ON THE GENERAL TERM IN THE REVERSION OF SERIES.

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IN reverting the series

$$y = a_0 x + a_1 x^2 + a_2 x^3 + \dots \qquad [a_0 \neq 0]$$

it is usual to assume a development for x in the form

$$x = A_0 y + A_1 y^2 + A_2 y^3 + \dots,$$

and then to substitute, and equate coefficients of like powers, thus determining $A_0, A_1, \dots$ in succession.

This method does not give any observable law for the independent formation of the expression for the coefficient of a given power of y.

A different method, however, based on Lagrange's series, furnishes the desired general term.

The first equation may be written

$$a_0 x = y - a_1 x^2 - a_2 x^3 - \dots,$$

or

$$x = z + b_1 x^2 + b_2 x^3 + \dots, \quad = z + \phi(x),$$

where

$$z = \frac{y}{a_0}, \quad b_1 = -\frac{a_1}{a_0}, \quad b_2 = -\frac{a_2}{a_0}, \quad \dots,$$

and

$$\phi(x) = b_1 x^2 + b_2 x^3 + \dots;$$

whence, by Lagrange's series,

$$x = z + \phi(z) + \frac{1}{2!}\frac{d}{dz}[\phi(z)]^2 + \frac{1}{3!}\frac{d^2}{dz^2}[\phi(z)]^3 + \dots$$

in which

$$\phi(z) = b_1 z^2 + b_2 z^3 + \ldots,$$

$$\frac{d}{dz}[\phi(z)]^2 = 4b_1^2 z^3 + 5(2b_1 b_2)z^4 + 6(b_2^2 + 2b_1 b_3)z^5 + \ldots,$$

$$\frac{d^2}{dz^2}[\phi(z)]^3 = 6 \,.\, 5b_1^3 z^4 + 7 \,.\, 6(3b_1^2 b_2)z^5$$
$$+ 8 \,.\, 7(3b_1 b_2^2 + 3b_1^2 b_3)z^6 + \ldots,$$

$$\ldots\ldots$$

$$\therefore x = z + b_1 z^2 + \left(b_2 + \frac{4}{2!}b_1^2\right)z^3 + \left(b_3 + \frac{5}{2!} \,.\, 2b_1 b_2 + \frac{5 \,.\, 6}{3!}b_1^3\right)z^4$$
$$+ \left[b_4 + \frac{6}{2!}(b_2^2 + 2b_1 b_3) + \frac{6 \,.\, 7}{3!} \,.\, 3b_1^2 b_2 + \frac{6 \,.\, 7 \,.\, 8}{4!}b_1^4\right]z^5$$
$$+ \ldots$$
$$+ \left[b_{n-2} + \frac{n}{2!}(2b_1 b_{n-3} + 2b_2 b_{n-4} + \ldots)\right.$$
$$+ \frac{n^{2|1}}{3!}(3b_1^2 b_{n-4} + 6b_1 b_2 b_{n-4} + \ldots)$$
$$\left. + \frac{n^{3|1}}{4!}(4b_1^3 b_{n-5} + 12b_1^2 b_2 b_{n-6} + \ldots) + \ldots\right]z^{n-1}$$
$$+ \ldots\ldots;$$

wherein

$$n^{r|1} = n(n+1)(n+2)\ldots(n+r-1),^*$$

and the parenthesis that $\frac{n^{r|1}}{(r+1)!}$ multiplies, is

$$\Sigma\frac{(p+q+\ldots)!}{p!\,q!\ldots}b_i^p b_j^q \ldots,$$

in which

$$p + q + \ldots = r + 1, \quad \text{and} \quad pi + qj + \ldots = n - 2;$$

* In accordance with the notation

$$n^{r|d} = n(n+d)(n+2d)\ldots[n+(r-1)d].$$

that is to say, the *order* of this parenthesis in the letters $b_1, b_2, \ldots$ is $r+1$, its *weight* is $n-2$, and the numerical coefficients are those of the polynomial theorem.

In terms of the original letters $y, a_0, a_1, \ldots$ the series may be written in the more homogeneous form

$$x = \frac{y}{a_0} + \frac{y^2}{a_0^3}(-a_1) + \frac{y^3}{a_0^5}\left(-a_0a_2 + \frac{4}{2!}a_1^2\right)$$

$$+ \frac{y^4}{a_0^7}\left(-a_0^2a_3 + \frac{5}{2!}\cdot 2a_0a_1a_2 - \frac{5^{2|1}}{3!}a_1^3\right)$$

$$+ \frac{y^5}{a_0^9}\left[-a_0^3a_4 + \frac{6}{2!}a_0^2(a_2^2 + 2a_1a_3) - \frac{6^{2|1}}{3!}a_0(3a_1^2a_2) + \frac{6^{3|1}}{4!}a_1^4\right]$$

$$+ \quad \cdot \quad \cdot \quad \cdot$$

$$+ \frac{y^{n-1}}{a_0^{2n-3}}\left[-a_0^{n-3}a_{n-2} + \frac{n}{2!}a_0^{n-4}(2a_1a_{n-3} + \ldots)\right.$$

$$\left. - \frac{n^{2|1}}{3!}a_0^{n-5}(3a_1^2a_{n-4} + \ldots) + \ldots + \frac{n^{n-3|1}}{(n-2)!}(-a_1)^{n-2}\right]$$

$$+ \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

It will be noticed that the coefficient of $\dfrac{y^{n-1}}{a_0^{2n-3}}$ is now a homogeneous function of $a_0, a_1, \ldots$, of order $n-2$, weight $n-2$; and that the "polynomial coefficients" involved in such terms as $\dfrac{n^{3|1}}{4!}a_0^{n-6}(12a_1^2a_2a_{n-6} + \ldots)$ are to be chosen without reference to the exponent of a_0; while the latter exponent is related to the outside coefficient $\dfrac{n^{3|1}}{4!}$, by an obvious rule.

If $a_0 = 0$, let $a_{m-1}x^m$ be the first term in the given series, then the relation between x and y may be written in the form $x = f[z + \phi(x)]$, where $z = \dfrac{y}{a_{m-1}}$, and $f(z) = z^{\frac{1}{m}}$; hence Laplace's theorem gives a development for x that can be arranged in powers of $z^{\frac{1}{m}}$. As the general term now involves both n and m, the law is not so simple as in the case above, for which $m = 1$.