

## THE LUNAR THEORY.

*Traité de Mécanique Céleste.* Vol. III. Perturbations des planètes d'après la méthode de Hansen; théorie de la lune. By F. TISSERAND. Paris, Gauthier-Villars, 1894. 4to, pp. 424.

It is somewhat strange that a subject like the lunar theory, which has received so much attention since its first principles were given by Newton, should be allowed to pass its second centenary before the appearance of a treatise like the present one. Every writer on the subject up to the present time has given a theory from one point of view only—either his own or further developments of some previous one. It has been reserved for M. Tisserand to give us in a collected form the principal methods which have been used to obtain a practical solution of a problem which can be stated in the very simplest terms, but which is perhaps one of the most difficult to master in the whole range of applied mathematics. And its difficulties are not merely those of simple analysis. The results obtained have been generally considered of little value unless they either assist us in the solution or are of value in the practical applications or finally help towards giving a geometrical (intuitive) idea of the meaning of the various expressions arrived at. The geometrical representations have always been kept prominently in the foreground, perhaps owing to Newton's influence, whose idea of the variable ellipse, developing itself naturally from his methods, has been used almost without exception by every writer since his time. In the last few years some doubts have been raised as to whether the ellipse is the most valuable starting-point, and of these I shall speak later in connection with the works of Adams, Gylden, and Hill. At the same time it must not be forgotten that the complete problem of the motion of the moon is a single one only in the sense that the erection of a house is so—we might use perhaps one kind of material only which may suffice for every portion, but which would hardly produce the most finished results. And thus M. Tisserand in making a collection of methods is not merely giving us a series of disconnected solutions, but furnishes the materials for a building which completed would comprise parts of perhaps every one. This is well illustrated by Dr. Hill's contributions in recent times. He uses in giving a new method of initial treatment Euler's plan of dividing up the various classes of inequalities: for the class of inequalities depending on the figure of the earth Delaunay's method of the variation of arbitrary constants, and, for the inequalities produced by the motion of the

ecliptic, Laplace's method. In each of these cases there is a special advantage depending on the part of the problem to be solved.

From the time of Newton to the beginning of the present century it is to the French that we owe most of the advance made in the development of the theory. The magnificent genius of Newton in presenting all his discoveries in a geometrical form and absolutely covering up all the traces of the analysis by which he reached them seems to have raised a barrier against advance in England. His successors, unable to proceed on geometrical lines and gifted with obstinacy in keeping to his methods without recognizing what underlay them, made no progress. In fact the method of Newton which was constructed in order that he might make himself understood and also avoid controversy was a somewhat costly one for England. It fulfilled its design, but the cure became a new disease, and as such remained until Babbage and Peacock brought in the calculus methods, long familiar to their Continental contemporaries.

The history of the lunar theory after Newton gave the "Principia" to the world in 1684-6 embraces several periods. It was almost untouched for over half a century until Clairaut, d'Alembert, and Euler, within a few years of one another, published theories on the motion of the moon. In these we have practically the first attempts to apply analysis in the modern sense and to determine definitely the principal inequalities from elliptic motion. This constitutes the second stage. In the third we must place first Laplace, who extended the theory in every direction, taking into account such inequalities as might be produced by the action of the planets, and in particular assigning a reason for the secular acceleration of the moon's mean motion. It is well known now how theory explains only half this acceleration, and the curious accident of the first approximation apparently accounting for the whole of it is one of those points in scientific history of which instances are not wanting elsewhere. His successors Poisson, Plana, de Pontécoulant, Lubbock, and Damoiseau aided in carrying the theory further, and of these perhaps the most notable was de Pontécoulant, who gave a complete analytical treatment of the motion as affected by the sun.

To the fourth period must belong Hansen, Delaunay, and many other names of lesser note. The two mentioned gave entirely new methods of treatment, each of which stands to this day unrivalled for the particular object pursued. Hansen, though he worked out his theory only numerically, did so with a completeness and accuracy which has satisfied most of the needs even of the present time. On the other hand, Delaunay conceived the idea of a complete analytical development,

the immense labor of which occupied him twenty years, and as a single monument of patient labor it stands without parallel. Though some of the coefficients are not perfectly satisfactory, their general agreement with Hansen's is so good that most of them may satisfy our requirements for some time to come.

The fifth and last period was opened in 1877 by the publication by Dr. Hill of his two papers, "Researches in the lunar theory" and "The part of the motion of the Lunar Perigee which is a function of the mean motions of the Sun and Moon." A new lease of life was given to the whole subject when these appeared. Adams, who had a great aversion to publishing any results except such as were in a very complete state, gave the value which he had obtained for the similar part of the motion of the node, working on somewhat the same lines. Since that time most of the investigations, chiefly by Poincaré, have turned on the convergence of the series employed and the legitimacy of the methods used.

M. Tisserand's plan may be briefly sketched. The main portion of the book consists of accounts of the various methods adopted to solve the problem of three bodies only as far as it refers to the sun, earth, and moon. Under these different methods are given, as appears most convenient, those portions of the perturbations of the moon which proceed from other causes, as the action of the planets, the influence of the figures of the earth and moon. The natural result of such a treatment is a certain want of continuity which, to a student reading the subject for the first time, would be bewildering. But for such a reader the book cannot be intended. Not one of the least difficulties to be encountered is the meaning of the various constants in which the results of a given theory are finally expressed and their connection with those of other methods, and this difficulty is hardly touched upon at all. Again, questions of purely mathematical interest are entirely omitted. It is not intended by these remarks to attempt to throw any discredit on M. Tisserand's labors, but rather to show that there is ample scope for a treatise on different lines more adapted to the needs of a student.

Chapters I and II are devoted to a short study of the equation

$$\frac{d^2x}{dt^2} + x(q^2 + 2q_1 \cos 2t + 2q_2 \cos 4t + \dots) = 0,$$

$q, q_1, q_2, \dots$  being constants. The first chapter is not altogether clear. It is complicated by the fact that M. Tisserand deals with the equation in three cases, namely, when  $q_3 = q_4 = \dots = 0$ , when the number of quantities  $q$  is finite, and

lastly when their number is infinite. And in some of the discussions it is not easy to see whether the conclusions for one case apply to the others, or indeed which case is for the moment being treated. The equation is integrated and expansions are given for the results. The second chapter is devoted to the case when the number of quantities  $q_i$  is infinite, and in it the expansion of the infinite determinant is explained with admirable clearness.

In chapter III we are given a few details of Newton's method of treating the lunar theory. Here M. Tisserand, in setting forth a general idea of the method, seems to have preferred to follow the analytical translations, chiefly using that of Laplace. The effect is perhaps to do Newton an injustice. It is not made clear that Newton in the first instance worked out his results by some kind of analytical method, and that the feat of the "Principia" was the geometrical portion. We lose thus much of its pleasant antique flavor. And though we are given one or two examples of Newton's geometry it seems hardly necessary to develop the results analytically, at any rate, not in this chapter. Newton's objections to analysis were rather from the point of view of mental development than from any lack of appreciation of its value as a method of research. He valued highly the habit of mind and intellectual grasp obtained by a "picture view" of any problem. In this perhaps in later times we have been somewhat negligent, though Riemann and others have done much to bring it to the front again. Perhaps a selected portion of the "Principia" would still be the most valuable way of initiation into the lunar theory.

The theories of Clairaut and d'Alembert occupy chapter III, the greater portion of which is devoted to that of the former. Clairaut was the first to publish a method founded on the integration of the differential equations of motion. He employed as independent variable the true longitude of the moon, and he first showed that the observed motion of the perigee could be fully accounted for by the law of Newton only. D'Alembert's method is similar in principle. He observed that we can express all the perturbations by means of sines and cosines of four arguments. M. Tisserand shows us how doubts still existed as to the sufficiency of Newton's law in accounting for all the principal perturbations. Clairaut had in fact found the effect of introducing a term depending on the inverse cube when it occurred to him to go to a further approximation with the Newtonian law only, and the apparent discordance between theory and observation in the motion of the perigee was soon accounted for.

In chapters V and VI we have the two theories of Euler published in 1753 and 1772, the former with an appendix in

which he outlined other methods of treating the subject. It is this appendix which M. Tisserand uses, and in it the method of the variation of arbitrary constants makes its first appearance. In the second theory we find rectangular coördinates in use with the mean anomaly of the sun as independent variable.

The next chapter contains an exposition of Laplace's method of proceeding as given in the classic "*Mécanique Céleste*." Though the method of using the true longitude as independent variable simplifies the equations of motion and gives developments which are somewhat more convergent than those in terms of the time, it possesses a fatal defect due to the fact that a reversion of series is necessary in order to express the coefficients directly in terms of the time. To the extent of a small number of approximations this defect is not very evident. The only text-book in English, Godfray's "*Lunar Theory*," has adopted this method.\* There can be little doubt that, as far as the solar perturbations are concerned, it is practically useless where great accuracy is required. In this chapter Laplace's discovery of the secular acceleration is given a slight treatment in preparation for a full explanation and development later.

With Laplace's theory we are given those of Damoiseau and Plana, the former of whom used Laplace's method with numerical values, the latter giving algebraic results and substituting numbers only at the conclusion. A table is given of one hundred and sixteen of Hansen's coefficients in longitude, with columns denoting the differences between his results and those of Damoiseau's theory, Damoiseau's tables (1828), de Pontécoulant, Plana, and Delaunay. Prof. Newcomb's transformation of Hansen published in vol. I of the "*Astronomical Papers*" is used.

Then follows an interesting chapter on a variety of topics which have been treated more recently by Laplace's method. We have Gyldén's "intermediary" orbit where the equation solved in chapter I arises. Such an orbit is taken as funda-

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\* Owing to our method of measuring time we have become so accustomed to regard it as a fundamental conception which is not capable of being replaced by any other that there exists a real mental difficulty in considering the time as a function of the true longitude—a difficulty which scarcely appears when we consider the true longitude as a function of the time. And in working out the perturbations of the time when expressed in the former manner it is not easy to recognize their meaning. For this reason Godfray's text-book lies under a great disadvantage from a beginner's point of view. The analytical developments are not difficult, but until the reversion of series has been made it is not easy to understand the geometrical meaning of the various inequalities. For teaching purposes a theory like that of de Pontécoulant would seem to be much more advantageous.

mental in the same sense as the elliptic one is in most of the other theories. M. Tisserand rightly thinks that the introduction of elliptic functions is unnecessary, and he touches a question in this matter that is of vital importance as far as the best mode of expression in the lunar theory is concerned. Take, for instance, two different intermediary orbits, the first an ellipse and the second Dr. Hill's periodic curve, depending on the ratio of the mean motions ( $m$ ); the coefficients of these are functions of  $e$  and  $m$  respectively. Either is well and easily expressed, the first by means of elliptic or Besselian functions and the second by sines and cosines with coefficients which are series with a fair degree of convergence. Yet if we try to combine these so as to express the terms depending on both  $e$  and  $m$  by means of these functions we meet with many difficulties. To express the results as a function of these two curves gives an analytical expression which would be of little use owing to its complicated quality, and we fall back on the old method of using one or the other as a primary orbit. The question narrows itself then down to the best choice of a primary or "intermediary" orbit. These orbits have been given some attention by Gylden and Hill.

To illustrate Poisson's method, that of the variation of arbitrary constants, M. Tisserand finds the inequalities due to the shape of the earth, the secular motion of the node and perigee due to the action of the sun, the influence of the displacement of the ecliptic, and a few long-period inequalities. They are worked out in one or two cases with sufficient practical accuracy. Then follows a chapter on Lubbock and de Pontécoulant which gives the method of the latter. We are shown the complete first approximation into which no serious difficulties enter. Any trouble that the method entails enters in the higher approximations. It seems as if we might have had one or two classes of inequalities only worked out for a first approximation and the space thus saved devoted to a second approximation of one class, say that depending on  $m$  and  $e$  or  $m$  and  $\gamma$ .

In chapters XI and XII Delaunay's theory is very fully developed. M. Tisserand uses for this purpose the "principal function," and by its means avoids some of the tedious transformations of Delaunay. These can be also managed easily by the use of the calculus of variations, and perhaps the results of Art. 94 might have been obtained more briefly in this way. It may be mentioned that Hayward\* obtained the primary set of equations immediately from geometry.

An extremely interesting chapter follows devoted to the

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\* *Quarterly Journal of Mathematics*, vol. 3, p. 22.

secular acceleration of the moon's mean motion. In giving a slight sketch of its discovery by observation M. Tisserand remarks how narrowly Lagrange missed making the discovery which was reserved for Laplace. On the ancient eclipses some remarks are made which put their value in such a clear light that I believe it worth while to reproduce them. "If," he says, "we pay attention to the known fact that in every eclipse of the sun the regions of the earth covered by the pure shadow of the moon (regions of totality) form a very narrow band; that in a given place total eclipses of the sun are very rare (at Paris, for instance, one only has been visible during the eighteenth and nineteenth centuries); it will be understood that if we know positively that an eclipse of the sun has been observed at a given place at a period not exactly fixed but nevertheless comprised between limits not too large, we can deduce exact conclusions about the values of certain elements of the lunar orbit at this period, and in particular about the corresponding mean motion. With a small variation in this latter quantity, the given eclipse *calculated by the tables* might not have been total at this place. At the same time there is room for doubt in other directions, namely, the question of the unknown date, the want of definiteness in giving the place of observation, the vagueness of the record, and the fables which always tend to enter in these cases, so that we cannot always be certain if there really was a total eclipse. Also the errors of the tables play an important part, and we can understand from all these considerations that the method gives in reality less than it seems to promise."

In relating the discovery of Adams as to the inexactness of the previous theoretical values, M. Tisserand relates graphically how Plana, who was most directly concerned, first recognized his error and then found a new value different from his own old one and also from that of Adams; how Delaunay, Lubbock, Cayley, Hansen successively proved that Adams was right, and how since then the theoretical value has been calculated with all the exactness which we can ask for. He then gives Delaunay's calculations for finding the term in  $m^4$  by Poisson's method, and finally, what is perhaps as useful as all that precedes, tells what attempts have been made to account for the still unexplained portion.

In chapter XIV we are given in full the periodic solution as found by Hill which depends on the mean motions of the sun and moon only. No reference is made to the portion of his work on moons of other lunations which perhaps have a theoretical rather than a practical interest. (The numerical values found are given correctly with one exception which might cause a misapprehension. On page 270, equation (34), in the value of  $r/a_0 \sin(L - nt - \epsilon)$ , the constant term

unity should be omitted.) Then follows a chapter containing the investigations of the same writer on the Perigee of the Moon. The celebrated equation

$$\frac{d^2 W}{d\tau^2} + \Theta W = 0$$

has been given a definite geometrical meaning by Adams, in his lectures delivered in 1885 and since independently by Prof. G. H. Darwin in his lectures of this year. It will probably be published with the collection of papers left by Adams now in course of preparation for the press. Adams obtains the equation in the form

$$\frac{d^2 \omega}{dt^2} + P\omega = 0,$$

where  $\omega$  is the displacement from the primary orbit in a direction along the normal to that orbit. When  $\omega$  has been found, the displacement along the tangent is obtained by a simple integration.

The work of Adams on the motion of the node and his two theorems on the constant part of the inverse of the radius vector occupy another chapter, while a further chapter is devoted to an explanation, or rather a development, of Hansen's theory which is not easy to follow. The large number of symbols used necessarily complicates it, and matters are not improved by the recurrence in two or three cases of the same symbol with different meanings. The theory is, however, given very fully, and concludes with Prof. Newcomb's transformations for comparing Delaunay and Hansen as given in the "Astronomical Papers."\*

Chapter XVIII treats of the planetary inequalities in the motion of the moon, and is developed mostly according to M. Radau's method. The principal long-period inequalities are given and discussed with some detail.

The last chapter on the actual state of the theory of the moon will be found the most interesting of any in the volume. A large portion of it is devoted to the secular acceleration as shown by ancient eclipses, and naturally is chiefly based on Prof. Newcomb's great investigation. There are also in this chapter, as in other parts of the volume, some of M. Tisse-

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\* It would be difficult to find four consecutive volumes of any publication concerned with a single science with more valuable results than those published at Washington under the title of "Astronomical Papers prepared for the use of American Ephemeris and Nautical Almanac."



rand's own contributions, and a single reading will show that it is written by one who has not merely made a compilation of the works of others, but who possesses in addition that intimate acquaintance with the subject which is only obtained by those who have advanced it at least in some directions. Whatever the defects of the book may be, it will take a high rank amongst the many classic treatises on celestial mechanics. If we might venture to make a suggestion, it would be that a full alphabetical index of subjects and names be appended to the fourth volume, thus adding greatly to the value of the series as books of reference. The increasing labor of finding out what has been done in a subject makes such an index almost necessary.

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### BACHMANN'S THEORY OF NUMBERS.

*Die Elemente der Zahlentheorie.* Von PAUL BACHMANN.  
Leipzig, Teubner, 1892. 8vo, xii + 264 pp.

*"The most beautiful theorems of higher arithmetic have this peculiarity, that they are easily discovered by induction, while on the other hand their demonstrations lie in exceeding obscurity, and can be ferreted out only by very searching investigations. It is precisely this which gives to higher arithmetic that magic charm which has made it the favorite science of the first mathematicians, not to mention its inexhaustible richness, wherein it so far excels all other parts of pure mathematics."*—GAUSS.

Interest in the theory of numbers has, perhaps naturally enough, not always remained at the high-water mark indicated in the above encomium, though there is not wanting a list of illustrious names, reaching down to the present generation, of those who have made important contributions to the theory. Within the last decade there have appeared translations of classic works of the highest importance, one at least grown well-nigh inaccessible in the original, as well as quite a number of text-books on the subject. During the last three years there have appeared text-books, each with its own points of merit, in the languages English, French, and German.\* All these recent publications point to a strong interest in the theory of numbers now existent, and at the same time give promise that the increased facilities for study will give a fresh impetus to research in this field.

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\* Respectively by Mathews, Lucas, and Bachmann.