ON THE NUMBER OF INSCRIPTIBLE REGULAR POLYGONS.

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LET us denote for brevity the phrase, a regular polygon which is geometrically inscriptible, by P. The P's are few in number compared with the non-P's. The number of P's up to 100 is 24; up to 300 is 37; up to 1000 is 52; up to 1,000,-000, only 206. Indeed, a P must have for the number of its sides a *prime* number of the form 2^x+1, or the product of a power of 2 by any number of different primes of that form. This was proven by Gauss; and, more simply, by myself in another paper.* Further, x here must be a power of 2. For if x contains an odd factor m, such that $lm = x$, then will 2^x+1 be divisible by 2^l+1. This is seen by writing y for 2^l in $2^{lm}+1$, which thus becomes y^m+1, a number divisible by $y+1$. But the inverse of this, that all numbers of the form $(2^{2^y}+1)$ are prime, is not generally true, as Fermat affirmed. Euler pointed out that this rule [true for $y=0$, 1, 2, 3, 4] fails for $y=5$. Again, it is stated in Lucas' *Théorie des Nombres*, pages 51 and 448, that it fails for $y=6$; but that we are still ignorant in the case of $y=7$. Hence, the numbers $(2^{32}+1)$ and $(2^{64}+1)$ do not give P's; while $(2^{128}+1)$ may or may not give a P. Thus the P's below $2^{32}+1$ are given by 2^x times one, or 2^x times the product of two or more different ones, of the primes $2^1+1, 2^2+1, 2^4+1, 2^8+1, 2^{16}+1$; that is, will fall under one of these 32 forms: 2^x, $3\cdot 2^x$, $5\cdot 2^x$, $3\cdot 5\cdot 2^x$, $17\cdot 2^x$, $3\cdot 17\cdot 2^x$, $5\cdot 17\cdot 2^x$, $3\cdot 5\cdot 17\cdot 2^x$, $257\cdot 2^x$, $3\cdot 257\cdot 2^x$, . . . $3\cdot 5\cdot 17\cdot 257\cdot 65537\cdot 2^x$. Further, all P's between $2^{32}+1$ and $2^{128}+1$ also fall under the same 32 forms; for the only *new* factors which could occur, $2^{32}+1$ and $2^{64}+1$, are ruled out, not being prime. We cannot proceed to P's of sides greater than $2^{128}+1$ (which requires 39 figures to express it).

The object of this paper is to give a general expression for the number of inscriptible regular polygons between certain arbitrary limits.

Theorem I. *The number of P's below 2^x+1 sides, where x is less than* 32, *is* $\dfrac{(x-1)(x+2)}{2}$. This follows from the fact that the numbers below $2^{32}+1$ giving P's have a definite

* See this volume of BULLETIN, p. 20.—ED.

positional relation to the powers of 2. Writing in order a few of these ; thus,

[3], 4, [5, 6], 8, [10, 12, 15], 16, [17, 20, 24, 30], 32, [34, etc.],

we see empirically that between the successive powers of 2 there lie 1, 2, 3, 4, 5, etc., P's. This may be proven by considering special ones of the above 32 forms and then generalizing; or directly by considering the most general expression for the number of sides of a P; viz., $(2^l+1)(2^m+1)\ldots(2^n+1)\cdot 2^x$. Thus we may prove that the numbers of this general form (which includes all the above 32 forms, and hence all P's below $2^{32}+1$) fall one and one only between the successive powers of 2, the first one falling between $2^{l+m\ldots+n}$ and $2^{l+m\ldots+n+1}$. Hence between any two successive powers of 2 lies one and only one number of each of the above 32 forms, the first one of any form falling just one interval later than the first one of the form preceding it. Hence, for these successive intervals, the numbers giving P's go up by the natural series of numbers; thus, omitting the powers of 2 themselves, n such numbers lie between 2^n and 2^{n+1}. Therefore, all such numbers below 2^x (x being less than 32) are included in the sum of the natural numbers up to $(x-1)$; i.e., $\frac{x(x-1)}{2}$. But the number of P's below 2^x+1 of the form 2^x is $(x-1)$. Hence the total number of P's of sides below 2^x+1, where x is less than 32, is $(x-1)+\frac{x}{2}(x-1)=\frac{(x-1)(x+2)}{2}$.

Theorem II. *The number of P's below* 2^x+1 *sides, where* x *is greater than* 32 *but less than* 128, *is* $(32x-497)$.

The number of P's up to $2^{32}+1$ sides is 527 by our last formula. But the number of those between 2^{32} and 2^{33} sides is 31 instead of 32, as we would expect ; for $2^{32}+1$ does not give a P. Similarly, the number of P's between 2^{33} and 2^{34} is 31 instead of 33 ; for $2(2^{32}+1)$ and $3(2^{32}+1)$ do not give P's. But also $(2^{64}+1)$ is ruled out. Hence, generally, the number of P's between any two successive powers of 2 (between 2^{32} and 2^{128}) is always 31. Hence, omitting the powers of 2, the number of P's between $2^{32}+1$ and 2^x+1 is $31(x-32)$; and including the $(x-32)$ powers of 2, is $32(x-32)$. Hence the total number of P's below 2^x+1, x being between 32 and 128, is $527+32(x-32)=32x-497$.

A more elegant proof of this is as follows: Below 2^x+1 sides, the form 2^x includes $(x-1)$ P's ; the form $3\cdot 2^x$ includes $(x-1)$; the form $5\cdot 2^x$ includes $(x-2)$; $3\cdot 5\cdot 2^x$ includes $(x-3)$; $17\cdot 2^x$ includes $(x-4)$; and so on for all 32 forms. Thus the whole number of P's below 2^x+1, x being less than 128 and greater than 32, is $(x-1)+\{(x-1)+(x-2)\ldots+(x-31)\}=x-1+31(x-16)=32x-497$.

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BIBLIOGRAPHY OF MATHEMATICAL DISSERTATIONS.

Catalogue des Thèses de Sciences soutenues en France de 1810 *à* 1890 *inclusivement.* Par ALBERT MARIE, Bibliothécaire universitaire. Paris, H. Welter, 1892. 8vo. xi+224 pp.

Verzeichnis der seit 1850 *an den deutschen Universitäten erschienenen Doctor-Dissertationen und Habilitationsschriften aus der reinen und angewandten Mathematik.* Herausgegeben auf Grund des für die Universitäts-Ausstellung in Chicago erschienenen Verzeichnisses. München, 1892. 8vo.

IT is generally a matter of considerable difficulty to obtain information in regard to doctors' dissertations. They appear unannounced, at irregular intervals, in many places, and are usually the productions of authors as yet unknown. Separate publications of no great size, relating to subjects of a highly special character, they are often of but little interest except to the author. Still they are sometimes of the highest scientific importance, and the two recently published lists of the dissertations of the French and German universities will be of great value.

The contents and arrangement of the French catalogue are described in its preface, from which the following notes are extracted.

The special schools of science established in 1802 conferred no degree, and the creation of the faculties of mathematical and physical sciences in 1808 marks the commencement in France of courses in science leading to the doctorate.

The total number of dissertations at Paris from 1811 to the end of 1890 is 701. These are classified as follows: mathematics 184, physics 281, natural science 236.

Those from the Departments* number 172, classified as follows: mathematics 44, physics 61, natural science 67.

The arrangement of the list is first according to location, and then chronological. Each entry gives the name of the author; place and date of birth, and of death (if dead); official positions held; title of dissertation, publisher, date o

* Besançon 8; Bordeaux 5; Caen 7; Clermont-Ferrand 1 (1858) Dijon 10; Grenoble 9; Lille 8; Lyon 14; Marseille 5; Metz 1 (1813) Montpellier 36; Nancy 9; Poitiers 2; Rennes 2; Strassburg 52 Toulouse 16.