THREE NOTES ON PERMUTATIONS.

BY PROFESSOR F. MORLEY.

1. *A plea for the chess-board in teaching determinants.*

Not a few writers on determinants, whether in works devoted to the subject or in an incidental treatment, dispose of the "rule of signs" with such a statement as: "The sign + or — is to be prefixed to a term according as it can be derived from the leading term by an even or odd number of interchanges of suffixes." It will, I think, be conceded that the rule, so stated, appears very arbitrary to a class; and I suppose that many teachers, like myself, have experienced "a certain gaspingness" in laying it down. Of course the difficulty disappears when the characters of even and odd permutations have been first explained, as is done, for example, by R. F. Scott in his treatise. The chief question for the teacher then is: which rule for determining the nature of a permutation shall I recommend—Cramer's or Cauchy's? And the question seems answered by the terms in which Jacobi (*Crelle,* vol. 22, p. 287) refers to these rules after proving them; he refers to Cramer's rule as "quam regulam," and to Cauchy's as "hanc pulchram regulam." See also the remarks of Muir, History of the Theory of Determinants, vol. 1, p. 247.

As the basis of this first note, Cauchy's writings, in particular the memoir which appeared in the *Comptes Rendus,* vol. 12, March 1841 (Works, series 1, vol. 6, p. 87; summarized in Muir's History, vol. 1, p. 234), must be especially referred to. I do not think that it has been sufficiently pointed out how to render the ideas of his abstract presentation intelligible to the average beginner, and I venture, in this first note, to sketch what seems to me the best way of presenting the whole matter to a class, namely, by fusing the idea of a term of a determinant with that of a permutation. I omit a proof which will be found in the section on substitutions in Serret's Algèbre Supérieure, or in Muir; the changes to be made being merely verbal.

We are not concerned with the size of the constituents, but only with certain elementary questions of permutations.* Only, instead of regarding the things to be permuted as restricted to a line, we regard them as material "men" actually placed on a material chess-board, in the first or standard case

* It is perhaps inadvisable to say much about substitutions in this connexion.

all on a diagonal.* We then make "moves"; regarding any two men as at opposite corners of a rectangle whose sides are parallel to those of the board, the move displaces these men to the two other corners of their rectangle.†

We can suppose the moves made always along columns; then if we look along the columns we see the men always in their original order, but if we "give a side-glance" we see the various permutations.

In the standard case every man is on the diagonal selected or is "home." A man so placed is a "cycle of the first order" or say a monad. The first move puts a in b's row—a and b being any two men—and b in a's. The two men, so placed, are a "cycle of the second order" or say a duad. Next move b and c; now a is in b's row, b in c's, c in a's; that is, we have a cyclic interchange. The resulting position of the three men is a triad. And so on, taking (to begin with) always one of the men already moved and a new monad, in order to show the formation of cycles of the order x, say of x-ads. When the idea is grasped, we have then to introduce the cardinal proposition: a move, when the two men are of different cycles, welds the two cycles into one; and a move, when the men are of the same cycle, splits the cycle into two.‡

Now let n be the whole number of men, n_x the number of x-ads, so that in any position

$$n_1 + 2n_2 + 3n_3 + \ldots = n,$$

or

$$\Sigma x n_x = n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In the standard position we have n monads, and it follows at once from the cardinal proposition that m, the number of moves from the standard position to any other assigned one, is congruent (mod 2) to the difference of their numbers of cycles, that is,

$$n - \Sigma n_x \equiv m, \text{ mod } 2. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Thus the number of moves is either essentially odd or essentially even according to the nature of the position. When m is even, we call the position or permutation an even one;

* The two-dimensional idea in permutations was used by Rothe in 1800. See Muir, History, vol. 1, p. 60.

† The move is of course an "interchange of suffixes" or a "transposition."

‡ Serret, vol. 2, p. 274; Muir's History, vol. 1, p. 262; see also Johnson, "On the fifteen puzzle," *American Journal*, vol. 2.

when m is odd, the position is called odd. The "rule of signs" is that the sign $+$ or $-$ is to be prefixed to a term according as it arises from an even position or from an odd one.

From (1) and (2) it follows that

$$n_2 + 2n_3 + 3n_4 + \ldots \equiv m;$$

in fact it requires only $x - 1$ moves to bring an x-ad home, so that the left side is the minimum of moves necessary to bring the assigned position home.

Again, the last congruence shows that

$$n_2 + n_4 + n_6 + \ldots \equiv m. \quad . \quad . \quad . \quad . \quad (3)^*.$$

2. *A special rule of signs.*

A glance at the determinant itself tells us the number of monads and duads in any selected position; and, when $n < 6$, the nature of the cycles is evident merely from the number of monads and duads. For example, when $n = 5$, we have the scheme of solutions of (1),

n_1	n_2	n_3	n	n_5
5	0	0	0	0
3	1	0	0	0
2	0	1	0	0
1	0	0	1	0
1	2	0	0	0
0	0	0	0	1
0	1	1	0	0

in which the first two columns determine the rest. Thus when the position is symmetrical † about the diagonal, so that it contains merely monads and duads,

$$\Sigma n_x = n_1 + n_2,$$

but in other cases

$$\Sigma n_x = n_1 + n_2 + 1.$$

Hence, when $n < 6$, we have from (2) the following rule: Understand by a "symmetry" either a monad (constituent on the leading diagonal) or a duad (two constituents symmetrical with respect to that diagonal). Glance along the dia-

* Lucas, Récréations mathématiques, vol. 1, p. 204.

† The symmetrical position is Rothe's "self-conjugate permutation"; positions which are reflexions of one another as to the diagonal are Rothe's "conjugate" and Jacobi's "reciprocal" permutations. Muir, pp. 59 and 245.

gonal and note the number of symmetries; only if the term consist altogether of symmetries, one of them is *forced* and is not to be counted. Let s be the number of unforced symmetries, n the number of constituents; then the term is to be taken positively or negatively according as $n - s$ is odd or even. The rule fails when $n = 6$, owing to the fact that we cannot infer, merely from the absence of symmetries, whether a position consists of two duads or a single hexad. But it does not seem worth while to complicate the rule to meet cases of comparatively infrequent occurrence, or to raise the further question whether the rule then affords any improvement on the direct application of (3).

3. *The enumeration of positions.*

When a solution of (1) has been selected, we may say that we have all positions of a given kind, and the next question is: how many positions are there of a given kind? This question affords a good exercise in combinations, but it is enough to refer to Muir (vol. 1, p. 238), Serret (vol. 2, p. 259), Netto (p. 22 of Cole's translation), or to the original sources, Cauchy (*op. cit.*) and Jacobi (*Crelle*, vol. 22, p. 372). The answer is

$$n\,!/n_1!\ 2^{n_2}.\ n_2!\ 3^{n_3}.\ n_3!\ \ldots,$$

and by summing this for all solutions of (1), that is, for all unrestricted partitions of n, we have the total number of positions, which is $n\,!$. Hence

$$\Sigma 1/n_1!\ 2^{n_2}.\ n_2!\ 3^{n_3}.\ n_3!\ldots = 1. \quad . \quad . \quad . \quad (4)$$

It may be remarked that Jacobi published this formula in the same year (1841) as Cauchy, in the paper just cited, at the end of the three memoirs on determinants. He gives two proofs, of which the second is combinational; from the first I wish to draw an inference or two.

We have

$$1/(1-t) = \exp\{-\log(1-t)\}$$
$$= \exp(t + t^2/2 + t^3/3 + \ldots).$$

Jacobi now expands the exponential and uses the polynomial theorem.*

* Again a coincidence may be noticed: Cauchy employs a similar device, though not in immediate connexion with formula (4), in the memoir which immediately followed the one already cited; see his works, series 1, vol. 6, in particular p. 112.

We may instead write

$$
\left.
\begin{array}{l}
1/(1-t) = \exp t \,.\, \exp t^2/2 \,.\, \exp t^3/3 \ldots \\[1em]
= 1 + t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \dfrac{t^4}{4!} + \dfrac{t^5}{5!} + \dfrac{t^6}{6!} + \ldots \\[1em]
\times 1 \quad + \dfrac{t^2}{2} \qquad + \dfrac{t}{2!\,2^2} \quad + \dfrac{t^6}{3!\,2^3} + \ldots \\[1em]
\times 1 \qquad\quad + \dfrac{t^3}{3} \qquad\qquad + \dfrac{t^6}{2!\,3^2} + \ldots \\[1em]
\times 1 \qquad\qquad\quad + \dfrac{t^4}{4} \qquad\qquad\quad + \ldots \\[1em]
\times \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{array}
\right\} \quad . \quad . \quad (5)
$$

whence the coefficient of t^n in the product of the power-series $=$ the coefficient of t^n in $(1-t)^{-1}, = 1$.

Now it seems to me worthy of notice that not merely does the coefficient of $t^n/n!$ in the above product give the whole number of positions in a determinant, but each term of the coefficient is the number of positions of a given kind. If, for example, we wish for 5 monads we must take the term $t^5/5!$ from the expansion of $\exp t$; if 3 duads, the term $t^6/3!\,2^3$ from $\exp(t^2/2)$; and so on.

Therefore the number of positions with no x-ad is found by excluding the series for $\exp(t^x/x)$; thus on the one hand it is $n!\,\Sigma'$ (where Σ' is the sum in (4), with the restriction that in the summation n_x is not to appear), and on the other hand it is the coefficient of $t^n/n!$ in $\exp(-t^x/x)/(1-t)$.

Thus, when $x = 1$, the number of positions with no monad is the coefficient of $t^n/n!$ in $\exp(-t)/(1-t)$, or

$$n!\{1 - 1 + 1/2! - 1/3! + \ldots + (-)^n/n!\};$$

a result already known from another stand-point.*

And similarly, when $x = 2$, the number of positions with no duad is

$$n!\{1 - 1/2 + 1/2!\,2^2 - 1/3!\,2^3 + \ldots\},$$

the series containing $n/2 + 1$ or $(n+1)/2$ terms according as n is even or odd—a result possibly new.

*Chrystal, Algebra, vol. 2, p. 25; Whitworth, Choice and Chance.

Suppose, to change the mode of illustration, n letters written and their n envelopes addressed, and let the letters be put, at random, one into each envelope. The chance that in no case will each of two people get the other's letter is

$$1 - 1/2 + 1/2!\,2^2 - 1/3!\,2^3 + \ldots,$$

which, when $n = \infty$, is $\exp(-1/2)$.

When $x > n/2$, the number of positions with no x-ad takes the simple form $n!(1 - 1/x)$, so that the number of positions with such an x-ad is $n!/x$, a result which can also be proved directly from (4). For if we denote the left side of (4) by Σ_n, the number sought is

$$n!\,\Sigma_{n-x}/x,$$

and

$$\Sigma_{n-x} = 1.$$

From the scheme (5) it is easy to prove the formula which Rothe (loc. cit.) stated without proof, namely, that if u_n be the number of symmetrical positions,

$$u_n = u_{n-1} + (n-1)u_{n-2}.$$

For since we are to have only monads and duads, u_n is the coefficient of $t^n/n!$ in the power-series for $\exp\,(t + t^2/2)$, that is,

$$\exp\,(t + t^2/2) = \Sigma u_n t^n/n!,$$

whence, after differentiation,

$$(1 + t)\Sigma u_n t^n/n! = \Sigma u_n t^{n-1}/(n-1)!,$$

and the required result is obtained by equating the coefficients of $t^{n-1}/(n-1)!$

Again, to determine the number of distinct terms in a symmetrical determinant, we observe that $2^{n_3 + n_4 + n_5 + \cdots}$ positions give terms which are not distinct, inasmuch as we can reflect any cycle with regard to the diagonal without altering the term (Cauchy, Works, series 1, vol. 6, p. 98). That is, when $x > 2$ each x-ad in any given kind of position is to halve the number of positions of that kind. Thus the required number is the coefficient of $t^n/n!$ in

$$\exp(t + t^2/2 + t^3/6 + t^4/8 + \ldots),$$

that is, in

$$\exp(t/2 + t^2/4) \cdot \exp\{-\tfrac{1}{2}\log(1-t)\},$$

that is, in

$$\exp(t/2 + t^2/4)/\sqrt{1-t},$$

which is the result obtained by Cayley. See Salmon's Higher Algebra, art. 45 of the third edition.

HAVERFORD COLLEGE, PA.

NOTES.

A REGULAR meeting of the NEW YORK MATHEMATICAL SOCIETY was held Saturday afternoon, February 3, at half-past three o'clock, the president, Dr. McClintock, in the chair. The following persons having been duly nominated, and being recommended by the council, were elected to membership: Professor L. C. Walker, St. Lawrence University; Miss Ruth Gentry, Bryn Mawr College; Miss Frances Hardcastle, University of Chicago. Professor Fine made some remarks upon the continuity of the number system, and Dr. Fiske described several different demonstrations of Weierstrass's theorem that only algebraic and periodic functions can possess an algebraic addition theorem.

THE address of Professor Newcomb on "Modern mathematical thought," which appeared in the BULLETIN for January, has been published in full in *Nature* of February 1, 1894, pp. 325–329.

A PORTION of the mathematical models and charts exhibited at Chicago by the German universities was secured by the department of mathematics of Columbia College. This portion, the principal part of which came originally from the institute of technology at Munich, is illustrative of theory of functions, analysis situs, plane curves and their singularities, surfaces, their singularities and curvature, and line-geometry.