ARTHUR CAYLEY.

BORN AUGUST 16TH, 1821. DIED JANUARY 26TH, 1895.

"*My subject is the life of a great artist.*"

WHILE it is fitting that the BULLETIN of the American Mathematical Society should make mention of the passing away of professed mathematicians, the death of one so great as Arthur Cayley demands more than a slight notice. My willingness to undertake the task is due, not to any sense of fitness on my part, but to my intense admiration for his work and for his personality, and to the fact that for the last fourteen years I have been privileged to know him and experience his kindness.

The facts of his uneventful life are given by his friend, Dr. Salmon, in *Nature* for September, 1883. He was the second son of an Englishman, a merchant in Russia, and was born in England during a visit his parents paid to their home. In 1829 they returned to England to live, so Professor Cayley was English by education as well as by nationality. He entered Trinity College, Cambridge, in 1838, and in 1842 was Senior Wrangler and first Smith's Prizeman. He entered the legal profession, and practised until 1863, when he returned to Cambridge to hold the Sadlerian Professorship of Pure Mathematics ; this he held until his death. He has published about 800 papers, of which the first appeared in 1841; the last may easily turn out to be the one contained in the January number of the BULLETIN, for this was written December 18th, 1894. It would have been a cause of special regret to all interested in the American Mathematical Society had the pages of the BULLETIN contained no contribution from our most distinguished member.

At the time when Cayley entered Cambridge the mathematical curriculum was very different from the present one. Text-books were few and soon exhausted; but in this there was the advantage that so much the sooner was the student thrown into direct contact with the works of the great mathematicians. As regards the majority of students, it may have been that they had nothing to draw with, and the well was deep ; but for one like Cayley, to whom the most abstract language of mathematical analysis was as his mother-tongue, the well was as a spring bubbling up. Nor was he alone in this direct communication with the fountain-head ; his contemporaries in Cambridge were such men as Leslie

Ellis, Stokes, Adams, Thomson. The time too was favorable. Whether or no one accepts Bacon's dictum, that knowledge after it has been systematized is less likely to increase than before, it is plain that periods of growth, creative periods, alternate with quieter times of reflection and systematization, critical periods. Now that the lines of mathematical research are so many and various, this sharp alternation is less noticeable; but the time we are considering was distinctly a creative period. Analytical methods had just succeeded in displacing the fluxional and geometrical methods which had held undisputed sway in England from the time of Newton, cutting off English mathematicians from intercourse with the Continent of Europe, and exercising a blighting effect that is frequently referred to ; see for instance Hankel's Entwickelung der Mathematik:—"Die Engländer, bei denen bis vor Kurzem aus Pietät gegen ihren grossen Landsmann Newton die Mathematik völlig still stand." This change had been brought about by the efforts of Peacock, Herschel, and Babbage, and was an accomplished fact by 1830. The influence that made the change possible was felt as a general quickening all along the line ; among other manifestations of it we may note the founding of the Cambridge Philosophical Society in 1819, and the establishment in 1837 of the *Cambridge Mathematical Journal.*

As regards the mathematical literature that was available, the works of the great French analysts were already classics; Gauss, Poncelet, Möbius, Chasles, Steiner, Plücker, Jacobi, were in the height of their activity; Abel's life had actually ended, but the influence of his too-lately-recognized genius was just beginning to be felt. During the twenty years immediately preceding Cayley's graduation there had appeared (among other works) the *Propriétés projectives* of Poncelet, the *Barycentrische Calcul* of Möbius, the *Systematische Entwickelung* of Steiner, Jacobi's *Fundamenta Nova,* Chasles' *Aperçu Historique,* the *Analytisch-geometrische Entwickelungen,* the *System der analytischen Geometrie,* and the *Theorie der algebraischen Curven,* of Plücker. Apart from any conjectures as to what Cayley read first, we know from his own words, in papers published while he was yet an undergraduate and shortly after, that before 1843 he was familiar with the works of Laplace and with a wide range of memoir literature in the journals of Liouville and Crelle.

Some of the most striking of Professor Cayley's earlier papers deal with the subject of Elliptic Functions, which had been treated in two very different ways by Abel and Jacobi. For the work of both these mathematicians Cayley had a great admiration, which as regards Abel, who died in 1829, at the age of 26, was intensified by keen interest in the man himself. No one who attended Professor Cayley's lectures on Abel's Theorem, delivered in Cambridge in 1881, could fail to be struck by the note of personal regret with which Abel was referred to. Abel's great paper, neglected by the French Academy of Sciences for fourteen years, was finally published in 1841, and probably this had brought vividly home to Cayley the pathetic story of the great Norwegian. The functions discussed by Jacobi are singly-infinite products, those of Abel's papers are doubly-infinite products; Cauchy had shown that the properties of the Elliptic Functions could be deduced from their definitions by means of Jacobi's singly-infinite products; Cayley showed that Abel's doubly-infinite products define functions which also satisfy the equations that define Jacobi's Elliptic Functions. In the note to the re-issue of these papers (Nos. 24 and 25 in the Collected Papers) Cayley remarks that the need for this investigation existed at the time, though it has since been superseded by the work of Weierstrass.

Many of the early papers exhibit a wide acquaintance with the works of various writers; perhaps the one whose influence is most plainly to be discerned then and at a later time is Plücker. But this influence seems to have been exhibited more as an impulse starting Cayley's genius along certain lines, and providing him with preliminary subject-matter, than in any modifying or formative effect. In their conviction of the inherent identity of algebraic and geometric operations Plücker and Cayley were at one; in the marvellous richness and plenitude of their ideas, their creative power, they resembled one another; but the elaborated carefulness of Cayley's work, in lectures as in memoirs, contrasts with the hasty intuitiveness of Plücker, to which Clebsch bears witness; Cayley's extensive acquaintance with mathematical literature, and the breadth of his interests, with Plücker's very small knowledge of what his contemporaries were doing or had done; and, for a most striking difference, Plücker's style is so repulsive that probably very few know him at first hand.

Plücker's works on Plane Geometry, though published at intervals from 1828–1839, were evidently not well known to Cayley until after 1843, though he refers to special papers; it is plain that Chasles' Aperçu Historique was first known to him; for in No. 5, a classical paper of three quarto pages, originally published in 1843, he ascribes to Chasles the demonstration of Pascal's Theorem by means of the theorem that every cubic through eight of the intersections of two cubics passes through the ninth; this is however to be found in Plücker, Entwickelungen, I., pp. 266–7, antedating Chasles by nine years. Cayley reproduces the proof, with the remark that it ought to be generally known, and after proving the nine-points theorem, which had simply been enunciated by Chasles, he passes on to give the theorem known by his name, which limits the number of the points determining a curve of order r that can be chosen common to two curves of orders m and n. It is a matter of some historical interest that Clebsch, writing in December, 1871, speaking of this memoir, says that with it the algebraic side of this investigation can be regarded as finally closed. It was, however, reopened in 1881 (*Mathematische Annalen*, 1886) by Bacharach, who points out a limitation which, though referring to a group of special cases, is yet of sufficient importance to be incorporated in the theorem. The theorem on the intersections of curves is fundamental in the fabric raised up by Brill, Nöther, and others.

Accepting the broad division of mathematical thought (given in the notice of Clebsch, *Mathematische Annalen*, vol. 7) into two main lines,—the one concerning itself with the precise defining and justifying of the fundamental conceptions of the science, manifesting itself in investigations in the Theory of Numbers and the general Theory of Functions, connecting itself in the first instance with the names of Gauss and Dirichlet; the other accepting a certain small number of fundamental conceptions and working from these, thus producing modern algebraic and geometrical investigations, and to be traced to the influence of Jacobi and Plücker,—there is no difficulty in assigning Professor Cayley to the latter division. He himself stated that he had not been able to feel any special sympathy with Riemann's ideas, and this is plainly indicated in his address as President of the British Association for the Advancement of Science, 1883, where after a mere passing reference to the general subject he left it and discussed *con amore* the special classes of Elliptic and Abelian Functions to which so much of his thought had been devoted. I never heard him assign any reason for this feeling, nor can I find in his papers anything to explain it; his interests were certainly wide enough to embrace this field, and it is well known with what open-minded warmth he welcomed new thoughts. Probably the explanation will be given by some of his intimate friends.

In speaking of Cayley's work, and attempting to give a reason for ranking him first among the mathematicians of the age, it is easiest to speak of his work on Invariants, for the reason that here we can point to a single theory, by universal consent due to Cayley, which as a matter of general knowledge has dominated one half of mathematics ever since its first acceptance. Every entity is endowed with properties inherent in itself, which cannot be affected by any change of base on our part, so long as this change of base is no more than a change of what we assume as fixed, or of our standards of measurement, or (speaking for the sake of definiteness of such entities only as are contained in a plane) of the angle under which we view the entity. In its most simple geometrical form this is identical with the theory of conical projection, and the geometrical permanence here exhibited was well known; in the algebraic form the theory is that quantics homogeneous in any number of variables have properties depending on certain allied expressions that are not affected by linear transformations. Some of these allied expressions had of necessity been noticed, since it is impossible even to solve an equation with literal coefficients without coming upon them; the great merit of Cayley's work is the recognition of this permanence of relation, this Invariance, as an attribute to be recognized for its own sake, and investigated *a priori*. The importance in result is twofold: it gives to algebraic forms a substantive existence, and it leads of necessity to the conception of groups of transformations. So thoroughly has this theory permeated all mathematical work, that it is hard to realize that the jubilee of the discovery occurs only this year, Cayley's original paper, "On the Theory of Linear Transformations," having been published in 1845.

Another conception that can be mentioned as entirely due to Cayley is that of the Absolute. The existence of the circular points in a plane was known; the fact that the angle between two lines was really a measure of their relation to the circular points had been shown by Laguerre. It is easy to show that in a plane the analytical expression of any property depending on exact measurements really expresses a relation of the figure to the circular points, and that the relations expressed are simply such relations of position as can be held by the figure to a conic; and that similarly in space of three dimensions the relations are such as can be held to a quadric surface. Cayley's conception is then that instead of discussing the metric properties of a system directly, the relations of position with respect to a quadric surface may be considered, the metric properties being deduced by properly choosing the quadric. To quote his own words, from the Sixth Memoir upon Quantics, 1858,—"The theory in effect is, that the metrical properties of a figure are not the properties of the figure considered *per se* apart from everything else, but its properties when considered in connection with another figure, the Absolute." This theory of Cayley's deals primarily with Euclidean space, giving a more general principle for dealing with measurements. If however the fundamental quadric be differently chosen, we are led to the theories of non-Euclidean space. Professor Cayley's concern was not with these; his object was to obtain a general conception of the nature of measurement in space, and thus to subject metric properties to descriptive treatment. The effect of this idea on the development of mathematical thought can be well seen, to name only one instance, in Professor Klein's "Review of Recent Researches in Geometry," especially in § 2. (See vol. 2 of this BULLETIN.)

In other matters it is not so easy to give an account of what Cayley has done; it is hardly possible to point out a region, complete in itself, as his; his work is interwoven with that of his time, and not until the history of the development of mathematics during the present century comes to be written by a competent hand will it be possible to form a just estimate of his overwhelming importance. Apart from his minute and extensive series of investigations in solid geometry, algebra, and questions of analysis, there are numerous detached papers that have put a question in its true light, and so served as starting-points for the researches of other mathematicians. For instance, Cayley by his investigations in the theory of Higher Singularities prepared the way for more recent discussions of the question. (See Nos. 343, 374, published in 1864 and 1866.) Plücker, using the dual generation of the curve by a moving point and a moving line, had arrived at the conclusions that a point at which two branches of a curve have contact of order g must be regarded as $g+1$ double points, and that there are also $g+1$ double tangents; and that a cusp of the first species is to be counted as one and a half nodes, a cusp of the second species as two and a half. While this result, as a numerical one, is correct to a certain extent, yet as a geometrical view it is essentially incorrect, for it assumes that cusps of different species are different in nature. Cayley showed that the apparent difference is due to the fact that the cusp of the second species is a cusp with an adjacent node; and that it contains also a double tangent, in addition to the inflexional tangent recognized by Plücker; he shows in general what is the nature of the branches that may occur, and explains how the number of ordinary cusps and nodes included in any singularity can be determined; he considers also the determination of the inflexional and double tangents. He does not consider the grouping of the double points into multiple points, nor the possibility of separating the component multiple points from one another; for present use these are important questions, and the process of Cayley, proceeding by expansions, is to a great extent replaced by the successive reduction by means of quadratic transformations, a process which reveals not only the number but also the inner arrangement of the elements of the singularity. But however the work be arranged, to Professor Cayley is due the fundamental theorem that any singularity that can occur on an algebraic curve can be fully accounted for by a definite and determinable number of cusps and nodes, inflexional and double tangents.

Moreover, to Professor Cayley we owe a debt not only as an explorer, but as one who followed out of set purpose the tracks often so faintly indicated by those that first passed that way. Where he found a mere mountain-path he left a broad road, steep it may be, but plainly marked out, and with no hidden pitfalls or lurking dangers. While it is his creative power that excites our admiration, it is this determination that others shall be enabled to follow that endears him to us. This he has done consistently for his own work as well as for that of others, and it is partly due to this that so large a proportion of the results and processes of his earlier papers have become the commonplaces of text-books. To this however another cause has powerfully contributed. With the single exception already noticed, Cayley was emphatically in touch with the mathematics of the day; his omnivorous reading, his rapid assimilation of new thoughts, his readiness to believe in others, his frank and genial response to any appeal for help, all combined to make him the ideal head of the mathematical world.

All this the historian of a century hence may know; it is written in the unintentional references of his contemporaries as well as in their deliberate tributes. But any sketch of Professor Cayley is self-condemned if it leaves out of account the childlike purity and simplicity of his nature, the entire freedom from the professional touchiness on the score of priority to which mathematicians are as liable as other men. He was ever ready to say what he was working at, to indicate the lines of thought, to state what difficulties he was encountering. It is not every mathematician that will lecture to a class of specialists on the incomplete investigation of the night before, and end up with the remark, obviously genuine, "Perhaps some of you may find this out before I do." To this engaging sweetness of his nature must be ascribed also the urbanity of manner which, as one of its many manifestations, would not allow him to leave unacknowledged the slightest mathematical paper sent to him. His little note of thanks would frequently contain a few words of comment, just enough to show that he had made time to glance over the paper. Anecdotes illustrating his absorption in his work, his disregard for appearances, are well known to all that knew him, and will doubtless be forthcoming in the reminiscences of his contemporaries; but his greatness and his simplicity cannot be enshrined in anecdotes.

My own acquaintance with Professor Cayley dates back to

1880, the first of the four years in which I attended his lectures. His subjects in the different years were Modern Algebra, the Abelian Functions, the Theory of Numbers, the Theory of Substitutions, the Theory of Seminvariants. To my great regret I never had the opportunity of hearing him lecture on Geometry. His lectures differed strikingly from his memoirs in that the subject was presented in less synthetic style. It was a recognized fact that he lectured on what he himself was working out at the time, and consequently his class was privileged to obtain some insight into the workings of his mind. The necessary connection of the different ideas presented was perhaps not always obvious at the time—at any rate to some that were not very familiar with what had already been done in that line; but it is my belief that Cayley had the power of communicating to his hearers, for the instant at any rate, something of his own clear vision of the essential nature of the truth he was expounding.

He has been spoken of as the greatest living master of algebra. In an interesting estimate of his genius and personality, contained in the *Bulletin des Sciences Mathématiques* for 1893, Darboux quotes and endorses the opinion of Bertrand, who compares him with Euler, instancing as one reason among many his extraordinary power and certainty in analysis. Some, passing lightly over the geometrical side of his investigations, and perhaps dazzled by his supremacy over the rest of the world in analysis, appear inclined to consider him only under this aspect. Professor Klein, in his Evanston Lectures, dividing mathematicians into logicians, formalists, and intuitionists, classes him as a formalist, a term not altogether pleasing, inasmuch as it conveys a shade of reproach, but which is immediately explained by the speaker. That Cayley did "excel in the skilful formal treatment of a given question, in devising for it an algorithm," is beyond question; but I doubt whether there will be equally general agreement in the statement that he excelled "mainly" in this. It has always seemed to me that while Cayley's processes were algebraic, since the language of algebra was simpler to him than the ordinary language of words, the color of his thought was essentially geometrical. But algebraic and geometric operations are so absolutely identical in his work, that the passages which convey to one mind the idea of a geometrical substratum may very well convey to another the idea that nothing existed for Cayley but the analysis. It is curious sometimes to note how little he realizes what a barrier extensive analytical operations are to the progress of many students. I have lying before me a note, dated December 1894, in which he urges his view that a complete knowledge of in-