^n \\ \alpha_1^2 & 2\alpha_1\alpha_2 & \alpha_2^2 & \ldots & 0 & 0 \\ 0 & \beta_1^2 & 2\beta_1\beta_2 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 2\epsilon_1\epsilon_2 & \epsilon_2^2 \end{vmatrix} \begin{matrix} \left.\begin{matrix} \\ \\ \end{matrix}\right\} 2 \text{ rows.} \\ \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix}\right\} n \text{ rows.} \end{matrix}$$

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This determinant exhibits plainly the "semicombinant" property of the resultant, viz., that it remains unaltered when the coefficients of f are replaced by those of F:

$$F = f + L\phi = A_x^n = a_x^n + \lambda_x^{n-2} \cdot \alpha_x^2$$

(in which formula the coefficients of λ_x^{n-2} are $n - 1$ arbitrary quantities). For this substitution of A_x^n for a_x^n merely increases the coefficients standing in the first and second rows, respectively, of the determinant R by arbitrary multiples of those in the $(n - 1)$ rows beginning with the third and the fourth rows respectively.

This derivation of the semicombinant property is so well known as to seem trivial. I adduce it here for the sake of showing, as a consequence of it, two properties of a certain covariant, derived by the evectant process from the resultant.

The evectant that I mean is that obtained by substituting, in every term of the developed determinant R for *one* of the two factors $a_1^i a_2^{n-i}$, or $a_1^k a_2^{n-k}$ the variable factor $x_2^i(-x_1)^{n-i}$, or $x_2^k(-x_1)^{n-k}$. If I make the substitution without expanding the determinant, it is done by substituting for the first row only, then for the second row only, in succession, the proper power-products of x_2 and $(-x_1)$, and adding the resulting terms. The covariant, of order n, so derived I will call $\overline{F}$, or $\overline{A}_x^n$ or $\overline{B}_x^n$. Now each of the two determinants composing $\overline{F}$ has obviously, like the resultant, the semicombinant property; therefore

(1) the covariant $\overline{F}$ is a semicombinant of f and ϕ.

But, further, if I replace in $\overline{F}$ the power-products of x_1, x_2 by their several coefficients in the symbolic product

$$(x_1 y_1 + x_2 y_2)^{n-2}(x_1 \alpha_2 - x_2 \alpha_1)^2,$$

linear and homogeneous in the coefficients of ϕ or α_x^2, being careful to drop in this substitution the numerical factor of each coefficient, the result vanishes identically, whatever the value of y_1, y_2. In other words, the second transvectant of $\overline{F}$ over ϕ vanishes identically,

$$(\overline{A}\alpha)^2 \overline{A}_y^{n-2} \equiv 0.$$

Introducing technical terms first used by Rosanes* and Reye,† I may say:

(2) the quantic $\overline{F}$ is *conjugate* or *apolar* to the quadric ϕ.

My present interest in the covariant $\overline{F}$ arises from the fact that, as $\overline{F}$ was derived from R by evection, so, inversely, the resultant itself can be obtained from $\overline{F}$ immediately by trans-

* *Journal für Math.*, vol. 75 (1873).
† " " " vol. 72 (1870).

vection over f n times. Accordingly I shall consider in their order the following questions:

(1) What symbolic products can enter in the expression of the resultant R by fundamental invariants?

(2) Consequently, what symbolic products can enter in the expression of the covariant $\overline{F}$ by fundamental covariants?

(3) In order that the quantic $\overline{F}$ may be *apolar* to the quadric ϕ, what must be the numerical multipliers of these symbolic products?—This amounts to the determination of arbitrary constants by means of a differential equation.

(4) What, finally, is the consequent expression for the resultant R?

For the sake of simplicity I will consider f as a quantic of *even* order: $f = a_x^n = a_x^{2p} = b_x^{2p}$.

The first question is answered by applying two elementary theorems: first, that if a symbolic factor (ab) or $(\alpha\beta)$ containing two symbols of the same quantic occur in any term, it must occur to the second or some other even power; second, that a factor $(a\alpha)$ containing symbols of different quantics can occur only to an even power, i.e. to the second power, for the only alternative is the appearance of the four factors $[(a\alpha)(b\alpha)(a\beta)(b\beta)]$, and this product is identically equal to the following:

$$\tfrac{1}{2}\{(a\alpha)^2(b\beta)^2 + (a\beta)^2(b\alpha)^2 - (ab)^2(\alpha\beta)^2\}.$$

Therefore R is composed of not more than these $\frac{n}{2}+1$ terms:

$$R = \left\{ \begin{array}{l} l_0 \cdot (ab)^{2p} \cdot [(\alpha\beta)^2]^p \\ + l_1 \cdot (ab)^{2p-2} \cdot (a\alpha)^2(b\beta)^2 \cdot [(\delta\gamma)^2]^{p-1} \\ + l_2 \cdot (ab)^{2p-4} \cdot (a\alpha)^2(a\beta)^2(b\gamma)^2(b\delta)^2 \cdot [(\epsilon\zeta)^2]^{p-2} \\ + \text{etc.} \\ + l_p \cdot (a\alpha)^2(a\beta)^2 \ldots (\text{to } p \text{ factors}) \cdot (b\gamma)^2 \cdot (b\delta)^2 \ldots (\text{to} \\ \qquad p \text{ factors}). \end{array} \right\}$$

The second question is answered, since this formula for R is symmetrical in symbols a and b, by replacing the b's with x's as heretofore. We find in this way:

$$\tfrac{1}{2} \cdot \overline{F} = \left\{ \begin{array}{l} l_0 \cdot a_x^{2p} \cdot [(\alpha\beta)^2]^p \\ + l_1 \cdot a_x^{2p-2}(a\alpha)^2 \cdot \beta_x^2 \cdot [(\gamma\delta)^2]^{p-1} \\ l_2 + \cdot a_x^{2p-4}(a\alpha)^2(a\beta)^2 \cdot \gamma_x^2\delta_x^2[(\epsilon\zeta)^2]^{p-2} \\ + \text{etc.} \\ + l_p \left\{ \begin{array}{l} (a\alpha)^2(a\beta)^2 \ldots (\text{to } p \text{ factors}). \\ \gamma_x^2 \cdot \delta_x^2 \ldots (\text{to } p \text{ factors}. \end{array} \right\} \end{array} \right\}$$

This formula contains, I may remark in passing, a theorem which ought to find a direct proof through skilful manipulation of the determinant expression for $\overline{F}$. It is this:

When the form f is of even order, the evectant $\overline{F}$ of its resultant with a quadric ϕ is expressible in the form

$$\overline{F} \equiv A \,.\, f + B \,.\, \phi,$$

where A is a rational integral function of coefficients of the quadric ϕ and independent of the variables, while B is of the order $(n - 2)$ in the variables, and likewise rational and integral in the coefficients of f and of ϕ.

Taking up the third question, we have only to apply combinatory enumeration and the identity

$$(\alpha\beta)(a\beta)\alpha_x = \tfrac{1}{2}(\alpha\beta)^2 a_x$$

in order to form directly from the development of $\overline{F}$ on p. 13 the second transvectant $(\overline{A}\alpha)^2\overline{A}_y^{\,n-2}$, whose identical vanishing yields the equations of condition:

$$l_0 + \frac{2}{n} l_1 = 0,$$

$$\frac{(n-2)(n-3)}{2} l_1 + 2(n-2) l_2 = 0,$$

$$\frac{(n-4)(n-5)}{2} l_2 + 3(n-3) l_3 = 0,$$

etc., etc.

From these, calling for convenience $l_0 = 1$, we find for the numerical constants these values:

$$l_1 = -\frac{n}{2} = p,$$

$$l_2 = +\frac{n(n-3)}{2^2 \,.\, 2 \,.\, 1},$$

$$l_3 = -\frac{n(n-4)(n-5)}{2^3 \,.\, 3 \,.\, 2 \,.\, 1},$$

$$\vdots$$

$$l_p = (-1)^p \,.\, \frac{2p \,.\, \underline{|p-1}}{2^p \,.\, \underline{|p}} = \frac{(-1)^p}{2^{p-1}}.$$

Thus we have the constants needed in writing as a covariant of a_x^{2p} and α_x^2 the quantic

$$\bar{A}_x^{2p} = a_x^{2p} + \lambda_x^{2p-2} \cdot \alpha_x^2,$$

which is required to be apolar (or conjugate) to the quadric α_x^2.

Fourthly, this gives for the resultant R exactly the expression found by Clebsch (*Binäre Formen*, p. 88). Writing, in our formula on p. 13, Δ for the discriminant $(\alpha\beta)^2$ of the quadric, and inserting the above-found numerical values of the constants $l_0, l_1, l_2, \ldots l_p$, I have the final formula (neglecting possibly a multiplicative constant)

$$\begin{aligned} R = {} & \Delta^p (ab)^{2p} \\ & - \frac{2p}{2} \cdot \Delta^{p-1} \cdot (ab)^{2(p-1)} (a\alpha)^2 (b\beta)^2 \\ & + \frac{2p(2p-3)}{2^2 \cdot 2!} \Delta^{p-2} \cdot (ab)^{2(p-2)} \cdot (a\alpha)^2 (a\beta)^2 (b\gamma)^2 (b\delta)^2 \\ & + \text{etc.} + \text{etc.} \\ & + \frac{(-1)^p}{2^{p-1}} \left\{ \begin{matrix} (a\alpha)^2 (a\beta)^2 \ldots \text{(to } p \text{ factors).} \\ (b\gamma)^2 (b\delta)^2 \ldots \text{(to } p \text{ factors).} \end{matrix} \right\} . \end{aligned}$$

Had we assumed the order of f as odd, the process would require but one principal change,—the substitution of the Jacobian of f and ϕ,

$$(a\alpha) a_x^{n-1} \alpha_x = g,$$

instead of f throughout the formulæ for R and $\bar{F}$, and so of course in the theorem mentioned (p. 13) as deserving of a more direct proof. This change affects the constants of the formula, which again come to agree with those given by Clebsch.

In closing I may add that the coincidence of the semicombinant property and the apolar property, which appeared quite incidentally in our discussion of the covariant $\bar{F}$ (p. 12), is demonstrably a necessary coincidence quite apart from the special form of the covariant. On this necessary union of the two properties in covariants of a certain order, indeed, is based the entire theory of semicombinant invariants and covariants of a system of quantics.

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