ON THE INTRODUCTION OF THE NOTION OF HYPERBOLIC FUNCTIONS.*

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The difficulties in the way of a satisfactory geometrical deduction of the fundamental formulae of the hyperbolic functions seem to be due to the lack of a definition of these functions which shall be independent of the particular position of the argument area. A general definition of this kind can, however, readily be found in terms of the ratios of certain areas, instead of lines. From this definition the addition-theorem and other characteristics can be easily deduced by the methods of analytic geometry; and the definitions hold, furthermore, not merely for the rectangular, but for any hyperbola.

I. The circular functions. In order to bring out clearly the analogy with the circular functions, I will first indicate briefly how the latter would be defined according to this method.

In a circle of radius \( a \) (Fig. 1) let \( \phi \) be the angle between the radii \( OP \) and \( OQ \), and let \( OP' \) be drawn perpendicular to

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$OP$. The following areas are either well known or easily found (the sign $\triangle$ denoting triangle):

\[
\text{sector } OPQ = \frac{1}{2}a^2\phi, \quad \triangle OPQ = \frac{1}{2}a^2 \sin \phi, \\
\triangle OPQ' = \frac{1}{2}a^2 \cos \phi, \quad \triangle OPP' = \frac{1}{2}a^2,
\]

from which follow immediately:

\[
\phi = \frac{\text{sector } OPQ}{\triangle OPP'}, \quad \sin \phi = \frac{\triangle OPQ}{\triangle OPP'}, \quad \cos \phi = \frac{\triangle OPQ'}{\triangle OPP'}.
\]

Similarly, if the tangent at $Q$ meet $OP$ in $T$ and $OP'$ in $T'$, it is not difficult to show that

\[
\tan \phi = \frac{\triangle OTQ}{\triangle OPP'}, \quad \ctn \phi = \frac{\triangle QT'}{\triangle OPP'},
\]

\[
\sec \phi = \frac{\triangle OTP'}{\triangle OPP'}, \quad \csc \phi = \frac{\triangle TQ}{\triangle OPP'}.
\]

Now, these formulae might be taken as definitions of the argument $\phi$ and of its various functions. They can then be immediately extended to any ellipse, the only modification necessary being that $OP$ and $OP'$ shall be conjugate semidiameters. The area of the triangle $OPP'$ is then $= \frac{1}{2}ab$, and $\phi$ is equal to the difference between the eccentric angles $P$ and $Q$.

II. Definition of the hyperbolic functions. Since of two conjugate diameters only one meets the hyperbola in real points, the conjugate hyperbola must be employed also, and $P'$ is the point where the diameter conjugate to $OP$ meets the conjugate hyperbola. We shall then define the argument $u$ and its functions in strict analogy with the preceding results, as follows (see Fig. 2):

\[
u = \frac{\text{sector } OPQ}{\triangle OPP'}, \quad \sinh u = \frac{\triangle OPQ}{\triangle OPP'}, \quad \cosh u = \frac{\triangle OPQ'}{\triangle OPP'},
\]

etc.

If now the hyperbola be referred to its principal axes as axes of coördinates, its equation may be written

\[
x^2 - \frac{y^2}{b^2} = 1.
\]
Let \( x_1, y_1 \) be the coordinates of \( P \), and \( x_2, y_2 \) those of \( Q \). Then the coordinates of \( P' \) will be \( \frac{ay_1}{b}, \frac{bx_1}{a} \), and the area of the triangle \( OPP' \) is equal to \( \frac{1}{2}ab \). The definitions of \( \sinh u \) and \( \cosh u \) become

\[
\sinh u = \frac{x_1y_2 - x_2y_1}{ab}, \quad \cosh u = \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2}.
\]

(2)

Interchanging the coordinates of \( P \) and \( Q \), we have

\[
\sinh (-u) = -\sinh u, \quad \cosh (-u) = \cosh u.
\]

(3)

Also, if \( P \) and \( Q \) coincide, \( u = 0 \) and

\[
\sinh 0 = 0, \quad \cosh 0 = 1.
\]

(4)

Let \( u_1 = \frac{\text{sector } OAP}{\triangle OPP'} \) and \( u_2 = \frac{\text{sector } OAQ}{\triangle OPQ} \), so that \( u = u_2 - u_1 \). The coordinates of \( A \) being \( a, 0 \), we have

\[
\sinh u_1 = \frac{y_1}{b}, \quad \cosh u_1 = \frac{x_1}{a}; \quad \sinh u_2 = \frac{y_2}{b}, \quad \cosh u_2 = \frac{x_2}{a}.
\]

(5)

Comparing with (1), we see at a glance that

\[
\cosh^2 u_1 - \sinh^2 u_1 = 1,
\]

(6)
and that this result is general may be instantly verified by (2), for
\[
\cosh^2 u - \sinh^2 u = \frac{(b^2 x_1 x_2 - a^2 y_1 y_2)^2 - a^2 b^2 (x_1 y_2 - x_2 y_1)^2}{a^4 b^4} = 1.
\]

III. The Addition-Theorem. Substituting (5) in the definitions of (2), we have
\[
\begin{align*}
\sinh (u_2 - u_1) &= \sinh u_2 \cosh u_1 - \cosh u_2 \sinh u_1, \\
\cosh (u_2 - u_1) &= \cosh u_2 \cosh u_1 - \sinh u_2 \sinh u_1.
\end{align*}
\]
Writing \(+ u\) instead of \( - u\), we have by (3)
\[
\begin{align*}
\sinh (u_2 + u_1) &= \sinh u_2 \cosh u_1 + \cosh u_2 \sinh u_1, \\
\cosh (u_2 + u_1) &= \cosh u_2 \cosh u_1 + \sinh u_2 \sinh u_1.
\end{align*}
\]
The generality of these formulae is easily verified in the manner just exemplified in formula (6).

IV. It is clear from (5) that the definitions we have given reduce to the ordinary form for the special position there considered. It remains to be shown that in the form given \(\sinh u\) and \(\cosh u\) are really functions of \(u\) alone. To this end let us choose the asymptotes as axes of coordinates. Let \(\alpha\) and \(\beta\) be the coordinates of any point, and let \(\omega\) be the angle between the asymptotes; the equation of the hyperbola is then
\[
\alpha \beta = \frac{1}{2} ab \csc \omega.
\]
The coordinates of \(P\) being \(\alpha, \beta, = OL\) and \(\beta, = LP\), those of \(Q\) being \(\alpha,\) and \(\beta,\) those of \(P'\) will be \(\alpha,\) and \(\beta,\), and we have
\[
\begin{align*}
\sinh u &= \frac{\alpha \beta_1 - \alpha_1 \beta_2}{ab \csc \omega}, \\
\cosh u &= \frac{\alpha \beta_1 + \alpha_1 \beta_2}{ab \csc \omega},
\end{align*}
\]
or, if we write \(\frac{\alpha^2}{\alpha_1} = \lambda\) and therefore \(\frac{\beta_1}{\beta_1} = \lambda^{-1}\),
\[
\begin{align*}
\sinh u &= \frac{1}{2}(\lambda - \lambda^{-1}), \\
\cosh u &= \frac{1}{2}(\lambda + \lambda^{-1}).
\end{align*}
\]
If we now apply to the plane the linear transformation

\[ \alpha' = k\alpha, \quad \beta' = k^{-1}\beta, \quad (12) \]

the hyperbola is transformed into itself. The points \( P \) and \( Q \) are moved to any arbitrary position on the curve, but the ratio \( \alpha : \alpha \) is unaltered. It is easily shown that in this transformation the area of any triangle, and hence the area of any figure in the plane, is unchanged; so that \( u \), sinh \( u \) and cosh \( u \) are unchanged. They are therefore all functions of the ratio \( \alpha : \alpha \), alone. Hence sinh \( u \) and cosh \( u \) are functions of \( u \) alone, and the definitions here given are a proper generalization of the usual definitions.

V. The exponential formula. The sector \( OPQ \) may be regarded as the limit of a circumscribed polygon and hence \( u \) may be regarded as the limit of the sum of a series of hyperbolic sines. To make each of the terms in this series equal, we have evidently only to put \( \lambda = \rho^n \), where \( n \) may be any whole number. Then, writing \( \ln \) briefly for \( \lim \),

\[ u = \frac{1}{2} \lim n(\rho - \rho^{-1}) = \lim \frac{n}{2}(\lambda^n - \lambda^{-n}) \quad (13) \]

\[ = \lim \frac{\lambda^n - 1}{2n} \lambda^{-\frac{1}{n}}, \text{ as } n \text{ increases indefinitely.} \]

The limit of \( \lambda^{-\frac{1}{n}} \) is equal to 1, and the limit of \( \frac{\lambda^n - 1}{2n} \) is the natural logarithm of \( \lambda \). Hence \( \lambda = e^n \). Introducing this value of \( \lambda \) in (12), we have

\[ \sinh u = \frac{1}{2}(e^u - e^{-u}), \quad \cosh u = \frac{1}{2}(e^u + e^{-u}). \quad (14) \]

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