APOLAR TRIANGLES ON A CONIC.

BY PROFESSOR F. MORLEY.

§ 1. *Apolar Triads.*

Take two triangles, or point-triads, in a plane, say T and T', where (attaching complex numbers to the points in the usual way) T is t_1, t_2, t_3, and T' is t_1', t_2', t_3'. Take the polar pair of t_1' as to T, and the polar point of t_2' as to this polar pair, and let this polar point be t_3'. The relation thus imposed on T and T' is symmetric both as to the points T, the points T', and the two triads T and T'; it is, in fact,

$$s_1 s_2' - s_2 s_1' + 3(s_3 - s_3') = 0, \qquad (1)$$

where $s_1 = \Sigma t_\lambda$, $s_2 = \Sigma t_\mu t_\nu$, $s_3 = t_1 t_2 t_3$, and similarly for T'. Compare Salmon, Higher Algebra, § 151. Or in the symbolic notation, if T and T' are given by $a_t^3 = 0$, $a'_{t'}{}^3 = 0$, the relation is $(aa')^3 = 0$. The triads are now said to be *apolar*.

This covariant notion of apolarity, stated above for triads, is the natural extension of the notion of harmonic pairs, and can be immediately generalized, as is well known.

If the points lie on a line, we can deal with them projectively. Joining them to any point we have two apolar triads of a pencil. Pass a conic through the vertex of the pencil; the triads of rays cut out apolar point-triads of the conic. Express the co-ordinates of any point of the conic parametrically, say by

$$x : y : z = t^2 : t : 1;$$

then the parameters of the two triads of the conic obey the relation (1), inasmuch as the parameter t is proportional to the ratio in which a side of the triangle of reference is divided by the line joining the opposite vertex of the triangle to the point t of the conic. And in general this is sufficient justification for the interpretation of binary forms and their covariants by means of points of a conic.

Now if in the treatment of covariants by means of complex numbers (which I will call for shortness the inversive method) we restrict our view to points of a circle, any interpretation of a covariant (or rather of its vanishing), so obtained, can be at once stated also projectively for a conic. For if we take the circle as having the equation

$$xz = 1,$$

where x, z are conjugate complex numbers, this equation is a special form of the equation

$$xz = y^2$$

which we selected for the conic. In the transition from the inversive construction to the projective one, pairs of points inverse as to the circle must be replaced, for projective purposes, by the imaginary points where their line of symmetry meets the circle. In particular the centre of the circle and the point ∞ must be replaced by the circular points.

This identity of the results obtained so long as we restrict ourselves on the one hand to points on a circle and the pairs of inverse points, on the other to the single conic and auxiliary lines, is illustrated as far as concerns the cubic by comparing Beltrami's constructions for the covariants (for which I may refer to my article On the covariant Geometry of the Triangle, *Quar. Jour.*, vol. 25) with Salmon's Conics, note, p. 387 of the sixth edition. The identity is evident so far as it goes; but there is not in Salmon any projective construction for the polar pair of a given point. By drawing the canonical figure, in which the conic is a circle, the cubic is represented by an equilateral triangle, and the Hessian points are therefore the circular points, we can verify at once the following statement:—

Let t_1, t_2, t_3 be the fundamental points, j_1, j_2, j_3 the Jacobian points, x any other point of the conic. Draw xy_λ harmonic with $t_\lambda j_\lambda$; then the lines $t_\lambda y_\lambda$ meet at the pole of the line required. This pole lies on the Hessian line, so that the line required passes through the intersection of the three lines $\overline{t_\lambda j_\lambda}$.

The inversive point of view is taken in the special problem of the next section; the rest of the article is projective, but the statements are made for the point as primary element, and are not repeated in the reciprocal form. And in accordance with this one-sided mode of statement, the word triangle is often loosely used here as meaning point-triad.

§ 2. *Feuerbach's Theorem.*

Many of the proofs which have been given of this remarkable and familiar theorem employ the method of ordinary inversion; but, so far as I know, none of them have stated the covariant aspect of the matter. The theorem is, for my purpose, as follows:

From a triangle T, and an auxiliary point x, form a new triangle T_x by taking the harmonic of x as to each pair of points of T. From T_x and an auxiliary point y form in the same way a third triangle T_{xy}. Let x and y be inverse points as to the circumcircle (T) of T. Then the circles (T) and (T_{xy}) touch at a point t.

To reduce this statement to the usual form, take for x the centre of (T), and therefore for y the point ∞. Then T_x is a triangle whose sides touch (T), and the middle points of these sides make up T_{xy}.

It is natural to suppose that *the points x, y, t are apolar with T*; this I have to verify.

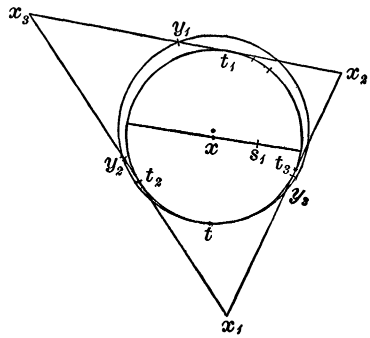

FIG. 1.

Taking $x = 0$, $y = \infty$, we have from (1)

$$t = s_2/s_1. \tag{2}$$

The harmonic of 0 as to t_μ and t_ν is

$$x_\lambda = 2t_\mu t_\nu/(t_\mu + t_\nu).$$

The harmonic of ∞ as to x_μ and x_ν is

$$\begin{aligned} y_\lambda &= \tfrac{1}{2}(x_\mu + x_\nu) \\ &= t_\lambda\left(\frac{t_\nu}{t_\lambda + t_\nu} + \frac{t_\mu}{t_\lambda + t_\mu}\right). \end{aligned}$$

Hence

$$\begin{aligned} y_\lambda - t_\lambda &= t_\lambda \cdot \frac{t_\mu t_\nu - t_\lambda^2}{(t_\lambda + t_\mu)(t_\lambda + t_\nu)}, \\ t - y_\lambda &= \frac{t_\mu t_\nu}{s_1}\left(1 - \frac{t_\lambda}{t_\lambda + t_\nu} - \frac{t_\lambda}{t_\lambda + t_\mu}\right) \\ &= \frac{t_\mu t_\nu}{s_1} \frac{t_\mu t_\nu - t_\lambda^2}{(t_\lambda + t_\mu)(t_\lambda + t_\nu)}, \end{aligned}$$

and therefore

$$\frac{y_\lambda - t_\lambda}{t - y_\lambda} = \frac{s_1 t_\lambda}{t_\mu t_\nu}. \tag{3}$$

Now introduce the condition that the origin is the centre of the circle (T); supposing this circle to have the radius 1, then

$$\left|\frac{y_\lambda - t_\lambda}{t - y_\lambda}\right| = \delta,$$

where $\delta = |s_1| =$ distance between the circumcentre and orthocentre of T.

Now if the stroke from t to y_λ meet (T) again at z_λ, we have, by elementary geometry (Euclid iii. 36),

$$\left|\frac{y_\lambda - t_\lambda}{t - y_\lambda}\right| = \left|\frac{z_\lambda - y_\lambda}{y_\lambda - t_\lambda}\right|.$$

Hence

$$\left|\frac{z_\lambda - y_\lambda}{t - y_\lambda}\right| = \delta^2,$$

that is, z_λ divides the stroke from t to y_λ in the ratio $1 - \delta^2 : \delta^2$. Since this ratio is independent of λ, the circles (T_{xy}) and (T) touch at t.

We have taken as origin the circumcentre. If also we take as real axis the line through the centroid, so that now s_1 is real and $= \delta$, then

$$1/t_1 + 1/t_2 + 1/t_3 = \delta,$$

and

$$s_2 = \delta s_3.$$

Therefore

$$t, = s_2/\delta, = s_3.$$

Thus the triangle T is given by the equation

$$t_\lambda^3 - t - \delta(t_\lambda^2 - t_\lambda t) = 0, \qquad (4)$$

and this for different values of δ represents an involution of triangles on a circle, any two of which are apolar, as appears on applying the condition (1). But it includes also triangles of which one point is on the circle, and the other two are inverse as to the circle, for when we write for t_λ and t their reciprocals, we get the same equation, and we can suppose either that t_λ is conjugate with $1/t_\lambda$, when $\lambda = 1, 2, 3$, or that t_μ is conjugate with $1/t_\nu$, in which case one pair of points is an inverse pair.

For three points on the circle we have, from $t = s_3$, the theorem that *the orientation of the radii to the three points is constant and equal to that of the radius to t*, the orientation being the sum of the angles made with the selected real axis.

§ 3. *Doubly Apolar Conics.*

An involution of triads of which each pair is apolar, such as was given by (4), may be called an *apolar involution.* The simplest view of it is that it consists of the polar triads of a self-apolar* tetrad. For, first, if we take the equations of any two triads, as

$$(a, b, c, d)(x, y)^3, \quad (b, c, d, e)(x, y)^3,$$

as in Salmon's Higher Algebra, § 203, then when the triads are apolar

$$ae - 4bd + 3c^2 = 0,$$

and the tetrad of which they are polar triads is self-apolar. And second, taking for the equation of the tetrad

$$x^4 + 4xy^3 = 0,$$

it is verified at once that any triad which is apolar with two polar triads is itself a polar triad.

Such an involution will now be considered in connexion with a special system of conics.

Inversively, we say that an equilateral triangle and the point ∞ are self-apolar. Hence projectively we can say that adjacent corners of a regular hexagon in a circle, and the circular points, are self-apolar points of the circle.

Now imagine a quilt formed of equal regular hexagons, say of side 1; draw circles round the hexagons, and take two visibly intersecting circles. Their four intersections are self-apolar points of each circle.

Referring to Salmon's Conics, p. 336, Ex. 3, we see that the intermediate invariants Θ and Θ' both vanish; for we have

$$r = r' = 1, \quad \alpha^2 + \beta^2 = 3.$$

Reye calls a conic in point co-ordinates and a conic in line co-ordinates apolar when the bilinear invariant—Θ or Θ', as the case may be—vanishes; hence our two circles are sufficiently characterized if we may say that they are doubly apolar.

Hence through four points—two real and two imaginary—two doubly apolar conics can be drawn; for these conics $\Theta = \Theta' = 0$. And it is easy to prove, conversely, that when $\Theta = \Theta' = 0$, the intersections of the conics are a self-apolar system on each. Cf. Clebsch-Lindemann-Benoist, vol. I. p. 375.

* The term *self-apolar* is used in preference to the clumsy word equianharmonic, which is, moreover, sometimes used with reference to any four points on a line and their projection.

§ 4. *Wolstenholme's Configuration.*

With two such conics (T) and (U), we can associate a third conic (V), which is equally the reciprocal of (T) as to (U), that of (U) as to (T), the envelope of lines divided harmonically by (T) and (U), and the locus of points divided harmonically by (T) and (U). Moreover, the relations of the three are entirely symmetrical. This system of three conics, of which each pair is doubly apolar, was studied by Wolstenholme, Problems, pp. 263–4; but his results are stated without reference to the theory of binary forms, and it is desirable to indicate the connexion.

The quilt arrangement is not so convenient, for the system of three conics, as the arrangement to which Wolstenholme's canonical equations lead. These equations are

$$x^2 + 2yz = 0, \quad y^2 + 2zx = 0, \quad z^2 + 2xy = 0,$$

for (T), (U), (V), respectively.

Taking the fundamental triangle equilateral and the co-ordinates areal (Fig. 2)*, these give three rectangular hyperbolas; the real intersections form a regular hexagon, and the real foci form an equal regular hexagon.

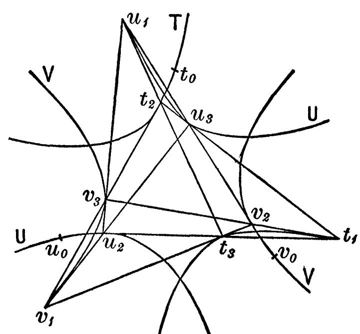

FIG. 2.

Selecting (T) as the conic on which apolar triangles are to be considered, we take for the co-ordinates of any point t on it,

$$x : y : z = t : -t^2 : \tfrac{1}{2}.$$

The intersections of (T) and (U) are points on (T) given by

$$t^4 + t = 0; \tag{5}$$

* How to do this appears from a remark of Clifford, Papers, p. 412.

these we take as the fundamental self-apolar tetrad on (T). Now taking any point t_0, its polar triad, t_λ, t_μ, t_ν, is given by

$$t_0(4t^3+1)+3t=0, \tag{6}$$

so that

$$\Sigma t_\lambda=0, \quad \Sigma t_\mu t_\nu = 3/4t_0, \quad t_\lambda t_\mu t_\nu = -1/4. \tag{7}$$

Rotating the figure round its centre through angles $2\pi/3$, the points t_α pass into the points u_α and v_α, where $\alpha = 0, 1, 2, 3$; the co-ordinates of all these points being given by the scheme:

$$\begin{array}{l} t_\alpha \text{ is } t_\alpha,\ -t_\alpha^2,\ \tfrac{1}{2}, \\ u_\alpha \text{ is } \tfrac{1}{2},\ t_\alpha,\ -t_\alpha^2, \\ v_\alpha \text{ is } -t_\alpha^2,\ \tfrac{1}{2},\ t_\alpha. \end{array}$$

Now the line joining t_μ and t_ν is

$$x(t_\mu+t_\nu)+y-2zt_\mu t_\nu=0,$$

or from (7)

$$-2xt_\lambda^2+2yt_\lambda+z=0,$$

and this is the tangent of (U) at u_λ.

So also the sides of the triangle u_1, u_2, u_3 touch (V) at v_1, v_2, v_3.

The triangle u_1, u_2, u_3 is harmonic as to (T). For two points x_1, y_1, z_1, and x_2, y_2, z_2 are harmonic as to $x^2+2yz=0$ when

$$x_1x_2+y_1z_2+y_2z_1=0;$$

and in this case the left side

$$\begin{aligned} &= \tfrac{1}{4} - t_\mu t_\nu(t_\mu+t_\nu) \\ &= 0. \end{aligned}$$

Thus Wolstenholme's main results are verified.

Next, *the three lines* $\overline{t_\lambda u_\lambda}$ *meet at the point* t_0, *with which we began.*

For the join of t_0 and t_λ is

$$x(t_0+t_\lambda)+y-2zt_0t_\lambda=0;$$

this passes through u_λ if

$$\tfrac{1}{2}(t_0+t_\lambda)+t_\lambda+2t_0t_\lambda^3=0,$$

which is equation (6).

Similarly, the join of t_0 and u_0 passes through the polar point of t_0.

The tangent at u_0 *meets* (T) *at the (imaginary) Hessian points of* t_1, t_2, t_3. For this tangent is

$$-2xt_0^2 + 2yt_0 + z = 0;$$

it meets (T) where

$$-2tt_0^2 - 2t_0t^2 + \tfrac{1}{2} = 0;$$

and the left is the Hessian of the left of (6).

The tangent at v_0 *meets* (T) *at the polar pair of* t_0. F this tangent is

$$x - 2yt_0^2 + 2zt_0 = 0;$$

it meets (T) where

$$t + t_0 + 2t^2t_0^2 = 0,$$

and the left is the second polar of t_0 as to the fundamental tetrad.

Lastly, *the lines* $\overline{v_0t_\lambda}$ *meet* (T) *again at the Jacobian of* t_1, t_2, t_3. For evidently $v_0, t_\lambda, v_\lambda$ are collinear; and v_λ is the pole of $\overline{t_\mu t_\nu}$ as to (T). Hence $\overline{v_0t_\lambda}$ and $\overline{t_\mu t_\nu}$ are harmonic as to (T); but harmonic lines meet a conic in harmonic pairs.

The configuration is determined when we take a conic and on it two apolar triangles. To return to the original case of a circle, and on it two triangles t_1, t_2, t_3, and t, 0, ∞, § 2, we project figure 2 so that the points where a tangent of (U) meets (T) pass into the circular points. (T) becomes the circle, (U) a parabola touching the sides of the system of triangles T, (V) a rectangular hyperbola on which lie all the points of the tangent triangles T_0.

§ 5. *The Complementary Line.*

In general a line which cuts two plane curves of degree 3, say $a_x^3 = 0$, $b_y^3 = 0$, in two apolar triads, envelops a curve of class 3, in Clebsch's notation $(abu)^3 = 0$. Compare Clebsch's Geometry, vol. I. p. 344 *et seq.* of the French edition.* Let the given cubics reduce to line-triads, and let these triads touch a conic and be apolar triads of that conic; then the class-cubic is clearly this conic and some complementary point. Reciprocally, then, in our case, where we begin with two

* Clebsch gives the method; the fact is stated in some lecture notes of Clifford, Works, p. 534. It appears that the equation by which Clifford defines apolar (or harmonic) triads should have only three terms on the left, as the second three terms are only a rewriting of the first three.

apolar point-triads of a conic, the locus of points which are divided apolarly by the two triads is the conic itself and some complementary line. It remains to identify this line.

In the study of harmonic pairs the degenerate case when one of the elements is arbitrary is of great use. Then we know the other elements all coincide. Here, in the study of apolar triads, *the Hessian pair of a triad and an arbitrary element are apolar with that triad.* For we know that the polar pair of an arbitrary point, as to a point-triad, is harmonic with the Hessian pair.

Now, taking the triangles t_1, t_2, t_3 and t, 0, ∞, where t is Feuerbach's point (§ 2), we know, first, that lines forming an equilateral triangle determine on the line infinity a triad whose Hessian pair is the circular points 0, ∞, so that three such lines on the one hand and an isotropic pair and an arbitrary line on the other cut the line infinity apolarly; and we know, secondly, from elementary geometry (*Quar. Jour.*, vol. 25, p. 190) that there are two points e, the lines from which to t_1, t_2, t_3 form a vanishing equilateral triangle—those points, namely, which have been called the equiangular points of the triangle. Hence the line-triads from e to t_1, t_2, t_3 and t, 0, ∞ are apolar; for the pencil is cut apolarly by the line infinity. The equiangular points do not lie on the circumcircle, hence *any point on the join of the equiangular points is divided apolarly by the two triads.*

Thus the complementary line is determined for this case. To pass to a covariant statement, we notice (*Q. J., loc. cit.*) that the line passes through the symmedian point of $t_1t_2t_3$, that is, through the pole of the Hessian line. Thus given two apolar triads on a conic, the points divided apolarly by them lie either on the conic itself or on the line through the poles of their Hessian lines, that is, on *the line which meets the conic at the Jacobian of the Hessians of the triads.*

In conclusion, it is hoped that the instance of Apolarity which has now been worked out may be useful to the student of Meyer's work, *Apolarität und Rationale Curven,* to which, above all, reference must be made for the projective development of the theory.

HAVERFORD COLLEGE.

AN INSTANCE WHERE A WELL-KNOWN TEST TO PROVE THE SIMPLICITY OF A SIMPLE GROUP IS INSUFFICIENT.

IN the December number of this journal (page 64, footnote) Professor Moore asks whether an instance is known where the test used by Klein in his "Vorlesungen über das