In any case where Lindstedt series are applicable there are no asymptotic solutions, and, where there are asymptotic solutions, Lindstedt's series would be illusory.

We owe much to M. Poincaré for having commenced the attack on this class of questions. But the mist which overhangs them is not altogether dispelled; there is room for further investigation.

KRONECKER'S LINEAR RELATION AMONG MINORS OF A SYMMETRIC DETERMINANT.

BY PROFESSOR HENRY S. WHITE.

AMONG the minors of any determinant there exist well-known identical relations; those of lowest order, the quadratic relations, being readily obtained by the expansion of a determinant in which at least one pair of rows or of columns are identical. If, however, the original determinant is symmetrical, there are identities of a lower order than the quadratic, the linear identities first formally noticed by Kronecker in 1882.* These linear relations, published with no hint as to the manner of their discovery, are suggestive of a certain formula in such constant use as to have become a commonplace in the transformations of the Theory of Invariants of linear substitutions. The latter formula, however, relates to products of two determinant-factors, while Kronecker's is linear; but the latter uses double indices for the constituents, and herein lies the resemblance. By virtue of the ordinary process of multiplication of two determinants, Kronecker's theorem is easily proved to be a consequence from the other identity. Both are equally general, hence it seems likely that the earlier may have been the source of the later. This theory I will develop inductively, using for the sake of brevity determinants of three rows, and obtaining a typical linear relation among three-rowed minors of a six-rowed symmetric determinant.

Form an array of three rows of six constituents each:

$$\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \end{matrix}$$

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* *Sitzungsberichte der Berliner Akademie*, 1882, p. 824. See also C. Runge: Die linearen Relationen zwischen den verschiedenen Subdeterminanten symmetrischer Systeme. *Jour. für r. u. a. Math.*, vol. 93 (1882), pp. 319–327.

Denote the determinant of any three columns in this array by the three indices in proper order, as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1\,2\,3).$$

The identity employed in transforming invariants will then be written as follows:

(1) $(1\,2\,3)\,(4\,5\,6) \equiv (1\,2\,4)\,(3\,5\,6) + (1\,2\,5)\,(4\,3\,6) + (1\,2\,6)\,(4\,5\,3).$

The correctness of this is seen upon developing in minors of the first three, and of the last three rows the vanishing determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ 0 & 0 & a_3 & a_4 & a_5 & a_6 \\ 0 & 0 & b_3 & b_4 & b_5 & b_6 \\ 0 & 0 & c_3 & c_4 & c_5 & c_6 \end{vmatrix} \equiv 0.$$

Multiplying the determinants of each pair in (1), column by column, we have in each product-determinant constituents all of the same form, which can be abbreviated thus:

$$a_1a_4 + b_1b_4 + c_1c_4 = 1,\,4,$$

or

$$a_ia_k + b_ib_k + c_ic_k = i,\,k.$$

Identity (1) will thus become:

$$(2)\quad \begin{vmatrix} 1,4 & 2,4 & 3,4 \\ 1,5 & 2,5 & 3,5 \\ 1,6 & 2,6 & 3,6 \end{vmatrix} \equiv \begin{vmatrix} 1,3 & 2,3 & 4,3 \\ 1,5 & 2,5 & 4,5 \\ 1,6 & 2,6 & 4,6 \end{vmatrix} + \begin{vmatrix} 1,4 & 2,4 & 5,4 \\ 1,3 & 2,3 & 5,3 \\ 1,6 & 2,6 & 5,6 \end{vmatrix} + \begin{vmatrix} 1,4 & 2,4 & 6,4 \\ 1,5 & 2,5 & 6,5 \\ 1,3 & 2,3 & 6,3 \end{vmatrix}$$

We may observe, from the definition of the symbol, that $i, k = k, i$. Accordingly if the 36 quantities i, k be arranged as a six-rowed determinant, in the order denoted by their double indices, that determinant will be symmetric; that is, if

$$A_{ik} = A_{ki} = (a_ia_k + b_ib_k + c_ic_k),$$

then, (3) $$D_6 = |A_{ik}|(i,\,k = 1,\,2,\,3,\,4,\,5,\,6)$$

will be a symmetric determinant, of which the determinants in (2) above will be third minors.

The identity (2) *is a Kronecker relation among three-rowed minors of the six-rowed symmetric determinant* D_6.

Such relations are thus derived for certain symmetric six-rowed determinants. D_6, however, is of a highly specialized type; it is the discriminant of the sum of three 6-ary squares. Is the relation (2), established for this special type, valid for all symmetric determinants of six rows? It is; for it involves no constituent from the principal diagonal, so that the 18 parameters of our 6×3 array are available for representing the 15 constituents $\left(\frac{6 \cdot 5}{1 \cdot 2} = 15\right)$, lying outside the principal diagonal of any given symmetric 6-rowed determinant.

The same consideration can be relied upon in adapting this proof to any Kronecker relation among m-rowed minors of a $2m$-rowed symmetric determinant, since always

$$\frac{2m(2m-1)}{2} < 2m^2.$$

For convenience of reference, I subjoin the general form of this relation as enunciated by Kronecker, *loc. cit.:*

$$|a_{gh}| = \sum_r |a_{ik}|,$$

$$\left(\begin{matrix} g=1, 2, \cdots, m; & h=m+1,\ m+2, \cdots, 2m \\ i=1, 2, \cdots, m-1, r; & k=m+1,\ m+2, \cdots, r-1,\ m,\ r+1 \cdots 2m \end{matrix}\right).$$

NORTHWESTERN UNIVERSITY,
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ON THE LISTS OF ALL THE SUBSTITUTION GROUPS THAT CAN BE FORMED WITH A GIVEN NUMBER OF ELEMENTS.

BY DR. G. A. MILLER.

T. P. KIRKMAN published in 1863 the first extensive list of all the transitive substitution groups that can be formed with a given number of letters. A number of interesting facts are associated with this list. Before entering upon its discussion we shall give a brief account of the more important direct steps towards the formation of such lists.

Paolo Ruffini published a work* in 1799 in which we do

* The complete title of this work in two volumes is, "*Teoria generale delle equazioni, in cui si dimostra impossibile la soluzione algebraica delle equazioni generali di grado superiore al quarto, di Paolo Ruffini,*" Bologna, 1799.