THE GROUP OF HOLOEDRIC TRANSFORMATION INTO ITSELF OF A GIVEN GROUP.

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§ 1.

Every group G_n of order n determines a substitution-group Γ^{n-1} of degree $n - 1$.

Given an (abstract) group G_n of order n, with the elements $s_1 =$ identity, $s_2, \dots s_n$. The group G_n is holoedrically isomorphic with itself if a 1-1 correspondence*

$$(1) \qquad (s_1, s_2, \dots s_n) \sim (s_{\iota_1}, s_{\iota_2}, \dots s_{\iota n})$$

can be set up amongst its elements of such a sort that whenever in G_n we have $s_f s_g = s_h$ we have also $s_{\iota_f} s_{\iota_g} = s_{\iota_h}$. Obviously corresponding elements must have the same period.

Attributing an *order* to the correspondence (1) we have the substitution

$$(2) \qquad \sigma = \begin{pmatrix} s_1, & s_2, & \dots & s_n \\ s_{\iota_1}, & s_{\iota_2}, & \dots & s_{\iota_n} \end{pmatrix}$$

which holoedrically transforms the group G_n into itself, that is, it transforms the system of product relations

$$(3) \qquad s_f s_g = s_h$$

into itself. *The totality of such substitutions σ constitutes a substitution-group Γ^n of degree n,* which I call *the group of holoedric transformation into itself of the original abstract group G_n of order n.*

If we transform the elements of the group G_n through any particular element of G_n we obtain such an holoedric transformation of G_n into itself. The totality of such transformations constitutes a sub-group of Γ_n.

The group Γ^n on the n elements $s_1, s_2, \dots s_n$ is certainly intransitive, since elements of the same period in G_n are permuted

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* This correspondence may always be established by setting properly chosen *generators* in 1-1 correspondence.

amongst themselves by the substitutions σ of Γ^n. Γ^n is isomorphic with various substitution-groups $\Gamma^{n_\iota}_{(p_\iota)}$, where a particular $\Gamma^{n_\iota}_{(p_\iota)}$ is a substitution-group on the, say, n_ι elements of period p_ι. It will on occasion happen that Γ^n, $\Gamma^{n_\iota}_{(p_\iota)}$ are *holoedrically* isomorphic, in which case the abstract-group-properties of the Γ^n may perhaps be more conveniently studied as they appear in the $\Gamma^{n_\iota}_{(p_\iota)}$. *It will always be desirable to replace the Γ^n by the Γ^{n-1} on the $n-1$ elements $s_2, s_3, \ldots s_n$,* since the identity-element s_1 is invariant.

Intending to discuss the general question somewhat further in a subsequent note, I proceed now to consider an instructive example.

§ 2.

The group Γ^7_{168} of holoedric transformation into itself of the Abelian group G_8 whose elements are the identity and seven commutative elements of period two.

The Abelian or commutative group G_8 has the utmost regularity of internal structure. We define it by the generating elements a, b, c, with the generating relations

$$(4) \qquad a^2 = b^2 = c^2 = 1, \quad ab = ba, \quad ac = ca, \quad bc = cb,$$

where 1 is the identity. The elements may have the following notation:

$$(5) \qquad \begin{array}{cccc} & a & bc = d & \\ 1 & b & ca = e & abc = g \\ & c & ab = f & \end{array}$$

Clearly, if $k \neq l$ and $kl = m$, the four elements, 1, k, l, m form a four-group. Three such elements k, l, m (whose order is immaterial) form, say, a *triple*. The seven elements $a \ldots g$ arrange themselves into seven triples $A \ldots G$, viz.,

$$(6) \qquad \begin{array}{lll} A = bcd & D = adg & \\ B = cae & E = beg & G = def \\ C = abf & F = cfg & \end{array}$$

This triple system Δ_7 so set up on the elements $a \ldots g$ is invariant under a certain substitution-group $\Gamma^7_{(\Delta_7)}$. This $\Gamma^7_{(\Delta_7)}$ is exactly the Γ^7 of holoedric transformation into itself of the G_8, since the Δ_7 fully defines the product relations of the G_8, at least when we adjoin the system of equations, $a^2=1, .. g^2=1$, which is invariant under every substitution on the $a .. g$.

The G_8 and the Δ_7 should be used together in studying the Γ^7. The extreme simplicity of the internal structure of the G_8 makes the properties of the Γ^7 almost immediately evident. A substitution σ of Γ^7 is determined by the elements a', b', c' which it makes correspond to the *generators* a, b, c. Clearly any one of the seven elements may be chosen for a', any other one for b', and any one except the product $a'b'$ (i.e., except the third element of the triple containing a', b') for c'. The Γ^7 has then the order $7 \cdot 6 \cdot 4 = 168$. The Γ^7_{168} is doubly transitive on the seven elements. I shall prove it simple and thereby identify it abstractly with *the only existent* simple * group of order 168.

An enumeration of the substitutions σ of the Γ^7_{168} by period and by type † shows the following distribution :

	Period	1	2	3	4	7
(7)	Type	1^7	$1^3 . 2^2$	$1 . 3^2$	$1 . 2 . 4$	7
	Number	1 +	21 +	56 +	42 +	48 = 168.

The 21 σ of period 2 are conjugate under the Γ^7_{168}; likewise the 56 of period 3, and the 42 of period 4. As to those of period 7 consider, say,

$$\tau = (abcfdge); \tag{8}$$

this τ does belong to Γ^7_{168} for the triples

$$abf,\ bcd,\ cfg,\ fde,\ dga,\ geb,\ eac \tag{9}$$

lie on its face cyclically and hence the triple system Δ_7 is invariant under τ. Obviously the triples must lie cyclically on the face of every substitution σ of period 7 of the Γ^7_{168}, not necessarily, however, exactly as in τ; there are two and only two possibilities, both of which occur in the list of powers of τ. In τ^1, τ^2, τ^4 the triples lie as in τ, in, say, the *forward* way; in τ^3, τ^5, τ^6 the triples lie as in

$$\tau^{-1} = \tau^6 = (aegdfcb), \tag{10}$$

in, say, the *backward* ‡ way. Thus the 48 σ of period 7 are half

* HOLDER: Die einfachen Gruppen im ersten und zweiten Hundert der Ordnungszahlen. *Mathematische Annalen*, vol. 40, pp. 55–88, 1892.

† The type of a substitution is the partition-symbol of its degree, arising from its complete expression in the cyclic notation. For instance, in the Γ^7_{168} $\sigma = (abcg)(df)$ has the type $1 . 2 . 4$.

‡ It will be noticed that the separation of the powers is according to the character of the exponent as a quadratic residue or non-residue of 7.

of the forward and half of the backward style. All those of the same style are conjugate under the Γ^7_{168}. For let

$$\tau_1 = (t_{11} t_{12} \,.\,.\, t_{17}) \text{ and } \tau_2 = (\tau_{21} \tau_{22} \,.\,.\, \tau_{27})$$

be two of the same style; the triples lie on the faces of τ_1, τ_2 in the same relative positions; the substitution

$$\text{(11)} \qquad \lambda = \begin{pmatrix} t_{11} t_{12} \,.\,.\, t_{17} \\ t_{21} t_{22} \,.\,.\, t_{27} \end{pmatrix}$$

transforms the triple system Δ_7 into itself and hence belongs to Γ^7_{168}, and it obviously transforms τ_1 into τ_2. Equally clearly those of different style are *not* conjugate under the Γ^7_{168}. Nevertheless the eight cyclic groups of order 7 are conjugate under the Γ^7_{168}.

Any self-conjugate sub-group Γ^7 of the Γ^7_{168} must then contain besides the identity all or none of the substitutions σ of each type of the list (7). The fact that the diophantine equation

$$\text{(12)} \qquad 1 + 21\alpha + 56\beta + 42\gamma + 48\delta = d,$$

where $\alpha, \beta, \gamma, \delta$ are respectively 0 or 1 and where d is a divisor of 168, has only the two solutions

$$\text{(13)} \quad (\alpha, \beta, \gamma, \delta;\ d) = (1, 0, 0, 0;\ 1), (1, 1, 1, 1;\ 168)$$

shows that the Γ^7_{168} is simple.* This identifies the group Γ^7_{168}; of which later.

The triple system Δ_7 is given in (5) (6) from the standpoint of the seven elements $a \,.\,.\, g$. I give it again from the standpoint of the seven triples $A \,.\,.\, G$:

$$\text{(14)} \qquad \begin{matrix} A & D & \\ B & E & G \\ C & F & \end{matrix}$$

$$\text{(15)} \qquad \begin{matrix} a = BCD & d = ADG & \\ b = CAE & e = BEG & g = DEF \\ c = ABF & f = CFG & \end{matrix}$$

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* This method of proving the simplicity of a simple group when its elements and cyclic sub-groups have been classified as to conjugacy under the group was used by KLEIN for the icosahedron G_{60} (Vorlesungen über das Ikosaeder, p. 18, 1884) and has, as is well known, wide application. Is an instance known in which it does not apply?

Obviously (14) (15) exhibit in the $A \,..\, G$ a triple system $\Delta_7(A \,..\, G)$ exactly* like the triple system $\Delta_7(a \,..\, g)$ exhibited by (5) (6).

Now a substitution σ of Γ^7_{168} permutes the $a \,..\, g$, leaving $\Delta_7(a \,..\, g)$ invariant; through the $a \,..\, g$ it permutes the $A \,..\, G$, but nevertheless leaves the $\Delta_7(A \,..\, G)$ invariant. Call $\bar{\sigma}$ the substitution on the $A \,..\, G$ which permutes the $A \,..\, G$ directly, just as σ does indirectly. The totality of these substitutions $\bar{\sigma}$ constitutes a group, the group $\bar{\Gamma}^7_{168}$ of the $\Delta_7(A \,..\, G)$, which is clearly on the one hand abstractly identical with the Γ^7_{168} of the $\Delta_7(a \,..\, g)$ and on the other hand holoedrically isomorphic with the Γ^7_{168} by the correspondence

$$(16) \qquad \sigma \sim \bar{\sigma}.$$

This holoedric isomorphism of the abstract Γ^7_{168} with itself is not that arising from a transformation of the Γ^7_{168} through one of its own elements. For such a transformation would change the substitution τ (8) of period 7 into another one of the forward style, while (16) gives the correspondence

$$(17) \qquad \tau = (abcfdge) \sim \bar{\tau} = (AFGDEBC),$$

where $\bar{\tau}$ is of the backward style.

The simple group Γ_{168} (abstractly, there is but one) first appeared as the substitution-group Γ^8_{168} connected with the modular equation for the transformation of elliptic functions of order 7. Galois discovered the existence of a resolvent of order 7. As to the development of the theory one may consult the introduction to a memoir of Mr. Gordan.†

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* A triple system Δ_n in n elements can exist only if n has the form $6m + 1$ or $6m + 3$, say the form t. All triple systems Δ_3 are of the same class; likewise for the Δ_7, Δ_9. Triple systems Δ_t for every t do exist, and indeed for every $t \geqq 13$ they fall into at least two distinct classes. As to these theorems these references:

NETTO: Zur Theorie der Tripelsysteme. *Mathematische Annalen*, vol. 42, pp. 143–152, 1893.

MOORE: Concerning Triple Systems. *Mathematische Annalen*, vol. 43, pp. 271–285, 1893.

JAN DE VRIES: Zur Theorie der Tripelsysteme. *Rendiconti del Circolo Matematico di Palermo*, vol. 8, pp. 222–226, 1894.

† GORDAN: Ueber Gleichungen siebenten Grades mit einer Gruppe von 168 Substitutionen. *Mathematische Annalen*, vol. 20, pp. 515–530, 1882.

The triple system as determining invariant of the group Γ^7_{168} was introduced by Mr. Noether.*

The sub-groups of the group † $\Gamma^{q+1}_{\frac{q(q^2-1)}{2}}$ connected ‡ with the modular equation for the transformation of prime order q have been completely enumerated by Mr. Gierster, § thus illuminating Galois' theorem that the modular equation has resolvents of degree q for $q = 5, 7, 11$, but for no $q > 11$, and Kronecker's theorem ‖ that the group $\Gamma^{q+1}_{\frac{q(q^2-1)}{2}}$ is holoedrically isomorphic with itself in two distinct ways, and only¶ in those ways.

The idea** of the group Γ^{n-1} of holoedric transformation into itself of a given group G_n is, so far as I know, new. In the preceding sketch of the Γ^7_{168} the perspicuous determination from the Abelian G_8 of the triple system Δ_7 and of the Γ^7_{168} is the only element of novelty.

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* NOETHER: Uber die Gleichungen achten Grades und ihr Auftreten in der Theorie der Curven vierter Ordnung. *Mathematische Annalen*, vol. 15, pp. 89–110, 1879).

† The $\Gamma^{q+1}_{\frac{q(q^2-1)}{2}}$ for $q > 3$ is simple.

‡ *Connected with* is *of*, if the numerical irrationality $\sqrt{(-1)^{\frac{q-1}{2}} \cdot q}$ is adjoined

§ GIERSTER: Die Untergruppen der Galois'schen Gruppe der Modulargleichungen für den Fall eines primzahligen Transformationsgrades. *Mathematische Annalen*, vol. 18, pp. 319–365, 1881.

‖ KRONECKER: *Monatsberichte der Berliner Akademie*, 1861.

¶ As GIERSTER adds, *loc. cit.* p. 356.

** This concept and an application of it to the Γ^7_{168} were communicated to the Mathematical Club of the University of Chicago in a paper presented January 19, 1893, entitled "An existence-proof of the simple group of order 168 as a group of substitutions on 7 letters."